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INTEGRAL CALCULUS. DIFFERENTIAL EQUATIONS. SERIES (ІНТЕҐРАЛЬНЕ ЧИСЛЕННЯ. ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ. РЯДИ)

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Викладаються основні поняття теорій невизначеного, визначеного і подвійного інтеґралів, звичайних диференціальних рівнянь першого і другого порядків та нормальних систем диференціальних рівнянь. Докладно розглядаються приклади розв’язання типових задач. Всі основні розділи супроводжуються англо-українсько-російськими термінологічними словниками. Дано завдання для самостійного розв’язання.

Велику допомогу в створенні посібника надали автору студенти факультету економіки і менеджменту ДонНТУ Мамічева В., Маринова К., Бородина Ю., Бердянська В., Костюк О., Полєнок Т., Прокопенко О., Рева О. , Фролофф Г. (впорядкування лекційних конспектів, редагування англомовного тексту, робота над термінологічним словником). Слід особливо відзначити роботу Галі Фролофф, яка ретельно перевірила всі математичні викладки, повторно розв”язала всі приклади і допомогла значно покращити текст посібника. Суттєвий внесок в написання посібника внесла старший викладач Слов”янського педагогічного університету Косолапова Н. В. (підготовка ілюстративного матеріалу, робота над англо-українсько-російським термінологічним словником). Всім своїм помічникам автор висловлює щиру подяку.

Для студентів і викладачів технічних вузів.

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INTEGRAL CALCULUS

LECTURE NO. 19. PRIMITIVE AND INDEFINITE INTEGRAL

POINT 1. PRIMITIVE

POINT 2. INDEFINITE INTEGRAL AND ITS PROPERTIES

POINT 3. INTEGRATION BY SUBSTITUTION (CHANGE OF A VARIABLE)

POINT 4. INTEGRATION BY PARTS

POINT 1. PRIMITIVE

The major problem of the differential calculus: to find the derivative $f'(x)$ or the differential $df(x) = f'(x)dx$ of a given function $f(x)$.

The major problem of the integral calculus is inverse one**:** to find a function $F(x)$ knowing its derivative $F'(x) = f(x)$ or its differential $dF(x) = F'(x)dx = f(x)dx$.

Ex. 1. Find the equation of the curve through the point $A(2; 3)$ such that at any point $M(x; y)$ of the curve the slope of the tangent is x^2.

Let $y = y(x)$ be a sought equation of the curve. By condition and geometrical sense of the derivative $y'(x) = x^2$. We must find a function $y(x)$ knowing its derivative x^2.

It's obviously that $y(x) = x^3/3 + C$ where C is a constant. We can find it from the condition $y(2) = 3$. Hence

$$3 = 2^3/3 + C, C = 1/3.$$

Therefore the curve has the equation

$$y(x) = x^3/3 + 1/3.$$

Def.1. A function $F(x)$ is called a primitive [a primitive function, an antiderivative] of a function $f(x)$ on a segment $[a, b]$ if for any $x \in [a, b]$ the derivative of the function $F(x)$ equals $f(x)$,

$$F'(x) = f(x) \tag{1}$$

Ex. 2. The function $F(x) = x^3/3$ in Ex. 1 is a primitive of the function $f(x) = x^2$, and functions

$$F_1(x) = \sin x, F_2(x) = \sin x - 5, F_3(x) = \sin x + 16$$

are primitives of a function $f(x) = \cos x$ on $\Re^1 = (-\infty, \infty)$ because of for any $x \in \Re^1$

$$F'(x) = x^2, F_1'(x) = F_2'(x) = F_3'(x) = \cos x = f(x).$$

Theorem 1 (existence of a primitive). If a function $y = f(x)$ is continuous one on a segment $[a, b]$, then it has a primitive on $[a, b]$.

We'll prove this theorem later.

Properties of primitives

1. If a function $F(x)$ is a primitive of a function $f(x)$, then for any constant C the sum $F(x) + C$ is also a primitive.

■Indeed, if $F'(x) = f(x)$, then for any constant C one has

$$(F(x) + C)' = F'(x) + C' = f(x) + 0 = f(x),$$

and the function $F(x) + C$ is a primitive of the function $f(x)$.■

2. If functions $F_1(x)$, $F_2(x)$ be two primitives of a function $f(x)$ on a segment $[a, b]$, then they differ only by a constant summand, that is their difference is constant one on $[a, b]$,

$$F_1(x) - F_2(x) = C = \text{const}.$$

■By condition $F_1'(x) = F_2'(x) = f(x)$ and so

$$(F_1(x) - F_2(x))' = F_1'(x) - F_2'(x) = f(x) - f(x) = 0$$

identically on $[a, b]$. By virtue of the corollary from Lagrange theorem (see Lecture 16, point 2) the difference $F_1(x) - F_2(x)$ is constant one on $[a, b]$.■

Corollary. This latter property permits to obtain the general form of a primitive of a function $f(x)$: each primitive can be represented in the form of the sum:

$$F(x) + C, \tag{2}$$

where $F(x)$ is some one primitive of $f(x)$ and C is an arbitrary constant.

Note 1. It can be said that the expression (2) represents the **set of all primitives** of the function $f(x)$. This set is a family of functions depending on one parameter C.

Note 2. Geometrically, the set of all primitives is that of parallel curves.

POINT 2. INDEFINITE INTEGRAL AND ITS PROPERTIES

Def. 2. The set of all primitives of a function $f(x)$ is called the **indefinite integral** of this function and is denoted by a symbol $\int f(x)dx$.

On the base of the definition and the Note 1

$$\int f(x)dx = F(x) + C, \qquad (3)$$

where $F(x)$ is some primitive of the function $f(x)$ and C is an arbitrary constant.

Ex. 3. We know that the function $F(x) = x^3/3$ is one of primitives of the function $f(x) = x^2$, so

$$\int x^2 dx = x^3/3 + C.$$

In general for each real number $\alpha \neq -1$ a function $F(x) = x^{\alpha+1}/(\alpha+1)$ is one of primitives of a function $f(x) = x^{\alpha}$, therefore

$$\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \alpha \neq -1.$$

In particular for $\alpha = 0$ one has

$$\int dx = \int x^0 dx = \frac{x^{0+1}}{0+1} = x + C.$$

In the case $\alpha = -1$ we have a function $f(x) = 1/x$ with one of primitives $F(x) = \ln|x|$, hence

$$\int \frac{dx}{x} = \ln|x| + C.$$

Def. 3. Finding the indefinite integral of a function $f(x)$ (or finding its primitive) is called **integration** of $f(x)$. To **integrate** a function means to find its indefinite integral (or its primitive).

Def. 4. The symbol $\int$ is called the integral sign; $f(x)$ is called the integrand (or the function under the integral sign, the function to be integrated); $f(x)dx$ the integrand (or the expression to be integrated, the expression under the integral sign, the element of integration, the integration element), x the variable of integration, C the constant of integration.

On the base of the definition of an indefinite integral and the table of the derivatives we can form the next table.

Table of simplest indefinite integrals

1. $\int x^{\alpha}dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \alpha \neq -1$;

1 a) $\int dx = x + C$; **1 b)** $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C$; **1c)** $\int \frac{dx}{x^2} = -\frac{1}{x} + C$.

2. $\int \frac{dx}{x} = \ln|x| + C$.

3. $\int e^x dx = e^x + C$; **3 a)** $\int e^{kx}dx = \frac{1}{k}e^{kx} + C,\ k$ - const.

4. $\int a^x dx = \frac{a^x}{\ln a} + C$; **4 a)** $\int a^{kx}dx = \frac{a^x}{k\ln a} + C,\ k$ - const.

5. $\int \cos x dx = \sin x + C$; **5a)** $\int \cos ax dx = \frac{1}{a}\sin ax + C$, a - const.

6. $\int \sin x dx = -\cos x + C$; **6a)** $\int \sin ax dx = -\frac{1}{a}\cos ax + C$, a - const.

7. $\int \frac{dx}{\cos^2 x} \equiv \tan x + C \equiv tgx + C$. **8.** $\int \frac{dx}{\sin^2 x} \equiv -\cot x + C \equiv -ctgx + C$.

9. $\int \frac{dx}{x^2+1} = \arctan x + C = arctgx + C$ **10.** $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C \equiv \frac{1}{a}arctg\frac{x}{a} + C$

11. $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$. **12.** $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$.

13. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$ **formula of a high logarithm.**

14. $\int \frac{dx}{\sqrt{x^2+a}} = \ln\left|x + \sqrt{x^2+a}\right| + C$ **formula of a long logarithm.**

It's useful to take into account the next integrales in applications

15. $\int \sqrt{x^2+a}\,dx = \frac{x}{2}\sqrt{x^2+a} + \frac{a}{2}\ln\left|x+\sqrt{x^2+a}\right| + C$.

16. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$.

17. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}(b\sin bx + a\cos bx)}{a^2+b^2} + C$.

18. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C$.

19. $\int chx\,dx \equiv \int \cosh x = shx + C \equiv \sinh x + C$.

20. $\int shx\,dx \equiv \int \sinh x = chx + C \equiv \cosh x + C$.

21. $\int \frac{dx}{ch^2 x} \equiv \int \frac{dx}{\cosh^2 x} = thx + C \equiv \tanh x + C$.

22. $\int \frac{dx}{sh^2 x} \equiv \int \frac{dx}{\sinh^2 x} = cthx + C \equiv \coth x + C$.

The formulas 1 – 12, 19 - 22 are evident. The rest of formulas will be proved below (or can be directly checked by differentiation). All integrals of the table are called those **tabular**.

Properties of indefinite integral

1. The derivative (the differential) of an indefinite integral is equal to the function (*corr*. to the expression) under the integral sign:

$$\left(\int f(x)dx\right)' = f(x), \quad d\left(\int f(x)dx\right) = f(x)dx.$$

$$\blacksquare \left(\int f(x)dx\right)' = (F(x)+C)' = F'(x) = f(x), d\left(\int f(x)dx\right) = \left(\int f(x)dx\right)' dx = f(x)dx \blacksquare$$

Corollary. Correctness of integration can be tested by differentiation.

2. The indefinite integral of the derivative (of the differential) of a function equals the sum of this function and an arbitrary constant:

$$\int F'(x)dx = \int dF(x) = F(x) + C.$$

Corollary. A function can be recovered from its derivative or differential with accuracy to an additive constant.

Ex. 4. $\int dx = x + C$.

3 (additivity). The indefinite integral of an algebraic sum of a finite number of functions equals the sum of their integrals, in particular

$$\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx .$$

■It's sufficient to prove that the derivatives of the left and right sides of the equality are equal. But by the property 1

$$\left(\int (f(x)+g(x))dx\right)' = f(x)+g(x),$$
$$\left(\int f(x)dx + \int g(x)dx\right)' = \left(\int f(x)dx\right)' + \left(\int g(x)dx\right)' = f(x)+g(x).$$ ■

4 (homogeneity). A constant factor can be taken out of the integral sign:

$$k - const,\ \int kf(x)dx = k\int f(x)dx .$$

Prove the property yourselves.

Corollary 1 (linearity). For any functions $f(x)$, $g(x)$ and constants k, l

$$\int (k\cdot f(x) + l\cdot g(x))dx = k\cdot \int f(x)dx + l\cdot \int g(x)dx .$$

Corollary 2. On the base of the linear property and the table of simplest integrals one often can fulfil so-called **direct integration**.

Ex. 5. $\displaystyle\int \frac{dx}{36+25x^2} = \int \frac{dx}{36+25x^2} = \int \frac{dx}{25(36/25+x^2)} = \frac{1}{25}\int \frac{dx}{36/25+x^2} =$

$$= \frac{1}{25}\int \frac{dx}{(6/5)^2+x^2} = \frac{1}{25}\cdot\frac{1}{6/5}\arctan\frac{x}{6/5} + C = \frac{1}{30}\arctan\frac{5x}{6} + C .$$

Ex. 6. $\displaystyle\int \frac{dx}{\sqrt{13-15x^2}} = \int \frac{dx}{\sqrt{15(13/15-x^2)}} = \frac{1}{\sqrt{15}}\int \frac{dx}{\sqrt{\left(\sqrt{13/15}\right)^2 - x^2}} =$

$$= \frac{1}{\sqrt{15}}\arcsin\frac{x}{\sqrt{13/15}} + C = \frac{1}{\sqrt{15}}\arcsin\frac{\sqrt{15}x}{\sqrt{13}} + C .$$

Ex. 7. $\displaystyle\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{(\sin^2 x + \cos^2 x)dx}{\sin^2 x\cos^2 x} = \int \frac{\sin^2 x dx}{\sin^2 x\cos^2 x} + \int \frac{\cos^2 x dx}{\sin^2 x\cos^2 x} =$

$$= \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x} = (\tan x + C_1) + (-\cot x + C_2) = \tan x - \cot x + (C_1 + C_2) =$$

$$= \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} + C = \frac{-2(\cos^2 x - \sin^2 x)}{2\sin x \cos x} + C = -2 \cdot \frac{\cos 2x}{\sin 2x} = -2\cot 2x + C,$$

where $C = C_1 + C_2$ is an arbitrary constant because of arbitrariness of C_1 and C_2.

Ex. 8. On the base of Ex. 5-7

$$\int \left(\frac{11}{36+25x^2} - \frac{9}{\sqrt{13-15x^2}} + \frac{7}{\sin^2 x \cos^2 x} \right) dx =$$

$$= 11 \cdot \int \frac{dx}{36+25x^2} - 9 \cdot \int \frac{dx}{\sqrt{13-15x^2}} + 7 \cdot \int \frac{dx}{\sin^2 x \cos^2 x} =$$

$$= 11 \cdot \left(\frac{1}{30} \arctan \frac{5x}{6} + C_1 \right) - 9 \cdot \left(\frac{1}{\sqrt{15}} \arcsin \frac{\sqrt{15}x}{\sqrt{13}} + C_2 \right) + 7 \cdot (-2\cot 2x + C_3) =$$

$$= \frac{11}{30} \arctan \frac{5x}{6} - \frac{9}{\sqrt{15}} \arcsin \frac{\sqrt{15}x}{\sqrt{13}} - 14\cot 2x + C,$$

where $C = 11C_1 - 9C_2 + 7C_3$ is an arbitrary constant by virtue of arbitrariness of the constants C_1, C_2, C_3.

In future we'll not introduce arbitrary constants for each indefinite integral, but we'll use at once unique arbitrary constant *C*.

POINT 3. INTEGRATION BY SUBSTITUTION (CHANGE OF A VARIABLE)

Theorem 2. Let functions $f(x), x = \varphi(t), \varphi'(t)$ be continuous in corresponding intervals and the function $x = \varphi(t)$ has a continuously differentiable inverse function $t = \phi(x)$. In this case the next formula (**formula of change of a variable**) is true

$$\int f(x)dx = \left| \begin{array}{c} x = \varphi(t) \\ \text{differentiation} \\ dx = \varphi'(t)dt \end{array} \right| = \int f(\varphi(t))\varphi'(t)dt \qquad (4)$$

The formula implies returning to preceding variable *x* after integration with respect to the variable *t*. The word "differentiation" always means finding the differential.

■The first method. The derivatives of the left and right sides of the formula (4) are equal because of

$$\left(\int f(x)dx\right)' = f(x);$$

$$\left(\int f(\varphi(t))\varphi'(t)dt\right)'_x = \left(\int f(\varphi(t))\varphi'(t)dt\right)'_t \cdot t'_x = f(\varphi(t))\varphi'(t)\frac{1}{x'_t} = f(\varphi(t))\varphi'(t)\frac{1}{\varphi'(t)} =$$
$$= f(\varphi(t)) = f(x).$$

The second method. If $F(x)$ is a primitive of the function $f(x)$, then the function $F(\varphi(t))$ is the primitive of the function $f(\varphi(t))\varphi'(t)$ on the strength of

$$(F(\varphi(t)))'_t = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t).$$

Therefore by virtue of definition of the indefinite integral

$$\int f(\varphi(t))\varphi'(t)dt = F(\varphi(t)) + C = F(x) + C = \int f(x)dx. \blacksquare$$

Note 3. The formula (4) is often applied "from the right to the left", and in this case it's useful to write it in the next form

$$\int f(\varphi(x))\varphi'(x)dx = \left| \begin{array}{c} \varphi(x) = t \\ \text{differentiation} \\ \varphi'(x)dx = dt \end{array} \right| = \int f(t)dt. \qquad (5)$$

The formula (5) means: if an integrand is represented as a product of some function *f* of a function $\varphi(x)$ and the derivative $\varphi'(x)$ of this latter then it's well to put $\varphi(x) = t$.

Note 4. We can use any letter instead *t* in the formula (5).

Ex. 9.

$$\int \cos axdx = \left| \begin{array}{c} \text{Let's put } ax = t, \\ d(ax) = dt, \\ adx = dt, dx = \frac{1}{a}dt \end{array} \right| = \int \cos t \cdot \frac{1}{a}dt = \frac{1}{a}\int \cos tdt = \frac{1}{a}\sin t + C = \frac{1}{a}\sin ax + C.$$

Ex. 10. Prove yourselves that $\int \sin axdx = -\frac{1}{a}\cos ax + C$.

Ex. 11. $\int \frac{dx}{x^2 + a^2} = \left| \begin{array}{c} \text{Let } x = at, \\ dx = d(at), \\ dx = adt \end{array} \right| = \int \frac{adt}{(at)^2 + a^2} = \int \frac{adt}{a^2(t^2+1)} = \frac{1}{a}\int \frac{dt}{t^2+1} =$

$$=\frac{1}{a}\arctan t+C=\frac{1}{a}\arctan\frac{x}{a}+C.$$

Ex. 12. Evaluate the indefinite integral $\int\frac{x}{\sqrt{4-x^2}}dx$.

The integrand $\frac{x}{\sqrt{4-x^2}}=\frac{1}{\sqrt{4-x^2}}\cdot x$ is a product of a function of $\varphi(x)=4-x^2$, namely $\frac{1}{\sqrt{\varphi(x)}}$, and the derivative of $\varphi(x)=4-x^2$ (up to a constant factor -2, for $\left(4-x^2\right)'=-2x$). On the base of the formula (5) we can put $\varphi(x)=4-x^2=t$ or better $4-x^2=t^2$. We suppose that $t>0$ and so $t=\sqrt{4-x^2}$. Therefore

$$\int\frac{xdx}{\sqrt{4-x^2}}=\left|\begin{array}{c}4-x^2=t^2, t>0,\\ d\left(4-x^2\right)=d\left(t^2\right),\\ -2xdx=2tdt,\\ xdx=-tdt\end{array}\right|=\int\frac{-tdt}{\sqrt{t^2}}=-\int\frac{tdt}{t}=-\int dt=-t+C=-\sqrt{4-x^2}+C.$$

Ex. 13. Calculate the indefinite integral $\int\frac{\sin 20xdx}{\sqrt{100-\cos^2 20x}}$.

The integrand $\frac{\sin 20x}{\sqrt{100-\cos^2 20x}}=\frac{1}{\sqrt{100-\cos^2 20x}}\cdot\sin 20x$ is a product of the function of $\varphi(x)=\cos 20x$ and (up to a factor -20) the derivative of $\cos 20x$. So, putting $\varphi(x)=\cos 20x=y$, we reduce the given integral to a tabular one (see the tabular formula (12) where we must take $a^2=100, a=10$)

$$\int\frac{\sin 20xdx}{\sqrt{100-\cos^2 20x}}=\left|\begin{array}{c}\cos 20x=y,\\ d(\cos 20x)=dy,\\ -20\sin 20xdx=dy,\\ \sin 20xdx=-\frac{1}{20}dy\end{array}\right|=\int\frac{-\frac{1}{20}dy}{\sqrt{10^2-y^2}}=-\frac{1}{20}\int\frac{dy}{\sqrt{10^2-y^2}}=$$

$$=-\frac{1}{20}\arcsin\frac{y}{10}+C=-\frac{1}{20}\arcsin\frac{\cos 20x}{10}+C.$$

Ex. 14. The case when an integrand is a fraction, the numerator of which is the derivative of the denominator. The integral is reduced to a tabular one. Namely,

$$\int \frac{f'(x)dx}{f(x)} = \left| \begin{matrix} \text{Let } f(x) = z, \\ d(f(x)) = dz, \\ f'(x)dx = dz \end{matrix} \right| = \int \frac{dz}{z} = \ln|z| + C = \ln|f(x)| + C \qquad (6)$$

For ex. a) $\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{adx}{ax+b} = \frac{1}{a}\int \frac{(ax+b)' dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C,$ (7)

in particular

$$\int \frac{dx}{x+b} = \int \frac{(x+b)' dx}{x+b} = \ln|x+b| + C. \qquad (7\,a)$$

b) $\int \frac{dx}{x^2-a^2} = \left| \frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right) \right| = \frac{1}{2a}\int\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx =$

$$= \frac{1}{2a}\left(\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right) = \frac{1}{2a}\left(\int \frac{(x-a)'}{x-a}dx - \int \frac{(x+a)' dx}{x+a}\right) =$$

$$= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C.$$

c) $\int \frac{x^2}{x+1}dx = \int \frac{x^2-1+1}{x+1}dx = \int \frac{(x+1)(x-1)+1}{x+1}dx = \int\left(x-1+\frac{1}{x+1}\right)dx =$

$$= \int xdx - \int dx + \int \frac{dx}{x+1} = \int xdx - \int dx + \int \frac{(x+1)' dx}{x+1} = \frac{x^2}{2} - x + \ln|x+1| + C.$$

d) $\int \frac{xdx}{ax^2+b} = \frac{1}{2a}\int \frac{2axdx}{ax^2+b} = \frac{1}{2a}\int \frac{(ax^2+b)' dx}{ax^2+b} = \frac{1}{2a}\ln|ax^2+b| + C$ (8)

Ex. 15. Prove that $\int \frac{xdx}{\sqrt{ax^2+b}} = \frac{1}{a}\sqrt{ax^2+b} + C$ (9)

Solution. Using the method of Ex. 12 we put $ax^2+b = z^2$, $z > 0$, $z = \sqrt{ax^2+b}$.

Hence, $\int \frac{xdx}{\sqrt{ax^2+b}} = \left| \begin{matrix} ax^2+b = z^2, \\ d(ax^2+b) = d(z^2) \\ 2axdx = 2zdz \\ xdx = \frac{1}{a}zdz \end{matrix} \right| = \int \frac{\frac{1}{a}zdz}{z} = \frac{1}{a}\int dz = \frac{1}{a}z + C = \frac{1}{a}\sqrt{ax^2+b} + C.$

Ex.16. Indefinite integrals

$$\int \frac{(Ax+B)dx}{ax^2+bx+c}, \int \frac{(Ax+B)dx}{\sqrt{ax^2+bx+c}} \tag{10}$$

which contain the quadratic trinomial ax^2+bx+c are reduced to sums of two integrals, namely of the type (8) or (9) and a tabular one, with the help of the substitution

$$\frac{1}{2}\left(ax^2+bx+c\right)' = t, \tag{11}$$

whence $$\frac{1}{2}(2ax+b)=t,\ ax+\frac{b}{2}=t,\ x=\frac{1}{a}\left(t-\frac{b}{2}\right),\ dx=\frac{1}{a}dt;$$

$$Ax+B=\frac{A}{a}\left(t-\frac{b}{2}\right)+B=\frac{A}{a}t+\left(B-\frac{Ab}{2a}\right);$$

$$ax^2+bx+c=a\cdot\frac{1}{a^2}\left(t-\frac{b}{2}\right)^2+\frac{b}{a}\left(t-\frac{b}{2}\right)+c=\frac{1}{a}\left(t^2-bt+\frac{b^2}{4}\right)+\frac{b}{a}t-\frac{b^2}{2a}+c=$$

$$=\frac{1}{a}\left(t^2-bt+bt+\frac{b^2}{4}-\frac{b^2}{2}+ac\right)=\frac{1}{a}\left(t^2-\frac{b^2}{4}+ac\right)=\frac{1}{a}\left(t^2-\frac{b^2-4ac}{4}\right)=\frac{1}{a}\left(t^2-\frac{D}{4}\right),$$

where $D=b^2-4ac$ is the discriminant of the quadratic trinomial.

For example: a) Using the substitution (11), the formula (8) and the tabular integral No.13 (the formula of a high logarithm), we get

$$\int\frac{(3x-2)dx}{4-8x-x^2}=\left|\begin{array}{l}\frac{1}{2}\left(4-8x-x^2\right)'=\frac{1}{2}(-8-2x)=-4-x=t,\\ x=-4-t,\ dx=-dt,\ 3x-2=3(-4-t)-2=\\ =-14-3t,\ 4-8x-x^2=4-8(-4-t)-\\ -(-4-t)^2==4+32+8t-\left(16+8t+t^2\right)\\ =20-t^2=-\left(t^2-20\right)\end{array}\right|=\int\frac{(-14-3t)(-dt)}{20-t^2}=$$

$$=-\int\frac{(3t+14)dt}{t^2-20}=-\left(3\int\frac{tdt}{t^2-20}+14\int\frac{dt}{t^2-20}\right)=-\left(\frac{3}{2}\ln\left|t^2-20\right|+\frac{14}{2\sqrt{20}}\ln\left|\frac{t-\sqrt{20}}{t+\sqrt{20}}\right|\right)+C=$$

$$=-\left(\frac{3}{2}\ln\left|-\left(4-8x-x^2\right)\right|+\frac{7}{2\sqrt{5}}\ln\left|\frac{-4-x-\sqrt{20}}{-4-x+\sqrt{20}}\right|\right)+C=$$

$$=-\frac{3}{2}\ln\left|4-8x-x^2\right|-\frac{7}{2\sqrt{5}}\ln\left|\frac{4+x+\sqrt{20}}{4+x-\sqrt{20}}\right|+C.$$

b) With the help of (11), (9) and the tabular integral No.12 we obtain

$$\int \frac{(3x-2)dx}{\sqrt{4-8x-x^2}} = \left| \begin{array}{c} \frac{1}{2}\left(4-8x-x^2\right)' = -4-x = t, \\ x=-4-t,\ dx=-dt,\ 3x-2=-14-3t, \\ 4-8x-x^2 = 20-t^2 \end{array} \right| = \int \frac{(-14-3t)(-dt)}{\sqrt{20-t^2}} =$$

$$= \int \frac{(3t+14)dt}{\sqrt{20-t^2}} = 3\int \frac{tdt}{\sqrt{20-t^2}} + 14\int \frac{dt}{\sqrt{20-t^2}} = -3\sqrt{20-t^2} + 14\arcsin\frac{t}{\sqrt{20}} + C =$$

$$= -3\sqrt{4-8x-x^2} + 14\arcsin\frac{-4-x}{2\sqrt{5}} + C.$$

Ex. 17. To evaluate the indefinite integral

$$\int \frac{dx}{\sqrt{3+2e^{5x}}}$$

we put $3+2e^{5x} = z^2$, $z>0$, whence (with the help of the tabular integral No. 13)

$$\int \frac{dx}{\sqrt{3+2e^{5x}}} = \left| \begin{array}{c} 3+2e^{5x} = z^2, \\ 10e^{5x}dx = 2zdz, \\ dx = \frac{zdz}{5e^{5x}} = \frac{2zdz}{5(z^2-3)} \end{array} \right| = \frac{2}{5}\int \frac{zdz}{z(z^2-3)} = \frac{2}{5}\int \frac{dz}{z^2-3} = \frac{2}{5}\int \frac{dz}{z^2-\left(\sqrt{3}\right)^2} =$$

$$= \frac{2}{5}\cdot\frac{1}{2\sqrt{3}}\ln\left|\frac{z-\sqrt{3}}{z+\sqrt{3}}\right| = \frac{1}{5\sqrt{3}}\ln\left|\frac{\sqrt{3+2e^{5x}}-\sqrt{3}}{\sqrt{3+2e^{5x}}+\sqrt{3}}\right| + C$$

Ex. 18. To prove the tabular formula No.14 we'll use so-called Euler's substitution $\sqrt{x^2+a} = t-x$ which permits to express x, dx and $\sqrt{x^2+a}$ in terms of t. Therefore

$$\int \frac{dx}{\sqrt{x^2+a}} = \left| \begin{array}{c} \sqrt{x^2+a} = t-x,\ x^2+a = t^2-2tx+x^2, \\ x = \frac{t^2-a}{2t},\ dx = \frac{t^2+a}{2t^2}dt,\ \sqrt{x^2+a} = t-\frac{t^2-a}{2t} = \frac{t^2+a}{2t} \end{array} \right| = \int \frac{\frac{t^2+a}{2t^2}dt}{\frac{t^2+a}{2t}} =$$

$$= \int \frac{dt}{t} = \ln|t| + C = \ln\left|\sqrt{x^2+a}+x\right| + C.$$

POINT 4. INTEGRATION BY PARTS

Theorem 3. Let $u = u(x),\ v = v(x)$ be two continuously differentiable functions. The next formula (**formula of integration by parts**) is true

$$\int u dv = uv - \int v du \qquad (12)$$

■It's known that the differential of a product of two functions $u = u(x),\ v = v(x)$ equals

$$d(uv) = udv + vdu .$$

Integrating this equality we obtain

$$\int d(uv) = \int udv + \int vdu,\ uv = \int udv + \int vdu,\ \int udv = uv - \int vdu .■$$

According to the formula (12) we represent the expression under the integral sign in the form of a product of two functions, namely *u* and *dv*. Then we differentiate the first function and integrate the second one.

Ex. 19.

$$\int \frac{xdx}{\cos^2 x} = \left| \begin{array}{c} \text{Let's put} \\ u = x,\ dv = \dfrac{dx}{\cos^2 x}; \\ \text{then} \\ du = dx,\ v = \int \dfrac{dx}{\cos^2 x} = \tan x \end{array} \right| = x\tan x - \int \tan x dx = x\tan x - \int \frac{\sin x}{\cos x} dx =$$

$$= x\tan x + \int \frac{-\sin x}{\cos x} dx = x\tan x + \int \frac{(\cos x)' dx}{\cos x} = x\tan x + \ln|\cos x| + C .$$

Ex. 20.

$$\int x\sin 3x dx = \left| \begin{array}{c} u = x,\ dv = \sin 3x dx; \\ du = dx,\ v = \int \sin 3x dx = -\dfrac{1}{3}\cos 3x \end{array} \right| = x \cdot \left(-\frac{1}{3}\cos 3x\right) - \int \left(-\frac{1}{3}\cos 3x\right) dx =$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x dx = -\frac{1}{3}x\cos 3x + \frac{1}{3}\cdot\frac{1}{3}\sin 3x + C = -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$$

Note 5. Integration by parts can be performed by necessity several times.

Ex. 21. Let's calculate the indefinite integral $\int (3x^2 - 5x + 9)e^{4x} dx$.

With the help of double integration by part we'll get

$$\int (3x^2-5x+9)e^{4x}dx=\left|\begin{array}{c} u=3x^2-5x+9,\ dv=e^{4x}dx; \\ du=(6x-5)dx,\ v=\int e^{4x}dx=\frac{1}{4}e^{4x} \end{array}\right|=\frac{1}{4}(3x^2-5x+9)e^{4x}-$$

$$-\frac{1}{4}\int (6x-5)e^{4x}dx=\left|\begin{array}{c} u=6x-5,\ dv=e^{4x}dx; \\ du=6dx,\ v=\frac{1}{4}e^{4x} \end{array}\right|=\frac{1}{4}(3x^2-5x+9)e^{4x}-$$

$$-\frac{1}{4}\left(\frac{1}{4}(6x-5)e^{4x}-\frac{6}{4}\int e^{4x}dx\right)=\frac{1}{4}(3x^2-5x+9)e^{4x}-\frac{1}{16}(6x-5)e^{4x}+\frac{3}{32}e^{4x}+C.$$

Ex. 22. $\int \ln^2 x dx=\left| Let\ u=\ln^2 x,\ dv=dx;\ then\ du=2\ln x\cdot\frac{dx}{x},\ v=\int dx=x\right|=$

$$=x\ln^2 x-\int x\cdot 2\ln x\cdot\frac{dx}{x}=x\ln^2 x-2\int \ln x dx=\left|\begin{array}{c} let\ u=\ln x,\ dv=x; \\ then\ du=dx/x;\ v=x \end{array}\right|=$$

$$=x\ln^2 x-2\left(x\ln x-\int x\cdot\frac{dx}{x}\right)=x\ln^2 x-2\left(x\ln x-\int dx\right)=x\ln^2 x-2(x\ln x-x)+C.$$

Note 6. Sometimes integration by parts leads to an equation in a required integral

Ex. 23. Let $I=\int e^{2x}\cos 3x dx$.

After double integration by parts we'll have

$$I=\int e^{2x}\cos 3x dx=\left|\begin{array}{c} u=e^{2x},dv=\cos 3x dx \\ du=2e^{2x}dx,v=\int \cos 3x dx=\frac{1}{3}\sin 3x \end{array}\right|=\frac{1}{3}e^{2x}\sin 3x-$$

$$-\frac{2}{3}\int e^{2x}\sin 3x dx=\left|\begin{array}{c} u=e^{2x},dv=\sin 3x dx \\ du=2e^{2x}dx,v=\int \sin 3x dx=-\frac{1}{3}\cos 3x \end{array}\right|=\frac{1}{3}e^{2x}\sin 3x-$$

$$-\frac{2}{3}\left(-\frac{1}{3}e^{2x}\cos 3x+\frac{2}{3}\int e^{2x}\cos 3x dx\right)=\frac{1}{3}e^{2x}\sin 3x+\frac{2}{9}e^{2x}\cos 3x-\frac{4}{9}I.$$

We've got the equation in the sought integral I and hence

$$I=\frac{1}{3}e^{2x}\sin 3x+\frac{2}{9}e^{2x}\cos 3x-\frac{4}{9}I,\quad \frac{13}{9}I=\frac{1}{3}e^{2x}\sin 3x+\frac{2}{9}e^{2x}\cos 3x,$$

$$I=\int e^{2x}\cos 3x dx=\frac{3}{13}e^{2x}\sin 3x+\frac{2}{13}e^{2x}\cos 3x+C=\frac{e^{2x}}{13}(3\sin 3x+2\cos 3x)+C$$

By the same method one proves the formulas for tabular integrals No. 17, 18.

Ex. 24.

$$I=\int\sqrt{x^2+a}\,dx=\left|\begin{array}{c} u=\sqrt{x^2+a},dv=dx \\ du=\dfrac{xdx}{\sqrt{x^2+a}},v=\int dx=x\end{array}\right|=x\sqrt{x^2+a}-\int\frac{x^2dx}{\sqrt{x^2+a}}=$$

$$=x\sqrt{x^2+a}-\int\frac{\left(x^2+a\right)-a}{\sqrt{x^2+a}}dx=x\sqrt{x^2+a}-\int\sqrt{x^2+a}\,dx+a\int\frac{dx}{\sqrt{x^2+a}}=$$

$$=x\sqrt{x^2+a}-I+a\ln\left|x+\sqrt{x^2+a}\right|. \text{ Therefore } 2I=x\sqrt{x^2+a}+a\ln\left|x+\sqrt{x^2+a}\right|,$$

$$I=\int\sqrt{x^2+a}\,dx=\frac{x}{2}\sqrt{x^2+a}+\frac{a}{2}\ln\left|x+\sqrt{x^2+a}\right|+C.$$

We've proved the formula for the tabular integral No. 15. Prove yourselves the formula for the integral No. 16.

Note 7. There are no general rules to choose u, dv. But in some cases one can give certain advices.

In the cases of integrals

$$\int P(x)e^{kx}dx,\quad \int P(x)\sin kxdx,\quad \int P(x)\cos kxdx,$$

where $P(x)$ is a polynomial, it's well to put $u=P(x)$.

In the cases of integrals

$$\int P(x)\ln xdx, \int P(x)\arcsin xdx, \int P(x)\arccos xdx, \int P(x)\arctan dx, \int P(x)arc\cot dx$$

it's well to put $dv=P(x)dx$.

Ex. 25. $\int\arcsin xdx=\left|\begin{array}{c} u=\arcsin x,\ dv=dx; \\ du=\dfrac{dx}{\sqrt{1-x^2}},v=\int dx=x\end{array}\right|=x\arcsin x-\int\dfrac{xdx}{\sqrt{1-x^2}}=$

$$=\left|\begin{array}{l}\text{by the formula (9)} \\ \text{with } a=-1,b=1\end{array}\right|=x\arcsin x+\sqrt{1-x^2}+C.$$

Ex. 26. $\int xarc\cot xdx=\left|u=arc\cot x,\ dv=xdx;\ du=-\dfrac{dx}{x^2+1},\ v=\dfrac{x^2}{2}\right|=$

$$=\frac{x^2}{2}arc\cot x+\frac{1}{2}\int\frac{x^2}{1+x^2}dx=\frac{x^2}{2}arc\cot x+\frac{1}{2}\int\frac{x^2+1-1}{1+x^2}dx=$$

$$=\frac{x^2}{2}arc\cot x+\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx=\frac{x^2}{2}arc\cot x+\frac{1}{2}\left(\int dx-\int\frac{1}{1+x^2}dx\right)=$$

$$=\frac{x^2}{2}arc\cot x+\frac{1}{2}(x-\arctan x)+C.$$

Note 8. Evaluating indefinite integrals one often can combine both methods of changing a variable and integration by parts.

Ex. 27. $\int\sin\sqrt{x}dx=\left|\begin{matrix}x=t^2,\ \text{let } t\ge 0,\\ dx=2tdt\end{matrix}\right|=\int\sin t\cdot 2tdt=2\int t\sin tdt=$

$$=\left|\begin{matrix}u=t,\ dv=\sin tdt,\\ du=dt,\ v=-\cos t\end{matrix}\right|==-t\cos t-\int(-\cos t)dt=-\sqrt{x}\cos\sqrt{x}+\int\cos tdt=$$

$$=-\sqrt{x}\cos\sqrt{x}+\sin t+C=-\sqrt{x}\cos\sqrt{x}+\sin\sqrt{x}+C.$$

Ex. 28. $\int e^{\sqrt[3]{x}}dx=\left|\begin{matrix}x=t^3\\ dx=3t^2dt\end{matrix}\right|=3\int t^2e^tdt=\left|\begin{matrix}u=t^2,\ dv=e^tdt\\ du=2tdt,\ v=e^t\end{matrix}\right|=3\left(t^2e^t-2\int te^tdt\right)=$

$$=3\left(t^2e^t-2\int te^tdt\right)=\left|\begin{matrix}u=t, & dv=e^tdt,\\ du=dt, & v=e^t\end{matrix}\right|=3\left(t^2e^t-2\left(te^t-\int e^tdt\right)\right)=$$

$$=3\left(t^2e^t-2\left(te^t-e^t\right)\right)=3\left(\sqrt[3]{x^2}e^{\sqrt[3]{x}}-2\left(\sqrt[3]{x}e^{\sqrt[3]{x}}-e^{\sqrt[3]{x}}\right)\right)+C=e^{\sqrt[3]{x}}\left(3\sqrt[3]{x^2}-6\sqrt[3]{x}+6\right)+C.$$

Ex. 29. $\int\frac{xe^xdx}{\sqrt{1+e^x}}=\left|\begin{matrix}u=x,\ dv=\frac{e^xdx}{\sqrt{1+e^x}},\ du=dx, v=\int\frac{e^xdx}{\sqrt{1+e^x}}=\\ =\left|\begin{matrix}1+e^x=t^2\\ e^xdx=2tdt\end{matrix}\right|=\int\frac{2tdt}{t}=2\int dt=2t=2\sqrt{1+e^x}\end{matrix}\right|=$

$$=2x\sqrt{1+e^x}-2\int\sqrt{1+e^x}dx=\left|\begin{matrix}1+e^x=t^2,\ e^xdx=2tdt\\ dx=\frac{2tdt}{e^x}=\frac{2tdt}{t^2-1}\end{matrix}\right|=2x\sqrt{1+e^x}-2\int t\cdot\frac{2tdt}{t^2-1}=$$

$$=2x\sqrt{1+e^x}-4\int\frac{t^2dt}{t^2-1}=2x\sqrt{1+e^x}-4\int\frac{(t^2-1+1)dt}{t^2-1}=2x\sqrt{1+e^x}-4\int\left(1+\frac{1}{t^2-1}\right)dt=$$

$$=2x\sqrt{1+e^x}-4\left(t+\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right)+C=2x\sqrt{1+e^x}-4\left(\sqrt{1+e^x}+\frac{1}{2}\ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right|\right)+C.$$

Ex. 30.

$$\int\frac{x^2e^{3x}dx}{(3x+2)^2}=\left|\begin{matrix}\text{let } u=x^2e^{3x},\ dv=dx/(3x+2)^2,\ \text{then } du=x(2+3x)e^{3x},\\ v=\int\frac{dx}{(3x+2)^2}=\left|3x+2=t,\ dx=\frac{1}{3}dt\right|=\frac{1}{3}\int\frac{dt}{t^2}=-\frac{1}{3t}=-\frac{1}{3(3x+2)}\end{matrix}\right|=$$

$$= -\frac{x^2 e^{3x}}{3(3x+2)} - \int \left(-\frac{1}{3(3x+2)} \right) x(2+3x) e^{3x} = -\frac{x^2 e^{3x}}{3(3x+2)} + \frac{1}{3}\int x e^{3x} dx =$$

$$= \left| \begin{array}{l} \text{if we put } u = x,\, dv = e^{3x}dx, \\ \text{we'll have } du = dx,\, v = \int e^{3x}dx = \frac{1}{3}e^{3x} \end{array} \right| = -\frac{x^2 e^{3x}}{3(3x+2)} + \frac{1}{3}\left(\frac{1}{3} x e^{3x} - \frac{1}{3}\int e^{3x} dx \right) =$$

$$= -\frac{x^2 e^{3x}}{3(3x+2)} + \frac{1}{3}\left(\frac{1}{3} x e^{3x} - \frac{1}{3}\cdot\frac{1}{3} e^{3x} \right) + C = \frac{e^{3x}}{3}\left(-\frac{x^2}{3x+2} + \frac{x}{3} - \frac{1}{9} \right) + C = \frac{(3x-2)e^{3x}}{27(3x+2)} + C.$$

Ex. 31. Evaluate the next indefinite integral: $\int (\arccos x)^2 dx$.

$$\int (\arccos x)^2 dx = \left| u = (\arccos x)^2,\, dv = dx;\, du = -\frac{2\arccos x dx}{\sqrt{1-x^2}},\, v = x \right| =$$

$$= x(\arccos x)^2 + 2\int \frac{x \arccos x dx}{\sqrt{1-x^2}} = \left| \begin{array}{l} \text{let's put } \arccos x = t \ (0 \le t \le \pi); \\ \text{then } \frac{dx}{\sqrt{1-x^2}} = -dt,\, x = \cos t,\, \sin t = \sqrt{1-x^2} \end{array} \right| =$$

$$= x(\arccos x)^2 - 2\int t \cos t dt = \left| u = t,\, dv = \cos t dt;\, du = dt,\, v = \sin t \right| =$$

$$= x(\arccos x)^2 - 2\left(t \sin t - \int \sin t dt \right) = x(\arccos x)^2 - 2(t \sin t + \cos t) + C =$$

$$= x(\arccos x)^2 - 2\left(\sqrt{1-x^2} \arccos x + x \right) + C.$$

LECTURE NO.20. CLASSES OF INTEGRABLE FUNCTIONS

POINT 1. RATIONAL FRACTIONS (RATIONAL FUNCTIONS)

POINT 2. TRIGONOMETRIC FUNCTIONS

POINT 3. IRRATIONAL FUNCTIONS

POINT 1. RATIONAL FUNCTIONS (RATIONAL FRACTIONS)

Def. 1. A rational function is called a function which can be represented in the form of a rational fraction that is as a ratio of two polynomials

$$R(x)=\frac{P_n(x)}{Q_m(x)}=\frac{a_n x^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0}{b_m x^m+b_{m-1}x^{m-1}+\ldots+b_1x+b_0},\ a_n\neq 0,\ b_m\neq 0. \qquad (1)$$

Def. 2. A rational fraction (1) is called **proper** one if $n<m$ and **improper** otherwise ($n\geq m$).

Theorem 1 (extraction of integer part of an improper rational fraction)**.** Every improper rational fraction can be represented as a sum of some polynomial (so-called **integer part**) and a proper rational fraction.

■Let $n\geq m$. Dividing the numerator $P_n(x)$ by the denominator $Q_m(x)$ we get

$$P_n(x)=Q_m(x)S(x)+r(x),\quad R(x)=\frac{Q_m(x)S(x)+r(x)}{Q_m(x)}=S(x)+\frac{r(x)}{Q_m(x)}$$

where polynomials $S(x), r(x)$ are respectively the quotient and the remainder and

$$r(x)/Q_m(x)$$

is a proper rational fraction.■

Ex. 1. Extract an integer part of an improper rational fraction

$$R(x)=\frac{x^2}{x+1}$$

a) The first (theoretical) way. After division of x^2 by $x+1$ we get

$$x^2=(x+1)(x-1)+1,\ S(x)=x-1,\ r(x)=1,\ \frac{x^2}{x+1}=\frac{(x+1)(x-1)+1}{x+1}=x-1+\frac{1}{x+1}$$

b) The second way. Subtracting and adding 1 in the numerator we'll have

$$\frac{x^2}{x+1}=\frac{x^2-1+1}{x+1}=\frac{(x^2-1)+1}{x+1}=x-1+\frac{1}{x+1}$$

There are partial [simplest, elementary] rational fractions of 1- 4 types.

1. $\dfrac{A}{ax+b}$;

2. $\dfrac{A}{(ax+b)^k}$, $k\in \mathbf{N}$;

3. $\dfrac{Ax+B}{ax^2+bx+c}$ $\quad (D=b^2-4ac<0)$;

4. $\dfrac{Ax+B}{(ax^2+bx+c)^k}$, $k\in \mathbf{N}$ $\quad (D=b^2-4ac<0)$.

We've integrated the fractions 1, 3 in the Point 2 of the preceding lecture (ex. 14, 16). To integrate the fraction 2 we can put $ax+b=t$. Integration of the fraction 4 with the help of the substitution

$$\frac{1}{2}(ax^2+bx+c)'=t$$

leads to a linear combination of a simple integral

$$\int\frac{tdt}{(t^2+m^2)^k}=\left|\begin{array}{l}t^2+m^2=z,\\ tdt=\dfrac{1}{2}dz\end{array}\right|=\int\frac{\frac{1}{2}dz}{z^k}=\frac{1}{2}\int z^{-k}dz=\frac{z^{(-k+1)}}{2(-k+1)}+C=\frac{(t^2+m^2)^{-k+1}}{2(-k+1)}+C$$

and the next one

$$\int\frac{dt}{(t^2+m^2)^k}.$$

As to evaluation of this latter see textbooks. For small values of k ($k=2, k=3$) one can use the next substitution: $t=m\tan z$.

Ex. 2.

$$\int\frac{dt}{(x^2+4)^2}=\left|\begin{array}{l}x=2\tan z,\ dx=\dfrac{2dz}{\cos^2 z},\\ x^2+4=4\tan^2 z+4=\\ =4(\tan^2 z+1)=\dfrac{4}{\cos^2 z}\end{array}\right|=\int\frac{\frac{2dz}{\cos^2 z}}{\left(\frac{4}{\cos^2 z}\right)^2}=\frac{1}{8}\int\cos^2 zdz=\frac{1}{8}\int\frac{1+\cos 2z}{2}dz=$$

$$=\frac{1}{16}\left(\int dz+\int\cos 2z dz\right)=\frac{1}{16}\left(z+\frac{1}{2}\sin 2z\right)+C=\frac{1}{16}\left(\arctan\frac{x}{2}+\frac{1}{2}\sin 2\arctan\frac{x}{2}\right)+C.$$

Thus we can say that we able to integrate the partial fractions 1 – 4.

Theorem 2 (a partial decomposition of a proper rational fraction)**.** Every proper rational fraction can be represented as a linear combination of partial fractions.

For example

$$\frac{Ax+B}{(ax+b)(cx+d)}=\frac{P}{ax+b}+\frac{Q}{cx+d};$$

$$\frac{Ax^2+Bx+C}{(ax+b)(cx^2+dx+c)}=\frac{P}{ax+b}+\frac{Qx+R}{cx^2+dx+c};$$

$$\frac{Ax^3+Bx^2+Cx+D}{(ax+b)^2(cx^2+dx+c)}=\frac{P}{(ax+b)^2}+\frac{Q}{ax+b}+\frac{Rx+S}{cx^2+dx+c}.$$

Here P, Q, R, S are some unknown numbers (undetermined coefficients), which one can find by so-called **method of undetermined coefficients**.

Corollary. Every rational function can be integrated by virtue of the linear property of indefinite integral.

Rule of integration of a rational function. To integrate a rational function it's necessary:

1. To extract its integer part if it is an improper rational fraction or contains an improper fraction.

2. To factorize the denominator of obtained proper fraction into a product of polynomials of degree not higher than two.

3. To make a partial decomposition of the proper fraction.

4. To integrate all the terms of the obtained algebraic sum.

5. To write an answer.

Ex. 3. $\int\frac{x^2}{x+1}dx=\int\frac{(x+1)(x-1)+1}{x+1}dx=\int\left(x-1+\frac{1}{x+1}\right)dx=\int xdx-\int dx+$

$$+\int\frac{1}{x+1}dx=\frac{x^2}{2}-x+\int\frac{(x+1)'}{x+1}dx=\frac{x^2}{2}-x+\ln|x+1|+C.$$

Ex. 4. Evaluate the indefinite integral $\int \frac{dx}{x^2 - a^2}$.

1 step (factorizing the denominator of the proper rational fraction).

$$x^2 - a^2 = (x-a)(x+a).$$

2 step (a partial decomposition of the proper fraction).

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} \Bigg| \cdot (x-a)(x+a), \quad 1 = A(x+a) + B(x-a) \quad (*)$$

Let's assign two particular values to x in (*), namely $x = a, x = -a$.

$$\begin{array}{r|l} x = a & 1 = 2aA, \quad A = \frac{1}{2a}; \\ x = -a & 1 = -2aB, \ B = -\frac{1}{2a}; \end{array} \Rightarrow \frac{1}{x^2 - a^2} = \frac{\frac{1}{2a}}{x-a} + \frac{-\frac{1}{2a}}{x+a} = \frac{1}{2a}\left(\frac{1}{x-a} - \frac{1}{x+a}\right).$$

3 step (integration of the given function by integration of its partial decomposition).

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\int\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx = \frac{1}{2a}\left(\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right) =$$

$$= \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right) + C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

Ex. 5. Calculate the next indefinite integral

$$I = \int \frac{(x^2 - 27x + 14)dx}{(x-3)(4 - 8x - x^2)}$$

1 step (a partial decomposition of the proper rational fraction with factorized denominator).

$$\frac{x^2 - 27x + 14}{(x-3)(4 - 8x - x^2)} = \frac{A}{x-3} + \frac{Bx + C}{4 - 8x - x^2} \ \Big| \cdot (x-3)(4 - 8x - x^2),$$

$$x^2 - 27x + 14 = A(4 - 8x - x^2) + (Bx + C)(x-3). \qquad (**)$$

Assigning tree arbitrary values to x in (**), for example $x = 3, x = 0, x = 1$, we get a system of linear equations in A, B, C,

$$\begin{array}{r|l} x = 3 & \left\{\begin{array}{ll} -58 = -29A, & A = 2, \\ 14 = 4A - 3C, & C = -2, \\ -12 = -5A - 2B - 2C; & B = 3; \end{array}\right. \\ x = 0 & \\ x = 1 & \end{array} \Rightarrow \frac{x^2 - 27x + 14}{(x-3)(4 - 8x - x^2)} = \frac{2}{x-3} + \frac{3x - 2}{4 - 8x - x^2}.$$

2 step (integration of all the terms of obtained partial decomposition of the integrand)

$$\int \frac{2dx}{x-3} = 2\int \frac{dx}{x-3} = 2\int \frac{(x-3)'\,dx}{x-3} = 2\ln|x-3| + C_1;$$

$$\int \frac{3x-2}{4-8x-x^2}\,dx = -\frac{3}{2}\ln\left|4-8x-x^2\right| - \frac{7}{2\sqrt{5}}\ln\left|\frac{4+x+\sqrt{20}}{4+x-\sqrt{20}}\right| + C_2 \text{ (see Ex. 16 of Lect. No. 19)}$$

Answer. $I = 2\ln|x-3| - \frac{3}{2}\ln\left|4-8x-x^2\right| - \frac{7}{2\sqrt{5}}\ln\left|\frac{4+x+2\sqrt{5}}{4+x-2\sqrt{5}}\right| + C$

POINT 2. TRIGONOMETRIC FUNCTIONS

In this point we study the methods of integration of a rational function

$$R(\cos x, \sin x) \tag{2}$$

or two arguments $\cos x$, $\sin x$.

Universal trigonometrical substitution

Theorem 3. Integration of a function (2) always reduces to that of a rational function of one variable t with the help of so-called **universal trigonometrical substitution (UTS)**

$$\tan\frac{x}{2} = t \tag{3}$$

■On the base of (3) we have

$$\cos x = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{1} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} = \left|\frac{:\cos^2\frac{x}{2}}{:\cos^2\frac{x}{2}}\right| = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2},$$

$$\sin x = \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{1} = \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} = \left|\frac{:\cos^2\frac{x}{2}}{:\cos^2\frac{x}{2}}\right| = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2},$$

$$\frac{x}{2} = \arctan t,\ x = 2\arctan t,\ dx = \frac{2dt}{1+t^2}.$$

Therefore,

$$\cos x = \frac{1-t^2}{1+t^2},\quad \sin x = \frac{2t}{1+t^2},\quad dx = \frac{2dt}{1+t^2}, \tag{4}$$

and

$$\int R(\cos x, \sin x)dx = \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)\frac{2dt}{1+t^2} = \int R_1(t)dt,$$

where a function of the argument t

$$R_1(t) = R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)\frac{2}{1+t^2}$$

is a rational one■

Ex. 6.

$$\int \frac{dx}{\sin^3 x} = \left| \begin{matrix} UTS \\ \tan\frac{x}{2} = t \end{matrix} \right| = \int \frac{\frac{2dt}{t^2+1}}{\left(\frac{2t}{t^2+1}\right)^3} = \frac{2}{8}\int \frac{(t^2+1)^3 dt}{t^3(t^2+1)} = \frac{1}{4}\int \frac{(t^2+1)^2}{t^3}dt = \frac{1}{4}\int \frac{t^4+2t^2+1}{t^3}dt =$$

$$= \frac{1}{4}\int\left(t + \frac{2}{t} + \frac{1}{t^3}\right)dt = \frac{1}{4}\left(\frac{t^2}{2} + 2\ln|t| + \frac{t^{-3+1}}{-3+1}\right) + C = \frac{\tan^2\frac{x}{2}}{8} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{8\tan^2\frac{x}{2}} + C =$$

$$= \frac{1}{8}\left(\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} - \frac{\cos^2\frac{x}{2}}{\sin^2\frac{x}{2}}\right) + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C = \frac{1}{8}\cdot\frac{\sin^4\frac{x}{2} - \cos^4\frac{x}{2}}{\sin^2\frac{x}{2}\cos^2\frac{x}{2}} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C =$$

$$= \frac{1}{2}\cdot\frac{\left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right)\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right)}{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C = -\frac{1}{2}\cdot\frac{\cos x}{\sin^2 x} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| + C.$$

Evaluation of a rather like integral

$$\int \frac{dx}{\cos^3 x}$$

with the help of UTS leads to very complicated integral. Indeed,

$$\int \frac{dx}{\cos^3 x} = \left| \begin{matrix} \text{УТП} \\ \tan\frac{x}{2} = t \end{matrix} \right| = \int \frac{\frac{2dt}{t^2+1}}{\left(\frac{1-t^2}{t^2+1}\right)^3} = 2\int \frac{\left(t^2+1\right)^2 dt}{\left(1-t^2\right)^3}.$$

Ex. 7. To calculate the integral

$$\int \frac{dx}{\cos^3 11x}$$

it's well to reduce it to the integral of the preceding example by changing a variable, namely

$$\int \frac{dx}{\cos^3 11x} = \int \frac{dx}{\sin^3\left(\frac{\pi}{2} - 11x\right)} = \left| \begin{matrix} \frac{\pi}{2} - 11x = y \\ -11dx = dy \\ dx = -\frac{1}{11}dy \end{matrix} \right| = -\frac{1}{11}\int \frac{dy}{\sin^3 y} = \left| \begin{matrix} UTS \\ \tan\frac{y}{2} = t \end{matrix} \right|$$

and so on (see Ex. 6).

Ex. 8. $$\int \frac{dx}{2 - 4\cos 5x + 5\sin 5x} = \left| \begin{matrix} 5x = y \\ 5dx = dy \\ dx = \frac{1}{5}dy \end{matrix} \right| = \frac{1}{5}\int \frac{dy}{2 - 4\cos y + 5\sin y} =$$

$$= \left| \begin{matrix} \tan\frac{y}{2} = t, dy = \frac{2dt}{t^2+1} \\ \cos y = \frac{1-t^2}{t^2+1}, \sin y = \frac{2t}{t^2+1} \end{matrix} \right| = \frac{1}{5}\int \frac{\frac{2dt}{t^2+1}}{2 - 4\frac{1-t^2}{t^2+1} + 5\cdot\frac{2t}{t^2+1}} = \left| \begin{matrix} \cdot\left(t^2+1\right) \\ \cdot\left(t^2+1\right) \end{matrix} \right| = \frac{1}{5}\int \frac{dt}{3t^2 + 5t - 1} =$$

$$= \left| \begin{matrix} \frac{1}{2}\left(3t^2 + 5t - 1\right)' = 3t + \frac{5}{2} = z, t = \frac{z}{3} - \frac{5}{6}, dt = \frac{dz}{3} \\ 3t^2 + 5t - 1 = 3\left(\frac{z}{3} - \frac{5}{6}\right)^2 + 5\left(\frac{z}{3} - \frac{5}{6}\right) - 1 = \frac{z^2}{3} - \frac{37}{12} \end{matrix} \right| = \frac{1}{5}\int \frac{\frac{dz}{3}}{\frac{z^2}{3} - \frac{37}{12}} = \frac{1}{5}\int \frac{dz}{z^2 - \left(\frac{\sqrt{37}}{2}\right)^2} =$$

$$= \frac{1}{5\sqrt{37}} \ln\left| \frac{z - \frac{\sqrt{37}}{2}}{z + \frac{\sqrt{37}}{2}} \right| + C = \left| \begin{matrix} z = 3t + \frac{5}{2} = \\ = 3\tan\frac{y}{2} + \frac{5}{2} = \\ = 3\tan\frac{5x}{2} + \frac{5}{2} \end{matrix} \right| = \frac{1}{5\sqrt{37}} \ln\left| \frac{6\tan\frac{5x}{2} + 5 - \sqrt{37}}{6\tan\frac{5x}{2} + 5 + \sqrt{37}} \right| + C$$

Other substitutions

I. If a function (2) is odd with respect to $\cos x$,

$$R(-\cos x, \sin x) = -R(\cos x, \sin x), \tag{5}$$

then it can be transformed to the next form:

$$R(\cos x, \sin x) = R_2(\sin x)\cos x,$$

where $R_2(\sin x)$ is a rational function of one variable $\sin x$. Substitution

$$\sin x = t \tag{6}$$

reduces integration of the given function to that of a rational function of *t*.

II. If a function (2) is odd with respect to $\sin x$,

$$R(\cos x, -\sin x) = -R(\cos x, \sin x), \tag{7}$$

one can bring it to the form

$$R(\cos x, \sin x) = R_3(\cos x)\sin x$$

($R_3(\cos x)$ is a rational function of $\cos x$) and apply the substitution

$$\cos x = t \tag{8}$$

III. If a function (2) is even with respect both to $\sin x$ and $\cos x$Если функция (2) четна относительно совокупности двух аргументов $\sin x$ и $\cos x$, то есть

$$R(-\cos x, -\sin x) = R(\cos x, \sin x), \tag{9}$$

it's transformable into a rational function $R_4(\tan x)$ of $\tan x$,

$$R(\cos x, \sin x) = R_4(\tan x),$$

and can be integrated with the help of one of substitutions

$$\tan x = t, \quad \cot x = t\,. \tag{10}$$

Ex. 9. Calculate the indefinite integral $\int \cos^5 7x \sin^6 7x dx$.

The integrand $\cos^5 7x \sin^6 7x$ is odd function with respect to $\cos 7x$, because of

$$(-\cos 7x)^5 \sin^6 7x = -\cos^5 7x \sin^6 7x,$$

and so

$$\int \cos^5 7x \sin^6 7x dx = \int \cos^4 7x \sin^6 7x \cos 7x dx = \int \left(\cos^2 7x\right)^2 \sin^6 7x \cos 7x dx =$$

$$= \int \left(1 - \sin^2 7x\right)^2 \sin^6 7x \cos 7x dx = \left| \begin{matrix} \sin 7x = t,\ 7\cos 7x dx = dt, \\ \cos 7x dx = \frac{1}{7} dt \end{matrix} \right| = \int \left(1 - t^2\right)^2 t^6 \frac{1}{7} dt =$$

$$= \frac{1}{7} \int \left(t^{10} - 2t^8 + t^6\right) dt = \frac{1}{7}\left(\frac{t^{11}}{11} - \frac{2t^9}{9} + \frac{t^7}{7}\right) + C = \frac{1}{7}\left(\frac{\sin^{11} 7x}{11} - \frac{2\sin^9 7x}{9} + \frac{\sin^7 7x}{7}\right) + C.$$

Ex. 10. $\displaystyle\int \frac{dx}{\cot^5 8x} = \int \tan^5 8x dx = \left| \begin{matrix} \tan 8x = z,\ 8x = \arctan z, \\ x = \frac{1}{8}\arctan z,\ dx = \frac{1}{8}\cdot\frac{dz}{1+z^2} \end{matrix} \right| = \frac{1}{8}\int \frac{z^5 dz}{z^2+1} =$

$$= \frac{1}{8}\int \frac{\left(z^3 - z\right)\left(z^2+1\right) + z}{z^2+1} dz = \frac{1}{8}\int \left(z^3 - z + \frac{z}{z^2+1}\right) dz =$$

$$= \frac{1}{8}\left(\frac{z^4}{4} - \frac{z^2}{2} + \frac{1}{2}\int \frac{2z dz}{z^2+1}\right) = \frac{1}{8}\left(\frac{\tan^4 8x}{4} - \frac{\tan^2 8x}{2} + \frac{1}{2}\ln\left(\tan^2 8x + 1\right)\right) + C.$$

Ex. 11. Find the indefinite integral $\displaystyle\int \frac{dx}{\sin^{10} 3x}$.

$$\int \frac{dx}{\sin^{10} 3x} = \int \frac{dx}{\sin^8 3x \sin^2 3x} = \int \left(\frac{1}{\sin^2 3x}\right)^4 \frac{dx}{\sin^2 3x} = \int \left(1 + \cot^2 3x\right)^4 \frac{dx}{\sin^2 3x} =$$

$$= \left| \begin{matrix} \cot 3x = t,\ -\frac{3dx}{\sin^2 3x} = dt, \\ \frac{dx}{\sin^2 3x} = -\frac{1}{3} dt \end{matrix} \right| = -\frac{1}{3}\int \left(t^2+1\right)^4 dt = -\frac{1}{3}\int \left(t^8 + 4t^6 + 6t^4 + 4t^2 + 1\right) dt = -\frac{1}{3}\left(\frac{t^9}{9} + \right.$$

$$\left. + \frac{4t^7}{7} + \frac{6t^5}{5} + \frac{4t^3}{3} + t\right) + C = -\frac{\cot^9 3x}{27} - \frac{4\cot^7 3x}{21} - \frac{6\cot^5 3x}{15} - \frac{4\cot^3 3x}{9} - \frac{1}{3}\cot 3x + C$$

Ex. 12. Calculate the indefinite integral $\displaystyle\int \frac{\sin 9x \cos 9x dx}{\sin^4 9x + \cos^4 9x}$.

The integrand $\dfrac{\sin 9x \cos 9x dx}{\sin^4 9x + \cos^4 9x}$ is even function with respect both to $\sin 9x$ and $\cos 9x$ because of

$$\frac{(-\sin 9x)(-\cos 9x)}{(-\sin 9x)^4 + (-\cos 9x)^4} = \frac{\sin 9x \cos 9x}{\sin^4 9x + \cos^4 9x}.$$

So

$$\int \frac{\sin 9x \cos 9x dx}{\sin^4 9x + \cos^4 9x} = \int \frac{\sin 9x \cos 9x dx}{\cos^4 9x\left(\tan^4 9x + 1\right)} = \int \frac{\tan 9x dx}{\cos^2 9x\left(\tan^4 9x + 1\right)} = \left| \begin{matrix} \tan 9x = t \\ \dfrac{dx}{\cos^2 9x} = \dfrac{1}{9}dt \end{matrix} \right| =$$

$$= \frac{1}{9}\int \frac{tdt}{t^4+1} = \left| \begin{matrix} t^2 = y, \\ tdt = \dfrac{1}{2}dy \end{matrix} \right| = \frac{1}{18}\int \frac{dy}{y^2+1} = \frac{1}{18}\arctan y + C = \frac{1}{18}\arctan\left(\tan^2 9x\right) + C$$

Note 1. Substitutions of this point can be applicable to some irrational functions of $\sin x$ and $\cos x$.

Ex. 13. Evaluate the indefinite integral $\int \sqrt[6]{\cos 13x}\sin 13x dx$.

$$\int \sqrt[6]{\cos 13x}\sin 13x dx = \left| \begin{matrix} \cos 13x = y^6,\ y \ge 0,\ y = \sqrt[6]{\cos 13x}, \\ -13\sin 13x dx = -6y^5 dy,\ \sin 13x dx = -\dfrac{6}{13}y^5 dy \end{matrix} \right| = -\frac{6}{13}\int y^6 dy =$$

$$= -\frac{6}{13}\cdot\frac{y^7}{7} + C = -\frac{6}{91}\left(\sqrt[6]{\cos 13x}\right)^7 + C = -\frac{6}{91}\sqrt[6]{\cos^7 13x} + C.$$

Ex. 14. For positive $\sin x, \cos x$

$$\int \frac{\sqrt{\sin^3 2x}}{\sin^5 x}dx = \int \frac{\sqrt{(2\sin x\cos x)^3}}{\sin^5 x}dx = \int \frac{\sqrt{8\sin^3 x\cos^3 x}}{\sin^5 x}dx = \int \frac{2\sqrt{2}\sqrt{\sin^3 x\cos^3 x}}{\sin^3 x}\frac{dx}{\sin^2 x} =$$

$$= 2\sqrt{2}\int \sqrt{\frac{\sin^3 x\cos^3 x}{\sin^6 x}}\frac{dx}{\sin^2 x} = 2\sqrt{2}\int \sqrt{\cot^3 x}\frac{dx}{\sin^2 x} = \left| \begin{matrix} \cot x = t, \\ \dfrac{dx}{\sin^2 x} = -dt \end{matrix} \right| = -2\sqrt{2}\int t^{\frac{3}{2}}dt =$$

$$= -2\sqrt{2}\frac{t^{5/2}}{5/2} + C = -\frac{4\sqrt{2}}{5}\cot^{5/2} x + C = C - \frac{4\sqrt{2}}{5}\sqrt{\cot^5 x} = C - \frac{4\sqrt{2}}{5}\cot^2 x\sqrt{\cot x}.$$

Some other methods

a) Application of power reduction formulas

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2} \qquad (11)$$

Ex. 15. $\int \sin^2 15x dx = \frac{1}{2}\int (1-\cos 30x)dx = \frac{1}{2}\left(x - \frac{1}{30}\sin 30x\right) + C$.

Ex. 16. $\int \cos^4 5x dx = \int \left(\cos^2 5x\right)^2 dx = \int \left(\frac{1+\cos 10x}{2}\right)^2 dx =$

$$=\frac{1}{4}\int\left(1+2\cos 10x+\cos^2 10x\right)dx==\frac{1}{4}\int\left(1+2\cos 10x+\frac{1+\cos 20x}{2}\right)dx=$$

$$=\frac{1}{8}\int\left(3+4\cos 10x+\cos 20x\right)dx==\frac{1}{8}\left(3x+\frac{2}{5}\sin 10x+\frac{1}{20}\sin 20x\right)+C.$$

Ex. 17.

$$\int\sin^6 3xdx=\int\left(\sin^2 3x\right)^3 dx=\int\left(\frac{1-\cos 6x}{2}\right)^3 dx=\frac{1}{8}\int\left(1-3\cos 6x+3\cos^2 6x-\right.$$

$$\left.-\cos^3 6x\right)dx=\frac{1}{8}\left(x-\frac{3}{6}\sin 6x+3\int\frac{1+\cos 12x}{2}dx-\int\left(1-\sin^2 6x\right)\cos 6xdx\right)=$$

$$=\left|\begin{matrix}\sin 6x=t\\ \cos 6xdx=\frac{1}{6}dt\end{matrix}\right|=\frac{1}{8}\left(x-\frac{1}{2}\sin 6x+\frac{3}{2}\left(x+\frac{1}{12}\sin 12x\right)-\frac{1}{6}\int\left(1-t^2\right)dt\right)=$$

$$\frac{1}{8}\left(\frac{5}{2}x-\frac{1}{2}\sin 6x+\frac{1}{8}\sin 12x-\frac{1}{6}\left(\sin 6x-\frac{\sin^3 6x}{3}\right)\right)+C=$$

$$=\frac{1}{8}\left(\frac{5}{2}x-\frac{2}{3}\sin 6x+\frac{1}{8}\sin 12x+\frac{1}{18}\sin^3 6x\right)+C.$$

b) Application of product formulas

$$1)\ \sin x\cdot\cos y=\frac{\sin(x+y)+\sin(x-y)}{2};\quad 2)\ \cos x\cdot\cos y=\frac{\cos(x+y)+\cos(x-y)}{2};$$

$$3)\ \sin x\cdot\sin y=\frac{\cos(x-y)-\cos(x+y)}{2} \qquad (12)$$

Ex. 18.

$$\int\sin 7x\cos 4xdx=\frac{1}{2}\int\left(\sin 11x+\sin 3x\right)dx=\frac{1}{2}\left(-\frac{1}{11}\cos 11x-\frac{1}{3}\cos 3x\right)+C.$$

Ex. 19. $$\int\cos^2 x\sin^2 2xdx=\int\frac{1+\cos 2x}{2}\frac{1-\cos 4x}{2}dx=$$

$$=\frac{1}{4}\int\left(1-\cos 4x+\cos 2x-\cos 2x\cos 4x\right)dx=\frac{1}{4}\left(x-\frac{1}{4}\sin 4x+\frac{1}{2}\sin 2x-\right.$$

$$\left.-\int\cos 2x\cos 4xdx\right)=\frac{1}{4}\left(x-\frac{1}{4}\sin 4x+\frac{1}{2}\sin 2x-\frac{1}{2}\int\left(\cos 6x+\cos 2x\right)dx\right)+C=$$

$$= \frac{1}{4}\left(x - \frac{1}{4}\sin 4x + \frac{1}{2}\sin 2x - \frac{1}{12}\sin 12x - \frac{1}{4}\sin 2x \right) + C =$$

$$= \frac{1}{4}\left(x - \frac{1}{4}\sin 4x - \frac{1}{4}\sin 2x - \frac{1}{12}\sin 12x \right) + C =$$

$$= \frac{1}{48}\left(12x - 3\sin 4x - 3\sin 2x - \sin 12x\right) + C.$$

POINT 3. IRRATIONAL FUNCTIONS

Linear and linear-fractional irrationalities

Evaluation of indefinite integrals of the type

$$\int R\left(x, \sqrt[n]{ax+b}\right)dx, \tag{13}$$

with so-called **linear irrationality** $\sqrt[n]{ax+b}$, reduces to integration of a rational function of one variable t with the help of the substitution

$$ax+b=t^n \tag{14}$$

■From (14) we get

$$x=\frac{1}{a}\left(t^n-b\right), dx=\frac{n}{a}t^{n-1}dt, \int R\left(x, \sqrt[n]{ax+b}\right)dx=\frac{n}{a}\int R\left(\frac{1}{a}\left(t^n-b\right), t\right)t^{n-1}dt=\frac{n}{a}\int R_1(t)dt.$$

We've got an integral of a rational function $R_1(t)=R\left(\frac{1}{a}\left(t^n-b\right), t\right)t^{n-1}$.■

Indefinite integrals of the type

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx \tag{15}$$

with a linear-fractional [homographic] irrationality $\sqrt[n]{\frac{ax+b}{cx+d}}$ is reduced to that of a rational function by the substitution

$$\frac{ax+b}{cx+d}=t^n \tag{16}$$

Prove this assertion yourselves.

Ex. 20.

$$\int\frac{dx}{\sqrt{2x+3}-2\sqrt[3]{2x+3}}=\left|\begin{array}{c}2x+3=t^6, t>0,\\ 2dx=6t^5dt, dx=3t^5dt\end{array}\right|=3\int\frac{t^5dt}{t^3-2t^2}=3\int\frac{t^3dt}{t-2}=$$

$$3\int\frac{t^3-8+8}{t-2}dt=3\int\left(t^2+2t+4+\frac{8}{t-2}\right)dt=3\left(\frac{t^3}{3}+2\frac{t^2}{2}+4t+8\ln|t-2|\right)+C=$$

$$=3\left(\frac{\left(\sqrt[6]{2x+3}\right)^3}{3}+2\frac{\left(\sqrt[6]{2x+3}\right)^2}{2}+4\sqrt[6]{2x+3}+8\ln\left|\sqrt[6]{2x+3}-2\right|\right)+C=$$

$$=3\left(\frac{\sqrt{2x+3}}{3}+2\frac{\sqrt[3]{2x+3}}{2}+4\sqrt[6]{2x+3}+8\ln\left|\sqrt[6]{2x+3}-2\right|\right)+C.$$

Ex. 21. $\displaystyle\int\sqrt{\frac{1+x}{1-x}}\frac{dx}{1-x}=\left|\begin{array}{l}\frac{1+x}{1-x}=t^2,\ t\ge 0,\ x=\frac{t^2-1}{t^2+1},\\ dx=\frac{4tdt}{\left(t^2+1\right)^2},\ \frac{1}{1-x}=\frac{t^2+1}{2}\end{array}\right|=\int t\cdot\frac{t^2+1}{2}\cdot\frac{4tdt}{\left(t^2+1\right)^2}=2\int\frac{t^2}{t^2+1}dt=$

$$=2\int\frac{t^2+1-1}{t^2+1}dt=2\int\left(1-\frac{1}{t^2+1}\right)dt=2(t-\arctan t)+C=2\left(\sqrt{\frac{1+x}{1-x}}-\arctan\sqrt{\frac{1+x}{1-x}}\right)+C$$

Quadratic irrationalities. Trigonometric substitutions

Indefinite integral of the type

$$\int R\left(x,\ \sqrt{a^2-x^2}\right)dx \tag{17}$$

reduces to that with an integrand, depending on $\sin x, \cos x$, by a trigonometric substitution

$$x=a\sin t. \tag{18}$$

The same is true for an integral

$$\int R\left(x,\ \sqrt{a^2+x^2}\right)dx \tag{19}$$

if one introduces a substitution

$$x=a\tan t, \tag{20}$$

and for an integral

$$\int R\left(x,\ \sqrt{x^2-a^2}\right)dx \tag{21}$$

provided a substitution

$$x=a\sec t=\frac{a}{\cos t} \tag{22}$$

■Let's consider the integral (17) and put $x=a\sin t$. We'll have $dx=a\cos tdt$,

$$a^2-x^2=a^2-a^2\sin^2 t=a^2\left(1-\sin^2 t\right)=a^2\cos^2 t,\ \sqrt{a^2-x^2}=a|\cos t|.$$

$$\int R\left(x,\ \sqrt{a^2-x^2}\right)dx=\int R(a\sin t,\ a|\cos t|)a\cos tdt=\int R_1(\sin t,\cos t)dt,$$

where $R_1(\sin t, \cos t) = R(a\sin t,\ a|\cos t|)a\cos t$. ■

Consider the integrals (19), (21) yourselves.

Ex. 22.

$$\int \frac{\sqrt{4-x^2}}{x^2}dx = \left| \begin{array}{l} x = 2\sin t, \text{ we suppose } 0 < t \le \frac{\pi}{2}, \\ dx = 2\cos t dt,\ 4 - x^2 = 4\cos^2 t \end{array} \right| = \int \frac{2|\cos t|}{4\sin^2 t}\cdot 2\cos t dt = \int \cot^2 t dt =$$

$$= \int\left(\frac{1}{\sin^2 t} - 1\right)dt = -\cot t - t + C = \left| \begin{array}{l} \sin t = \frac{x}{2},\ \cos t = \\ = \sqrt{1-\left(\frac{x}{2}\right)^2} = \frac{\sqrt{4-x^2}}{2} \end{array} \right| = -\frac{\sqrt{4-x^2}}{x} - \arcsin\frac{x}{2} + C.$$

Ex. 23.

$$\int \frac{dx}{\left(\sqrt{25+x^2}\right)^3} = \left| \begin{array}{l} x = 5\tan t, \text{ we suppose } 0 \le t < \frac{\pi}{2}, \\ dx = \frac{5dt}{\cos^2 t},\ 25 + x^2 = 25\left(1+\tan^2 t\right) = \frac{25}{\cos^2 t} \end{array} \right| = \int \frac{\frac{5dt}{\cos^2 t}}{\left(\frac{5}{\cos t}\right)^3} =$$

$$= \frac{1}{25}\int \cos t dt = \frac{1}{25}\sin t + C = \left| \sin t = \frac{\tan t}{\sqrt{1+\tan^2 t}} = \frac{x}{\sqrt{25+x^2}} \right| = \frac{x}{25\sqrt{25+x^2}} + C.$$

Ex. 24.

$$\int \frac{dx}{x^2\sqrt{x^2-9}} = \left| \begin{array}{l} x = \frac{3}{\cos t}, \text{ we suppose } 0 \le t < \frac{\pi}{2}, dx = \frac{3\sin t dt}{\cos^2 t} \\ x^2 - 9 = 9\left(\frac{1}{\cos^2 t} - 1\right) = 9\left(\frac{1-\cos^2 t}{\cos^2 t}\right) = 9\tan^2 t \end{array} \right| = \int \frac{\frac{3\sin t dt}{\cos^2 t}}{\frac{9}{\cos^2 t}\cdot 3\tan t} =$$

$$= \frac{1}{9}\int \cos t dt = \frac{1}{9}\sin t + C = \left| \cos t = \frac{3}{x},\ \sin t = \sqrt{1-\cos^2 t} = \frac{\sqrt{x^2-9}}{x} \right| = \frac{\sqrt{x^2-9}}{9x} + C.$$

Quadratic irrationalities (general case)

Indefinite integral of the form

$$\int R\left(x, \sqrt{ax^2+bx+c}\right)dx \qquad (23)$$

can be reduced to one of integrals (17), (19), (21) with the help of the substitution

$$\frac{1}{2}\left(ax^2+bx+c\right)' = t \qquad (24)$$

There are many other methods of evaluating the integral (23). For example one can reduce integration to that of rational function with the help of Euler substitutions.

The first Euler substitution (if $a > 0$):

$$\sqrt{ax^2 + bx + c} = \pm\sqrt{a}x + t; \quad (25)$$

the second Euler substitution (if $c > 0$):

$$\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c}; \quad (26)$$

the third Euler substitution (if the trinomial $ax^2 + bx + c$ has two real roots x_1, x_2):

$$\sqrt{ax^2 + bx + c} = (x - x_1)t. \quad (27)$$

Ex. 25.

$$I = \int \frac{dx}{x + \sqrt{x^2 + x + 1}} = \left| \begin{array}{c} \sqrt{x^2 + x + 1} = t - x,\ x^2 + x + 1 = t^2 - 2tx + x^2, \\ x = \frac{t^2 - 1}{2t + 1},\ dx = \frac{2(t^2 + t + 1)dt}{(2t + 1)^2},\ x + \sqrt{x^2 + x + 1} = t \end{array} \right| = 2\int \frac{(t^2 + t + 1)dt}{t(2t + 1)^2}$$

$$\frac{t^2 + t + 1}{t(2t + 1)^2} = \frac{A}{t} + \frac{B}{(2t + 1)^2} + \frac{C}{2t + 1} \Big| \cdot t(2t + 1)^2,\ t^2 + t + 1 = A(2t + 1)^2 + Bt + Ct(2t + 1);$$

$$\begin{array}{c} t = 0 \\ t = -1/2 \\ t = -1 \end{array} \left| \left\{ \begin{array}{l} 1 = A,\ A = 1; \\ 3/4 = -1/2\,B,\ B = -3/2; \\ 1 = A - B + C,\ C = -3/2; \end{array} \right. \right. \quad \frac{t^2 + t + 1}{t(2t + 1)^2} = \frac{1}{t} - \frac{3/2}{(2t + 1)^2} - \frac{3/2}{2t + 1};$$

$$I = 2\int \left(\frac{1}{t} - \frac{3/2}{(2t + 1)^2} - \frac{3/2}{2t + 1} \right) dt = 2\int \frac{dt}{t} - 3\int \frac{dt}{(2t + 1)^2} - 3\int \frac{dt}{2t + 1} = \left| \begin{array}{c} 2t + 1 = y \\ 2dt = dy \end{array} \right| =$$

$$= 2\ln|t| - \frac{3}{2}\int \frac{dy}{y^2} - \frac{3}{2}\int \frac{dy}{y} = 2\ln|t| + \frac{3}{2y} - \frac{3}{2}\ln|y| + C = 2\ln|t| + \frac{3}{2(2t + 1)} - \frac{3}{2}\ln|2t + 1| + C =$$

$$= \frac{3}{2(2t + 1)} + \frac{1}{2}\ln\frac{t^4}{|2t + 1|^3} + C = \frac{3}{2\left(2\left(\sqrt{x^2 + x + 1} + x\right) + 1\right)} + \frac{1}{2}\ln\frac{\left(\sqrt{x^2 + x + 1} + x\right)^4}{\left|2\left(\sqrt{x^2 + x + 1} + x\right) + 1\right|^3} + C$$

There are many indefinite integrals which **can't be expressed in terms of elementary functions**. For example:

$$\int e^{-x^2}dx,\ \int \frac{x}{\ln x}dx,\ \int \frac{\sin x}{x}dx,\ \int \frac{\cos x}{x}dx,\ \int \sin x^2 dx,\ \int \cos x^2 dx.$$

INDEFINITE INTEGRAL: Basic Terminology RUE

1. быть выразимым с помощью элементарных функций	бути виразним за допомогою елементарних функцій	can be expressed in terms of èleméntary functions
2. быть разложенным на множители	бути розкладеним на множники	be fáctorized [expán-ded into fáctors]
3. быть разложимой в сумму простейших дробей вида … (о правильной рациона-льной дроби)	бути розвивним в суму найпростіших дробів вигляду … (про правильний ра-ціональний дріб)	be dècompósable into (a sum of) símplest fráctions of the form…(of a próper rátional fráction)
4. быть разложимым в про-изведение многочленов степени не выше второй (о многочлене)	бути розкладним в добуток многочленів степеня не вище другого (про многочлен)	be fáctorable into (can be written as) a próduct of pòlynómials of degrée not hígher than two (of a pòly-nómial)
5. быть/являться элемен-тарной функцией	бути елементарною функцією	be an èleméntary function
6. ввести новую перемен-ную	ввести/запровадити нову змінну	ìntrodúce a new váriable
7. внеинтегральный член	позаінтеґральний член	term óutsíde the íntegral, íntegrated term
8. возвратиться к (старой, первоначальной, предыдущей) переменной *x*	повернутися до (старої, початкової, попередньої) змінної *x*	retúrn to the oríginal/inítial/precéding/prévious váriable *x*
9. выделение целой части неправильной рациона-льной дроби	виділення цілої частини неправильного раціонального дробу	extráction of an íntegral part of an impróper fráction
10.выделить целую часть неправильной дроби	виділити цілу частину неправильного дробу	extráct an íntegral part of the impróper fráction
11.выражаться с помощью элементарных функций	виражатися за допомогою елементарних функцій, через елементарні функції	be expréssed in terms of èleméntary functions
12.выразить (ко)синус че-рез тангенс половинного аргумента	виразити (ко)синус через танґенс половинно-го арґументу	expréss (co)sine in terms of the half árgument (half-angle) tángent
13.выразить (ко)синус че-рез тангенс того же	виразити (ко)синус через танґенс того ж арґу-	expréss (co)sine in terms of the tángent of the same

аргумента	менту	árgument
14.вычисление	обчислення	compúing/evàluátion/ eváluating/fínding/cálcula-tion
15.вычислить	обчислити	compúte/cálculate/ eválu-ate
16.давать/приписывать (ча-стные) значения ар-гмен-ту	давати/приписувати (ча-стинні) значення арґу-менту	give/assígn partícular válues to the árgument
17.дробно-линейная ир-ра-циональность	дробово-лінійна ірраціо-нальність	línear-fráctional irrà-tionálity
18.замена переменной	заміна змінної	chánge (of) a váriable
19.заменить переменную	замінити змінну	chánge a váriable
20.знак (неопреде-лённого) интеграла	знак (невизначеного) ін-теґрала	indéfinite íntegral sign
21.интеграл (не) может быть выражен с помо-щью элементарных функций	інтеґрал (не) може бути виражений за допомо-гою елементарних функ-цій	íntegral can(n't) be ex-pressed in terms of èle-méntary fúnctions
22.интеграл вычислен дво-йным (тройным) ин-тегрированием по час-тям	інтеґрал обчислено под-війним (потрійним) інте-ґруванням частинами	íntegral is compúted/ eváluated/cálculated/ fóund by double (triple) integrátion by parts
23.интегрирование	інтеґрування	ìntegrátion
24.интегрирование по час-тям	інтеґрування частинами	ìntegrátion by parts
25.интегрирование под-ста-новкой, заменой	інтеґрування підстанов-кою, заміною	ìntegrátion by a sùbs-titútion
26.интегрировать	інтеґрувати	íntegrate
27.интегрировать по час-тям	інтеґрувати частинами	íntegrate by parts
28.интегрировать под-становкой	інтеґрувати підстанов-кою	íntegrate by a sùbstitú-tion
29.иррациональная функция	іраціональна функція	irrátional fúnction
30.иррациональность	іраціональність	irràtionálity
31.квадратичная ирра-циональность	квадратична іраціональ-ність	quadrátic irràtionálity
32.линейная иррацио-наль-ность	лінійна іраціональність	línear irràtionálity
33.линейность	лінійність	lìneárity
34.метод интегрирова-ния	метод інтеґрування	méthod of ìntegrátion
35.метод неопреде-	метод невизначених ко-	méthod of úndetérmined

лённых коэффициентов	ефіцієнтів	còeffícients
36.многочлен	многочлен	pòlynómial
37.многочлен *n*-ой степени	многочлен *n*-го степеня	*n*-th degrée pòlynómial
38.множество всех перво-образных	множина всіх первісних	set of all prímitives [ántiderívatives]
39.называть, назвать	називати	call
40.неопределённый интег-рал	невизначений інтеґрал	indéfinite íntegral
41.неправильная дробь	неправильний дріб	impróper fráction
42.нечётная функция относительно (ко)синуса	непарна функція відносно (ко)синуса	odd fúnction with respéct to (co)sine
43.обозначать	позначати	denóte/désignate
44.обозначаться	позначатися	be denóted/désignated
45.обратная задача для ...	обернена задача для ...	ínvérse/cónverse próblem of...
46.обратная операция для ...	обернена операція для ...	ínvérse/cónverse operátion of...
47.обратная функция	обернена функція	ìnvérse fúnction
48.основная задача	основна задача	májor/máin/príncipal próblem
49.отношение двух много-членов	відношення двох многочленів	rátio of two pòlynómials
50.первообразная	первісна	prímitive [ántideríva-tive]
51.перейти к переменной *t*	перейти до змінної *t*	pass to the váriable *t*
52.переменная интегриро-вания	змінна інтеґрування	váriable of ìntegrátion
53.подстановка, замена	підстановка, заміна	sùbstitútion
54.подынтегральная функ-ция	підінтеґральна функція	íntegrand, fúnction to be íntegrated [under the íntegral sign], sùbíntegral fúnction
55.подынтегральное выра-жение	підінтеґральний вираз	íntegrand, expréssion to be íntegrated, exprés-sion under the íntegral sign
56.получать окончательно	діставати остаточно	get/recéive/obtáin fí-nally
57.получать табличный интеграл	діставати табличний інтеґрал	get/recéive/obtáin a tábular íntegral
58.получать/давать систему уравнений относительно (для нахождения) неопределённых коэф-	отримувати/давати систему рівнянь відносно (для знаходження) невизначених коефіцієнтів	get/set up/form/deríve/ obtáin/give the sýstem of equátions in (to detérmine) the úndetérmined còeffí-

фициентов		cients
59.пользоваться *чем-то* ещё раз, несколько раз	користуватися *чимось* ще раз, декілька разів	make use (once) again (some times) of *smth*
60.постоянная интегри-ро-вания	стала інтеґрування	cónstant of ìntegrátion
61.правильная дробь	правильний дріб	próper fráction
62.приводить дроби к об-щему знаменателю	зводити дроби до спіль-ного знаменника	redúce the fráctions to the cómmon denóminator
63.приравнивание (чис-ли-телей, коэффи-циентов при одинаковых степенях *x*)	прирівнювання (чисель-ників, коефіцієнтів при однакових степенях *x*)	équalize [equáting/ équal-izátion/sétting équal/ idèn-tificátion/idéntifying] (the numerators, the còef-fícients of équal (of the same) degrées/pówers (in like pówers) of *x*)
64.приравни-вать/отождест-влять (числители, коэф-фи-циенты при одинаковых степенях *x*)	прирівнювати/ототож-нювати (чисельники, ко-ефіцієнти при однакових степенях *x*)	equáte/set équal/idéntify (the númerators, còef-fícients of équal (of the same) degrées/pówers (in like pówers) of *x*)
65.проверить правиль-ность интегрирования дифферен-цированием	перевірити правиль-ність інтеґрування дифе-ренціюванням	check/vérify corréct-ness/truth of ìntegrátion by dìfferentiátion
66.произвести под-становку, замену $x=\varphi(t)$, $t=\psi(x)$	виконати, зробити, здій-снити підстановку, за-міну $x=\varphi(t)$, $t=\psi(x)$	make, cárry out, réalize, fulfíl, ímplemènt, éxe-cùte the sùbstitútion $x=\varphi(t)$, $t=\psi(x)$
67.произволь-ная/аддитив-ная посто-янная	довільна/адитивна стала	árbitrary/ádditive cónstant
68.простейшая ра-циональная (эле-ментарная) дробь перво-го (второго, третьего, четвёртого) типа	найпростіший раціона-льний (елементарний) дріб першого (другого, третього, четвертого) ти-пу	pártial/símplest/èle-méntary rátional fráction of the first (second, third, fourth) type/kind
69.прямое (непосредст-венное) интегрирование	пряме (безпосереднє) інтеґрування	diréct ìntegrátion
70.разложение мно-гочлена на множители	розклад многочлена на множники	fàctorizátion/fáctoring/ expánsion of the pòlynó-mial (into fáctors)
71.разложение пра-вильной рациональной	розвинення правильного раціонального дробу на	rèsolútion/expánsion/ devéloppement/dècompo-

дроби на простейшие дроби (в сумму простейших дробей)	найпростіші дроби (в суму найпростіших дробів)	sítion of a próper rational fráction into (a sum of) pártial/símplest/èleméntary fráctions [partial fráction dècomposítion of...]
72.разложить в сумму простейших дробей	розвинути в суму найпростіших дробів	resólve/expánd/devélop/dècompóse into (a sum of) símplest fráctions
73.разложить многочлен на линейные и квадратичные (с отрицательными дискриминантами) множители	розкласти многочлен на лінійні й квадратичні (з від"ємними дискримінантами) множники	fáctor/fáctorize/expánd the pòlynómial into línear and quadrátique (with négative discríminants) fáctors
74.рациональная дробь	раціональний дріб	rátional fráction
75.рациональная функция	раціональна функція	rátional fúnction
76.сводить интегрирование к интегрированию рациона-льной функции	зводити інтеґрування до інтеґрування раціональної функції	redúce the ìntegrátion to that of a rátional fúnction
77.сводить неопределённый интеграл к…	зводити невизначений інтеґрал до …	redúce the indéfinite íntegral to …
78.свойство линейности, линейное свойство	властивість лінійності, лінійна властивість	lìneárity próperty
79.семейство функций (за-висящее от единственной постоянной)	сім"я функцій (яка залежить від єдиної сталої)	fámily of fúnctions (which depénds on a single cónstant)
80.существование	існування	exístence
81.существовать	існувати	exíst
82.таблица простейших интегралов	таблиця найпростіших інтеґралів	táble of símplest íntegrals
83.табличный интеграл	табличний інтеґрал	tábular íntegral
84.универсальная тригоно-метрическая подстановка	універсальна тригонометрична підстановка	ùnivérsal trìgonométric(al) sùbstitútion
85.функция интегрируется посредством (с помощью) элементарных функций	функція інтеґрується в елементарних функціях (за допомогою елементарних функцій)	fúnction is íntegrated by means of èleméntary fúnctions
86.целая часть неправильной дроби	ціла частина неправильного дробу	íntegral part of the impróper fráction
87.чётная функция относительно (ко)синуса	парна функція відносно (ко)синуса	éven fúnction with respéct to (co)sine

INDEFINITE INTEGRAL: Basic Terminology ERU

1. árbitrary/ádditive cónstant	произвольная/аддитивная постоянная	довільна/адитивна стала
2. be an èleméntary function	быть/являться элементарной функцией	бути елементарною функцією
3. be dècompósable into (a sum of) símplest fráctions of the form…(of a próper rátional fráction)	быть разложимой в сумму простейших дробей вида … (о правильной рациона-льной дроби)	бути розвивним в суму найпростіших дробів вигляду … (про правильний ра-ціональний дріб)
4. be denóted/désignated	обозначаться	позначатися
5. be expréssed in terms of èleméntary functions	выражаться с помощью элементарных функций	виражатися за допомогою елементарних функцій, через елементарні функції
6. be fáctorable into (can be written as) a próduct of pòlynómials of degrée not hígher than two (of a pòlynómial)	быть разложимым в произведение многочленов степени не выше второй (о многочлене)	бути розкладним в добуток многочленів степеня не вище другого (про многочлен)
7. be fáctorized [expánded into fáctors]	быть разложенным на множители	бути розкладеним на множники
8. call	называть, назвать	називати
9. can be expressed in terms of èleméntary functions	быть выразимым с помощью элементарных функций	бути виразним за допомогою елементарних функцій
10.chánge (of) a váriable	замена переменной	заміна змінної
11.chánge a váriable	заменить переменную	замінити змінну
12.check/vérify corréctness/truth of ìntegrátion by dìfferentiátion	проверить правильность интегрирования дифферен-цированием	перевірити правиль-ність інтеґрування диференціюванням
13.compúing/evàluátion/eváluating/fínding/cálculation	вычисление	обчислення
14.compúte/cálculate/eváluate	вычислить	обчислити
15.cónstant of ìntegrátion	постоянная интегрирования	стала інтеґрування
16.denóte/désignate	обозначать	позначати
17.diréct ìntegrátion	прямое (непосредственное) интегрирование	пряме (безпосереднє) інтеґрування
18.pòlynómial	многочлен	многочлен

19.équalize [equáting/ équalizátion/sétting équal/ idèntificátion/idéntifying] (the numerators, the còefficients of équal (of the same) degrées/pówers (in like pówers) of x)	приравнивание (числителей, коэффициентов при одинаковых степенях x)	прирівнювання (чисельників, коефіцієнтів при однакових степенях x)
20.equáte/set équal/idéntify (the númerators, còefficients of équal (of the same) degrées/pówers (in like pówers) of x)	приравнивать/отождествлять (числители, коэффициенты при одинаковых степенях x)	прирівнювати/ототожнювати (чисельники, коефіцієнти при однакових степенях x)
21.éven fúnction with respéct to (co)sine	чётная функция относительно (ко)синуса	парна функція відносно (ко)синуса
22.exíst	существовать	існувати
23.exístence	существование	існування
24.expréss (co)sine in terms of the half árgument (half-angle) tángent	выразить (ко)синус через тангенс половинного аргумента	виразити (ко)синус через танґенс половинно-го арґументу
25.expréss (co)sine in terms of the tángent of the same árgument	выразить (ко)синус через тангенс того же аргумента	виразити (ко)синус через танґенс того ж арґументу
26.extráct an íntegral part of the impróper fráction	выделить целую часть неправильной дроби	виділити цілу частину неправильного дробу
27.extráction of an íntegral part of an impróper fráction	выделение целой части неправильной рациональной дроби	виділення цілої частини неправильного раціонального дробу
28.fáctor/fáctorize/expánd the pòlynómial into línear and quadrátique (with négative discríminants) fáctors	разложить многочлен на линейные и квадратичные (с отрицательными дискриминантами) множители	розкласти многочлен на лінійні й квадратичні (з від"ємними дискримінантами) множники
29.fàctorizátion/fáctoring/ expánsion of the pòlynómial (into fáctors)	разложение многочлена на множители	розклад многочлена на множники
30.fámily of fúnctions (which depénds on a single cónstant)	семейство функций (зависящее от единственной постоянной)	сім"я функцій (яка залежить від єдиної сталої)
31.fúnction is íntegrated by means of èleméntary fúnctions	функция интегрируется посредством (с помощью) элементарных функций	функція інтеґрується в елементарних функціях (за допомогою елементарних функцій)
32.get/recéive/obtáin a	получать табличный ин-	діставати табличний ін-

tábular íntegral	теграл	теґрал
33. get/recéive/obtáin fínally	получать окончательно	діставати остаточно
34. get/set up/form/deríve/obtáin/give the sýstem of equátions in (to detérmine) the úndetérmined còeffícients	получать/давать систему уравнений относительно (для нахождения) неопределённых коэффициентов	отримувати/давати систему рівнянь відносно (для знаходження) невизначених коефіцієнтів
35. give/assígn partícular válues to the árgument	давать/приписывать (частные) значения аргменту	давати/приписувати (частинні) значення арґументу
36. impróper fráction	неправильная дробь	неправильний дріб
37. indéfinite íntegral	неопределённый интеграл	невизначений інтеґрал
38. indéfinite íntegral sign	знак (неопределённого) интеграла	знак (невизначеного) інтеґрала
39. íntegral can(n't) be expressed in terms of èleméntary fúnctions	интеграл (не) может быть выражен с помощью элементарных функций	інтеґрал (не) може бути виражений за допомогою елементарних функцій
40. íntegral is compúted/eváluated/cálculated/fóund by double (triple) integrátion by parts	интеграл вычислен двойным (тройным) интегрированием по частям	інтеґрал обчислено подвійним (потрійним) інтеґруванням частинами
41. íntegral part of the impróper fráction	целая часть неправильной дроби	ціла частина неправильного дробу
42. íntegrand, expréssion to be íntegrated, expréssion under the íntegral sign	подынтегральное выражение	підінтеґральний вираз
43. íntegrand, fúnction to be íntegrated [under the íntegral sign], sùbíntegral fúnction	подынтегральная функция	підінтеґральна функція
44. íntegrate	интегрировать	інтеґрувати
45. íntegrate by a sùbstitútion	интегрировать подстановкой	інтеґрувати підстановкою
46. íntegrate by parts	интегрировать по частям	інтеґрувати частинами
47. ìntegrátion	интегрирование	інтеґрування
48. ìntegrátion by a sùbstitútion	интегрирование подстановкой, заменой	інтеґрування підстановкою, заміною
49. ìntegrátion by parts	интегрирование по частям	інтеґрування частинами

50.ìntrodúce a new vári-able	ввести новую перемен-ную	ввести/запровадити нову змінну
51.ìnvérse fúnction	обратная функция	обернена функція
52.ínvérse/cónverse ope-rátion of...	обратная операция для ...	обернена операція для ...
53.ínvérse/cónverse pró-blem of...	обратная задача для ...	обернена задача для ...
54.irrátional fúnction	иррациональная функ-ция	іраціональна функція
55.irràtionálity	иррациональность	іраціональність
56.línear irràtionálity	линейная иррациональ-ность	лінійна іраціональність
57.línear-fráctional irrà-tionálity	дробно-линейная ирра-циональность	дробово-лінійна іраціо-нальність
58.lìneárity	линейность	лінійність
59.lìneárity próperty	свойство линейности, линейное свойство	властивість лінійності, лінійна властивість
60.májor/máin/príncipal próblem	основная задача	основна задача
61.make use (once) again (some times) of *smth*	пользоваться *чем-то* ещё раз, несколько раз	користуватися *чимось* ще раз, декілька разів
62.make, cárry out, réal-ize, fulfíl, ímplemènt, éxe-cùte the sùbstitútion $x=\varphi(t)$, $t=\psi(x)$	произвести подстановку, замену $x=\varphi(t)$, $t=\psi(x)$	виконати, зробити, здій-снити підстановку, за-міну $x=\varphi(t)$, $t=\psi(x)$
63.méthod of ìntegrátion	метод интегрирования	метод інтеґрування
64.méthod of úndetérmi-ned còeffícients	метод неопределённых коэффициентов	метод невизначених ко-ефіцієнтів
65.n-th degrée pòlynómial	многочлен n-ой степени	многочлен n-го степеня
66.odd fúnction with re-spéct to (co)sine	нечётная функция отно-сительно (ко)синуса	непарна функція віднос-но (ко)синуса
67.pártial/símplest/èle-méntary rátional fráction of the first (second, third, fourth) type/kind	простейшая рациональ-ная (элементарная) дробь первого (второго, третьего, четвёртого) типа	найпростіший раціона-льний (елементарний) дріб першого (другого, третього, четвертого) ти-пу
68.pass to the váriable t	перейти к переменной t	перейти до змінної t
69.prímitive [ántiderívative]	первообразная	первісна
70.próper fráction	правильная дробь	правильний дріб
71.quadrátic irràtionálity	квадратичная иррацио-нальность	квадратична іраціональ-ність
72.rátio of two pòlynó-mials	отношение двух много-членов	відношення двох много-членів

73.rátional fráction	рациональная дробь	раціональний дріб
74.rátional fúnction	рациональная функция	раціональна функція
75.redúce the fráctions to the cómmon denóminator	приводить дроби к общему знаменателю	зводити дроби до спільного знаменника
76.redúce the indéfinite íntegral to …	сводить неопределённый интеграл к…	зводити невизначений інтеґрал до …
77.redúce the ìntegrátion to that of a rátional fúnction	сводить интегрирование к интегрированию рациона-льной функции	зводити інтеґрування до інтеґрування раціональної функції
78.rèsolútion/expánsion/ devéloppement/dècomposítion of a próper rational fráction into (a sum of) pártial/símplest/èleméntary fráctions [partial fráction dècomposítion of...]	разложение правильной рациональной дроби на простейшие дроби (в сумму простейших дробей)	розвинення правильного раціонального дробу на найпростіші дроби (в суму найпростіших дробів)
79.resólve/expánd/devélop/dècompóse into (a sum of) símplest fráctions	разложить в сумму простейших дробей	розвинути в суму найпростіших дробів
80.retúrn to the oríginal/ inítial/precéding/prévious váriable *x*	возвратиться к (старой, первоначальной, предыдущей) переменной *x*	повернутися до (старої, початкової, попередньої) змінної *x*
81.set of all prímitives [ántiderívatives]	множество всех первообразных	множина всіх первісних
82.sùbstitútion	подстановка, замена	підстановка, заміна
83.táble of símplest íntegrals	таблица простейших интегралов	таблиця найпростіших інтеґралів
84.tábular íntegral	табличный интеграл	табличний інтеґрал
85.term óutsíde the íntegral, íntegrated term	внеинтегральный член	позаінтеґральний член
86.ùnivérsal trìgonométric(al) sùbstitútion	универсальная тригонометрическая подстановка	універсальна тригонометрична підстановка
87.váriable of ìntegrátion	переменная интегрирования	змінна інтеґрування

LECTURE NO. 21. DEFINITE INTEGRAL

POINT 1. PROBLEMS LEADING TO THE NOTION OF A DEFINITE INTEGRAL

POINT 2. DEFINITE INTEGRAL

POINT 3. PROPERTIES OF A DEFINITE INTEGRAL

POINT 4. DEFINITE INTEGRAL AS A FUNCTION OF ITS UPPER VARIABLE LIMIT

POINT 5. NEWTON-LEIBNIZ FORMULA

POINT 6. MAIN METHODS OF EVALUATION A DEFINITE INTEGRAL

POINT 1. PROBLEMS LEADING TO THE CONCEPT OFA DEFINITE INTEGRAL

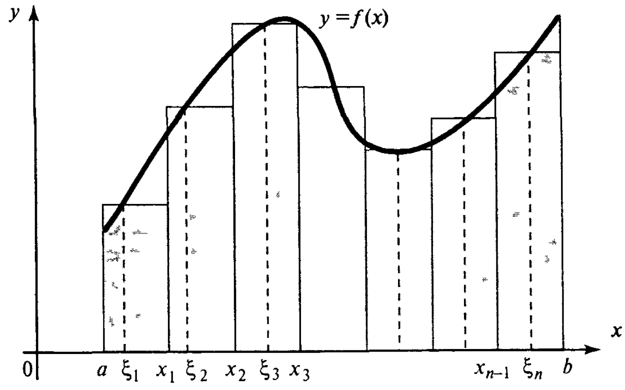


Fig. 1

Problem 1. Area of a curvilinear trapezium

Def. 1. Curvilinear trapezium on the xOy-plane is called a figure bounded by two straight lines $x = a$, $x = b$ $(a < b)$, Ox-axis and a curve $y = f(x)\,(f(x) \geq 0)$ (fig.1). It's useful to denote a curvilinear trapezium as the next point set on the xOy-plane

$$\{(x; y): a \leq x \leq b, 0 \leq y \leq f(x)\}. \qquad (1)$$

To define the notion of the area of a curvilinear trapezium (1), fig.1, we carry out the next construction.

1. With the help of points

$$a = x_0 < x_1 < x_2 < \ldots < x_{i-1} < x_i < \ldots < x_n = b$$

we divide the segment $[a, b]$ into *n* parts (subintervals)

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]$$

of the lengths

$$\Delta x_1 = x_1 - x_0,\ \Delta x_2 = x_2 - x_1, \ldots, \Delta x_i = x_i - x_{i-1}, \ldots, \Delta x_n = x_n - x_{n-1},$$

and let $\lambda = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_i, \ldots, \Delta x_n\}$.

2. We take an arbitrary point ξ_i in every part $[x_{i-1}, x_i]$, $i = \overline{1,n}$, find the value of the function at this point and multiply it by $\Delta x_i = x_i - x_{i-1}$.

3. Adding all these products $f(\xi_i)\Delta x_i$ we get a sum

$$\sigma = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \ldots + f(\xi_i)\Delta x_i + \ldots + f(\xi_n)\Delta x_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i \qquad (2)$$

- the area of a step-type figure generated by rectangles with bases Δx_i, $i = \overline{1,n}$, and altitudes $f(\xi_i)$, $i = \overline{1,n}$.

4. Let $\lambda = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_i, \ldots, \Delta x_n\}$ tend to zero. If there exists the limit of the sum (2), it is called the area of the curvilinear trapezium (1) (fig.1) and is denoted

$$S = S_{curv\_trap} = \lim_{\lambda \to 0} \sigma = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i . \qquad (3)$$

Problem 2. Produced quantity

Let $f(t)$ be a labour productivity of some factory at a time moment *t*. Find its produced quantity U during a time interval $[0, T]$.

If $f(t) = const = F$, then $U = F \cdot T$.

But as a rule $f(t) \neq const$, and we do as follows.

1. We divide the time segment [*0, T*] into *n* parts

$$[t_0, t_1], [t_1, t_2], \ldots, [t_{i-1}, t_i], \ldots, [t_{n-1}, t_n],\ t_0 = 0,\ t_n = T;\ \Delta t_i = t_i - t_{i-1},\ i = \overline{1,n},$$

and put

$$\lambda = \max\{\Delta t_1, \Delta t_2, \ldots, \Delta t_i, \ldots, \Delta t_n\}.$$

2. We take an arbitrary point τ_i in every part $[t_{i-1}, t_i]$, $i = \overline{1,n}$, find the value of the function $f(t)$ at this point and multiply it by $\Delta t_i = t_i - t_{i-1}$.

3. Adding all the products $\Delta U_i = f(\tau_i)\Delta t_i$ we find an approximate value of the produced quantity U during [0, *T*], that is

$$U = \sum_{i=1}^{n} \Delta U_i \approx \sigma = \sum_{i=1}^{n} f(\tau_i)\Delta t_i \ . \qquad (4)$$

4. Tending λ to zero we find the exact value of the produced quantity

$$U = \lim_{\lambda \to 0} \sigma = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\tau_i)\Delta t_i \ . \qquad (5)$$

Problem 3. Length path.

Find the length path *L* traveled by a material point with a given velocity $v(t)$ during a time interval *T* (from the time moment *t* = 0).

If $v(t) = const = v$ then $L = v \cdot T$.

For a variable velocity $v(t)$ we do by the same way as in preceding problems.

1. We divide the interval [*0, T*] into *n* parts

$$[t_0, t_1], [t_1, t_2], ..., [t_{i-1}, t_i], ..., [t_{n-1}, t_n], \ t_0 = 0, \ t_n = T; \ \Delta t_i = t_i - t_{i-1}, \ i = \overline{1, n}$$

and put

$$\lambda = \max\{\Delta t_1, \Delta t_2, ..., \Delta t_i, ..., \Delta t_n\}.$$

2. In every time interval $[t_{i-1}, t_i]$, $i = \overline{1, n}$, we take arbitrary moment τ_i, find the value of the velocity at this moment and multiply it by $\Delta t_i = t_i - t_{i-1}$.

3. Adding all the products $\Delta L_i = v(\tau_i)\Delta t_i$ we find an approximate value of the length path *L* traveled by a material point during [0, *T*], that is

$$L = \sum_{i=1}^{n} \Delta L_i \approx \sigma = \sum_{i=1}^{n} v(\tau_i)\Delta t_i \ . \qquad (6)$$

4. Tending λ to zero we find the exact value of the length path *L*

$$L = \lim_{\lambda \to 0} \sigma = \lim_{\lambda \to 0} \sum_{i=1}^{n} v(\tau_i)\Delta t_i \ . \qquad (7)$$

POINT 2. DEFINITE INTEGRAL

Def. 2. Let a function $y = f(x)$ be given on a segment [*a*, *b*] (fig. 2).

1. We divide the segment into *n* parts (subintervals)

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]$$

by points (division points)

$$a = x_0 < x_1 < x_2 < \ldots < x_{i-1} < x_i < \ldots < x_n = b;$$

let

$$\Delta x_1 = x_1 - x_0,\ \Delta x_2 = x_2 - x_1, \ldots,\ \Delta x_i = x_i - x_{i-1}, \ldots,\ \Delta x_n = x_n - x_{n-1},$$

$$\lambda = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_i, \ldots, \Delta x_n\}.$$

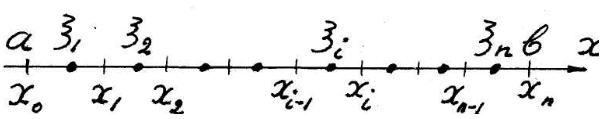


Fig. 2

2. We take an arbitrary point ξ_i in every subinterval $[x_{i-1}, x_i]$, $i = \overline{1, n}$, find the value of the function $f(x)$ at this point and multiply this value by the length $\Delta x_i = x_i - x_{i-1}$ of the subinterval.

3. Adding all these products $f(\xi_i)\Delta x_i$ we get a sum (**Cauchy**[1]**-Riemann**[2] **integral sum**)

$$\sigma = f(\xi_1)\Delta x_1 + f(\xi_2)\Delta x_2 + \ldots + f(\xi_i)\Delta x_i + \ldots + f(\xi_n)\Delta x_n = \sum_{i=1}^{n} f(\xi_i)\Delta x_i. \quad (8)$$

4. If there exists the limit of the integral sum (8) as $\lambda \to 0$, this limit is called the **definite integral** of the function $f(x)$ over the segment [*a*, *b*] and is denoted

$$\int_a^b f(x)dx = \lim_{\lambda \to 0} \sigma = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i. \quad (9)$$

We read the left side of (9) as “definite integral from *a* to *b* of $f(x)\,dx$”.

$f(x)$, $f(x)dx$, x have the same names as for indefinite integral; *a* is called the lower limit of integration, *b* the upper limit of integration.

Def. 3. A function $f(x)$ is called **integrable** on [over] the segment [*a*, *b*], if its

[1] Cauchy, A.L. (1780 - 1859), an eminent French mathematician
[2] Riemann G.F.B. (1826 - 1866), an eminent German mathematician

definite integral (9) exists.

Theorem 1 (*existence theorem***).** If a function $f(x)$ is continuous one on the segment [a, b] then it is integrable over this segment.

Geometric sense of a definite integral. If a function is non-negative, $f(x) \geq 0$, then by (2), (3) its definite integral is the area of a curvilinear trapezium (1), fig. 1,

$$S = S_{curv\_trap} = \lim_{\lambda \to 0} \sigma = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \int_a^b f(x)dx \qquad (10)$$

Economical sense of a definite integral. If a function $f(t)$ is a labour productivity of some factory, then its produced quantity *U* during a time interval [0, *T*] by virtue of (4), (5) is represented by a definite integral,

$$U = \lim_{\lambda \to 0} \sigma = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(\tau_i)\Delta t_i = \int_0^T f(t)dt\,. \qquad (11)$$

Physical sense of a definite integral. If a function $v(t)$ is the velocity of a material point then, on the base of (6), (7), the length path *L* traveled by the point during a time interval from *t* = 0 to *t* = *T* is given by a definite integral

$$L = \lim_{\lambda \to 0} \sigma = \lim_{\lambda \to 0} \sum_{i=1}^{n} v(\tau_i)\Delta t_i = \int_0^T v(t)dt \qquad (12)$$

Ex. 1. Prove that

$$\int_a^b dx = b - a \qquad (13)$$

■The integrand $f(x) \equiv 1$, and so the integral sum (8) equals the length of the segment $[a, b]$, that is

$$\sigma = \Delta x_1 + \Delta x_2 + \ldots + \Delta x_i + \ldots + \Delta x_n = \sum_{i=1}^{n} \Delta x_i = b - a\,,$$

therefore its limit, which is the integral (13), equals $b - a$.■

Note 1. Definite integral doesn't depend on a variable of integration that is

$$\int_a^b f(x)dx = \int_a^b f(y)dy = \int_a^b f(z)dz = \int_a^b f(t)dt = \ldots \qquad (14)$$

Def. 4 (definite integral with equal limits of integration)**.**

$$\int_a^a f(x)dx = 0 \qquad (15)$$

Def. 5 (interchanging limits of integration)**.**

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \qquad (16)$$

POINT 3. PROPERTIES OF A DEFINITE INTEGRAL

1 (homogeneity). A constant factor k can by taken outside the integral sign,

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx .$$

■ Integral sums for the left and right sides are equal, because of

$$\sigma_{kf(x)} = \sum_{i=1}^{n} kf(\xi_i)\Delta x_i = k\sum_{i=1}^{n} f(\xi_i)\Delta x_i = k\sigma_{f(x)},$$

therefore their limits are also equal.следовательно, их пределы также равны.■

2 (additivity with respect to an integrand). If $f_1(x)$, $f_2(x)$ be two integrable functions, then

$$\int_a^b (f_1(x) + f_2(x))dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx .$$

Prove this property yourselves.

Corollary (linearity). For any two integrable functions $f_1(x)$, $f_2(x)$ and arbitrary constants k_1, k_2

$$\int_a^b (k_1 f_1(x) + k_2 f_2(x))dx = k_1\int_a^b f_1(x)dx + k_2\int_a^b f_2(x)dx .$$

3 (additivity with respect to an interval of integration). For any a, b, c

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

if all three integrals exist.

■1) Let's suppose at first $c \in (a, b)$. We form an integral sum such that c be a division point. In this case (notations are clear)

$$\sigma_{[a,\, b]} = \sigma_{[a,\, c]} + \sigma_{[c,\, b]},$$

and the passage to limit as $\lambda \to 0$ gives the property in question.

2) Now let a disposition of points a, b, c be arbitrary, for example $a < b < c$. Using the first case and the definitions 4, 5 we'll have

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^b f(x)dx - \int_c^b f(x)dx,$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \,.■$$

Integration of inequalities

4. If $a < b$ and an integrand is non-negative ($f(x) \ge 0$) on the segment $[a, b]$, then

$$\int_a^b f(x)dx \ge 0\,.$$

The integral is strictly positive if a function $f(x)$is continuous on the segment $[a, b]$ and if it doesn't equal zero identically.

■ Nonnegativity of the integral immediately follows from nonnegativity of the integral sum for the function $f(x)$. Its strict positivity, as can be proved by more complicated reasonings, is the result of continuity of the function.■

5. If $a < b$ and $f(x) \le g(x)$ on $[a, b]$, then

$$\int_a^b f(x)dx \le \int_a^b g(x)dx\,.$$

The integrals are connected by strict inequality in the case of continuity of the functions $f(x)$, $g(x)$ on the segment $[a, b]$ and if these functions don't equal identically.

■It's sufficient to apply the preceding property to a difference $f(x) - g(x)$■

Ex. 2. $\int\limits_0^{\frac{\pi}{2}} \sin^2 x dx \le \int\limits_0^{\frac{\pi}{2}} \sin x dx$, because of $\sin^2 x \le \sin x$ on the segment $\left[0, \frac{\pi}{2}\right]$.

6. If $a < b$ then

$$\left|\int\limits_a^b f(x)dx\right| \le \int\limits_a^b |f(x)|dx \qquad (17)$$

■ It's sufficient to apply the property 5 to the inequality

$$-|f(x)| \le f(x) \le |f(x)| \blacksquare$$

7 (two-sided estimate of a definite integral). Let $a < b$, and a function $f(x)$ be continuous on a segment $[a, b]$. Then a double inequality is valid

$$m(b-a) \le \int\limits_a^b f(x)dx \le M(b-a), \text{ where } m = \min_{[a,\, b]} f(x),\ M = \max_{[a,\, b]} f(x). \qquad (18)$$

■Proving follows from the property 5, the inequality $m \le f(x) \le M$ on $[a, b]$ and the integral (13).■

Ex. 3. Estimate the integral $I = \int\limits_1^4 (3x^2 - 12x + 14)dx$.

$$f(x) = 3x^2 - 12x + 14, f'(x) = 6x - 12, f'(x) = 0 \text{ if } x = 2;\ f(1) = 5, f(2) = 2, f(4) = 14;$$
$$m = \min_{[1,\,4]} f(x) = f(2) = 2,\ M = \max_{[1,\,4]} f(x) = f(4) = 14,\ a = 1,\ b = 4,\ b - a = 3,$$

and by the formula (18)

$$2 \cdot 3 \le \int\limits_1^4 (3x^2 - 12x + 14)dx \le 14 \cdot 3,$$

$$6 \le \int\limits_1^4 (3x^2 - 12x + 14)dx \le 42.$$

8. ***Mean-value theorem*****.** If a function $f(x)$ is continuous on a segment $[a, b]$ then there exists a point $c \in (a, b)$ such that

$$\int\limits_a^b f(x)dx = f(c)(b-a) \qquad (19)$$

■ For example let $a < b$. After dividing of both sides of the inequality (18) by the po-

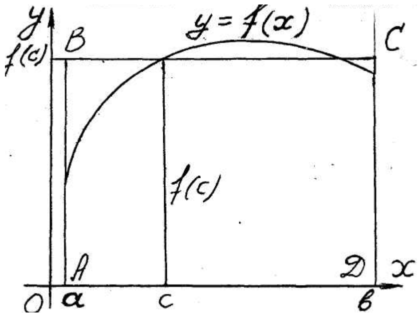


Fig. 3

sitive number $b-a$ we get

$$m \le \frac{1}{b-a}\int_a^b f(x)dx \le M .$$

By Bolzano[1]–Cauchy theorem for a function, continuous on a segment $[a, b]$, there is a point $c \in (a, b)$ such that

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c) \Rightarrow \int_a^b f(x)dx = f(c)(b-a).$$

The case $a > b$ is studied by the same way. Do it yourselves.■

Geometric sense of mean-value theorem (fig. 3). The area of a curvilinear trapezium (1) equals the area of the rectangle *ABCD* with the same base *AD*=[*a, b*] and the altitude $f(c)$.

Def 6. The expression

$$f_{mean} = f_{av} = \frac{1}{b-a}\int_a^b f(x)dx \qquad (20)$$

is called the **mean value** (the **average value**) of the function $f(x)$ on the segment [*a, b*].

POINT 4. DEFINITE INTEGRAL AS A FUNCTION OF ITS UPPER VARIABLE LIMIT

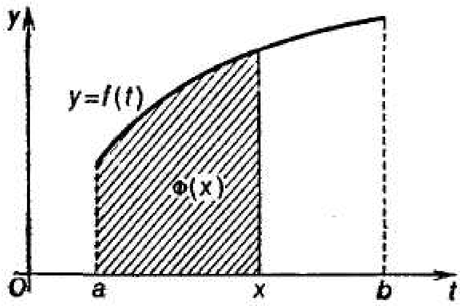


Fig. 4

Let $x \in [a, b]$. We consider a function

$$\Phi(x) = \int_a^x f(t)dt, \qquad (21)$$

that is a definite integral with variable upper limit *x*. Geometrically (for $f(t) \ge 0$) this integral represents the area of a part of a curvilinear trapezium

[1] Bolzano, B. (1781 - 1848), a Czech mathematician, philosopher, and logician

$$\{(t, y): a \le t \le b, 0 \le y \le f(t)\}$$

which lies between straight lines $t = a, t = x$ (fig. 4).

Theorem 2. If a function $f(x)$ is continuous on a segment [*a*, *b*] then for any $x \in [a, b]$ the derivative of the integral (21) equals

$$\Phi'(x) = \left(\int_a^x f(t)dt \right)' = f(x), \qquad (22)$$

that is the derivative of a definite integral with a variable upper limit *x*, with respect to this limit *x*, equals the value of the integrand at the point *x*.

■By definition of the derivative

$$\Phi'(x) = \lim_{\Delta x \to 0} \frac{\Delta\Phi(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Phi(x + \Delta x) - \Phi(x)}{\Delta x}.$$

Using the additivity of a definite integral with respect to an interval of integration we get

$$\Phi(x + \Delta x) = \int_a^{x+\Delta x} f(t)dt = \int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt, \; \Delta\Phi(x) = \Phi(x + \Delta x) - \Phi(x) = \int_x^{x+\Delta x} f(t)dt.$$

For example let $\Delta x > 0$. By virtue of the mean-value theorem there is a point c in the interval $(x, x + \Delta x)$ such that

$$\int_x^{x+\Delta x} f(t)dt = f(c)((x + \Delta x) - x) = f(c)\Delta x .$$

This point $c \to x$ as $\Delta x \to 0$. Taking into account continuity of the function *f* we get

$$\Phi'(x) = \lim_{\Delta x \to 0} \frac{\Delta\Phi(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} f(c) = f(x). \blacksquare$$

***Corollary* (*Fundamental theorem of the integral calculus*).** Every function which is continuous one on a segment [*a, b*] has a primitive on [*a, b*].

■One of such primitives is a definite integral (21) with variable upper limit *x*.■

Ex. 4. The derivative of the function $\int_{\sin x}^{\cos x} t^2 dt$ equals

$$\left(\int\limits_{\sin x}^{\cos x} t^2 dt\right)' = \left(\int\limits_0^{\cos x} t^2 dt - \int\limits_0^{\sin x} t^2 dt\right)' = \frac{d}{d\cos x}\left(\int\limits_0^{\cos x} t^2 dt\right)\cdot(\cos x)' - \frac{d}{d\sin x}\left(\int\limits_0^{\sin x} t^2 dt\right)\cdot(\sin x)' =$$

$$= -\cos^2 x\cdot\sin x - \sin^2 x\cdot\cos x = -\sin x\cos x(\cos x + \sin x) = -\frac{1}{2}\sin 2x(\cos x + \sin x).$$

POINT 5. NEWTON-LEIBNIZ FORMULA

Theorem 3. If a function $f(x)$ is continuous one on a segment [*a, b*], and *F*(*x*) is one of its primitives, then **Newton[1]-Leibniz[2] formula** for evaluation of the definite integral of the function over the segment [*a, b*] is true

$$\int\limits_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) \qquad (23)$$

■ We have two primitives: $F(x)$ and the integral (21) with upper variable limit *x*. By corresponding property of primitives the difference

$$\Phi(x) - F(x) = \int\limits_a^x f(t)dt - F(x) = C = \text{const}.$$

To find the value of the constant *C* we put $x = a$. So

$$\Phi(a) - F(a) = \int\limits_a^a f(t)dt - F(a) = 0 - F(a) = C,\ C = -F(a),$$

and therefore

$$\int\limits_a^x f(t)dt = F(x) - F(a).$$

Substituting *x* by *b* and *t* by *x* we obtain the formula (23).■

Note 1. The expression

$$F(x)\Big|_a^b,$$

which means the action $F(b) - F(a)$, is often called the **double substitution**.

---

[1] Newton, I. (1642 - 1727), the great English scientist
[2] Leibniz, G. (1646 – 1717), the great German philosopher and mathematician

Ex. 5. Calculate the definite integral $\int_0^{\pi/2} \cos x dx$.

A primitive of $\cos x$ is $\sin x$ and therefore by Newton-Leibniz formula

$$\int_0^{\pi/2} \cos x dx = \sin x \Big|_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1.$$

Ex. 6. Find the area of a figure bounded by the next lines $y = x^2$, $x = 2, y = 0$ (fig. 5).

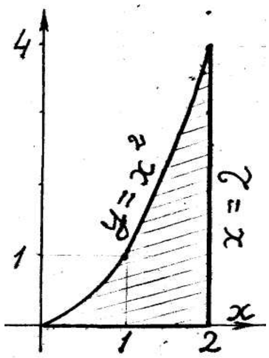


Fig. 5

The figure in question is a curvilinear trapezium, and so its area by the formula (10) equals the definite integral

$$S = \int_0^2 x^2 dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3} \approx 2.67.$$

Ex. 7. A particle moves in a straight line and *t* sec. after passing a point *O* the velocity of the particle is $v(t) = \left(1 - 1/4\,t^2\right)$ ft. per sec. Find the distance of the particle from *O* after 2 sec.

On the base of the formula (12) the distance in question equals

$$L = \int_0^2 v(t)dt = \int_0^2 \left(1 - 1/4\,t^2\right)dt = \left(t - 1/12\,t^3\right)\Big|_0^2 = \left(2 - 1/12 \cdot 2^3\right) = 4/3 \text{ ft.}$$

Ex. 8. Find the mean value of the function $y = \sin x$ on the segment $[0, \pi/2]$.

On the base of the formula (20)

$$y_{mean} = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} \cos x dx = \frac{2}{\pi} \cdot \sin x \Big|_0^{\pi/2} = \frac{2}{\pi}(\sin \pi/2 - \sin 0) = \frac{2}{\pi}(1 - 0) = \frac{2}{\pi}.$$

Ex. 9. Find the mean value of the velocity $v(t) = \left(1 - 1/4\,t^2\right)$ of the particle of the example 7 during the time interval from $t = 0$ to $t = 2$.

By (20) (and taking into account the result of integration in ex. 7) we get

$$v_{mean} = \frac{1}{2 - 0}\int_0^2 v(t)dt = \frac{1}{2}\int_0^2 \left(1 - 1/4\,t^2\right)dt = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}.$$

POINT 6. MAIN METHODS OF EVALUATION A DEFINITE INTEGRAL

Change of a variable (substitution method)

Theorem 4. Let: 1) a function $f(x)$ is continuous on a segment [*a*, *b*]; 2) a function $x=\varphi(t)$ is continuous with its derivative on a segment [*α*, *β*]; 3) φ (*α*) = *a*, φ (*β*) = *b*. Then the next formula (**formula of change of a variable**) is true

$$\int_a^b f(x)dx=\left|\begin{array}{c} x=\varphi(t), dx=\varphi'(t)dt, \\ \begin{array}{c|c|c} x & a & b \\ \hline t & \alpha & \beta \end{array}\end{array}\right|=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt. \qquad (24)$$

■Let $F(x)$ be some primitive of a function $f(x)$. Then $F(\varphi(t))$ is the primitive of the function $f(\varphi(t))\varphi'(t)$. By Newton-Leibniz formula

$$\int_a^b f(x)dx=F(x)\Big|_a^b=F(b)-F(a);$$

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt=F(\varphi(t))\Big|_\alpha^\beta=F(\varphi(\beta))-F(\varphi(\alpha))=F(b)-F(a).■$$

Note 2. As distinct from an indefinite integral it isn't necessary to return to the preceding variable after integration by the formula (24).

Ex. 9. Calculate the definite integral $I=\int_{-\pi/2}^{\pi/2}\frac{\cos xdx}{4+\sin^2 x}$.

Let's put $\sin x=t$. Then $\cos xdx=dt$, $\begin{array}{c|c|c} x & -\pi/2 & \pi/2 \\ \hline t & -1 & 1 \end{array}$, so

$$I=\int_{-\pi/2}^{\pi/2}\frac{\cos xdx}{4+\sin^2 x}=\int_{-1}^{1}\frac{dt}{4+t^2}=\frac{1}{2}\arctan\frac{t}{2}\Big|_{-1}^{1}=\frac{1}{2}\left(\arctan\frac{1}{2}-\arctan\left(-\frac{1}{2}\right)\right)=\arctan\frac{1}{2}$$

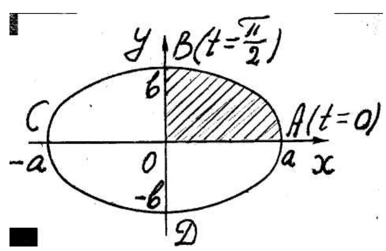


Fig. 6

Ex. 10. Find the area of a figure bounded by an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (fig. 6).

It's sufficient to find the quadruplicated area of the part *OAB* of the figure.

The first way. From the equation of the ellipse $y=\frac{b}{a}\sqrt{a^2-x^2}$, and so

$$S=4S_{OAB}=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx=\left|\begin{array}{c} x=a\sin t, dx=a\cos t dt,\\ \sqrt{a^2-x^2}=a\cos t,\ \begin{array}{c|c|c} x & 0 & a\\ \hline t & 0 & \pi/2\end{array}\end{array}\right|=4ab\int_0^{\pi/2}\cos^2 t dt=$$

$$=4ab\int_0^{\pi/2}\frac{1+\cos 2t}{2}dt=2ab\left(\int_0^{\pi/2}dt+\int_0^{\pi/2}\cos 2t dt\right)=2ab\left(t\Big|_0^{\pi/2}+\frac{1}{2}\sin 2t\Big|_0^{\pi/2}\right)=\pi ab$$

The second way. It's better to pass to parametric equations of the ellipse, namely $x=a\cos t, y=b\sin t$ [1]. In this case

$$S=4S_{OAB}=4\int_0^a ydx=\left|\begin{array}{c} x=a\cos t, y=b\sin t\\ dx=-a\sin t dt\ \begin{array}{c|c|c} x & 0 & a\\ \hline t & \pi/2 & 0\end{array}\end{array}\right|=-4ab\int_{\pi/2}^0\sin^2 t dt=4ab\int_0^{\pi/2}\sin^2 t dt=$$

$$=4ab\int_0^{\pi/2}\frac{1-\cos 2t}{2}dt=2ab\left(\int_0^{\pi/2}dt-\int_0^{\pi/2}\cos 2t dt\right)=2ab\left(t\Big|_0^{\pi/2}-\frac{1}{2}\sin 2t\Big|_0^{\pi/2}\right)=\pi ab.$$

Integration by parts

Theorem 5. If functions $u=u(x), v=v(x)$ are continuous with their derivatives on a segment $[a,b]$, then the next formula (**formula of integration by parts**) is true

$$\int_a^b udv=(uv)\Big|_a^b-\int_a^b vdu \qquad (25)$$

■To prove this formula it's sufficient to integrate from *a* to *b* both parts of the identity

$$udv=d(uv)-vdu$$

and apply Newton-Leibniz formula for the integral of the expression $d(uv)$■

Ex. 11. Evaluate the definite integral $\int_0^1\arctan x dx$.

[1] See Lect. No. 8, Point 4, Ex. 12

$$\int_0^1 \arctan x dx = \left| \begin{matrix} u = \arctan x,\ dv = dx \\ du = \dfrac{dx}{1+x^2},\ v = x \end{matrix} \right| = \left(x \arctan x\right)\Big|_0^1 - \int_0^1 \frac{xdx}{1+x^2} = \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{2xdx}{1+x^2} =$$

$$= \frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{\left(1+x^2\right)' dx}{1+x^2} = \frac{\pi}{4} - \frac{1}{2}\ln\left(1+x^2\right)\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2 \approx 0.785 - 0.347 \approx 0.44$$

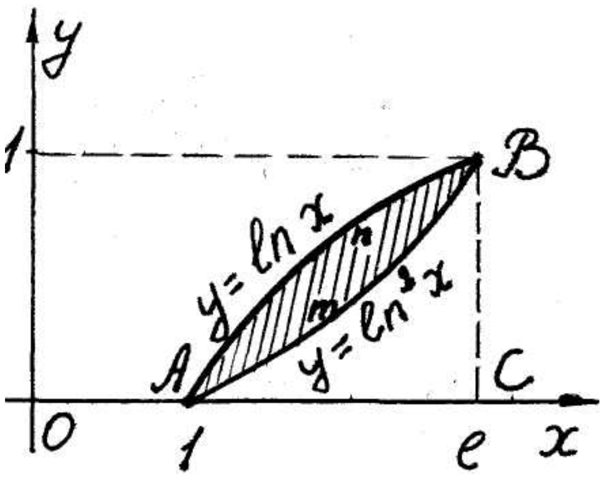


Fig. 7

Ex. 12. Find the area of a figure bounded by two curves $y = \ln x, y = \ln^2 x$ (see fig. 7).

The curves $y = \ln x, y = \ln^2 x$ intersect at the points $A(1;\ 0), B(e;\ 1)$ and form a given figure $AmBnA$. Its area is equal to the difference of the areas of two curvilinear trapeziums $AnBC, AmBC$.

$$S = S_{AnBC} - S_{AmBC} = \int_1^e \left(\ln x - \ln^2 x\right)dx = \left| \begin{matrix} u = \ln x - \ln^2 x,\ dv = dx, \\ du = \left(\dfrac{1}{x} - \dfrac{2\ln x}{x}\right)dx,\ v = x \end{matrix} \right| = \left(x\left(\ln x - \ln^2 x\right)\right)\Big|_1^e -$$

$$-\int_1^e \left(\frac{1}{x} - \frac{2\ln x}{x}\right)xdx = \int_1^e (2\ln x - 1)dx = \left| \begin{matrix} u = 2\ln x - 1,\ dv = dx, \\ du = \dfrac{2dx}{x},\ v = x \end{matrix} \right| = \left(x(2\ln x - 1)\right)\Big|_1^e - 2\int_1^e dx =$$

$$= e - (-1) - 2(e-1) = 3 - e \approx 0.28.$$

Ex. 13. Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx,\ J_n = \int_0^{\frac{\pi}{2}} \cos^n x dx,\ n = 0, 1, 2, 3, \ldots$ Prove that

$$I_n = \frac{n-1}{n} I_{n-2},\ J_n = \frac{n-1}{n} J_{n-2}.$$

$$\blacksquare\ I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \left| \begin{matrix} u = \sin^{n-1} x,\ dv = \sin x dx \\ du = (n-1)\sin^{n-2} x \cos x dx, v = -\cos x \end{matrix} \right| = \left(-\sin^{n-1} x \cos x\right)\Big|_0^{\frac{\pi}{2}} +$$

$$+ (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\left(1 - \sin^2 x\right)dx = (n-1)I_{n-2} - (n-1)I_n,$$

$$I_n + (n-1)I_n = (n-1)I_{n-2},\ nI_n = (n-1)I_{n-2},\ I_n = \frac{n-1}{n} I_{n-2} \blacksquare$$

For example $\int_0^{\frac{\pi}{2}} \sin^5 x dx = I_5 = \frac{4}{5} I_3 = \frac{4}{5}\cdot\frac{2}{3} I_1 = \frac{8}{15}\int_0^{\frac{\pi}{2}} \sin x dx = \frac{8}{15}\cdot 1 = \frac{8}{15}$.

$\int_0^{\frac{\pi}{2}} \cos^6 x dx = J_6 = \frac{5}{6} J_4 = \frac{5}{6}\cdot\frac{3}{4} J_2 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2} J_0 = \frac{15}{48}\int_0^{\frac{\pi}{2}} dx = \frac{15}{48}\cdot\frac{\pi}{2} \approx 0.49$.

Ex. 14. Find the remainder $R_n(x)$ of Taylor formula in Lagrange form.

Let, for example, $n = 1$, and

$$\Delta f(x_0) = f(x) - f(x_0) = f'(x_0)(x - x_0) + R_1(x).$$

By Lagrange formula

$$\varphi(x) - \varphi(x_0) = \varphi'(c)(x - x_0).$$

Taking $\varphi(x) = f'(x)$ we get

$$f'(x) - f'(x_0) = f''(c)(x - x_0),$$

and after integration over the segment $[x_0, x]$

$$\int_{x_0}^{x} f'(x)dx - f'(x_0)\int_{x_0}^{x} dx = f''(c)\int_{x_0}^{x}(x - x_0)dx,\ f(x) - f(x_0) - f'(x_0)(x - x_0) = \frac{f''(c)}{2}(x - x_0)^2,$$

$$\Delta f(x_0) = f(x) - f(x_0) = f'(x_0)(x - x_0) + \frac{f''(c)}{2}(x - x_0)^2.$$

Therefore

$$R_1(x) = \frac{f''(c)}{2}(x - x_0)^2 = \frac{f''(c)}{1\cdot 2}(x - x_0)^2 = \frac{f''(c)}{2!}(x - x_0)^2.$$

To get $R_n(x)$ for any *n* we write

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2 + \ldots + f^{(n)}(x_0)(x - x_0)^n + R_n(x),$$

then we put $\varphi(x) = f^{(n)}(x)$ and integrate *n* times over $[x_0, x]$. As the result we'll get

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

LECTURE NO.22. APPLICATIONS OF DEFINITE INTEGRAL

POINT 1. PROBLEM – SOLVING SCHEMES. AREAS.
POINT 2. ARC LENGTH
POINT 3. VOLUMES
POINT 4. ECONOMIC APPLICATIONS

POINT 1. PROBLEM – SOLVING SCHEMES. AREAS

There are two schemes for finding some quantity Q.

1) Setting integral sum, expression Q as the limit of this integral sum, that is in the form of a definite integral.

Ex. 1. See three problems in P. 1 of the 21-th lecture and corresponding formulas (10), (11), (12) in P. 2 of the same lecture.

2) Finding an element (in fact the differential) dQ of Q and then expression Q as a sum of all these elements. This procedure leads by the other way to a definite integral to evaluate Q.

To illustrate this second scheme we'll consider the same three problems as in the lecture No. 21.

Ex. 2. The area of a curvilinear trapezium

$$\{(x; y): a \le x \le b, 0 \le y \le f(x)\} \text{ (fig. 1).}$$

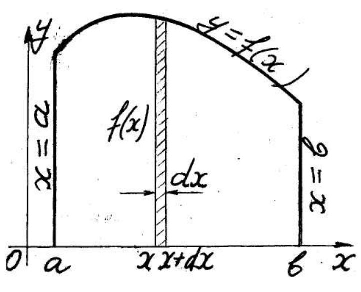


Fig. 1

Element (differential) dS of the area S (the area of a hatched strip with the base $[x, x+dx]$ on fig.1) equals

$$dS = f(x)dx.$$

Adding all these elements from a to b we get the required area as known definite integral

$$S = \int_a^b f(x)dx \qquad (1)$$

Ex. 3. Produced quantity U of a factory during a time interval $[0, T]$. Let $f(t)$

be a labour productivity of the factory at an arbitrary time moment t. The element dU of the produced quantity during an infinitely small time interval $[t, t+dt]$ equals

$$dU = f(t)dt.$$

Adding all these elements from 0 to T we get the sought produced quantity U, namely

$$U = \int_0^T f(t)dt \qquad (2)$$

Ex. 4. The length path L traveled by a material point during a time interval from $t = 0$ to $t = T$ with the velocity $v(t)$.

An element dL of the length path traveled during an infinitely small time interval $[t, t+dt]$ is equal to

$$dL = v(t)dt,$$

and the sum of all these elements from 0 to T gives the length path L to be found

$$L = \int_0^T v(t)dt. \qquad (3)$$

Certainly, results (1), (2), (3) coincide with those (10), (11), (12) of the preceding lecture.

Additional remarks about the areas of plane figures

a) The case of **nonpositive function**. The area of a figure

$$\{(x, y): a \le x \le b, f(x) \le y \le 0\}, f(x) \le 0,$$

represented of the fig. 2, equals

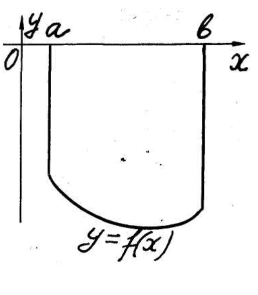


Fig. 2

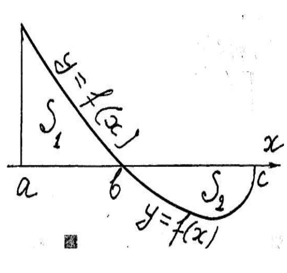


Fig. 3

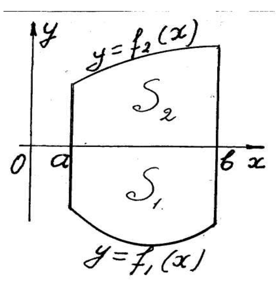


Fig. 4

$$S = -\int_a^b f(x)dx. \qquad (4)$$

b) The case when a function has **different signs** on various intervals. Let a function $f(x)$ be nonnegative on the interval $[a,b]$ and nonpositive on the interval $[b,c]$. The area of a corresponding figure, represented on the fig. 3, equals

$$S = S_1 + S_2 = \int_a^b f(x)dx - \int_b^c f(x)dx. \qquad (5)$$

c) The case when a figure lies **between two curves**. The area of a figure

$$\{(x,y): a \le x \le b,\, f_1(x) \le y \le f_2(x)\},$$

represented on the fig. 4, equals

$$S = \int_a^b (f_2(x) - f_1(x))dx. \qquad (6)$$

Ex. 5. Find the area enclosed by two curves $y = 4 - x^2$, $y = x^2 - 2$ (fig. 5).

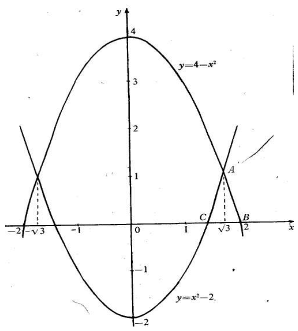


Fig. 5

Intersection points of the curves have abscises $\pm\sqrt{3}$. The figure is symmetric about the *Oy*-axis; we may find double area of its right part.

$$S = 2\int_0^{\sqrt{3}} \left((4 - x^2) - (x^2 - 2)\right)dx = 2\int_0^{\sqrt{3}} (6 - 2x^2)dx = 2\left[6x - \frac{2}{3}x^3\right]_0^{\sqrt{3}} =$$

$$= 2\left(6\sqrt{3} - \frac{2}{3}\cdot 3\sqrt{3}\right) = 8\sqrt{3} \approx 13.9.$$

d) Figures **oriented with respect to the *Oy*-axis**. Areas of figures

$$\{(x,y): c \le y \le d,\, 0 \le x \le g(y)\} \text{ (see fig. 6)},$$

$$\{(x,y): c \le y \le d,\, g(y) \le x \le 0\},\ g(y) \le 0 \text{ (see fig. 7)},$$

$$\{(x,y): c \le y \le d,\, 0 \le x \le g(y)\} \cup \{(x,y): d \le y \le e,\, g(y) \le x \le 0\} \text{ (see fig. 8)},$$

$$\{(x,y): c \le y \le d,\, g_1(y) \le x \le g_2(y)\} \text{ (see fig. 9)},$$

are equal respectively

$$S = \int_c^d g(y)dy, \qquad (7)$$

$$S = -\int_c^d g(y)dy, \tag{8}$$

$$S = S_1 + S_2 = \int_c^d g(y)dy - \int_d^e g(y)dy, \tag{9}$$

$$S = \int_c^d (g_2(y) - g_1(y))dy. \tag{10}$$

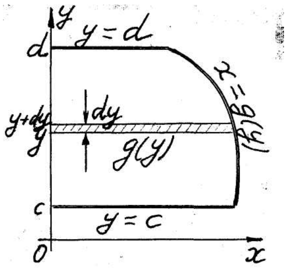


Fig. 6

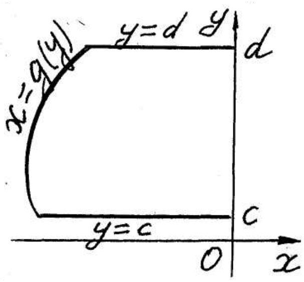


Fig. 7

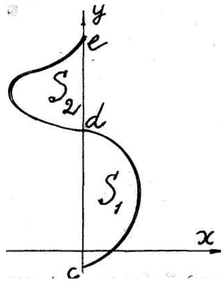


Fig. 8

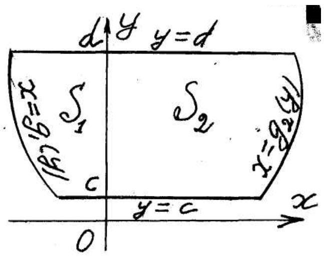


Fig. 9

Ex. 6. Find the area of a figure bounded by curves $y^2 = 4x + 4$, $y^2 = -2x + 4$ (fig. 10).

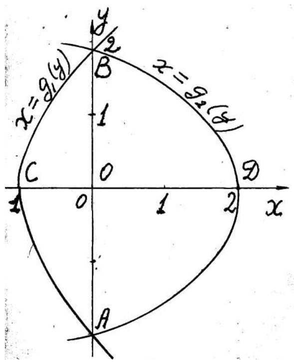


Fig. 10

It's well to write the equations of the curves in the next form

$$x = g_1(y) = \frac{y^2 - 4}{4} \ (ACB), \ x = g_2(y) = \frac{4 - y^2}{2} \ (ADB)$$

and to utilize the formula (10).

By virtue of symmetry of the figure with respect to the *Ox*-axis

$$S = 2S_{BCD} = 2\int_0^2 (g_2(y) - g_1(y))dy = 2\int_0^2 \left(\frac{4 - y^2}{2} - \frac{y^2 - 4}{4}\right)dy = \frac{1}{2}\int_0^2 (12 - 3y^2)dy = 2.$$

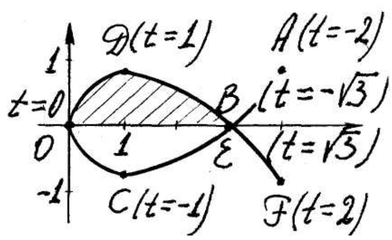


Fig. 11

e) The case of **parametrically represented curve**. Let, for example, be given a curvilinear trapezium $\{(x; y): a \le x \le b, 0 \le y \le f(x)\}$, (fig. 1), but a curve $y = f(x)$ is determined by parametric equations

$$x = x(t),\ y = y(t)$$

($x = a$ for $t = \alpha$ and $x = b$ for $t = \beta$).

To find the area of such the trapezium we change a variable in the integral (1), namely

$$S = \int_a^b f(x)dx = \left| \begin{array}{l} x = x(t),\ f(x) = y(t) \\ dx = x'(t)dt, \begin{array}{c|c|c} x & a & b \\ \hline t & \alpha & \beta \end{array} \end{array} \right| = \int_\alpha^\beta y(t)x'(t)dt .$$

Ex. 7. Find the area of the loop of a curve $x = t^2, y = t - \dfrac{t^3}{3}$ (fig. 11).

To construct the curve point by point and to see the loop we equate to zero the expressions $x = t^2,\ y = t - \dfrac{t^3}{3}$, $x' = 2t,\ y' = 1 - t^2$ and then form the next table

t	-2	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	2
x	4	3	1	0	1	3	4
y	$\frac{2}{3}$	0	$-\frac{2}{3}$	0	$\frac{2}{3}$	0	$-\frac{2}{3}$
Point	A	B	C	O	D	E	F

From the fig. 11 we see that $S = 2S_{ODEO}$, and so

$$S = 2S_{ODEO} = 2\int_0^3 f(x)dx = \left| \begin{array}{l} x = t^2, f(x) = t - \dfrac{t^3}{3} \\ dx = 2tdt, \begin{array}{c|c|c} x & 0 & 3 \\ \hline t & 0 & \sqrt{3} \end{array} \end{array} \right| = 2\int_0^{\sqrt{3}} \left(t - \frac{t^3}{3} \right) 2tdt = 4\int_0^{\sqrt{3}} \left(t^2 - \frac{t^4}{3} \right) dt = \frac{8\sqrt{3}}{5} .$$

f) The area **in polar coordinates**.

Given a curvilinear sector (or a curvilinear triangle) that is a plane figu-

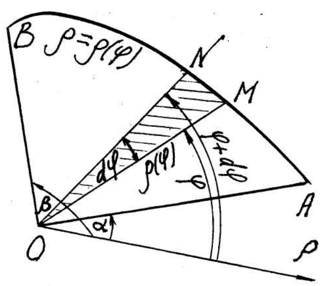


Fig. 12

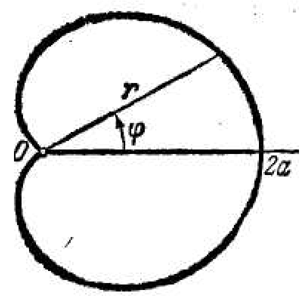


Fig. 13

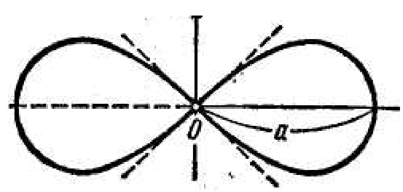


Fig. 14

re, bounded by two rays $OA:\varphi=\alpha, OB:\varphi=\beta$ $(\alpha<\beta)$ and a curve AB given in polar coordinates $\rho=f(\varphi)$ (fig. 12). Find the area of the curvilinear sector.An element dS of the area is the area of a hatched circular sector with the radius $OM=\rho=f(\varphi)$ and a central angle $d\varphi$,

$$dS=\frac{\pi\cdot OM^2}{2\pi}\cdot d\varphi=\frac{1}{2}OM^2\cdot d\varphi=\frac{1}{2}\rho^2\cdot d\varphi=\frac{1}{2}f^2(\varphi)\cdot d\varphi.$$

Adding all these elements from α to β we get the area in question

$$S=\frac{1}{2}\int_\alpha^\beta \rho^2 d\varphi=\frac{1}{2}\int_\alpha^\beta f^2(\varphi)d\varphi. \qquad (11)$$

Ex. 8. Find the area of a figure bounded by a cardioid $\rho=a(1+\cos\varphi)$[1] (fig. 13).

Desired area equals the twofold area of the upper part of the figure, which is a circular sector with the rays $\varphi=0, \varphi=\pi$.

$$S=2\cdot\frac{1}{2}\int_0^\pi \rho^2 d\varphi=\int_0^\pi (a(1+\cos\varphi))^2 d\varphi=a^2\int_0^\pi (1+2\cos\varphi+\cos^2\varphi)d\varphi=$$

$$=a^2\int_0^\pi (1+2\cos\varphi+\frac{1+\cos 2\varphi}{2})d\varphi=a^2\left(\frac{3}{2}\varphi+2\sin\varphi+\frac{1}{4}\sin 2\varphi\right)\Bigg|_0^\pi=\frac{3}{2}\pi a^2.$$

Ex. 9. Find the area of a figure bounded by Bernoulli lemniscate[2]

$$\left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right).$$

The figure is symmetric with respect to the *Ox*-, *Oy*-axes and lies in the angles determined by the straight lines $y=\pm x$ $(|y|\le|x|)$ (fig. 14). Passing to polar coordinates $x=\rho\cos\varphi, y=\rho\sin\varphi, x^2+y^2=\rho^2$, we write the equation of the lemniscate in more suitable form

$$\left((\rho\cos\varphi)^2+(\rho\sin\varphi)^2\right)^2=a^2\left((\rho\cos\varphi)^2-(\rho\sin\varphi)^2\right), \rho^4\left(\sin^2\varphi+\cos^2\varphi\right)=$$
$$=a^2\rho^2\left(\cos^2\varphi-\sin^2\varphi\right), \rho^2=a^2\cos 2\varphi, \rho=a\sqrt{\cos 2\varphi}.$$

Then we calculate the quadruplicated area of the circular sector bounded by the lem-

[1] See Lecture NO. 4, Point 2, Ex. 4

[2] Ib., Ex. 5

niscate and the rays $\varphi = 0$ $\varphi = \pi/4$. We'll obtain

$$S = 4 \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} \rho^2 d\varphi = 2 \int_0^{\frac{\pi}{4}} \left(a\sqrt{\cos 2\varphi}\right)^2 d\varphi = 2a^2 \int_0^{\frac{\pi}{4}} \cos 2\varphi d\varphi = a^2 \sin 2\varphi \Big|_0^{\frac{\pi}{4}} = a^2 .$$

POINT 2. ARC LENGTH

Let $\breve{AB}$ be an arc of some curve, and it's necessary to find its length.

The first method. We divide the arc $\breve{AB}$ into *n* parts by points $M_0 = A, M_1, M_2, \ldots, M_{n-1}, M_n = B$ and inscribe the poly-gonal line $M_0 M_1 M_2 \ldots M_{n-1} M_n$ in $\breve{AB}$ (fig. 15). Let

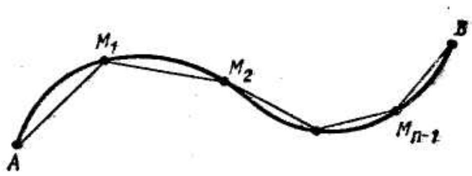


Fig. 15

$$P_n = \sum_{i=1}^{n} M_{i-1}M_i = M_0M_1 + M_1M_2 + \ldots + M_{n-1}M_n \quad (12)$$

is the perimeter of the polygonal line and $\lambda = \max_i M_{i-1}M_i$. If there exists the limit

$$L = \lim_{\lambda \to 0} P_n \quad (13)$$

it is called the length of the arc $\breve{AB}$.

Let an arc $\breve{AB}$ of a curve be determined in Cartesian coordinates by an equation

$$y = f(x) \quad (14)$$

on a segment $[a, b]$, and x_i, y_i $\left(i = \overline{0, n}\right)$ be the coordinates of the point M_i, $M_i(x_i, y_i)$, $x_0 = a, x_n = b$. In this case

$$M_{i-1}M_i = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} ,$$

and by Lagrange theorem there is a point $c_i \in [x_{i-1}, x_i]$ such that

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}).$$

Denoting $x_i - x_{i-1} = \Delta x_i$ we get $M_{i-1}M_i = \sqrt{1 + f'^2(c_i)}\Delta x_i$ and therefore

$$P_n = \sum_{i=1}^{n} \sqrt{1 + f'^2(c_i)}\Delta x_i .$$

Passage to the limit gives the desired arc length as a definite integral from *a* to *b*,

$$L = \int_a^b \sqrt{1 + f'^2(x)}dx. \qquad (15)$$

The arc length *L* exists if a function $f(x)$ is continuous with the first derivative on the segment $[a, b]$.

The second method. We find at first an element (or the differential) *ds* of the desired arc length and then the arc length as the sum of all the elements.

By Pythagorean theorem $ds^2 = dx^2 + dy^2$, and

$$ds = \sqrt{dx^2 + dy^2}. \qquad (16)$$

For an arc $\breve{}AB$ determined by an equation (14)

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + f'^2(x)}dx,$$

and the sum of all the elements from *a* to *b* leads to the same formula (15).

If an arc $\breve{}AB$ of a curve is determined parametrically by equations

$$x = x(t),\ y = y(t),\ \alpha \le t \le \beta, \qquad (17)$$

we have from (16)

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{x'^2(t) + y'^2(t)}dt,$$

and therefore

$$L = \int_\alpha^\beta \sqrt{x'^2(t) + y'^2(t)}dt. \qquad (18)$$

If an arc $\breve{}AB$ of a curve is given in polar coordinates by an equation

$$\rho = \rho(\varphi),\ \alpha \le \varphi \le \beta \qquad (19)$$

we pass to parametrical equations of the arc

$$x = \rho\cos\varphi,\ y = \rho\sin\varphi,\ \alpha \le \varphi \le \beta \qquad (20)$$

and apply the formula (18). Since

$$dx = (\rho'\cos\varphi - \rho\sin\varphi)d\varphi, dy = (\rho'\sin\varphi + \rho\cos\varphi)d\varphi, dx^2 + dy^2 = (\rho^2 + \rho'^2)d\varphi,$$

$$ds = \sqrt{\rho^2 + \rho'^2}d\varphi$$

the formula (18) gives

$$L = \int_{\alpha}^{\beta} \sqrt{\rho^2 + \rho'^2}\, d\varphi . \qquad (21)$$

Ex. 10. Find the arc length of a curve $y = \ln x$ for $1 \le x \le \sqrt{3}$.

By virtue of the formula (15) we have

$$L = \int_{0}^{\sqrt{3}} \sqrt{1 + (\ln x)'^2}\, dx = \int_{1}^{\sqrt{3}} \sqrt{1 + \frac{1}{x^2}} dx = \int_{1}^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x}\, dx = \left| \begin{array}{l|c|c|c} 1 + x^2 = t^2, & x & 1 & \sqrt{3} \\ xdx = tdt, & t & \sqrt{2} & 2 \end{array} \right\| =$$

$$= \int_{\sqrt{2}}^{2} \frac{\sqrt{t^2} \cdot tdt}{x \cdot x} = \int_{\sqrt{2}}^{2} \frac{t^2 dt}{t^2 - 1} = \int_{\sqrt{2}}^{2} \frac{\left(t^2 - 1 + 1\right)dt}{t^2 - 1} = \int_{\sqrt{2}}^{2} \left(1 + \frac{1}{t^2 - 1}\right) dt = \left(t + \frac{1}{2} \ln\left|\frac{t-1}{t+1}\right|\right)\Bigg|_{\sqrt{2}}^{2} \approx 0.04$$

Ex. 11. Find the length of the loop of the curve $x = t^2,\ y = t - \frac{t^3}{3}$ (fig. 11).

By the formula (18)

$$L = 2L_{ODE} = 2\int_{0}^{\sqrt{3}} \sqrt{\left(t^2\right)'^2 + \left(t - \frac{t^3}{3}\right)'^2}\, dt = 2\int_{0}^{\sqrt{3}} \sqrt{(2t)^2 + \left(1 - t^2\right)^2} dt = 2\int_{0}^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1} dt =$$

$$= 2\int_{0}^{\sqrt{3}} \sqrt{\left(t^2 + 1\right)^2} dt = 2\int_{0}^{\sqrt{3}} \left(t^2 + 1\right) dt = 2\left(\frac{t^3}{3} + t\right)\Bigg|_{0}^{\sqrt{3}} = 4\sqrt{3} \approx 6.93 .$$

Ex. 12. Find the length of the cardioid $\rho = a(1 + \cos\varphi)$ (fig. 13).

With the help of the formula (21)

$$L = 2\int_{0}^{\pi} \sqrt{\rho^2 + \rho'^2}\, d\varphi = 2\int_{0}^{\pi} \sqrt{a^2\left(1 + 2\cos\varphi + \cos^2\varphi\right) + a^2 \sin^2\varphi}\, d\varphi = 2a\int_{0}^{\pi} \sqrt{1 + \cos\varphi}\, d\varphi =$$

$$= 2a\int_{0}^{\pi} \sqrt{2(1 + \cos\varphi)}\, d\varphi = 2a\int_{0}^{\pi} \sqrt{4\cos^2\frac{\varphi}{2}}\, d\varphi = 4a\int_{0}^{\pi} \cos\frac{\varphi}{2}\, d\varphi = 8a\left(\sin\frac{\varphi}{2}\right)\Bigg|_{0}^{\pi} = 8a .$$

Ex. 13. Find the length of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Parametrical equations of the ellipse $x = a\cos t,\ y = b\sin t,\ 0 \le t \le 2\pi$, and by the formula (18) $L = \int_{0}^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt$.

The integral can be found in a finite form only for some particular values of a and b for example if $a=b$. In general case we can find only approximate values of L for concrete values of a and b because of a primitive of the integrand is inexpressible in terms of elementary functions.

Ex. 14. Prove that the length of Bernoulli lemniscate $\left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right)$ (fig. 14) can be represented by the next integral

$$L=4a\int_0^{\pi}\frac{d\varphi}{\sqrt{\cos 2\varphi}}=4a\int_0^{\pi}\frac{d\varphi}{\sqrt{1-2\sin^2\varphi}}.$$

A primitive of the integrand is also inexpressible in terms of elementary functions .

POINT 3. VOLUMES

Volume of a body with known areas of its parallel cross-sections

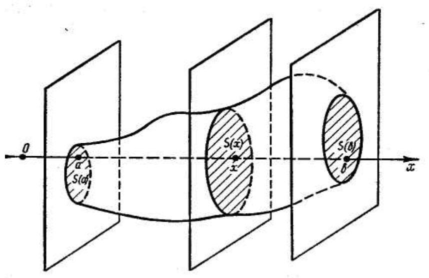


Fig.16

Let some body be situated between the planes $x=a$, $x=b$, and for any $x\in(a,b)$ the area $S(x)$ of its cross-section by a plane perpendicular to Ox-axis is known (see fig. 16). The volume of the body equals the integral

$$V=\int_a^b S(x)dx. \qquad (22)$$

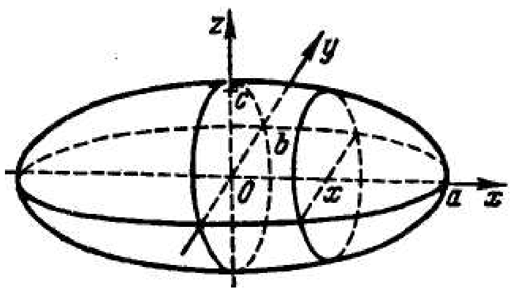


Fig. 17

■ An element dV of the volume is the volume of a right cylinder with the base $S(x)$ and the altitude dx, $dV=S(x)dx$. Adding all these elements we get the required volume representted by the formula (22).■

Ex. 15. Find the volume of the triaxial ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (fig. 17)

It's evident that $|x| \le a, -a \le x \le a$. For any $x \in (-a, a)$ the cross-section of the body perpendicular to the *Ox*-axis is the ellipse

$$\frac{y^2}{b^2\left(1-\frac{x^2}{a^2}\right)}+\frac{z^2}{c^2\left(1-\frac{x^2}{a^2}\right)}=1$$

with the semi-axes

$$b_x = b\sqrt{1-\frac{x^2}{a^2}},\ c_x = c\sqrt{1-\frac{x^2}{a^2}}$$

and the area

$$S(x) = \pi b_x c_x = \pi b\sqrt{1-x^2/a^2}\cdot c\sqrt{1-x^2/a^2} = \pi bc\left(1-x^2/a^2\right).$$

Therefore the volume of the ellipsoid by the formula (22) equals

$$V = \int_{-a}^{a} S(x)dx = \pi bc\int_{-a}^{a}\left(1-x^2/a^2\right)dx = \pi bc\left(x-\frac{x^3}{3a^2}\right)\Bigg|_{-a}^{a} = \frac{4}{3}\pi abc.$$

Let's remark that for $a = b = c$ we get the volume of the sphere $x^2 + y^2 + z^2 = a^2$ namely

$$V = \frac{4}{3}\pi a^3$$

Volume of a body of rotation

A curvilinear trapezium $\{(x, y): a \le x \le b, 0 \le y \le f(x)\}$ (see fig. 1) rotates about Ox-axis. Prove that the volume of the corresponding body (**body of rotation**, fig 18) equals the definite integral

$$V = V_{rotOx} = \pi\int_a^b f^2(x)dx. \qquad (23)$$

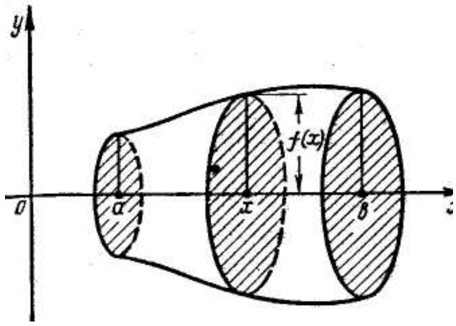


Fig. 18

■For any $x \in (a, b)$ a cross-section of the body of rotation by a plane perpendicular to *Ox*-axis is a circle of radius $f(x)$ (fig. 18). Therefore its area $S(x)=\pi f^2(x)$, and by the formula (22) the volume of the body in question is given by the formula (23).■

Let the same curvilinear trapezium $\{(x,y): a \le x \le b, 0 \le y \le f(x)\}$ (fig. 1) rotate about Oy-axis which doesn't pass through the interior of the trapezium[1]. Prove that the volume of the body of its rotation is represented by the next integral:

$$V = V_{rotOy} = 2\pi \int_a^b x f(x) dx . \qquad (24)$$

Instructions. As an element of the volume one can take the volume of a part of the body generated by rotation of the rectangle with the sides $y = f(x)$, dx about the Oy-axis. Then the element of the volume is $dV = 2\pi x f(x)$ whence it follows the formula (24).

Ex. 16. Let the arc of a sinusoid $y = \sin x, 0 \le x \le \pi$ rotate about the Ox-, Oy-axes. Calculate the volumes of corresponding bodies of rotation.

With the help of the formulas (23), (24) we get

$$V_{rotOx} = \pi \int_0^\pi f^2(x) dx = \pi \int_0^\pi \sin^2 x dx = \frac{\pi}{2} \int_0^\pi (1 - \cos 2x) dx = \frac{\pi}{2}\left(x - \frac{1}{2}\sin 2x\right)\Bigg|_0^\pi = \frac{1}{2}\pi^2 .$$

$$V_{rotOy} = 2\pi \int_0^\pi x f(x) dx = 2\pi \int_0^\pi x \sin x dx = \begin{vmatrix} u = x, dv = \sin x dx, \\ du = dx, v = -\cos x \end{vmatrix} = 2\pi\left(-(x\cos x)\Big|_0^\pi + \int_0^\pi \cos x dx \right) =$$

$$2\pi\left(\pi + \sin x \Big|_0^\pi \right) = 2\pi^2 .$$

Let's regard now a curvilinear trapezium

$$\{(x,y): c \le y \le d, 0 \le x \le g(y)\}$$

which is oriented with respect to the Oy-axis (see fig. 6).

Let such the curvilinear trapezium rotate about Oy-axis. Prove that the volume of the corresponding body of rotation is represented by the integral completely analogous to (23)

[1] That is $a \ge 0$ or $b \le 0$.

$$V = V_{rotOy} = \pi \int_c^d g^2(y)dy. \qquad (25)$$

Ex. 17. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ rotates about the *Ox*-axis, and then about the *Oy*-axis. Calculate the volumes of corresponding bodies of rotation.

From the equation of the ellipse

$$y^2 = f^2(x) = b^2\left(1 - \frac{x^2}{a^2}\right) = \frac{b^2}{a^2}\left(a^2 - x^2\right), -a \le x \le a;$$

$$x^2 = g^2(y) = a^2\left(1 - \frac{y^2}{b^2}\right) = \frac{a^2}{b^2}\left(b^2 - y^2\right), -b \le y \le b$$

and by the formulas (23), (25) we'll have

$$V_{rotOx} = \pi \int_{-a}^{a} y^2 dx = \pi \int_{-a}^{a} f^2(x)dx = \frac{\pi b^2}{a^2} \int_{-a}^{a} \left(a^2 - x^2\right)dx = \frac{8\pi ab^2}{3},$$

$$V_{rotOy} = \pi \int_{-b}^{b} x^2 dy = \pi \int_{-b}^{b} g^2(y)dy = \frac{\pi a^2}{b^2} \int_{-b}^{b} \left(b^2 - y^2\right)dy = \frac{8\pi a^2 b}{3}.$$

POINT 4. ECONOMIC APPLICATIONS

Problem 1 (produced quantity). Let $f(t)$ be the labour productivity of a some factory at a time moment *t*. It is known (see the formula (11) of the point 2 of prceding lecture) that its produced quantity *U* during the time interval [0, *T*] equals

$$U = \int_0^T f(t)dt.$$

Ex. 18. Let the labour productivity of a factory be $f(t) = (1-t)^2, 0 \le t \le 2$. Then its produced quantity

$$U = \int_0^2 (1-t)^2 dt = \left| \begin{array}{c} 1 - t = y,\ dt = -dy, \\ \begin{array}{c|c|c} t & 0 & 2 \\ \hline y & 1 & -1 \end{array} \end{array} \right| = -\int_1^{-1} y^2 dy = \int_{-1}^{1} y^2 dy = \left.\frac{y^3}{3}\right|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}.$$

Problem 2 (costs of conservation of goods). Let $f(t)$ be the quantity of goods in the storage at a time moment *t*, and a constant quantity *h* represents the costs of conservation of unit of goods per unit of time. Then the costs of conservation of goods during a time interval $[t, t+dt]$ (or an element of the costs of conservation) equals

$$dQ = hf(t)dt.$$

Adding all these elements from $t = 0$ to $t = T$ we get the costs of conservation of goods during the time interval $[0, T]$, that is

$$Q = \int_0^T hf(t)dt.$$

Let, for example, we study the case of uniform consumption of all goods from P at a time $t = 0$ to 0 at a time $t = T$. In this case the quantity of goods at a time moment *t* is

$$f(t) = P - P \cdot \frac{t}{T},$$

and the costs of conservation of goods equals

$$Q = \int_0^T hf(t)dt = hP\int_0^T \left(1 - \frac{t}{T}\right)dt = hP\left(t - \frac{t^2}{2T}\right)\Bigg|_0^T = \frac{hPT}{2}.$$

LECTURE NO. 23. DEFINITE INTEGRAL: ADDITIONAL QUESTIONS

POINT 1. APPROXIMATE INTEGRATION

POINT 2. IMPROPER INTEGRALS

POINT 3. EULER Γ-FUNCTION

POINT 1. APPROXIMATE INTEGRATION

We'll study the case of nonnegative function $f(x) \ge 0$ when a definite integral

$$\int_a^b f(x)dx$$

represents the area of a curvilinear trapezium $\{(x, y): a \le x \le b, 0 \le y \le f(x)\}$ bounded by straight lines $x = a$, $x = b$, the Ox-axis and a graph of the function $y = f(x)$. The same results remain valid in general case.

Rectangular Formulas

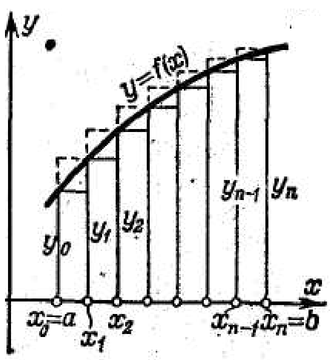


Fig. 1

We divide the segment $[a, b]$ into n equal parts of the length $h = \dfrac{b-a}{n}$ by points

$$x_0 = a, x_1 = x_0 + h, x_2 = x_1 + h, ..., x_n = x_{n-1} + h = b.$$

Straight lines $x = x_1, x = x_2, x = x_3, ..., x = x_{n-1}$ divide the curve $y = f(x)$ into n parts . Let's denote by

$$y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2), ..., y_n = f(x_n)$$

the values of the function $y = f(x)$ at the division points (see fig. 1)

a) Substituting all parts of the curve $y = f(x)$ by the segments of straight lines $y = y_0, y = y_1, ..., y = y_{n-1}$ we substitute the curvilinear trapezium by the set of rectangles with total area

$$S_1 = y_0 \cdot h + y_1 \cdot h + \ldots + y_{n-1} \cdot h = \frac{b-a}{n} \cdot \left(y_0 + y_1 + \ldots + y_{n-1}\right).$$

Therefore

$$\int_a^b f(x)dx \approx S_1 = h \cdot \left(y_0 + y_1 + \ldots + y_{n-1}\right) = \frac{b-a}{n} \cdot \left(y_0 + y_1 + \ldots + y_{n-1}\right) \qquad (1)$$

b) Similarly, substituting all parts of the curve $y = f(x)$ by the segments of straight lines $y = y_1, y = y_2, \ldots, y = y_n$, we'll get

$$\int_a^b f(x)dx \approx S_2 = h \cdot \left(y_1 + y_2 + \ldots + y_n\right) = \frac{b-a}{n} \cdot \left(y_1 + y_2 + \ldots + y_n\right) \qquad (2)$$

Absolute error of the formulas (1), (2) has the order $1/n$, that is

$$\left|\int_a^b f(x)dx - S_i\right| \le \frac{M}{n},\ i = 1, 2; M = \frac{(b-a)^2}{2} \max_{[a,b]} \left|f'(x)\right|.$$

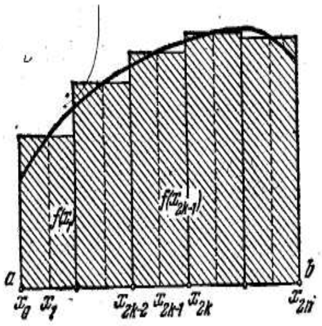


Fig.2

c) Dividing the segment $[a, b]$ into $2n$ equal parts of the length $h = \frac{b-a}{2n}$ by points

$$x_0 = a, x_1, x_2, \ldots, x_{2n} = b \text{ (fig. 2)},$$

we substitute the curvilinear trapezium by the set of rectangles with bases $2h$, altitudes y_1, y_3,..., y_{2n-1} and total area

$$S_3 = y_1 \cdot 2h + y_3 \cdot 2h + \ldots + y_{2n-1} \cdot 2h = 2 \cdot \frac{b-a}{2n} \cdot \left(y_1 + y_3 + \ldots + y_{2n-1}\right) =$$
$$= 2h \cdot \left(y_1 + y_3 + \ldots + y_{2n-1}\right) = \frac{b-a}{n} \cdot \left(y_1 + y_3 + \ldots + y_{2n-1}\right).$$

Hence

$$\int_a^b f(x)dx \approx S_3 = 2h \cdot \left(y_1 + y_3 + \ldots + y_{2n-1}\right) = \frac{b-a}{n} \cdot \left(y_1 + y_3 + \ldots + y_{2n-1}\right) \qquad (3)$$

Absolute error of the formula (3) has the order $1/n^2$, that is

$$\left|\int_a^b f(x)dx - S_3\right| \le \frac{N}{n^2}, \quad N = \frac{(b-a)^3}{24} \max_{[a,b]} \left|f''(x)\right|.$$

It means that the formula (3) is more exact than both (1) and (2).

Trapezium Formula

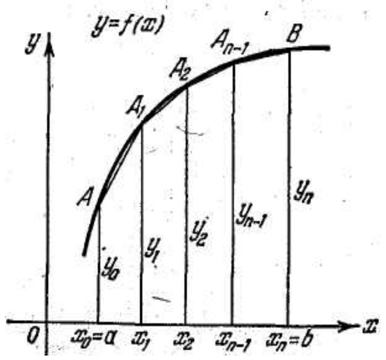


Fig. 3

After dividing the segment $[a, b]$ into n equal parts of the length $h = (b-a)/n$ by points

$$A = A(x_0, y_0),\ A_1(x_1, y_1), \ldots, A_n(x_n, y_n) = B$$

(fig. 3) we divide the graph of the function $y = f(x)$ into n arcs. Substituting all arcs of the graph by segments

$$AA_1,\ A_1A_2, \ldots, A_{n-1}B$$

we substitute the curvilinear trapezium by the set of trapeziums with total area

$$S_4 = \frac{y_0 + y_1}{2} \cdot h + \frac{y_1 + y_2}{2} \cdot h + \ldots + \frac{y_{n-1} + y_n}{2} \cdot h = \frac{b-a}{n} \cdot \left(\frac{y_0 + y_n}{2} + y_1 + y_2 + \ldots + y_{n-1} \right).$$

So we get the next approximate formula (trapezium formula)

$$\int_a^b f(x)dx \approx S_4,$$

$$\int_a^b f(x)dx \approx h \cdot \left(\frac{y_0 + y_n}{2} + y_1 + y_2 + \ldots + y_{n-1} \right) = \frac{b-a}{n} \cdot \left(\frac{y_0 + y_n}{2} + y_1 + y_2 + \ldots + y_{n-1} \right) \quad (4)$$

Its absolute error has the order $1/n^2$, that is

$$\left| \int_a^b f(x)dx - S_4 \right| \le \frac{P}{n^2}, \quad P = \frac{(b-a)^3}{12} \max_{[a,b]} |f''(x)|.$$

It means that the formulas (3) and (4) have the same order of exactness.

Simpson[1] formula (parabolic formula)

Let's divide (by points $x_0 = a, x_1, x_2, \ldots, x_{2n} = b$) the segment $[a, b]$ into an *even* number $2n$ of equal parts of the length $h = \dfrac{b-a}{2n}$ and let

$$M = M(x_0, y_0),\ M_1(x_1, y_1),\ M_2(x_2, y_2), \ldots, M_{2n-1}(x_{2n-1}, y_{2n-1}),\ M_{2n}(x_{2n}, y_{2n})$$

[1] Simpson, T. (1710 - 1761), an English mathematician

be points of the curve $y = f(x)$ corresponding to the division points (see fig. 4 for the case $2n = 6$).

At first we draw a quadratic parabola

$$y = Ax^2 + Bx + C$$

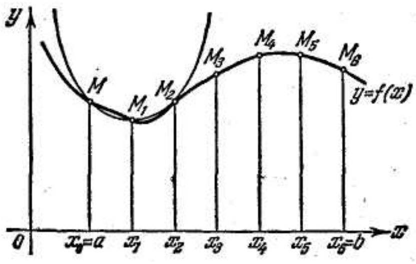


Fig. 4

through points M, M_1, M_2 (see fig. 4, 5). One can prove that the area of the figure between the arc MM_1M_2 of the parabola and the segment $[x_0, x_2]$ of the Ox-axis equals

$$\int_{x_0}^{x_2} \left(Ax^2 + Bx + C\right)dx = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right),$$

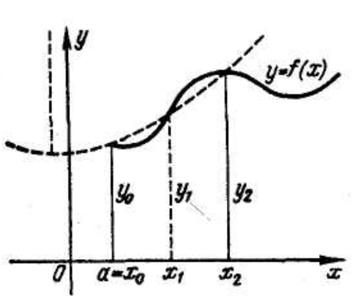


Fig. 5

and we can write

$$\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} \left(Ax^2 + Bx + C\right)dx = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right).$$

Carrying out the same procedure on the point triplets $M_2M_3M_4$, $M_4M_5M_6$, …, $M_{2n-2}M_{2n-1}M_{2n}$ we get

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \int_{x_4}^{x_6} f(x)dx + \ldots + \int_{x_{2n-2}}^{x_{2n}} f(x)dx \approx S_5 =$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) + \frac{h}{3}\left(y_2 + 4y_3 + y_4\right) + \frac{h}{3}\left(y_4 + 4y_5 + y_6\right) + \ldots + \frac{h}{3}\left(y_{2n-2} + 4y_{2n-1} + y_{2n}\right) =$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + y_4 + 4y_5 + y_6 + \ldots + y_{2n-2} + 4y_{2n-1} + y_{2n}\right) =$$

$$= \frac{h}{3}\left(\left(y_0 + y_{2n}\right) + 4\left(y_1 + y_3 + y_5 + \ldots + y_{2n-1}\right) + 2\left(y_2 + y_4 + y_6 + \ldots + y_{2n-2}\right)\right),\ h = \frac{b-a}{2n}.$$

Finally we get Simpson's formula for approximate integration

$$\int_a^b f(x)dx \approx S_5 = \frac{b-a}{6n}\left(\left(y_0 + y_{2n}\right) + 4\left(y_1 + y_3 + \ldots + y_{2n-1}\right) + 2\left(y_2 + y_4 + \ldots + y_{2n-2}\right)\right) \quad (5)$$

Simpson's formula (5) is the most exact in comparison with (1) – (4). Indeed, its absolute error has the order $1/n^4$ that is

$$\left|\int_a^b f(x)dx - S_5\right| \le \frac{Q}{n^4}, \quad Q = \frac{(b-a)^5}{2880}\max_{[a,b]}\left|f^{(4)}(x)\right|.$$

For example in the case $n = 3$, $2n = 6$ (fig. 4) Simpson's formula has the next form

$$\int_a^b f(x)dx \approx \frac{b-a}{18}\left((y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right).$$

Ex. 1. Calculate approximately the integral $I = \int_0^{1.6} \sin x^2 dx$.

Let's form the next table of values of the argument and the function:

i	x_i	x_i^2	$y_i = \sin x_i^2$
0	$x_0 = 0.0$	0.00	$y_0 = 0.0000$
1	$x_1 = 0.2$	0.04	$y_1 = 0.0400$
2	$x_2 = 0.4$	0.16	$y_2 = 0.1593$
3	$x_3 = 0.6$	0.36	$y_3 = 0.3523$
4	$x_4 = 0.8$	0.64	$y_4 = 0.5972$
5	$x_5 = 1.0$	1.00	$y_5 = 0.8415$
6	$x_6 = 1.2$	1.44	$y_6 = 0.9915$
7	$x_7 = 1.4$	1.96	$y_7 = 0.9249$
8	$x_8 = 1.6$	2.56	$y_8 = 0.5487$

It corresponds to division of the segment $[0, 1.6]$ into $n = 8$ parts of the length

$$h = \frac{b-a}{8} = \frac{1.8 - 0.0}{8} = 0.2.$$

By the formula (1)

$$I \approx 0.2 \cdot (y_0 + y_1 + ... + y_7) = 0.2 \cdot 3.9067 = 0.7813 \approx 0.78 \approx 0.8.$$

By the formula (2)

$$I \approx 0.2 \cdot (y_1 + ... + y_7 + y_8) = 0.2 \cdot 4.2488 = 0.8911 \approx 0.89 \approx 0.9.$$

Using the formula (3) we take $2n = 8$, $n = 4$, $h = \frac{b-a}{n} = \frac{1.6-0}{4} = 0.4$, and so

$$I \approx 0.4 \cdot (y_1 + y_3 + y_5 + y_7) = 0.4 \cdot 2.1587 = 0.8635 \approx 0.86.$$

By the formula (4)

$$I \approx 0.2 \cdot \left(\frac{y_0 + y_8}{2} + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 \right) = 0.2 \cdot 4.1811 = 0.8362 \approx 0.84.$$

We'll apply the formula (5) two times.

At first we divide the segment $[0, 1.6]$ into $2n = 4$ parts, $x_0 = 0.0$, $x_1 = 0.4$, $x_2 = 0.8$, $x_3 = 1.2$, $x_4 = 1.6$, correspondingly $y_0 = 0.0000$, $y_1 = 0.1593$, $y_2 = 0.5972$, $y_3 = 0.9915$, $y_4 = 0.5487$, $h = \dfrac{b-a}{2n} = \dfrac{1.6-0.0}{4} = 0.4$, and therefore

$$I \approx \frac{h}{3}\left((y_0 + y_4) + 4 \cdot (y_1 + y_3) + 2 \cdot y_2\right) = \frac{0.4}{3} \cdot \left((0.0000 + 0.5487) + 4 \cdot (0.1593 + 0.9915) + \right.$$
$$\left. + 2 \cdot 0.5972\right) = \frac{0.4}{3} \cdot 6.3463 = 0.8462 \approx 0.846.$$

Dividing now the segment $[0, 1.6]$ into $2n = 8$ parts, $h = \dfrac{b-a}{2n} = \dfrac{1.6-0.0}{8} = 0.2$, we have

$$I \approx \frac{h}{3}\left((y_0 + y_8) + 4 \cdot (y_1 + y_3 + y_5 + y_7) + 2 \cdot (y_2 + y_4 + y_6)\right) =$$
$$= \frac{0.2}{3} \cdot 12.6795 = 0.8453 \approx 0.845.$$

It's useful to compare all these results with known approximate value of the same integral up to 10^{-10}, namely

$$I = \int_0^{1.6} \sin x^2 dx \approx 0.8452689707.$$

POINT 2. IMPROPER INTEGRALS

Improper integrals of the first kind

Def. 1. Let a function $f(x)$ be continuous on an infinite interval $[a;+\infty]$. If there exists a finite limit

$$\lim_{b\to+\infty}\int_a^b f(x)dx<\infty \qquad (6)$$

then we say that the next integral (the **improper integral of the first kind**)

$$\int_a^{+\infty} f(x)dx \qquad (7)$$

converges (exists, is convergent).

Thus by the definition 1

$$\int_a^{+\infty} f(x)dx=\lim_{b\to+\infty}\int_a^b f(x)dx. \qquad (8)$$

Def. 2. If the limit (6) is infinite one or doesn't exist we say that the improper integral (7) **diverges** (doesn't exist, is divergent).

By the same way we can define the next two improper integrals of the first kind

Def. 3.

$$\int_{-\infty}^{a} f(x)dx=\lim_{c\to-\infty}\int_c^a f(x)dx \qquad (9)$$

if a function $f(x)$ is continuous on an infinite interval $(-\infty, a]$.

Def. 4.

$$\int_{-\infty}^{+\infty} f(x)dx=\int_{-\infty}^{a} f(x)dx+\int_a^{+\infty} f(x)dx=\lim_{\substack{b\to+\infty\\ c\to-\infty}}\int_c^b f(x)dx \qquad (10)$$

if a function $f(x)$ is continuous on the set of all reals.

The improper integral (9) is called convergent if a finite limitexists in (9) and divergent otherwise. The same is concerned to the improper integral (10).

Def. 5. Principle value of the improper integral (10) is called the next limit

$$p.v.\int_{-\infty}^{+\infty} f(x)dx=\lim_{b\to+\infty}\int_{-b}^{b} f(x)dx. \qquad (11)$$

If an integral (10) converges then its principal value also converges. But there are cases when the integral (10) diverges but its principal value converges.

Ex. 2. Improper integrals

$$\int_a^{+\infty}\frac{dx}{x^p}\,(a>0),\ \int_{-\infty}^{a}\frac{dx}{x^p}\,(a<0),\ p\in\Re \qquad (12)$$

are convergent for $p>1$ and divergent for $p\le 1$.

■Let's consider the first integral.

a) If $p>1$ we can suppose $p=1+\alpha$, where $\alpha>0$, and so

$$\int_a^{+\infty}\frac{dx}{x^p}=\int_a^{+\infty}\frac{dx}{x^{1+\alpha}}=\lim_{b\to+\infty}\int_a^b\frac{dx}{x^{1+\alpha}}=\lim_{b\to+\infty}\int_a^b x^{-1-\alpha}dx=\lim_{b\to+\infty}\left.\frac{x^{-\alpha}}{-\alpha}\right|_a^b=-\frac{1}{\alpha}\lim_{b\to+\infty}\left(b^{-\alpha}-a^{-\alpha}\right)=$$

$$=-\frac{1}{\alpha}\lim_{b\to+\infty}\left(\frac{1}{b^\alpha}-\frac{1}{a^\alpha}\right)=-\frac{1}{\alpha}\left(0-\frac{1}{a^\alpha}\right)=\frac{1}{\alpha a^\alpha}=\frac{1}{(p-1)a^{p-1}}<\infty,$$

that is the integral converges for $p>1$.

b) Let $p=1$. In this case

$$\int_a^{+\infty}\frac{dx}{x}=\lim_{b\to+\infty}\int_a^b\frac{dx}{x}=\lim_{b\to+\infty}\ln x\Big|_a^b=\lim_{b\to+\infty}(\ln b-\ln a)=+\infty.$$

The integral diverges.

c) If $p<1$ we put $p=1-\alpha$, where $\alpha>0$, and

$$\int_a^{+\infty}\frac{dx}{x^p}=\int_a^{+\infty}\frac{dx}{x^{1-\alpha}}=\lim_{b\to+\infty}\int_a^b\frac{dx}{x^{1-\alpha}}=\lim_{b\to+\infty}\int_a^b x^{-1+\alpha}dx=\lim_{b\to+\infty}\left.\frac{x^{\alpha}}{\alpha}\right|_a^b=\frac{1}{\alpha}\lim_{b\to+\infty}\left(b^{\alpha}-a^{\alpha}\right)=+\infty.$$

The integral diverges.■

Ex. 3. Prove that the integral

$$\int_{-\infty}^{+\infty}\sin x\,dx$$

diverges but its principal value converges (to zero).

■On the base of the formula (10)

$$\int_{-\infty}^{+\infty}\sin x\,dx=\lim_{\substack{b\to+\infty\\ c\to-\infty}}\int_c^b\sin x\,dx=-\lim_{\substack{b\to+\infty\\ c\to-\infty}}\cos x\Big|_c^b=-\lim_{\substack{b\to+\infty\\ c\to-\infty}}(\cos b-\cos c)=-\lim_{b\to+\infty}\cos b+\lim_{c\to-\infty}\cos c.$$

Both limits don't exist and therefore the integral diverges. On the other hand the principal value of the integral converges to zero (or equals zero), because of by the formula (11)

$$p.v.\int_{-\infty}^{+\infty}\sin x dx=\lim_{b\to+\infty}\int_{-b}^{b}\sin x dx=-\lim_{b\to+\infty}\cos x\Big|_{-b}^{b}=-\lim_{b\to+\infty}\left(\cos b-\cos(-b)\right)=$$

$$=-\lim_{b\to+\infty}\left(\cos b-\cos b\right)=0\,.\blacksquare$$

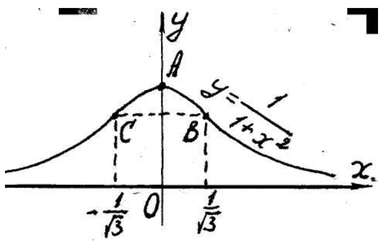


Fig. 6

Ex. 4. Find the area of an infinite figure bounded by Agnesi[1] witch $y=\dfrac{1}{1+x^2}$ and its asymptote (fig. 6).

The straight line $y=0$ (the Ox-axis) is a horizontal asymptote of Agnesi witch because of

$$\lim_{x\to\pm\infty}\frac{1}{1+x^2}=\lim_{x\to\pm\infty}\frac{1}{x^2}=0\,.$$

The figure is symmetric with respect to the Oy-axis, and therefore its area equals

$$S=2\int_{0}^{+\infty}\frac{dx}{1+x^2}=2\lim_{b\to+\infty}\int_{0}^{b}\frac{dx}{1+x^2}=2\lim_{b\to+\infty}\arctan x\Big|_{0}^{b}=2\lim_{b\to+\infty}\arctan b=2\cdot\frac{\pi}{2}=\pi\,.$$

Note 1 (Newton-Leibniz formula for improper integrals**).**

Let a function $F(x)$ be a primitive of a function $f(x)$. Denoting

$$F(+\infty)=\lim_{x\to+\infty}F(x)$$

we can represent evaluation of the improper integral (8) in the form of Newton-Leibniz formula, namely

$$\int_{a}^{+\infty}f(x)dx=\lim_{b\to+\infty}\int_{a}^{b}f(x)dx=\lim_{b\to+\infty}F(x)\Big|_{a}^{b}=\lim_{b\to+\infty}\left(F(b)-F(a)\right)=\lim_{b\to+\infty}F(b)-F(a)=$$

$$=\lim_{x\to+\infty}F(x)-F(a)=F(+\infty)-F(a)=F(x)\Big|_{a}^{+\infty}.$$

$$\int_{a}^{+\infty}f(x)dx=F(x)\Big|_{a}^{+\infty}\qquad(13)$$

The same Newton-Leibniz formula can be written for the other improper integrals.

[1] Agnesi, M.G. (1718 - 1799), an Italian mathematician

Ex. 5. $\int_1^{+\infty}\frac{dx}{x^2}=-\frac{1}{x}\Big|_1^{+\infty}=-\left(\lim_{x\to+\infty}\frac{1}{x}-1\right)=-(0-1)=1$.

Ex. 6. Calculate the next improper integral $\int_{-\infty}^{+\infty}\frac{dx}{4+x^2}$.

$$\int_{-\infty}^{+\infty}\frac{dx}{4+x^2}=\frac{1}{2}\arctan\frac{x}{2}\Big|_{-\infty}^{+\infty}=\frac{1}{2}\left(\lim_{x\to+\infty}\arctan\frac{x}{2}-\lim_{x\to-\infty}\arctan\frac{x}{2}\right)=\frac{1}{2}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=\frac{\pi}{2}.$$

Ex. 7. $\int_0^{+\infty}e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}$, $\int_0^{+\infty}e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2}$ for $a>0$

■For example $\int_0^{+\infty}e^{-ax}\cos bx\,dx=\frac{e^{-ax}(b\sin bx-a\cos bx)}{a^2+b^2}\Big|_0^{+\infty}=$

$$=\lim_{x\to\infty}\left(\frac{e^{-ax}(b\sin bx-a\cos bx)}{a^2+b^2}-\frac{-a}{a^2+b^2}\right)=0+\frac{a}{a^2+b^2}=\frac{a}{a^2+b^2}.$$ ■

Note 2 (**change of a variable** and **integration by parts** in improper integrals of the first kind)**.** In process of evaluation of improper integrals of the first kind we can use change of a variable and integration by parts.

Ex. 8. Evaluate the next improper integral or state its divergence.

$$\int_{-\infty}^{+\infty}\frac{dx}{e^x+e^{-x}}=\int_{-\infty}^{+\infty}\frac{e^x dx}{e^{2x}+1}=\left|\begin{array}{l|c|c} e^x=y, & x & -\infty & +\infty\\ e^x dx=dy & y & 0 & +\infty\end{array}\right|=\int_0^{+\infty}\frac{dy}{1+y^2}=\arctan y\Big|_0^{+\infty}=\frac{\pi}{2}-0=\frac{\pi}{2}.$$

Ex. 9. Evaluate the next improper integral or state its divergence.

$$\int_{\sqrt{2}}^{+\infty}\frac{dx}{x\sqrt{x^2-1}}=\left|\begin{array}{l|c|c} x^2-1=y^2,\ y>0, & x & \sqrt{2} & +\infty\\ xdx=ydy,\ x^2=1+y^2 & y & 1 & +\infty\end{array}\right|=\int_{\sqrt{2}}^{+\infty}\frac{xdx}{x^2\sqrt{x^2-1}}=\int_1^{+\infty}\frac{ydy}{(1+y^2)y}=$$

$$=\int_1^{+\infty}\frac{dy}{1+y^2}=\arctan y\Big|_1^{+\infty}=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}.$$

Let's calculate this integral by the other substitution.

$$\int_{\sqrt{2}}^{+\infty}\frac{dx}{x\sqrt{x^2-1}}=\left|\begin{array}{l|c|c} x=\frac{1}{\cos t},\ dx=-\frac{-\sin tdt}{\cos^2 t}=\frac{\sin tdt}{\cos^2 t}, & x & \sqrt{2} & +\infty\\ x^2-1=\frac{1}{\cos^2 t}-1=\frac{1-\cos^2 t}{\cos^2 t}=\frac{\sin^2 t}{\cos^2 t} & t & \pi/4 & \pi/2\end{array}\right|=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\frac{\sin tdt}{\cos^2 t}}{\frac{1}{\cos t}\cdot\frac{\sin t}{\cos t}}=$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} dt = t \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

After changing the variable we've got a usual definite intefral.

Ex. 10. Evaluate the next improper integral or state its divergence.

$$\int_{e}^{+\infty} \frac{dx}{x\sqrt{\ln^3 x}} = \left| \begin{matrix} \ln x = y, \\ \frac{dx}{x} = dy, \end{matrix} \quad \begin{array}{c|c|c} x & e & +\infty \\ \hline y & 1 & +\infty \end{array} \right| = \int_{1}^{+\infty} \frac{dy}{y^{3/2}} = -\frac{2}{\sqrt{y}} \Big|_{1}^{+\infty} = 2.$$

Improper integrals of the second kind

Def 6. Let a function $f(x)$ be continuous on one of these three sets: a) $[a,b)$, b) $(a,b]$, c) $[a,c)\cup(c,b]$ with discontinuity point *a*, *b*, *c* respectively. One introduces the next three **improper integrals of the second kind** (**integrals of discontinuous functions over a finite interval**) namely

$$\int_{a}^{b} f(x)dx = \lim_{\varepsilon \to 0+0} \int_{a}^{b-\varepsilon} f(x)dx; \quad (14)$$

$$\int_{a}^{b} f(x)dx = \lim_{\varepsilon \to 0+0} \int_{a+\varepsilon}^{b} f(x)dx; \quad (15)$$

$$\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx = \lim_{\varepsilon_1,\varepsilon_2 \to 0+0} \left(\int_{a}^{c-\varepsilon_1} f(x)dx + \int_{c+\varepsilon_2}^{b} f(x)dx \right). \quad (16)$$

Notions of convergence or divergence are introduced in the same way that for improper integrals of the first kind.

Def. 7. The **principal value** of the integral (16) is called the next limit

$$p.v.\int_{a}^{b} f(x)dx = \lim_{\varepsilon \to 0+0} \left(\int_{a}^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^{b} f(x)dx \right). \quad (16)$$

Ex. 11. Improper integrals

$$I_0 = \int_{0}^{a} \frac{dx}{x^p},\ I_1 = \int_{a}^{b} \frac{dx}{(b-x)^p},\ I_2 = \int_{a}^{b} \frac{dx}{(x-a)^p},\ I_3 = \int_{a}^{b} \frac{dx}{(x-c)^p}\ (a<c<b) \quad (17)$$

are convergent for $p<1$ and divergent for $p \geq 1$.

■Let's study the first integral I_0.

a) If $p=1$ we have $I_0=\int\limits_0^a \frac{dx}{x}=\lim\limits_{\varepsilon\to 0+0}\int\limits_\varepsilon^a \frac{dx}{x}=\lim\limits_{\varepsilon\to 0+0}\left(\ln|a|-\ln\varepsilon\right)=\infty$ (divergence);

b) In the case $p \neq 1$

$$I_0=\int\limits_0^a x^{-p}dx=\lim_{\varepsilon\to 0+0}\int\limits_\varepsilon^a x^{-p}dx=\frac{1}{-p+1}\lim_{\varepsilon\to 0+0}\left(a^{-p+1}-\varepsilon^{-p+1}\right)=\begin{cases}\dfrac{a^{1-p}}{1-p}\neq\infty & \text{if } p<1,\\ \infty & \text{if } p>1\end{cases}$$

(convergence for $p<1$ and divergence for $p>1$)■

Ex. 12. Investigate the integral $\int\limits_0^2 \frac{dx}{x^2-4x+3}$ for convergence ($x=1$ is discontinuity point).

Let's represent the integral in the form

$$\int\limits_0^2 \frac{dx}{x^2-4x+3}=\int\limits_0^2 \frac{dx}{(x-1)(x-3)}=\frac{1}{2}\int\limits_0^2\left(\frac{1}{x-3}-\frac{1}{x-1}\right)dx=\frac{1}{2}\left(\int\limits_0^2\frac{dx}{x-3}-\int\limits_0^2\frac{dx}{x-1}\right).$$

The first integral is usual one because of its integrant is continuous on the segment $[0, 2]$, and the second is divergent improper integral $(p=1)$. Therefore the given improper integral diverges.

Ex. 13. Find the principal value of the next divergent integral $\int\limits_{-2}^1 \frac{dx}{x}$.

$$p.v.\int\limits_{-2}^1 \frac{dx}{x}=\lim_{\varepsilon\to 0+0}\left(\int\limits_{-2}^{-\varepsilon}\frac{dx}{x}+\int\limits_{\varepsilon}^{1}\frac{dx}{x}\right)=\lim_{\varepsilon\to 0+0}\left(\ln|x|\Big|_{-2}^{-\varepsilon}+\ln|x|\Big|_{\varepsilon}^{1}\right)=\lim_{\varepsilon\to 0+0}\left(\ln|-\varepsilon|-\ln|-2|+\ln 1-\ln\varepsilon\right)=$$
$$=\lim_{\varepsilon\to 0+0}\left(\ln\varepsilon-\ln 2-\ln\varepsilon\right)=-\lim_{\varepsilon\to 0+0}\ln 2=-\ln 2.$$

Ex. 14. Find the area of an infinite figure bounded by the lines $y=1/\sqrt{x}$, $x=0$, $x=1$, $y=0$ (fig. 7).

$$S=\int\limits_0^1\frac{dx}{\sqrt{x}}=\lim_{\varepsilon\to 0+0}\int\limits_\varepsilon^1\frac{dx}{\sqrt{x}}=2\lim_{\varepsilon\to 0+0}\sqrt{x}\Big|_\varepsilon^1=2\left(1-\lim_{\varepsilon\to 0+0}\sqrt{\varepsilon}\right)=2.$$

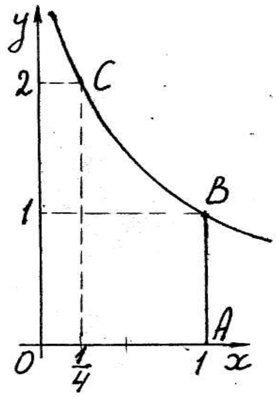


Fig. 7

Note 3 (**Newton-Leibniz formula**). Evaluation of improper integrals of the second kind can be represented in the form of Newton-Leibniz formula. For example let a function $f(x)$ be continuous on an interval $[a, b)$ and for any its primitive $F(x)$ we denote

$$F(b)=\lim_{x\to b-0} F(x).$$

Then

$$\int_a^b f(x)dx=\lim_{\varepsilon\to 0+0}\int_a^{b-\varepsilon} f(x)dx=\lim_{\varepsilon\to 0+0} F(x)\Big|_a^{b-\varepsilon}=\lim_{\varepsilon\to 0+0} F(b-\varepsilon)-F(a)=$$

$$=\lim_{x\to b-0} F(x)-F(a)==F(b)-F(a)=F(x)\Big|_a^b,$$

$$\int_a^b f(x)dx=F(x)\Big|_a^b.$$

Ex. 15. $\int_0^1 \frac{dx}{\sqrt{x}}=2\sqrt{x}\Big|_0^1=2\left(1-\lim_{x\to 0+0}\sqrt{x}\right)=2(1-0)=2$.

Ex. 16. For any positive number a

$$\int_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}\Big|_{-a}^{a}=\lim_{x\to a-0}\arcsin\frac{x}{a}-\lim_{x\to -a+0}\arcsin\frac{x}{a}=$$

$$=\lim_{x\to a}\arcsin\frac{x}{a}-\lim_{x\to -a}\arcsin\frac{x}{a}=\arcsin 1-\arcsin(-1)=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi .$$

Note 4 (**change of a variable** and **integration by parts).** Evaluating improper integrals of the second kind we can use change of a variable and integration by parts.

Ex. 17. Integrals I_1, I_2 of the example 7 can be reduced by change of a variable to the integral I_0 of the same example. In particular

$$I_1=\int_a^b\frac{dx}{(b-x)^p}=\left|\begin{array}{l|c|c} b-x=y,\ x & a & b \\ \hline dx=-dy,\ y & b-a & 0\end{array}\right|=-\int_{b-a}^{0}\frac{dy}{y^p}=\int_0^{b-a}\frac{dy}{y^p}=I_0\ \begin{array}{l}\text{converges for } p<1,\\ \text{diverges for } p\ge 1.\end{array}$$

Ex. 18. With the help of integration by parts we'll have

$$\int_0^1 x\ln x dx=\left|\begin{array}{l}\ln x=u,\ xdx=dv,\\ du=\dfrac{dx}{x},\ v=\dfrac{x^2}{2}\end{array}\right|=\left(\frac{x^2}{2}\ln x\right)\Big|_0^1-\int_0^1\frac{x^2}{2}\cdot\frac{dx}{x}=-\lim_{x\to 0+0}\left(\frac{x^2}{2}\ln x\right)-\frac{x^2}{4}\Big|_0^1=-\frac{1}{4}.$$

Convergence tests

We'll state and prove some important theorems for an improper integral of the first kind

$$\int_a^{+\infty} f(x)dx,$$

but they are valid for all other improper integrals.

Theorem 1 (comparison test for non-negative functions)**.** Let for continuous on $[a,+\infty)$ non-negative functions $f(x), g(x)$ and sufficient large x

$$0 \le f(x) \le g(x).$$

If the improper integral of the function $g(x)$ over the interval $[a,+\infty)$ converges, then the integral of the function $f(x)$ over the same interval also converges. If the integral of the function $f(x)$ diverges, then the integral of the function $g(x)$ also diverges.

■ For example let the integral of the function $g(x)$ converge, that is

$$\int_a^{+\infty} g(x)dx = \lim_{b\to+\infty}\int_a^b g(x)dx = I < \infty.,$$

and the inequality in question holds for all $x \ge a$. It follows that for any $b > a$

$$\int_a^b f(x)dx \le \int_a^b g(x)dx \le I$$

and so there exists the limit

$$\lim_{b+\to\infty}\int_a^b f(x)dx \le I.$$

It means that the integral of the function $f(x)$ converges.■

Ex. 19. On the base of theorem 1 the improper integral

$$\int_1^{+\infty}\frac{dx}{x^3+1}$$

converges because of for all x such that $x \ge 1$ the next inequality

$$\frac{1}{x^3+1} \le \frac{1}{x^3}$$

holds and the integral

$$\int_1^{+\infty} \frac{dx}{x^3}$$

converges $(p = 3 > 1)$.

Ex. 20. The integral $\int_1^{+\infty} \frac{x^4+1}{x^5} dx$ diverges on the base of the same theorem, because of for any $x \ge 1$

$$\frac{x^4+1}{x^5} > \frac{x^4}{x^5} = \frac{1}{x}$$

and the integral

$$\int_1^{+\infty} \frac{dx}{x} \quad (p = 1)$$

diverges.

Ex. 21 Prove convergence of the integral

$$\int_{-\infty}^{+\infty} e^{-x^2} dx.$$

■ Let's represent the given integral as the sum of three integrals

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \int_{-\infty}^{-1} e^{-x^2} dx + \int_{-1}^{1} e^{-x^2} dx + \int_{1}^{+\infty} e^{-x^2} dx.$$

The first and third integrals are those improper. They converge by the theorem 1, because of

$$e^{x^2} > 1 + x^2 \Rightarrow e^{-x^2} = \frac{1}{e^{x^2}} < \frac{1}{1+x^2} < \frac{1}{x^2},$$

and the integrals

$$\int_{-\infty}^{-1} \frac{dx}{x^2}, \int_{1}^{+\infty} \frac{dx}{x^2} \quad (p = 2 > 1)$$

converge. The second integral is usual (proper) one. Therefore the given integral converges.■

Ex. 22. Prove yourselves divergence of the integral

$$\int_1^{+\infty} \frac{\ln(x^2+1)}{\sqrt{x}}dx.$$

Hint. $\ln(x^2+1) \geq \ln 2$ for $x \geq 1$.

Ex. 23. Investigate for convergence the next integral

$$\int_1^{+\infty} \frac{dx}{\ln(x+1)}.$$

It's known that for any positive x

$$\ln(x+1) < x \Rightarrow 1/\ln(x+1) > 1/x,$$

and the given integral diverges on account of divergence of the integral

$$\int_1^{+\infty} \frac{dx}{x} \quad (p=1).$$

Theorem 2. Let for continuous on $[a, +\infty)$ functions $\varphi(x), f(x), \phi(x)$ and sufficient large x

$$\varphi(x) \leq f(x) \leq \phi(x).$$

If the integrals of the functions $\varphi(x), \phi(x)$ converge over $[a, +\infty)$, then the integral of the function $f(x)$ also converges.

■Validity of the theorem follows from the inequality

$$0 \leq f(x) - \varphi(x) \leq \phi(x) - \varphi(x)$$

and preceding theorem.■

Theorem 3 (absolute convergence of an improper integral)**.** If for a function $f(x)$, which is continuous on an interval $[a, +\infty)$, the integral of its absolute value

$$\int_a^{+\infty} |f(x)| dx$$

converges, then the integral

$$\int_a^{+\infty} f(x)dx$$

of the same function $f(x)$ also converges and is called **absolutely convergent**.

■Proving follows from the inequality

$$-|f(x)| \le f(x) \le |f(x)|$$

and the theorem 2.■

Ex. 24. Prove absolute convergence of the improper integral

$$\int_1^{+\infty} \frac{\sin x dx}{x^2}.$$

The improper integral of the absolute value of the integrand converges because of inequality

$$\left|\frac{\sin x}{x^2}\right| = \frac{|\sin x|}{x^2} \le \frac{1}{x^2}$$

and convergence of the integral

$$\int_1^{+\infty} \frac{dx}{x^2} \quad (p = 2 > 1).$$

Therefore the given integral absolutely converges.

Ex. 25. Investigate for absolute convergence the next improper integrals

$$\int_1^{+\infty} \frac{\cos ax dx}{x\sqrt{x}}, \int_0^1 \frac{\sin bx dx}{\sqrt[3]{x}}.$$

POINT 3. EULER Γ- FUNCTION

Def. 8. Euler Γ-function is called the next improper integral

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx \qquad (18)$$

It can be proved that the Γ-function (18) is continuous and has continuous derivatives of all orders for $\alpha > 0$.

Ler's state some properties of the Γ-function.

1) $\Gamma(1)=1$.

$$■\,\Gamma(1)=\int_0^{+\infty} e^{-x}dx=-e^{-x}\Big|_0^{+\infty}=-\left(\lim_{x\to+\infty} e^{-x}-1\right)=1.\,■$$

2) $\Gamma(\alpha+1)=\alpha\cdot\Gamma(\alpha)$.

$$■\,\Gamma(\alpha+1)=\int_0^{+\infty} x^{\alpha}e^{-x}dx=\left|\begin{matrix} u=x^{\alpha},\ dv=e^{-x}dx, \\ du=\alpha x^{\alpha-1}dx,\ v=-e^{-x}\end{matrix}\right|=-\left(x^{\alpha}e^{-x}\right)\Big|_0^{+\infty}-\int_0^{+\infty}\alpha x^{\alpha-1}\left(-e^{-x}\right)dx=$$

$$=\alpha\int_0^{+\infty} x^{\alpha-1}e^{-x}dx=\alpha\cdot\Gamma(\alpha)\,■$$

3) For natural value $\alpha=n\in\aleph$ $\Gamma(n+1)=n!$

$$■\ \begin{matrix}\Gamma(n+1)=n\Gamma(n)=n(n-1)\Gamma(n-1)=n(n-1)(n-2)\Gamma(n-2)=...= \\ =n(n-1)(n-2)\cdot...\cdot 3\cdot 2\cdot 1\cdot\Gamma(1)=n!\cdot 1=n!\end{matrix}\ ■$$

Def. 8. $\Gamma(\alpha+1)=\alpha!$

In accordance with definition 8 Γ-function is an extension on the set of all positive real numbers of the factorial-function $n!$, which is defined on the set of all natural numbers.

Ex. 26. $0!=1$.

■By virtue of the definition 8 $0!=\Gamma(1)=1$.■

DEFINITE INTEGRAL: Basic Terminology RUE

1. абсолютно сходящийся несобственный интеграл	абсолютно збіжний невластивий інтеґрал	ábsolutely convérgent impróper íntegral
2. аддитивность (по функции, по интер-валу)	адитивність (за функцією, за інтервалом)	àdditívity (with respéct to a fúnction, to an ínterval)
3. длина дуги	довжина дуги	arc length
4. порождаться, образовываться	породжуватися, утворюватися	be génerated (by…)
5. декартовы координаты	декартові координати	cartésian coórdinates
6. главное значение (коши) расходящего-ся интеграла	головне значення (коші) розбіжного інтеґрала	cauchy's príncipal válue of a divérgent íntegral
7. изменить пределы интегрирования	змінити межі [границі] інтеґрування	change the límits of ìntegrátion
8. круговой сектор	круговий сектор	círcular séctor
9. сходиться (абсолютно, условно)	збігатися (абсолютно, умовно)	convérge (ábsolutely, condítionally)
10. абсолютно сходиться	абсолютно збігатися	convérge ábsolutely
11. сходящийся несобственный интеграл	збіжний невластивий інтеґрал	convérgent impróper íntegral
12. площадь поперечного сечения	площа поперечного перерізу	cross-séction área [tránsverse séction área]
13. криволинейный сектор	криволінійний сектор	cùrvilínear séctor
14. криволинейная трапеция	криволінійна трапеція	cùrvilínear trapézium (*pl* trapéziums, trapézia) [trápezoid]
15. определённый интеграл как сумма элементов	визначений інтеґрал як сума элементів	défínite íntegral as a sum of élements
16. определённый интеграл от *a* до *b* функции эф от икс де икс	визначений інтеґрал від *a* до *b* функції эф від ікс де ікс	défínite íntegral from *a* to *b* [betwéen *a* and *b*] of a fúnction *f* of *x dx*
17. определённый интеграл по отрезку [по интервалу]	визначений інтеґрал по відрізку [по інтервалу]	défínite íntegral óver a ségment [an ínterval]
18. определённый интеграл с перемен-ным верхним преде-лом	визначений інтеґрал з змінною верхньою межею [границею]	défínite íntegral with the váriable/várying úpper límit/bóund
19. производная опре-	похідна визначеного	derívative of a défínite

делённого интеграла по переменному верхнему пределу	інтеґрала за змінною верхньою межою	íntegral with respect to its váriable/várying úpper límit/bóund
20.искомая величина как сумма элементов	шукана величина як сума елементів	desíred [sought, ùnknówn] quántity [quántity in quéstion] as a sum of elements
21.дифференциал длины дуги кривой	дифференціал довжини дуги кривої	dìfferéntial of an arc length
22.расходиться	розбігатися	divérge
23.расходящийся несобственный инте-грал	розбіжний невластивий інтеґрал	divérgent imprо́per íntegral
24.разбиение/подразделение интервала на части	розбиття/поділ інтервалу на частини	divísion [dècomposí-tion, partítion, split, splítting, súbdivìsion] an ínterval into párts
25.точка деления отрезка на части	точка поділу відрізка на частини	divísion/partítion/dècomposítion/súbdivísion póint of a ségment into párts
26.двойная подстановка	подвійна підстановка	dóuble sùbstitútion [pèrmutátion, replácement]
27.элемент искомой величины	елемент шуканої величини	élement of a(n) desíred [sought, ùnknówn] quántity [of a quántity in quéstion]
28.элемент интегрирования	елемент інтеґрування	élement of ìntegrátion
29.элементарный	елементарний	èleméntary
30.оценка определённого интеграла	оцінка/оцінювання визначеного інтеґрала	éstimate/èstimátion of définite íntegral
31.найти новые пределы интегрирования	знайти нові межі [границі] інтеґрування	find [detérmine] the new límits of ìntegrátion
32.образовывать, составлять интегральную сумму	утворити, скласти інтеґральну суму	form, make up, compóse an íntegral sum
33.обобщение поня-тия определённого интеграла	узагальнення поняття визначеного інтеґрала	gèneralizátion [extén-sion] of nótion of dé-finite íntegral
34.обобщить понятие определённого интеграла	узагальнити поняття визначеного інтеґрала	géneralize/exténd the nótion of définite ínte-gral
35.порождать, образовы-	породжувати, утворюва-	génerate

вать	ти	
36.геометрический смысл определённого интеграла	геометричний сенс визначеного інтеґрала	geométric(al) méaning [sense] of définite íntegral
37.однородность	однорідність	hòmogenéity
38.несобственный интеграл первого/вто-рого рода	невластивий інтеґрал першого/другого роду	impróper íntegral of the first/second kind
39.несобственный интеграл по конечно-му интервалу от раз-рывной [неограничен-ной] функции	невластивий інтеґрал по скінченному інтервалу від розривної [необмеженої] функ-ції	impróper íntegral on/ óver a fínite ínterval of a(n) dìscontínuous [un-límited, unbóunded] function
40.несобственный интеграл с бесконеч-ным пределом [с бесконечными предела-ми] от непрерывной функции	невластивий інтеґрал з нескінченною межею [границею] з нескінченними межами/ границями від неперервної функції	impróper íntegral with ínfinite límit [límits] of a còntínuous function
41.вписать ломаную линию (в кривую)	вписати ламану лінію (в криву)	inscríbe a polýgonal/ bróken line (in/into a curve)
42.вписанная лома-ная линия (в кривую)	вписана ламана лінія (в криву)	inscríbed polýgonal/ bróken line (in/into a curve)
43.интегрируемость	інтеґровність	íntegrabílity
44.интегрируемая функция	інтеґровна функція	íntegrable fúnction
45.интегральная сумма	інтеґральна сума	íntegral sum
46.подынтегральная функция	підінтеґральна функція	íntegrand, fúnction ún-der the íntegral sign [fúnction to be íntegra-ted], sùbíntegral fúnc-tion
47.подынтеграль-ное выражение	підінтеґральний вираз	íntegrand; expréssion únder the íntegral sign [expréssion to be íntegrated]; sùbíntegral expréssion
48.интегрировать	інтеґрувати	íntegrate
49.интегрирование	інтеґрування	ìntegrátion
50.интегрирование заменой переменной [подстановкой]	інтеґрування заміною змінної [підстановкою]	ìntegrátion by chánge/ súbstitútion of a váriable [by súbstitútion]

51.интегрирование по частям	інтеґрування частинами	ìntegrátion by parts
52.поменять мес-тами верхний и ниж-ний пре-делы интегри-рования	поміняти місцями верх-ню і нижню межі [границі] інтеґрування	ínterchánge the úpper and lówer límits of ìn-tegrátion
53.предел интегрирова-ния (верхний, нижний)	границя [межа] інтеґру-вання (верхня, нижня)	limit of ìntegrátion (úp-per, lówer [inférior])
54.предел интегра-льной суммы	границя інтеґральної су-ми	límit of the íntegral sum
55.линейность	лінійність	lìneárity
56.нижний предел ин-тегрирования	нижня межа [границя] інтеґрування	lówer [inférior] límit of ìntegrátion
57.среднее значе-ние функции на ин-тервале	середнє значення функції на інтервалі	méan/áverage válue of a function on an ínter-val
58.теорема о сред-нем	теорема про середнє	méan-válue théorem
59.механический смысл определённого интеграла	механічний сенс ви-значеного інтеґрала	mechánical méaning [sense] of définite íntegral
60.формула нью-тона-лейбница	формула ньютона-лейбніца	newton-leibniz fórmu-la (*pl* fórmulas, fórmu-lae)
61.часть интервала	частина інтервалу	part of an ínterval, súb-ìnterval
62.частичный интервал	частковий інтервал	pártial ínterval, súbìn-terval
63.перейти к пределу	перейти до границі	pass to the límit
64.перейти к поля-рным координатам	перейти до полярних ко-ординат	pass to the pólar coór-dinates
65.предельный пе-реход, переход к пре-делу	граничний перехід, перехід до границі	pássage to the límit
66.физический смысл определённо-го интегра-ла	фізичний сенс визна-ченого інтеґрала	phýsical méaning [sen-se] of définite íntegral
67.плоская об-ласть	плоска область	pláne domáin [région]
68.полярный угол	полярний кут	pólar ángle
69.полярная ось	полярна вісь	pólar áxis (*pl* áxes)
70.полярная сис-тема координат	полярна система ко-ординат	pólar coórdinate sýs-tem, sýstem of pólar coórdi-nates
71.полярные коор-динаты	полярні координати	pólar coórdinates
72.полярное урав-нение кривой	полярне рівняння кривої	pólar equátion of a cur-ve
73.полярный ра-диус	полярний радіус	pólar rádius (*pl* rádii)
74.полюс в поляр-ной системе коорди-нат	полюс в полярній си-стемі координат	póle in a pólar coórdi-nate sýstem

75.ломаная (линия)	ламана (лінія)	polígonal/bróken line
76.свойство аддитивности	властивість адитивності	próperty of àdditívity
77.свойство ли-нейности	властивість лінійності	próperty of lìneárity
78.выполнить, осуществить, произвести двойную подстановку	виконати, зробити, здійснити подвійну підстановку	réalize [cárry out, ef-féct, make] the dóuble sùbstitútion [pèrmutátion, replácement]
79.пересчёт пределов интегрирования	перелічування/перераховування меж [границь] інтеґрування	rècàlculátion/rècóunt límits of ìntegrátion
80.пересчитывать пределы интегрирования	перелічувати/перераховувати межі [границі] інтеґрування	rècóunt límits of íntegration
81.результат суммирования элементов	результат підсумовування елементів	resúlt of summátion of élements
82.вращаться вокруг оси [прямой]	обертатися навколо осі [прямої]	revólve [rotáte, turn] about an áxis (*pl* áxes) [a stráight line]
83.прямой круговой цилиндр	прямий (круговий) циліндр	right (círcular) cýlinder
84.схема примене-ния	схема застосування	schéme of àpplicátion, próblem-sólving schéme
85.тело	тіло	sólid [bódy]
86.тело вращения	тіло обертання	sólid [bódy] of rèvolútion [rotátion]
87.подстановка	підстановка	sùbstitútion
88.суммирование элементов	підсумовування елементів	summátion of élements
89.брать, выбирать точку	брати, вибирати (точку)	take, choos (chose; chosen) (a póint)
90.признак сходи-мости несобственного интеграла	ознака збіжності невластивого інтеґрала	test for convérgence of impróper íntegral
91.поперечное сечение	поперечний переріз	cross-séction
92.трактовать, рассматривать, интерпретировать искомую величину как сумму (как результат суммирования) элементов	трактувати, розгляда-ти, інтерпретувати шукану величину як суму (як результат підсумовування) елементів	tréat [consíder, regárd, intérpret] a(n) desíred [sought, ùnknówn] quántity [quántity in quéstion] as a sum (as a resúlt of summátion) of élements
93.верхний предел интегрирования	верхня межа [границя] інтеґрування	úpper límit of ìntegrá-tion
94.переменная ин-	змінна інтеґрування	váriable of ìntegrátion

тегрирования		
95.объём тела по известным площадям параллельных попе-речных сечений (перпендикулярных фиксированной прямой)	об"єм тіла по відомих площах паралельних поперечних перерізів (перпендикулярних фіксованій прямій)	vólume of a sólid/bódy from known áreas of parallel séctions [cross-séctions, tránsverse séctions] (which/that are pèrpendícular to a fixed straight line); vólume by slícing
96.объём тела вращения	об"єм тіла обертання	vólume of a sólid/bódy of rèvolútion/rotátion

DEFINITE INTEGRAL: Basic Terminology ERU

1. ábsolutely convérgent impróper íntegral	абсолютно збіжний невластивий інтеґрал	абсолютно сходя-щийся несобственный интеграл
2. àdditívity (with respéct to a fúnction, to an ínter-val)	адитивність (за фун-кцією, за інтервалом)	аддитивность (по функ-ции, по интер-валу)
3. arc length	довжина дуги	длина дуги
4. be génerated (by…)	породжуватися, утво-рюватися	порождаться, образовы-ваться
5. cartésian coórdinates	декартові координати	декартовы координаты
6. cauchy's príncipal válue of a divérgent ínte-gral	головне значення (коші) розбіжного інтеґрала	главное значение (коши) расходящего-ся интегра-ла
7. change the límits of ìntegrátion	змінити межі [границі] інтеґрування	изменить пределы ин-тегрирования
8. círcular séctor	круговий сектор	круговой сектор
9. convérge (ábsolutely, condítionally)	збігатися (абсолютно, умовно)	сходиться (абсолютно, условно)
10.convérge ábsolutely	абсолютно збігатися	абсолютно сходиться
11.convérgent impróper íntegral	збіжний невластивий ін-теґрал	сходящийся не-собственный интеграл
12.cross-séction área [tránsverse séction área]	площа поперечного перерізу	площадь попе-речного сечения
13.cùrvilínear séctor	криволінійний сектор	криволинейный сектор
14.cùrvilínear trapézium (*pl* trapéziums, trapézia) [trápezoid]	криволінійна трапеція	криволинейная трапеция
15.définite íntegral as a sum of élements	визначений інтеґрал як сума элементів	определённый интеграл как сумма элементов
16.définite íntegral from *a* to *b* [betwéen *a* and *b*] of a fúnction *f* of *x dx*	визначений інтеґрал від *a* до *b* функції эф від ікс де ікс	определённый интеграл от *a* до *b* функции эф от икс де икс
17.définite íntegral óver a ségment [an ínterval]	визначений інтеґрал по відрізку [по інтервалу]	определённый интеграл по отрезку [по интерва-лу]
18.définite íntegral with the váriable/várying úpper límit/bóund	визначений інтеґрал з змінною верхньою ме-жею [границею]	определённый интеграл с перемен-ным верхним преде-лом
19.derívative of a définite íntegral with respect to its váriable/várying úpper lí-mit/bóund	похідна визначеного інтеґрала за змінною верхньою межою	производная опре-делённого интеграла по переменному верхнему пределу

20.desíred [sought, ùnknówn] quántity [quántity in quéstion] as a sum of elements	шукана величина як сума елементів	искомая величина как сумма элементов
21.dìfferéntial of an arc length	дифференціал довжини дуги кривої	дифференциал длины дуги кривой
22.divérge	розбігатися	расходиться
23.divérgent impróper íntegral	розбіжний невластивий інтеґрал	расходящийся несобст-венный инте-грал
24.divísion [dècomposí-tion, partítion, split, splít-ting, súbdivìsion] an ínter-val into párts	розбиття/поділ інтер-валу на частини	разбиение/под-разделение интервала на части
25.divísion/partítion/dè-composítion/súbdivísion póint of a ségment into párts	точка поділу відрізка на частини	точка деления отре-зка на части
26.dóuble sùbstitútion [pèrmutátion, replácement]	подвійна підстановка	двойная подстановка
27.élement of a(n) desíred [sought, ùnknówn] quántity [of a quántity in quéstion]	елемент шуканої величини	элемент искомой величины
28.élement of ìntegrátion	елемент інтеґрування	элемент интегрирования
29.èleméntary	елементарний	элементарный
30.éstimate/èstimátion of définite íntegral	оцінка/оцінювання визначеного інтеґрала	оценка определённого интеграла
31.find [detérmine] the new límits of ìntegrátion	знайти нові межі [границі] інтеґрування	найти новые пределы интегрирования
32.form, make up, compóse an íntegral sum	утворити, скласти інтеґральну суму	образовывать, составлять интегральную сумму
33.gèneralizátion [exténsion] of nótion of définite íntegral	узагальнення поняття визначеного інтеґрала	обобщение поня-тия определённого интеграла
34.géneralize/exténd the nótion of définite ínte-gral	узагальнити поняття визначеного інтеґрала	обобщить понятие определённого интег-рала
35.génerate	породжувати, утворювати	порождать, образовывать
36.geométric(al) méaning [sense] of définite íntegral	геометричний сенс визначеного інтеґрала	геометрический смысл определённого интеграла
37.hòmogenéity	однорідність	однородность
38.impróper íntegral of the	невластивий інтеґрал	несобственный интеграл

first/second kind	першого/другого роду	первого/вто-рого рода
39.impróper íntegral on/ óver a fínite ínterval of a(n) dìscontínuous [un-límited, unbóunded] func-tion	невластивий інтеґрал по скінченному інтервалу від розривної [необмеженої] функ-ції	несобственный интеграл по конечно-му интерва-лу от раз-рывной [неог-раничен-ной] функции
40.impróper íntegral with ínfinite límit [límits] of a còntínuous function	невластивий інтеґрал з нескінченною межею [границею] з не-скінченними межами/ границями від непе-рервної функції	несобственный интеграл с бесконеч-ным преде-лом [с бес-конечными предела-ми] от непре-рывной функции
41.inscríbe a polýgonal/ bróken line (in/into a curve)	вписати ламану лінію (в криву)	вписать ломаную линию (в кривую)
42.inscríbed polýgonal/ bróken line (in/into a curve)	вписана ламана лінія (в криву)	вписанная лома-ная ли-ния (в кривую)
43.íntegrabílity	інтеґровність	интегрируемость
44.íntegrable fúnction	інтеґровна функція	интегрируемая функция
45.íntegral sum	інтеґральна сума	интегральная сумма
46.íntegrand, fúnction ún-der the íntegral sign [fúnc-tion to be íntegra-ted], sùbíntegral fúnc-tion	підінтеґральна функція	подынтегральная функ-ция
47.íntegrand; expréssion únder the íntegral sign [expréssion to be ínte-grated]; sùbíntegral exp-réssion	підінтеґральний вираз	подынтеграль-ное выра-жение
48.íntegrate	інтеґрувати	интегрировать
49.ìntegrátion	інтеґрування	интегрирование
50.ìntegrátion by chánge/ súbstitútion of a váriable [by súbstitútion]	інтеґрування заміною змінної [підстановкою]	интегрирование заменой переменной [подстанов-кой]
51.ìntegrátion by parts	інтеґрування частинами	интегрирование по час-тям
52.ínterchánge the úpper and lówer límits of ìn-tegrátion	поміняти місцями верх-ню і нижню межі [границі] інтеґрування	поменять мес-тами верх-ний и ниж-ний пределы интегри-рования
53.limit of ìntegrátion (úpper, lówer [inférior])	границя [межа] інтеґру-вання (верхня, нижня)	предел интегрирования (верхний, нижний)
54.límit of the íntegral	границя інтеґральної су-	предел интегра-льной

sum	ми	суммы
55.lìneárity	лінійність	линейность
56.lówer [inférior] límit of ìntegrátion	нижня межа [границя] інтеґрування	нижний предел интегрирования
57.méan/áverage válue of a function on an ínter-val	середнє значення функції на інтервалі	среднее значе-ние функции на ин-тервале
58.méan-válue théorem	теорема про середнє	теорема о сред-нем
59.mechánical méaning [sense] of définite íntegral	механічний сенс визначеного інтеґрала	механический смысл определённого интеграла
60.newton-leibniz fórmula (*pl* fórmulas, fórmu-lae)	формула ньютона-лейбніца	формула нью-тона-лейбница
61.part of an ínterval, súbìnterval	частина інтервалу	часть интервала
62.pártial ínterval, súbìnterval	частковий інтервал	частичный интервал
63.pass to the límit	перейти до границі	перейти к пределу
64.pass to the pólar coórdinates	перейти до полярних координат	перейти к поля-рным координатам
65.pássage to the límit	граничний перехід, перехід до границі	предельный пе-реход, переход к пре-делу
66.phýsical méaning [sense] of définite íntegral	фізичний сенс визначеного інтеґрала	физический смысл определённо-го интеграла
67.pláne domáin [région]	плоска область	плоская об-ласть
68.pólar ángle	полярний кут	полярный угол
69.pólar áxis (*pl* áxes)	полярна вісь	полярная ось
70.pólar coórdinate sýstem, sýstem of pólar coórdinates	полярна система координат	полярная сис-тема координат
71.pólar coórdinates	полярні координати	полярные коор-динаты
72.pólar equátion of a curve	полярне рівняння кривої	полярное урав-нение кривой
73.pólar rádius (*pl* rádii)	полярний радіус	полярный ра-диус
74.póle in a pólar coórdinate sýstem	полюс в полярній системі координат	полюс в поляр-ной системе коорди-нат
75.polígonal/bróken line	ламана (лінія)	ломаная (линия)
76.próperty of àdditívity	властивість адитивності	свойство адди-тивности
77.próperty of lìneárity	властивість лінійності	свойство ли-нейности
78.réalize [cárry out, efféct, make] the dóuble sùbstitútion [pèrmutátion, replácement]	виконати, зробити, здійснити подвійну підстановку	выполнить, осуществить, произвести двойную подстановку
79.rècàlculátion/rècóunt límits of ìntegrátion	перелічування/перераховування меж [границь] інтеґрування	пересчёт пределов интегрирования

80.rècóunt límits of íntegration	перелічувати/перерахо-вувати межі [границі] інтеґрування	пересчитывать пределы интегрирования
81.resúlt of summátion of élements	результат підсумову-вання елементів	результат сум-мирования элементов
82.revólve [rotáte, turn] about an áxis (*pl* áxes) [a stráight line]	обертатися навколо осі [прямої]	вращаться вокруг оси [прямой]
83.right (círcular) cýlinder	прямий (круговий) ци-ліндр	прямой круговой ци-линдр
84.schéme of àpplicátion, próblem-sólving schéme	схема застосування	схема примене-ния
85.sólid [bódy]	тіло	тело
86.sólid [bódy] of rèvolútion [rotátion]	тіло обертання	тело вращения
87.sùbstitútion	підстановка	подстановка
88.summátion of élements	підсумовування еле-ментів	суммирование элементов
89.take, choos (chose; chosen) (a póint)	брати, вибирати (точку)	брать, выбирать точку
90.test for convérgence of impróper íntegral	ознака збіжності не-властивого інтеґрала	признак сходи-мости не-собственного интеграла
91.cross-séction	поперечний переріз	поперечное сечение
92.tréat [consíder, regárd, intérpret] a(n) desíred [sought, ùnknówn] quántity [quántity in quéstion] as a sum (as a resúlt of summátion) of élements	трактувати, розгляда-ти, інтерпретувати шукану величину як суму (як ре-зультат підсумовування) елементів	трактовать, рас-сматривать, интерпре-тировать искомую ве-личину как сумму (как результат сумми-рования) элементов
93.úpper límit of ìntegrá-tion	верхня межа [границя] інтеґрування	верхний предел интегри-рования
94.váriable of ìntegrátion	змінна інтеґрування	переменная ин-тегрирования
95.vólume of a sólid/bódy from known áreas of pá-rallel séctions [cross-séc-tions, tránsverse séctions] (which/that are pèrpendí-cular to a fixed straight line); vólume by slícing	об"єм тіла по відомих площах паралельних по-перечних перерізів (пер-пендикулярних фіксованій прямій)	объём тела по известным площадям параллельных попе-речных сечений (пер-пендикулярных фик-сированной прямой)
96.vólume of a sólid/bódy of rèvolútion/rotátion	об"єм тіла обертання	объём тела вращения

LECTURE NO. 24. DOUBLE INTEGRAL

POINT 1. DOUBLE INTEGRAL

POINT 2. EVALUATION OF A DOUBLE INTEGRAL IN CARTESIAN COORDINATES

POINT 3. IMPROPER DOUBLE INTEGRAL. POISSON'S FORMULA

POINT 1. DOUBLE INTEGRAL

Def. 1. Let a function of two variables $z = f(M) = f(x, y)$ be given in a some domain *D* of the xOy-plane (fig. 1).

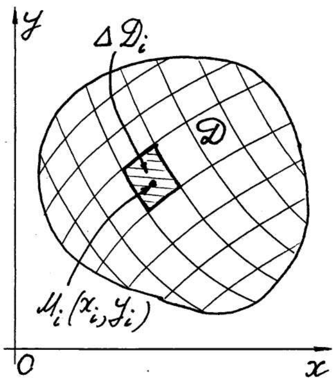


Fig. 1

1. We divide the domain into *n* parts ΔD_i, $i = \overline{1, n}$, with areas ΔS_i and diameters $\lambda_i = \max\limits_{M,N \in \Delta D_i} |MN|$.

2. We take arbitrary point $M_i(x_i; y_i)$ in every part ΔD_i, find the value of the function at this point and multiply it by the area ΔS_i of ΔD_i.

3. We add all these products $f(M_i)\Delta S_i = f(x_i, y_i)\Delta S_i$ and get an integral sum

$$\sigma = \sum_{i=1}^{n} f(M_i)\Delta S_i = \sum_{i=1}^{n} f(x_i, y_i)\Delta S_i .$$

4. Let $\lambda = \max\limits_{i=\overline{1,n}} \{\lambda_i\}$ and $\lambda \to 0$. If there exists the limit of the integral sum σ, then this limit is called the double integral of the function $z = f(M) = f(x, y)$ over the domain *D* and is denoted by

$$\iint\limits_D f(M)dS = \iint\limits_D f(x, y)dxdy = \lim_{\lambda \to 0} \sigma = \lim_{\lambda \to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta S_i \qquad (1)$$

We can consider the double integral as the sum of elements $f(x, y)dS$ where $dS = dxdy$ is an element of the area.

Theorem 1 (existence of a double integral)**.** If a function $z = f(M) = f(x, y)$ is continuous in a domain *D* then its double integral over *D* exists.

It is evident that for $f(M) = f(x, y) \equiv 1$ a double integral gives the area of the domain *D*,

$$S = S_D = \iint_D dxdy. \qquad (2)$$

Mechanic sense of a double integral. If $\gamma(M) = \gamma(x, y) \geq 0$ is the surface density of a plate $D \subset xOy$, then its mass equals the next double integral

$$m = \iint_D \gamma(M) dS = \iint_D \gamma(x, y) dxdy \qquad (3)$$

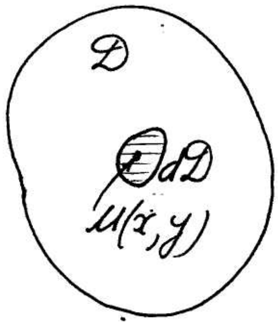


Fig. 2

■An element of the mass

$$dm = \gamma(M) dS = \gamma(x, y) dS;$$

it is the mass of the element $dD \subset D$ with the area dS and with a constant surface density $\gamma(M) = \gamma(x, y), M(x, y) \in dD$ (fig. 2). Sum of all these elements gives the mass of the plate which is represented by a double integral (3).■

Def. 2. A cylindrical body [a curvilinear cylinder] is called a body bounded:

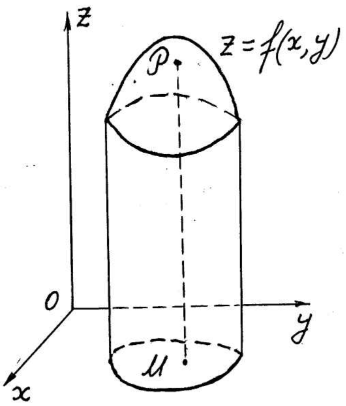


Fig. 3

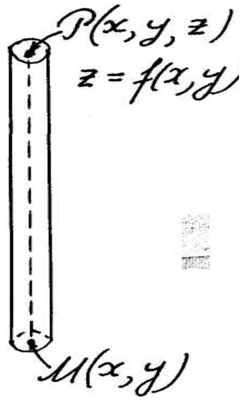


Fig. 4

a) above by a surface $z = f(x, y) \geq 0$;

b) below by a domain *D* of the *xOy*-plane;

c) aside by a cylindrical surface with the generatrix parallel to Oz-axis and the diréctrix which is the boundary of the domain *D* (fig. 3).

Geometric sense of a double integral. The volume of a cylindrical body equals the double integral

$$V = \iint_D f(x, y) dxdy. \qquad (4)$$

■An element of the volume

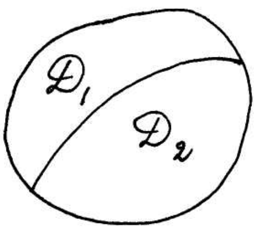
$$dV = f(M)dS = f(x,y)dS$$

is the volume of a right cylinder with the base $dD \subset D$ of the area dS and the height $f(M) = f(x,y),\ M(x,y) \in dD$ (fig. 4). The volume of a cylindrical body is the sum of all these elements and is represented by the double integral (4).■

Properties of a double integral are analogous to those of a definite integral. For example:

1 (linearity). For any functions $f_1(x,y),\ f_2(x,y)$ and any constants k_1, k_2

$$\iint_D (k_1 f_1(x,y) + k_2 f_2(x,y))dxdy = k_1 \iint_D f_1(x,y)dxdy + k_2 \iint_D f_2(x,y)dxdy.$$



2 (additivity with respect to a domain of integration). If a domain is divided into two disjoint parts $D = D_1 \cup D_2,\ \ D_1 \cap D_2 = \varnothing$ (fig. 5), then

$$\iint_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy.$$

Fig. 5

POINT 2. EVALUATION OF A DOUBLE INTEGRAL IN CARTESIAN COORDINATES

Def. 3. A domain D is called a **domain of the first type** if it is bounded (see fig. 6):

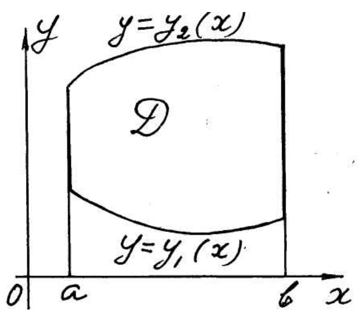


Fig. 6

a) from the left by a straight line $x = a$;

b) from the right by a straight line $x = b$;

c) below by a line $y = y_1(x)$;

d) above by a line $y = y_2(x)$,

$$D = \{(x,y):\ a \le x \le b,\ \forall x \in (a,b)(y_1(x) \le y \le y_2(x))\}.$$

A double integral over a domain of the first type is calculated by a formula

$$\iint_D f(x,y)dxdy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x,y)dy. \qquad (5)$$

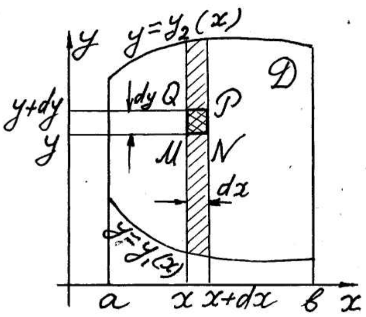


Fig. 7

Correspondingly to this formula we integrate at first with respect to *y* from $y_1(x)$ to $y_2(x)$, that is calculate an **inner integral**

$$\int_{y_1(x)}^{y_2(x)} f(x,y)dy,$$

and then we integrate the result with respect to *x* from *a* to *b*.

■We'll prove the formula (5) proceeding from the mechanic sense of a double integral. Let the integrand $f(M)=f(x,y)\geq 0$ be the surface density of a plate, defined by the figure $D=\{(x,y): a\leq x\leq b, \forall x\in(a,b)(y_1(x)\leq y\leq y_2(x))\}$ (fig. 6). Hence the mass of the plate equals the double integral

$$m=\iint_D f(M)dS=\iint_D f(x,y)dxdy.$$

Now we'll find the same mass by the other way and compare the results. The mass of the element $MNPQ$ of the plate between $x, x+dx$ and $y, y+dy$ (fig. 7) equals $f(M)\Delta x\Delta y=f(x,y)\Delta x\Delta y$. Adding all such the masses from $y_1(x)$ to $y_2(x)$ we find the mass of the hatched strip (fig. 7), namely

$$dm=\int_{y_1(x)}^{y_2(x)} f(x,y)dxdy=dx\int_{y_1(x)}^{y_2(x)} f(x,y)dy.$$

Adding finally the masses of all such the strips from $x=a$ to $x=b$ we find the mass of the whole plate that is

$$m=\int_a^b\left(dx\int_{y_1(x)}^{y_2(x)} f(x,y)dy\right)=\int_a^b dx\int_{y_1(x)}^{y_2(x)} f(x,y)dy.$$

Comparing of two results of the mass calculating proves validity of the formula (5).■

Note 1. Integral with respect to *x* from *a* to *b* is called that **exterior** [external]. The right side of the formula (5) is called the **repeated integral**.

Def. 4. A domain *D* is called a **domain of the second type** if it is bounded (see fig. 8):

a) below by a straight line $y=c$;

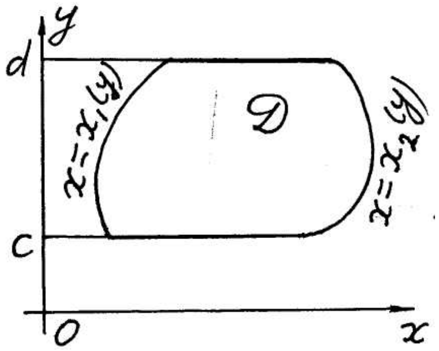


Fig. 8

b) above by a a straight line $y = d$;

c) from the left by a line $x = x_1(y)$;

d) from the right by a line $x = x_2(y)$,

$D = \{(x, y): c \le y \le d, \forall y \in (c, d)(x_1(y) \le x \le x_2(y))\}$.

The double integral over a domain D of the second type is calculated by a formula

$$\iint_D f(x, y)dxdy = \int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x, y)dx. \qquad (6)$$

At first we calculate an **inner integral**

$$\int_{x_1(y)}^{x_2(y)} f(x, y)dx,$$

that is integrate with respect to x from $x_1(y)$ to $x_2(y)$, and then we integrate the result with respect to y from c to d.

■Prove this formula yourselves.■

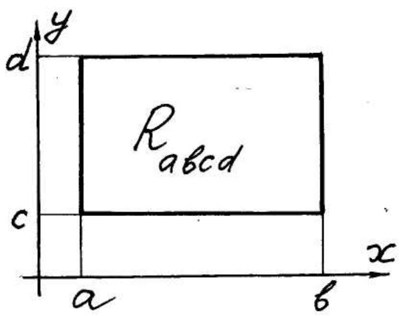


Fig. 9

Ex. 1. Let a domain of integration be a rectangle

$$R_{abcd} = \{(x, y): a \le x \le b, c \le y \le d\}$$

with the sides parallel to Ox-, Oy-axes (fig. 9). The rectangle is the domain of the first and second types, therefore we can use both formulas (5) and (6),

$$\iint_{R_{abcd}} f(x, y)dxdy = \int_a^b dx \int_c^d f(x, y)dy = \int_c^d dy \int_a^b f(x, y)dx. \qquad (7)$$

The formula (6) means that in the case of the rectangle R_{abcd} we can integrate in any order. But in practice one of orders of integration can lead to easier calculations than the other one.

Ex. 2. Find the mass of a plate

$$R = \left\{(x, y): \frac{1}{6} \le x \le \frac{1}{3}, \ln 3 \le y \le \ln 4\right\} \text{ (fig. 10)}$$

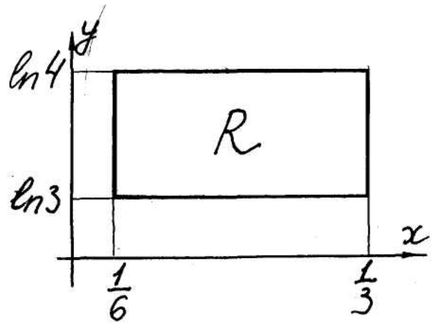


Fig. 10

if the surface density of the plate equals $\gamma(x,y) = 12ye^{6xy}$.

We find the mass by the formula (3). A domain of integration R is a rectangle with the sides parallel to Ox-, Oy-axes. Using the formula (7) we get

$$m = \iint_R \gamma(x,y)dxdu = \iint_R 12ye^{6xy}dxdy = 12\int_{ln3}^{ln4} dy \int_{\frac{1}{6}}^{\frac{1}{3}} ye^{6xy}dx = 12\int_{ln3}^{ln4} ydy \int_{\frac{1}{6}}^{\frac{1}{3}} e^{6xy}dx =$$

$$= \left| \int_{\frac{1}{6}}^{\frac{1}{3}} e^{6xy}dx = \frac{1}{6y}e^{6xy}\Big|_{\frac{1}{6}}^{\frac{1}{3}} = \frac{1}{6y}\left(e^{2y} - e^{y}\right) \right| = 12\int_{ln3}^{ln4} y\frac{1}{6y}\left(e^{2y} - e^{y}\right)dy = 2\int_{ln3}^{ln4}\left(e^{2y} - e^{y}\right)dy =$$

$$= 2\left(\frac{1}{2}e^{2y} - e^{y}\right)\Big|_{\ln 3}^{\ln 4} = 2\left(\frac{1}{2}\left(e^{2\ln 4} - e^{2\ln 3}\right) - \left(e^{\ln 4} - e^{\ln 3}\right)\right) = 2\left(\frac{1}{2}(16-9) - (4-3)\right) = 5.$$

The other order of integration isn't well (verify!).

Ex. 3. Evaluate by two ways the integral $\iint_D \sin(3x+2y)dxdy$ if the domain D is defined by inequalities $0 \le x \le \frac{\pi}{2}, 0 \le y \le 3x$ (fig. 11).

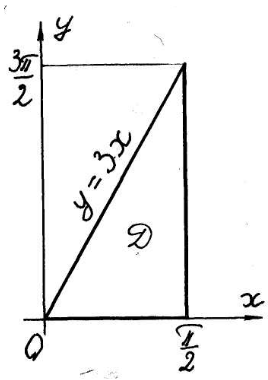


Fig. 11

The domain D is that of the first and second types.

The first way. We consider D as a domain of the first type,

$$D = \left\{(x,y): 0 \le x \le \frac{\pi}{2}, \forall x \in \left(0, \frac{\pi}{2}\right)(0 \le y \le 3x)\right\},$$

that is D is bounded

a) from the left by the straight line $x = 0$;

b) from the right by the straight line $x = \frac{\pi}{2}$;

c) below by the line $y = 0$;

d) above by the line $y = 3x$.

Using the formula (5) we have

$$\iint\limits_D \sin(3x+2y)dxdy = \int\limits_0^{\frac{\pi}{2}} dx \int\limits_0^{3x} \sin(3x+2y)dy =$$

$$= \left| \int\limits_0^{3x} \sin(3x+2y)dy = \left| \begin{array}{c|c|c} \multicolumn{3}{c}{3x+2y=t,\ 2dy=dt,} \\ y & 0 & 3x \\ \hline t & 3x & 9x \end{array} \right| = \frac{1}{2}\int\limits_{3x}^{9x} \sin t dt = -\frac{1}{2}\cos t \Big|_{3x}^{9x} = \frac{1}{2}(\cos 3x - \cos 9x) \right| =$$

$$= \frac{1}{2}\int\limits_0^{\frac{\pi}{2}} (\cos 3x - \cos 9x)dx = \frac{1}{2}\left(\frac{1}{3}\sin 3x - \frac{1}{9}\sin 9x\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{1}{2}\left(\frac{1}{3}\sin\frac{3\pi}{2} - \frac{1}{9}\sin\frac{9\pi}{2}\right) = -\frac{2}{9}.$$

In the second way we treat D as a domain of the second type,

$$D = \left\{(x,y): 0 \le y \le \frac{3\pi}{2}, \forall y \in \left(0, \frac{3\pi}{2}\right)\left(\frac{y}{3} \le x \le \frac{\pi}{2}\right)\right\},$$

that is D is bounded

a) below by the straight line $y = 0$;

b) above by the straight line $y = \frac{3\pi}{2}$;

c) from the left by the line $x = \frac{y}{3}$;

d) from the right by the line $x = \frac{\pi}{2}$.

Therefore with the help of the formula (6) we write

$$\iint\limits_D \sin(3x+2y)dxdy = \int\limits_0^{\frac{3\pi}{2}} dy \int\limits_{\frac{y}{3}}^{\frac{\pi}{2}} \sin(3x+2y)dx.$$

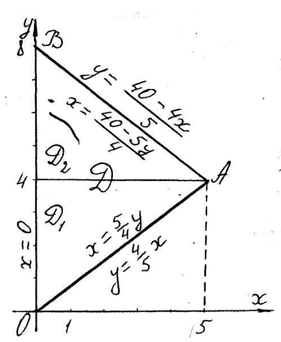


Fig. 12

Fulfill evaluation of the integral yourselves.

Ex. 4. Set the limits of integration in a double integral over a triangular domain D with the vertices $0(0;0)$, $A(5;4)$, $B(0;8)$ (fig. 12)

At first we compile the equations of the straight lines OA and AB.

OA: $y = kx;\ A(5;\,4) \in OA \Rightarrow 4 = 5k, k = \frac{4}{5}, y = \frac{4}{5}x, x = \frac{5}{4}y.$

AB: $\frac{x-0}{5-0} = \frac{y-8}{4-8}, 4x + 5y - 40 = 0,\ y = \frac{40-4x}{5}, x = \frac{40-5y}{4}.$

The first way. The domain D is that of the first type because of it's bounded from the left by a straight line $x = 0$, from the right by a straight line $x = 5$, below by a line $y = \frac{4}{5}x$, above by a line $y = \frac{40-4x}{5}$, hence on the base of the formula (5)

$$\iint_D f(x,y)dxdy = \int_0^5 dx \int_{\frac{4}{5}x}^{\frac{40-4x}{5}} f(x,y)dy.$$

The second way. To apply the formula (6) we divide the domain D into two domains D_1, D_2 of the second type by a straight line $y = 4$ (fig. 12). If we describe them by two double inequalities, namely

$$D_1 = \left\{(x,y): 0 \le y \le 4;\ \forall y \in (0,4)\left(0 \le x \le \frac{5}{4}y\right)\right\},$$

$$D_2 = \left\{(x,y): 4 \le y \le 8;\ \forall y \in (4,8)\left(0 \le x \le \frac{40-5y}{4}\right)\right\},$$

we'll get

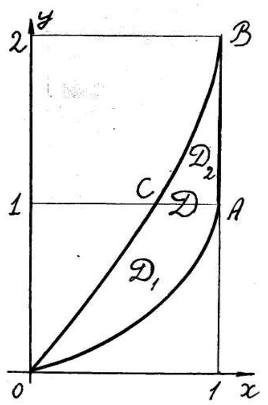


Fig. 13

$$\iint_D f(x,y)dxdy = \iint_{D_1} f(x,y)dxdy + \iint_{D_2} f(x,y)dxdy =$$

$$= \int_0^4 dy \int_0^{\frac{5}{4}y} f(x,y)dx + \int_4^8 dy \int_0^{\frac{40-5y}{4}} f(x,y)dx.$$

Ex. 5. Evaluate the double integral

$$\iint_D \left(x^2 + y^2\right)dxdy$$

over the domain

$$D = \left\{(x,y): 0 \le x \le 1, \forall x \in (0,1)\left(\sqrt{x} \le y \le 2\sqrt{x}\right)\right\} \text{ (fig. 13)}.$$

The domain D is that of the first type. It can be divided into two domains of the second type $D_1 = OAC, D_2 = ABC$ by the straight line

$y=1$ (fig. 13),

$$D = D_1 \cup D_2,$$

$$D_1 = \left\{ (x,y) : 0 \le y \le 1, \forall y \in (0,1) \left(\frac{y^2}{4} \le x \le y^2 \right) \right\},$$

$$D_2 = \left\{ (x,y) : 1 \le y \le 2, \forall y \in (1,2) \left(\frac{y^2}{4} \le x \le 1 \right) \right\}.$$

The integral in question equals the sum of two integrals. It's well to calculate the first one over the domain *D* and the second one as the sum of integrals over D_1 and D_2.

$$\iint_D (x^2+y^2)dxdy = \iint_D x^2 dxdy + \iint_D y^2 dxdy = \int_0^1 x^2 dx \int_{\sqrt{x}}^{2\sqrt{x}} dy + \iint_{D_1} y^2 dxdy + \iint_{D_2} y^2 dxdy =$$

$$= \int_0^1 x^2 dx \int_{\sqrt{x}}^{2\sqrt{x}} dy + \int_0^1 y^2 dy \int_{\frac{y^2}{4}}^{y^2} dx + \int_1^2 y^2 dy \int_{\frac{y^2}{4}}^{1} dx = \int_0^1 x^2 \sqrt{x} dx + \frac{3}{4} \int_0^1 y^4 dy + \int_1^2 y^2 \left(1 - \frac{y^2}{4} \right) dy =$$

$$= \frac{2}{7} x^{\frac{7}{2}} \Big|_0^1 + \frac{3}{20} y^5 \Big|_0^1 + \frac{1}{3} y^3 \Big|_1^2 - \frac{1}{20} y^5 \Big|_1^2 = \frac{2}{7} + \frac{3}{20} + \frac{7}{3} - \frac{31}{20} = \frac{128}{105} \approx 1.22.$$

POINT 3. IMPROPER DOUBLE INTEGRALS. POISSON FORMULA

We'll limit ourselves to improper double integrals of the first kind that is integrals of continuous functions over unbounded domains. As such the domains we'll consider: the first quadrant

$$R = \{(x,y) : 0 \le x < \infty, 0 \le y < \infty\},$$

an infinite rectangle

$$R_{ab} = \{(x,y) : -\infty < x \le a, -\infty < y \le b\}$$

and the xOy-plane

$$\Re^2 = \{(x,y) : -\infty < x < \infty, -\infty < y < \infty\}.$$

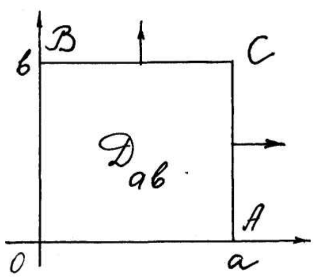


Fig. 14

Let R be the first quadrant and let $D_{ab} = OACB = \{(x, y): 0 \le x \le a, 0 \le y \le b\}$ be a finite rectangle with sides a and b (fig. 14). We define an improper integral over R as the next limit

$$\iint_D f(x, y)dxdy = \int_0^\infty\int_0^\infty f(x, y)dxdy = \lim_{\substack{a\to\infty \\ b\to\infty}} \iint_R f(x, y)dxdy = \lim_{\substack{a\to\infty \\ b\to\infty}} \int_0^a dx \int_0^b f(x, y)dy =$$

$$= \lim_{\substack{a\to\infty \\ b\to\infty}} \int_0^b dy \int_0^a f(x, y)dx = \int_0^\infty dx \int_0^\infty f(x, y)dy = \int_0^\infty dy \int_0^\infty f(x, y)dx .$$

As the corollary we get the formula of passing from a double integral to that repeating (the formula of **interchange of integrations**)

$$\int_0^\infty\int_0^\infty f(x, y)dxdy = \int_0^\infty dx \int_0^\infty f(x, y)dy = \int_0^\infty dy \int_0^\infty f(x, y)dx . \quad (8)$$

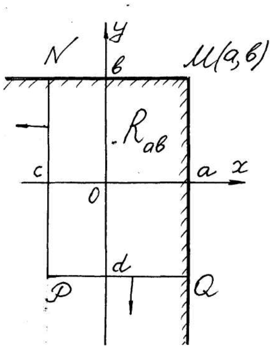


Fig. 15

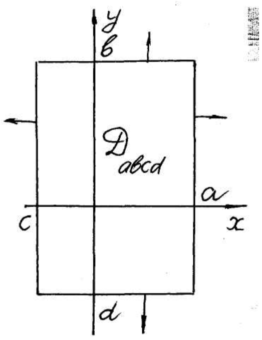


Fig. 16

An improper integral over the rectangle $R_{ab} = \{(x, y): -\infty < x \le a, -\infty < y \le b\}$ (see fig. 15) we define as the limit of the integral over a finite rectangle $\{(x, y): c \le x \le a, d \le y \le b\}$ as $c \to -\infty, d \to -\infty$, and an improper integral over the xOy-plane we define as the limit of the integral over the same rectangle (fig. 16) as $a \to \infty$, $b \to \infty$ and simultaneously $c \to -\infty, d \to -\infty$. As the result we get the next two formulas

$$\iint_{R_{ab}} f(x, y)dxdy = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x, y)dxdy = \int_{-\infty}^{a} dx \int_{-\infty}^{b} f(x, y)dy = \int_{-\infty}^{b} dy \int_{-\infty}^{a} f(x, y)dx , \quad (9)$$

$$\iint_{\Re^2} f(x, y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(x, y)dy = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x, y)dx . \quad (10)$$

Ex. 6. **Poisson integrals** $I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}, J = \int_0^\infty e^{-x^2} dx = \frac{1}{2} I = \frac{\sqrt{\pi}}{2}$. (11)

■ $$J=\int_0^\infty e^{-x^2}dx=\left|\begin{array}{l} x=yt,\ y\ge 0,\ t\ge 0 \\ dx=ydt \quad \begin{array}{c|c|c} x & 0 & \infty \\ \hline t & 0 & \infty \end{array}\end{array}\right|=\int_0^\infty e^{-y^2t^2}ydt,\ J\cdot e^{-y^2}dy=e^{-y^2}dy\int_0^\infty e^{-y^2t^2}ydt.$$

Let's integrate with respect to y over the interval $[0,\infty)$,

$$J\cdot\int_0^\infty e^{-y^2}dy=J^2=\int_0^\infty e^{-y^2}dy\int_0^\infty e^{-y^2t^2}ydt=\int_0^\infty dy\int_0^\infty e^{-y^2}\cdot e^{-y^2t^2}ydt=\int_0^\infty dy\int_0^\infty e^{-y^2\left(1+t^2\right)}ydt=$$

$$=\int_0^\infty dt\int_0^\infty e^{-y^2\left(1+t^2\right)}ydy=\left|\begin{array}{l} -y^2\left(1+t^2\right)=z,-2y\left(1+t^2\right)dy=dz \\ ydy=-\dfrac{dz}{2\left(1+t^2\right)} \quad \begin{array}{c|c|c} y & 0 & \infty \\ \hline z & 0 & -\infty \end{array}\end{array}\right|=\int_0^\infty dt\int_0^{-\infty} e^z\left(-\frac{1}{2\left(1+t^2\right)}\right)dz=$$

$$=\frac{1}{2}\int_0^\infty\frac{dt}{1+t^2}dt\int_{-\infty}^0 e^z dz=\frac{1}{2}\int_0^\infty\frac{dt}{1+t^2}dt\cdot\int_{-\infty}^0 e^z dz=\frac{1}{2}\arctan t\Big|_0^\infty\cdot e^z\Big|_{-\infty}^0=\frac{1}{2}\left(\frac{\pi}{2}-0\right)(1-0)=\frac{\pi}{4}.$$

We've proved that $J^2=\dfrac{\pi}{4}$, and therefore $J=\dfrac{\sqrt{\pi}}{2}$, $I=\sqrt{\pi}$.■

POINT 4. DOUBLE INTEGRAL IN POLAR COORDINATES

Let's suppose that we study a double integral

$$\iint_D f(x,y)dS=\iint_D f(x,y)dxdy$$

over a domain D of the xOy-plane, and we pass to polar coordinates

$$x=\rho\cos\varphi,\ y=\rho\sin\varphi,\ x^2+y^2=\rho^2, \tag{12}$$

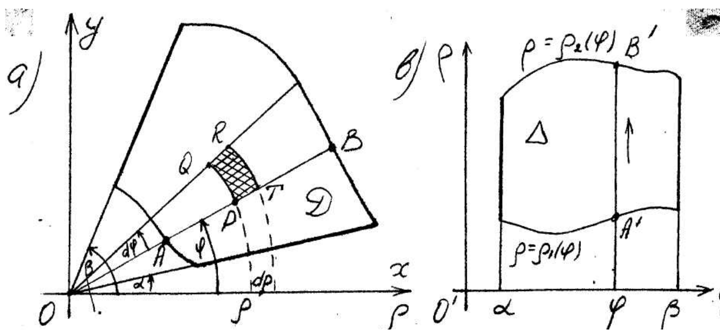


Fig. 17

coinciding the pole O with the origin $O(0;0)$ and the polar axis $O\rho$ with positive semiaxis of the Ox-axis of Cartesian coordinate system. The domain D transforms into some domain Δ of the $\varphi O'\rho$-plane and the double integral passes in that over the domain Δ.

To show up how the element dS of the area changes we generate an element dD of the domain D by two circles of radii $\rho,\ d\rho$ centered at the origin and by two rays starting from the pole under angles $\varphi, \varphi + d\varphi$ to the polar axis (fig. 17 a). We can consider *dD* as a curvilinear rectangle *PQRT* with the area

$$dS = S_{PQRT} = PT \cdot PQ = d\rho \cdot \rho d\varphi .$$

Therefore

$$dS = \rho d\rho d\varphi ,$$

and the formula of passing to polar coordinates in a double integral can be written as

$$\iint_D f(x, y)dxdy = \iint_\Delta f(\rho\cos\varphi, \rho\sin\varphi)\rho d\rho d\varphi . \qquad (13)$$

In applications we often meet the case of a domain *D* bounded by two rays

$$\varphi = \alpha, \varphi = \beta \ (\alpha < \beta) \qquad (14)$$

and two lines with polar equations

$$\rho = \rho_1(\varphi), \rho = \rho_2(\varphi), \ (\rho_1(\varphi) \le \rho_2(\varphi)) \qquad (15)$$

(fig. 17 a). One can describe such the domain by two double inequalities

$$\alpha \le \varphi \le \beta, \quad \forall \varphi \in (\alpha, \beta): \rho_1(\varphi) \le \rho \le \rho_2(\varphi), \qquad (16)$$

whence it follows that a domain $\Delta \subseteq \varphi O'\rho$ (fig. 17 b), in which *D* transforms after passage to polar coordinates, is that of the first type. Therefore, by the formula (5)

$$\iint_D f(x, y)dxdy = \int_\alpha^\beta d\varphi \int_{\rho_1(\varphi)}^{\rho_2(\varphi)} f(\rho\cos\varphi, \rho\sin\varphi)\rho d\rho . \qquad (17)$$

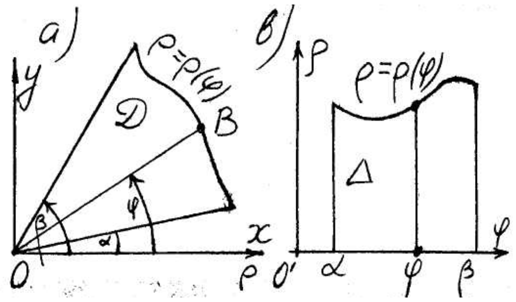


Fig. 18

If a line $\rho = \rho_1(\varphi)$ degenerates in the pole *O* we get a curvilinear sector *D* bounded by two rays

$$\varphi = \alpha, \varphi = \beta \ (\alpha < \beta) \qquad (18)$$

and a line with a polar equation

$$\rho = \rho(\varphi) \qquad (19)$$

(fig. 18 a). We describe it by the inequalities

$$\alpha \le \varphi \le \beta, \quad \forall \varphi \in (\alpha, \beta): O \le \rho \le \rho(\varphi), \qquad (20)$$

whence it follows that a domain $\Delta \subseteq \varphi O'\rho$ (fig. 18 b) is also that of the first type. Therefore, by the formula (5)

$$\iint\limits_D f(x,y)dxdy = \int\limits_\alpha^\beta d\varphi \int\limits_0^{\rho(\varphi)} f(\rho\cos\varphi, \rho\sin\varphi)\rho d\rho. \quad (21)$$

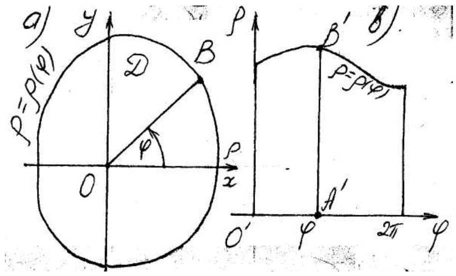


Fig. 19

Let a domain D contain the pole O, and every ray $\varphi = const$ intersect the boundary of the domain in unique point (fig. 19 a). If (19) is its polar equation, then $\quad 0 \le \varphi \le 2\pi, \quad \forall \varphi \in (0, 2\pi): 0 \le \rho \le \rho(\varphi) \quad (22)$ (fig. 19 b), and therefore

$$\iint\limits_D f(x,y)dxdy = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\rho(\varphi)} f(\rho\cos\varphi, \rho\sin\varphi)\rho d\rho. \quad (23)$$

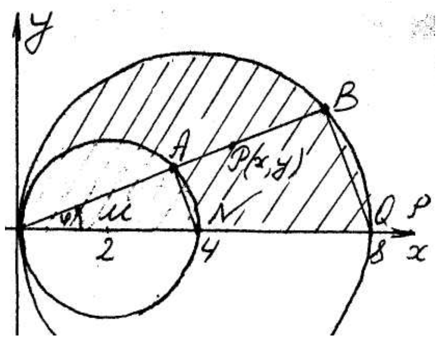


Fig. 20

Ex. 7. Evaluate the mass of a plate D containing between two curves

$$l_1: x^2 - 4x + y^2 = 0, \quad l_2: x^2 - 8x + y^2 = 0$$

for $y \ge 0$ (fig. 20), if its surface density $\gamma(P) = \gamma(x,y)$ at any point $P(x;y) \in D$ is proportional to the polar radius OP of this point and equals 8 at the point $N(4;0)$.

The surface density of the plate D

$$\gamma(P) = \gamma(x,y) = k \cdot OP = k\sqrt{x^2+y^2};\ \gamma(N) = \gamma(4;0) = k\sqrt{4^2+0^2} = 4k = 8 \Rightarrow k = 2,$$
$$\gamma(P) = \gamma(x,y) = 2\sqrt{x^2+y^2},$$

and by virtue of the mechanic sense of a double integral (see (3)) we have to calculate a double integral

$$m = \iint\limits_D \gamma(P)dS = \iint\limits_D \gamma(x,y)dxdy = 2\iint\limits_D \sqrt{x^2+y^2}\,dxdy.$$

Completing the squares we make sure that the curves l_1, l_2 are circles with radii 2 and 4 and centres $M(2;0), N(4;0)$ correspondingly:

$$x^2 - 4x + 4 + y^2 = 4, (x-2)^2 + y^2 = 2^2, x^2 - 8x + 16 + y^2 = 16, (x-4)^2 + y^2 = 4^2.$$

If we carry out the transition (12) to polar coordinates, we'll get the polar equations of l_1, l_2

$$l_1: x^2 + y^2 = 4x,\ \rho^2 = 4\rho\cos\varphi,\ \rho = 4\cos\varphi;\ l_2: x^2 + y^2 = 8x,\ \rho = 8\cos\varphi$$

and describe the domain D by two double inequalities

$$0 \le \varphi \le \frac{\pi}{2}, \forall \varphi \in \left(0, \frac{\pi}{2}\right)(4\cos\varphi \le \rho \le 8\cos\varphi).$$

Therefore by the formula (17)

$$m = 2\int_0^{\frac{\pi}{2}} d\varphi \int_{4\cos\varphi}^{8\cos\varphi} \sqrt{\rho^2}\rho d\rho = 2\int_0^{\frac{\pi}{2}} d\varphi \int_{4\cos\varphi}^{8\cos\varphi} \rho^2 d\rho = \frac{2}{3}\int_0^{\frac{\pi}{2}} d\varphi\left(\rho^3\right)\Big|_{4\cos\varphi}^{8\cos\varphi} = \frac{2}{3}\left(8^3 - 4^3\right)\int_0^{\frac{\pi}{2}} \cos^3\varphi d\varphi =$$

$$= \frac{896}{3}\int_0^{\frac{\pi}{2}} \left(1 - \sin^2\varphi\right)\cos\varphi d\varphi = \left| \begin{array}{l|c|c} \sin\varphi = t, \quad \varphi & 0 & \frac{\pi}{2} \\ \hline \cos\varphi d\varphi = dt, \quad t & 0 & 1 \end{array} \right| = \frac{896}{3}\int_0^1 \left(1 - t^2\right)dt = \frac{1792}{9} \approx 199.$$

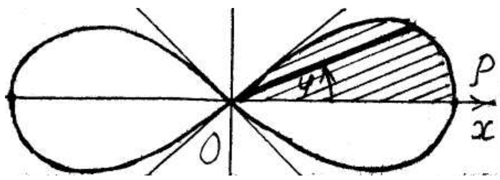


Fig. 21

Ex. 8. Find the area of a figure bounded by a curve (Bernoulli lemniscate, fig. 21)

$$\left(x^2 + y^2\right)^2 = 4a^2\left(x^2 - y^2\right).$$

We have studied this curve in the Lecture 8, Point 2. Its polar equation is

$$\rho = 2a\sqrt{\cos 2\varphi}.$$

Making use of the formula (2) we can write

$$S = 4\iint_D dxdy$$

where a domain D is the shaded curvilinear sector on the fig. 21. It's evident that

$$0 \le \varphi \le \frac{\pi}{4}, \forall \varphi \in \left(0, \frac{\pi}{4}\right)\left(0 \le \rho \le 2a\sqrt{\cos 2\varphi}\right).$$

Hence in correspondence with the formula (20) the area in question equals

$$S = 4\int_0^{\frac{\pi}{4}} d\varphi \int_0^{2a\sqrt{\cos 2\varphi}} \rho d\rho = 4\int_0^{\frac{\pi}{4}} d\varphi\left(\frac{\rho^2}{2}\Bigg|_0^{2a\sqrt{\cos 2\varphi}}\right) = 8a^2\int_0^{\frac{\pi}{4}} \cos 2\varphi d\varphi = 4a^2 \sin 2\varphi\Big|_0^{\frac{\pi}{4}} = 4a^2$$

In generale case the substitution $x = x(u, v), y = y(u, v)$, when a domain D of the plane xOy changes in a domain Δ of a plane $uO'v$, there is the next formula

$$\iint_D f(x, y)dxdy = \iint_\Delta f(x(u,v), y(u,v))|J(u,v)|dudv, \; where \; J(u,v) = \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix}.$$

DOUBLE INTEGRAL: Basic Terminology RUE

1. интеграл, распространенный на область	інтеґрал, поширений на область	íntegral exténded/tá-ken over a domáin/région
2. быть областью первого/второго типа	бути областю першого/другого типу	be a domáin/région of the first/sécond týpe [be the týpe i/ii domáin/ré-gion]
3. внешний интеграл	зовнішній інтеґрал	extérior/extérnal ínteg-ral
4. внутренний интеграл	внутрішній інтеграл	ínner íntegral
5. двойной интеграл по области	подвійний інтеґрал по області	dóuble íntegral over the domáin [région]
6. изменить порядок интегрирования	змінити порядок інтеґрування	change/pèrmutátion the órder of ìntegrátion
7. изменять порядок интегрирования	змінювати порядок інтеґрування	change/invért the órder of ìntegrátion
8. интегрировать сначала по *x* (y), а потом по y (*x*)	інтеґрувати спочатку по *x* (y), а потім по y (*x*)	íntegrate (at) first with respect to *x* (y) and then with respect to y (*x*)
9. криволинейный цилиндр	криволінійний циліндр	curvilinear cýlinder
10.масса (не)однородной фигуры	маса (не)однорідної фігури	mass of a(n) (in)hòmogéneous [(nòn)hòmogéneous] fígure
11.описать область интегрирования	описати область інтегрування	descríbe a domáin/ré-gion of ìntegrátion
12.определение пределов интегрирования	визначення меж [границь] інтеґрування	detèrminátion [placing, sétting] límits of ìntegrátion
13.определить пределы интегрирования	визначити межі [границі] інтеґрування	determine/place/set lí-mits of ìntegrátion
14.пластинка	платівка	pláte, mémbrane, disk
15.повторный интеграл	повторний інтеґрал	repéated/íterated ínte-gral
16.подынтегральная функция	підінтеґральна функція	íntegrand, fúnction ún-der the íntegral sign, fúnction to be íntegrated
17.подынтегральное выражение	підінтеґральний вираз	íntegrand, expression únder the íntegral sign, expréssion to be íntegrated
18.порядок интегрирования	порядок інтеґрування	órder of ìntegrátion
19.последовательно интегрировать	послідовно інтегрувати	succéssively [consécutively] íntegrate
20.пределы интегрирования	межі [границі] повторного інтеґрала	límits of íntegration of a(n) repéated/íterated ínte-

		gral
21.прямоугольная область интегрирования	прямокутна область інтеґрування	rectángular domáin/région of ìntegrátion
22.прямоугольник со сторонами, параллельными координатным осям	прямокутник з сторонами, паралельними до координатних осей	réctàngle with sides (which are) párallel to the coórdinate áxes
23.разделить область интегрирования на две области первого/второго типа	поділити область інтеґрування на дві області першого/другого типу	decompóse/divíde/sùb-divíde a domáin/région of integrátion into two domains/régions of the first/ second type
24.сверху	вище	above, from above
25.свести вычисление двойного интеграла к вычислению повторного интеграла	звести обчислення подвійного інтеґрала до обчислення повторного інтеґрала	redúce the càlculátion/ evàluátion of dóuble íntegral to that of repéated/íterated one
26.свести вычисление двойного интеграла к последовательному вычислению двух определенных интегралов	звести обчислення подвійного інтеґрала до послідовного обчислення двох визначених інтеґралів	redúce the càlculátion/ evàluátion of dóuble íntegral to consécutive/succéssive càlculátion/evàluátion of two définite íntegrals
27.свести двойной интеграл к повторному	звести подвійний інтеґрал до повторного	redúce the dóuble ín-tegral to that repéated/íte-rated
28.слева	зліва	to/at/from/on the left
29.снизу	знизу	belów, from belów
30.справа	справа	to/on/from the right
31.схема применения интеграла	схема застосування інтеґрала	schéme of àpplicátion of an íntegral, próblemsólving schéme
32.точка входа в область	точка входу в область	póint of éntrance in a domáin/région
33.точка выхода из области	точка виходу з області	póint of éxit [´eksit] from a domáin/région
34.цилиндрический брус	циліндричний брус	cylíndrical beam/bar
35.цилиндрическое тело (относительно оси z)	циліндричне тіло (відносно осі z)	cylíndrical bódy/sólid (with respéct to z-áxis)
36.элемент площади	елемент площі	área élement, élement of the area

DOUBLE INTEGRAL: Basic Terminology ERU

1. above, from above	вище	сверху
2. área élement, élement of the area	елемент площі	элемент площади
3. be a domáin/région of the first/sécond týpe [be the týpe i/ii domáin/région]	бути областю першого/другого типу	быть областью первого/второго типа
4. belów, from belów	знизу	снизу
5. change/invért the órder of ìntegrátion	змінювати порядок інтеґрування	изменять порядок интегрирования
6. change/pèrmutátion the órder of ìntegrátion	змінити порядок інтеґрування	изменить порядок интегрирования
7. curvilinear cýlinder	криволінійний циліндр	криволинейный цилиндр
8. cylíndrical beam/bar	циліндричний брус	цилиндрический брус
9. cylíndrical bódy/sólid (with respéct to z-áxis)	циліндричне тіло (відносно осі z)	цилиндрическое тело (относительно оси z)
10. decompóse/divíde/sùbdivíde a domáin/région of integrátion into two domains/régions of the first/second type	поділити область інтеґрування на дві області першого/другого типу	разделить область интегрирования на две области первого/второго типа
11. descríbe a domáin/région of ìntegrátion	описати область інтегрування	описать область интегрирования
12. detèrminátion [placing, sétting] límits of ìntegrátion	визначення меж [границь] інтеґрування	определение пределов интегрирования
13. determine/place/set límits of ìntegrátion	визначити межі [границі] інтеґрування	определить пределы интегрирования
14. dóuble íntegral over the domáin [région]	подвійний інтеґрал по області	двойной интеграл по области
15. extérior/extérnal íntegral	зовнішній інтеґрал	внешний интеграл
16. ínner íntegral	внутрішній інтеґрал	внутренний интеграл
17. íntegral exténded/táken over a domáin/région	інтеґрал, поширений на область	
18. íntegrand, expression únder the íntegral sign, expréssion to be íntegrated	підінтеґральний вираз	подынтегральное выражение
19. íntegrand, fúnction únder the íntegral sign, fúnction to be íntegrated	підінтеґральна функція	подынтегральная функция
20. íntegrate (at) first with	інтеґрувати спочатку по	интегрировать сначала

respect to x (y) and then with respect to y (x)	x (y), а потім по y (x)	по x (y), а потом по y (x)
21.límits of íntegration of a(n) repéated/íterated íntegral	межі [границі] повторного інтеґрала	пределы интегрирования
22.mass of a(n) (in)hòmogéneous [(nòn)hòmogéneous] fígure	маса (не)однорідної фігури	масса (не)однородной фигуры
23.órder of ìntegrátion	порядок інтеґрування	порядок интегрирования
24.pláte, mémbrane, disk	платівка	пластинка
25.póint of éntrance in a domáin/région	точка входу в область	точка входа в область
26.póint of éxit [ˊeksit] from a domáin/région	точка виходу з області	точка выхода из области
27.réctàngle with sides (which are) párallel to the coórdinate áxes	прямокутник з сторонами, паралельними до координатних осей	прямоугольник со сторонами, параллельными координатным осям
28.rectángular domáin/région of ìntegrátion	прямокутна область інтеґрування	прямоугольная область интегрирования
29.redúce the càlculátion/evàluátion of dóuble íntegral to consécutive/succéssive càlculátion/evàluátion of two définite íntegrals	звести обчислення подвійного інтеґрала до послідовного обчислення двох визначених інтеґралів	свести вычисление двойного интеграла к последовательному вычислению двух определенных интегралов
30.redúce the càlculátion/evàluátion of dóuble íntegral to that of repéated/íterated one	звести обчислення подвійного інтеґрала до обчислення повторного інтеґрала	свести вычисление двойного интеграла к вычислению повторного интеграла
31.redúce the dóuble íntegral to that repéated/íterated	звести подвійний інтеґрал до повторного	свести двойной интеграл к повторному
32.repéated/íterated íntegral	повторний інтеґрал	повторный интеграл
33.schéme of àpplicátion of an íntegral, próblem-sólving schéme	схема застосування інтеґрала	схема применения интеграла
34.succéssively [consécutively] íntegrate	послідовно інтегрувати	последовательно интегрировать
35.to/at/from/on the left	зліва	слева
36.to/on/from the right	справа	справа

DIFFERENTIAL EQUATIONS

LECTURE NO.25. FIRST AND SECOND ORDER DIFFERENTIAL EQUATIONS

POINT 1. GENERAL NOTIONS

POINT 2. INTEGRABLE TYPES OF THE FIRST ORDER DIFFERENTIAL EQUATIONS

POINT 3. ORDER REDUCING SECOND ORDER DIFFERENTIAL EQUATIONS

POINT 1. GENERAL NOTIONS

Def. 1. Equation with respect to an unknown function $y(x)$ is called differential one if it contains the derivative or derivatives of this function.

Def. 2. Order of a differential equation is called the highest order of derivatives of the unknown function which are contained in the equation.

$$F(x, y, y') = 0 \qquad (1)$$

is the general form of the first order differential equation in $y(x)$.

$$F(x, y, y', y'') = 0 \qquad (2)$$

is the general form of the second order differential equation in $y(x)$.

Def. 3. Solution of a differential equation is called a function $y = \varphi(x)$ which satisfies this equation that is turns it into identity.

For example a function $y = \varphi(x)$ is the solution of the first order differential equation (1) if $F(x, \varphi(x), \varphi'(x)) \equiv 0$.

Def. 4. Integral curve of a differential equation is called the graph of its solution.

Every differential equation has infinitely many solutions.

Ex. 1. Set of all solutions of a differential equation $y' = x$ is given by the expression $y = x^2/2 + C$ where C is an arbitrary constant.

To choose some certain solution of a differential equation one gives additional conditions.

There are **boundary** and **initial** conditions.

Ex. 2. Find a solution of a differential equation

$$y'' + k^2 y = 0 \quad (a < x < b)$$

which satisfies the next boundary conditions

$$y(a) = y_1,\ y(b) = y_2.$$

In our lectures we'll study as the rule differential equations with initial conditions.

For the first order differential equation (1) we have one initial condition namely

$$y(x_0) = y_0. \tag{3}$$

For the second order differential equation (2) we have two initial conditions

$$y(x_0) = y_0,\ y'(x_0) = y_0'. \tag{4}$$

Def. 5. Problem of finding solutions of a differential equation which satisfy an initial condition or initial conditions is called the **initial-value problem** or **Cauchy problem**.

For the first order differential equation (1) we have Cauchy problem (1), (3), and for the second order differential equation (2) we have Cauchy problem (2), (4).

Geometrical sense of Cauchy problem (1), (3): find integral curves which pass through a given point $M_0(x_0, y_0)$.

Geometrical sense of Cauchy problem (2), (4): find integral curves which pass through a given point $M_0(x_0, y_0)$ and have at this point the given slope of a tangent.

Ex. 3. Find a curve through a point $M_0(-1;\ 2)$ if the slope of the tangent at its arbitrary point $M(x; y)$ equals $2x$.

We must solve Cauchy problem for the first order differential equation

$$y' = 2x$$

with an initial condition

$$y(-1)=2.$$

The equation gives

$$y = x^2 + C,$$

and from the initial condition we find the value of *C*,

$$y(-1)=(-1)^2 + C = 2, C = 2-1 = 1,$$

and therefore

$$y = x^2 + 1.$$

Theory of differential equations establishes conditions for one-valued solvability of Cauchy problems.

Let's we study the first order differential equation which is resolved with respect to the derivative y' that is

$$y' = f(x, y) \tag{5}$$

Theorem 1. If a function $f(x, y)$ in the first order differential equation (5) and its partial derivative $f'_y(x, y)$ with respect to *y* are continuous in some domain *D* of the xOy-plane, then for any point $M_0(x_0, y_0) \in D$ Cauchy problem (5), (3) has unique solution.

Def. 6. General solution of the first order differential equation (5) in the domain *D* of the theorem 1 is called a function $y = \varphi(x, C)$ which contains an arbitrary constant C and satisfies two conditions: a) it is a solution of the equation for any value of C; b) for any initial condition (3) one can find a value C_0 of C such that the function $\varphi(x, C_0)$ satisfies this condition.

Def. 7. If an arbitrary constant *C* in the general solution $y = \varphi(x, C)$ of a differential equation (5) takes on some particular value C_0 then the corresponding function $\varphi(x, C_0)$ is called a particular solution of this equation.

For example the solution of Cauchy problem is a particular one.

Let's now we study the second order differential equation resolved with respect to the second derivative y''

$$y'' = f(x, y, y') \tag{6}$$

Theorem 2. If a function $f(x, y, y')$ in the differential equation (6) and its partial derivatives $f'_y(x, y, y'), f'_{y'}(x, y, y')$ with respect to y, y' are continuous in some domain *D* of the $Oxyz$-space, then for any point $M_0(x_0, y_0, y'_0) \in D$ Cauchy problem (6), (4) has unique solution.

Def. 8. General solution of the second order differential equation (6) in the domain *D* of the theorem 2 is called a function $y = \varphi(x, C_1, C_2)$ which contains two arbitrary constants C_1, C_2 and satisfies two conditions: a) it is a solution of the equation for any values of C_1, C_2; b) for any initial conditions (4) one can find values C_{01}, C_{02} of C_1, C_2 such that the function $y = \varphi(x, C_{01}, C_{02})$ satisfies these conditions.

For any particular values C_{01}, C_{02} of C_1, C_2 we have a particular solution $y = \varphi(x, C_{01}, C_{02})$ of the equation (6), for example the solution of Cauchy problem (6), (4).

Note 1. The general solution or a particular solution of a differential equation can be expressed implicitly. In such case they can be called respectively the general integral or a particular integral of this equation.

Note 2. Analogous definitions and theorems are introduced for *n*th order differential equations (if $n > 2$).

In the future the expression "to solve (or to integrate) a differential equation" means: a) to find its general solution or b) to solve Cauchy problem for the equation.

POINT 2. INTEGRABLE TYPES OF THE FIRST ORDER DIFFERENTIAL EQUATIONS (of DE - 1)

1. Separated DE-1 (DE-1 with separated variables)

Def. 9. The first order differential equation of the form

$$M(x)dx + N(y)dy = 0, \tag{7}$$

where the variables x, y are separated, is called that **in separated variables** (or **separated differential equation** of the first order).

Theorem 3. The general solution of the equation (7) has the next form:

$$\int M(x)dx + \int N(y)dy = C \tag{8}$$

where $\int M(x)dx, \int N(y)dy$ mean some primitives of the functions $M(x), N(x)$ correspondingly.

■a) Let a function $y = \varphi(x)$ be a solution of the equation (7) that is

$$M(x)dx + N(\varphi(x))d\varphi(x) \equiv 0,\ M(x)dx + N(\varphi(x))\varphi'(x)dx \equiv 0 .$$

Integrating the last identity with respect to x we get the equality (8), namely

$$\int M(x)dx + \int N(\varphi(x))\varphi'(x)dx \equiv C, \left| \begin{matrix} Let\ \varphi(x) = y, \\ \varphi'(x)dx = dy \end{matrix} \right|, \int M(x)dx + \int N(y)dy \equiv C .$$

b) Inversely, let a function $y = \varphi(x)$ satisfy the equality (8) that is

$$\int M(x)dx + \int N(\varphi(x))\varphi'(x)dx \equiv C .$$

Differentiating this identity we obtain

$$d\left(\int M(x)dx\right) + d\left(\int N(\varphi(x))\varphi'(x)dx\right) = 0,\ M(x)dx + N(\varphi(x))\varphi'(x)dx = 0,$$

and so the function $y = \varphi(x)$ is a solution of the equation (7).

Therefore every solution of the equation (7) satisfies the equality (8) and vice versa.

c) To prove possibility of choice a value of C to satisfy an initial condition

$$y(x_0) = y_0 \tag{9}$$

let's rewrite both primitives in the equality (8) in the form of definite integrals,

$$\int_a^x M(x)dx + \int_b^y N(y)dy = C .$$

By the initial condition (9) we get

$$C = \int_a^{x_0} M(x)dx + \int_b^{y_0} N(y)dy .■$$

Note 3. Taking $a = x_0, b = y_0$ we'll get $C = 0$. Therefore the solution of Cauchy problem for the equation (7) with an initial condition (9) can be written in the next simplest form

$$\int_{x_0}^{x} M(x)dx + \int_{y_0}^{y} N(y)dy = 0. \tag{10}$$

Note 4. Integration of a differential equation (7) is reduced to solving of more simple problem namely to evaluation of primitives. It is of no importance that at least one of these primitives can't be expressed in terms of elementary functions.

Ex. 4. The differential equation $e^{x^2}dx = \frac{dy}{\ln y}$ is that in separated variables. Its general solution is given by the next formula

$$\int e^{x^2}dx = \int \frac{dy}{\ln y} + C.$$

Both primitives do not express in terms of elementary functions.

2. Separable DE-1 (DE-1 with separable variables)

Def. 10. The first order differential equation is called that **with separable variables** (or **separable differential equation** of the first order) if it can be reduced to an equation in separated variables.

A differential equation

$$y' = f_1(x)f_2(y) \tag{11}$$

is that separable provided $f_2(y) \neq 0$. It's sufficient to represent y' as $\frac{dy}{dx}$, multiply by dx and divide by $f_2(y)$ both sides of the equation,

$$\frac{dy}{dx} = f_1(x)f_2(y) \left| \frac{dx}{f_2(y)}, \right. \frac{dy}{f_2(y)} = f_1(x)dx.$$

Therefore the general solution of the equation (11) is

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx + C. \qquad (12)$$

To the same type of separable differential equations concerns the next one

$$P(x)Q(y)dx + R(x)S(y)dy = 0 \qquad (13)$$

if $R(x) \neq 0, Q(y) \neq 0$. We get a separated differential equation dividing both sides of the given equation by $R(x)Q(y)$,

$$P(x)Q(y)dx + R(x)S(y)dy = 0, \left| \frac{1}{R(x)Q(y)}, \frac{P(x)dx}{R(x)} + \frac{S(y)dy}{Q(y)} = 0, \right.$$

and the general solution of the equation is given by the formula

$$\int \frac{P(x)}{R(x)}dx + \int \frac{S(y)}{Q(y)}dy = C. \qquad (14)$$

Ex. 5. A differential equation $e^y(1+x^2)dy - 2x(1+e^y)dx = 0$ has the form (13) ($P(x) = 1+x^2, Q(y) = e^y, R(x) = 2x, S(y) = 1+e^y$) and so it is separable one. Dividing both sides by $(1+x^2)(1+e^y)$ we get

$$e^y(1+x^2)dy - 2x(1+e^y)dx = 0 \left| \frac{1}{(1+x^2)(1+e^y)}, \frac{e^y}{1+e^y}dy - \frac{2x}{1+x^2}dx = 0, \right.$$

hence

$$\int \frac{e^y dy}{1+e^y} - \int \frac{2xdx}{1+x^2} = C, \ln(1+e^y) - \ln(1+x^2) = C, \ln\frac{1+e^y}{1+x^2} = C.$$

Note 5. For the sake of more simplicity we can represent an arbitrary constant C in different forms.

Let's take for example $\ln|C_1|$ instead C in the preceding ex. 5, so

$$\ln\frac{1+e^y}{1+x^2} = \ln|C_1|, \frac{1+e^y}{1+x^2} = |C_1|.$$

Putting finally $C = |C_1|$, we get the general solution of the equation in the next more simple form

$$1+e^y = C(1+x^2)$$

or

$$y=\ln\left(C\left(1+x^2\right)-1\right).$$

Ex. 6 (economical problem). Current inventory level of some company is 1680 items and it is producing with the rate of 900 items per month (i.p.m.). Demand is currently with the rate 800 i.p.m. and is dropping with the rate of 10 i.p.m. The company would like to reduce production with the rate *n* i.p.m. to sell all the production during the year. Find the value of *n*.

Let $I(t)$ be the inventory level at a time moment *t*, and $I(0)=1680$. Rate of its producing at a moment *t* is

$$V(t)=\lim_{\Delta t\to 0}\frac{I(t+\Delta t)-I(t)}{\Delta t}=I'(t).$$

It is known that

$$V(t)=S(t)-D(t),$$

where $S(t)$ is the rate of production, $D(t)$ is the rate of selling. By conditions of the problem

$$S(t)=900-nt,\ D(t)=800-10t,\ V(t)=100+10t-nt=100+(10-n)t,$$

or

$$I'(t)=100+(10-n)t.$$

We get the first order separable differential equation in $I(t)$ with an initial condition

$$I(0)=1680,$$

that is we must solve Cauchy problem. From the equation

$$I(t)=100t+(10-n)\cdot\frac{t^2}{2}+C,$$

by the initial condition $C=1680$, and

$$I(t)=100t+(10-n)\cdot\frac{t^2}{2}+1680.$$

To sell all the production during the year we must have

$$I(12)=100\cdot 12+(10-n)\cdot\frac{12^2}{2}+1680=0,$$

whence it follows that $n=50$.

Answer: its necessary to reduce the production with the rate of $n = 50$ i.p.m.

Ex. 7 (demographic problem). The rate of growth of population at arbitrary moment of time is proportional to the number of population at this moment (coefficient of proportionality equals k). Find the number of population at arbitrary moment t if it equals L_0 at the initial moment $t = 0$.

Let $L(t)$ be the number of population at a moment t (it is obvious that $L(t) > 0$). The rate of growth of population at this moment

$$v(t) = \lim_{\Delta t \to 0} \frac{L(t + \Delta t) - L(t)}{\Delta t} = L'(t).$$

By the condition

$$v(t) = kL(t).$$

Therefore,

$$L'(t) = kL(t),\ L(0) = L_0,$$

and we have to solve Cauchy problem.

The differential equation of the problem is separable one,

$$\frac{dL}{dt} = kL\left|\frac{dt}{L}, \frac{dL}{L} = kdt,\ \ln|L| = kt + \ln|C|,\ \ \ln|L| - \ln|C| = kt,\ \ln\left|\frac{L}{C}\right| = kt,\ \left|\frac{L}{C}\right| = e^{kt},\right.$$

$$|L(t)| = |C|e^{kt},\ L(t) = |C|e^{kt},\ L(0) = L_0 = |C|e^0,\ |C| = L_0,\ L(t) = L_0 e^{kt}.$$

Thus we've got exponential growth of population provided there are not opposed factors (decline in living standards, measures for limitation of birth rate *etc.*).

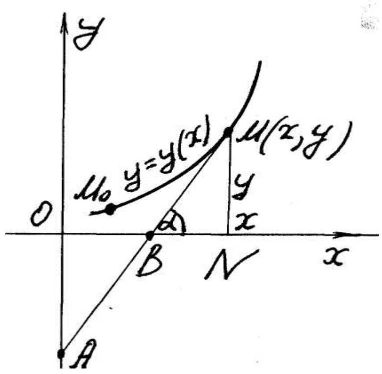


Fig. 1

Ex. 8 (geometric problem). Find a curve through a point $M_0(1; 2)$ if a segment of its arbitrary tangent between the tangency point and the Oy-axis is divided by the intersection point of the segment and the Ox-axis in the given ratio 5 : 8, counting from the Oy-axis (see fig. 1).

It follows from the conditions of the problem that an unknown curve can't intersect the coordinate axes and therefore lies in the first quadrant.

Let $y = y(x)$ be an unknown equation of a curve in question, $M(x;y)$ an arbitrary point of the curve which is the tangency point, $\alpha = \angle MBN$. Then

$$ON = x\ (x>0),\ NM = y\ (y>0),\ \tan\alpha = y' = \frac{NM}{BN} = \frac{y}{BN},\ y' = \frac{y}{BN}\ \text{(fig. 1)}.$$

By condition

$$AB : BM = 5:8 = OB : BN,\ OB = x - BN,\ (x - BN) : BN = 5:8 \Rightarrow BN = 8/13\,x,$$

$$y' = \frac{y}{(8/13)\cdot x},\quad \begin{cases} y' = \dfrac{13y}{8x}, \\ y(1) = 2 \end{cases}$$

We've Cauchy problem for a separable differential equation. Separating the variables we get

$$\frac{dy}{dx} = \frac{13y}{8x}\ \left|\frac{dx}{y}\right.,\ \frac{dy}{y} = \frac{13}{8}\frac{dx}{x},\ \int\frac{dy}{y} = \frac{13}{8}\int\frac{dx}{x} + \frac{13}{8}\ln|C|,\ \ln|y| = \frac{13}{8}\left(\ln|x| + \ln|C|\right),,$$

$$\ln|y| = \frac{13}{8}\ln|C||x|,\ \ln y = \frac{13}{8}\ln|C|x,\ \ln y = \ln(|C|x)^{\frac{13}{8}},\ y = |C|^{\frac{13}{8}}x^{\frac{13}{8}}.$$

Using the initial condition we find the corresponding value of C and the solution of Cauchy problem,

$$y(1) = 2,\ y(1) = |C|^{\frac{13}{8}},\ |C|^{\frac{13}{8}} = 2,\ y = 2x^{\frac{13}{8}}.$$

Remark. There is the other way to get a differential equation of the problem proceeding from the equation of the tangent to a desired curve at its arbitrary point $M(x;y)$.

Let's denote coordinates of an arbitrary point of the tangent by ξ, η then the equation of the tangent will get the next form

$$\eta = y(x) + y'(x)(\xi - x).$$

Putting $\eta = 0$ in this equation we'll get

$$0 = y(x) + y'(x)(\xi - x),\ y'(x)(\xi - x) = -y(x),\ \xi - x = -y(x)/y'(x),$$

$$\xi = OB = x - \frac{y(x)}{y'(x)} \Rightarrow BN = ON - OB = x - \left(x - \frac{y(x)}{y'(x)}\right) = \frac{y(x)}{y'(x)}.$$

Making use of the conditions of the problem we obtain

$$\frac{AB}{BM}=\frac{OB}{BN}=\frac{5}{8}\Rightarrow\left(x-\frac{y(x)}{y'(x)}\right)\Big/\left(\frac{y(x)}{y'(x)}\right)=\frac{5}{8},\ x-\frac{y(x)}{y'(x)}=\frac{5}{8}\cdot\frac{y(x)}{y'(x)},\ x=\frac{13}{8}\cdot\frac{y(x)}{y'(x)},$$

hence

$$y'(x)=\frac{13}{8}\cdot\frac{y(x)}{x},\text{ or briefly } y'=\frac{13y(x)}{8x}.$$

3. Homogeneous DE-1

Def. 11. The first order differential equation is called **homogeneous** one (with respect to the variables x, y) if it can be represented in the next form

$$y'=\varphi\left(\frac{y}{x}\right). \qquad (15)$$

Theorem 4. Homogeneous differential equation (15) as a rule reduces to that separable by introducing a new unknown function

$$\frac{y}{x}=u(x). \qquad (16)$$

■Finding y' and substituting its value in the equation we have

$$y=xu(x),\ y'=u+xu',\ u+xu'=\varphi(u),\ xu'=\varphi(u)-u.$$

The obtained equation is a separable one provided $\varphi(u)-u\neq 0$. Indeed, in this case

$$x\frac{du}{dx}=\varphi(u)-u\left|\frac{dx}{x(\varphi(u)-u)}\right.,\ \frac{du}{\varphi(u)-u}=\frac{dx}{x},\ \int\frac{du}{\varphi(u)-u}=\int\frac{dx}{x}+C.■$$

Note 6. It can be proved that differential equation

$$y'=f(x,y), \qquad (17)$$

is that homogeneous if for any λ

$$f(\lambda x,\lambda y)=f(x,y).$$

A more general differential equation

$$M(x,y)dx+N(x,y)dy=0 \qquad (18)$$

is homogeneous one if for any λ there exists a number k such that simultaneously

$$M(\lambda x,\lambda y)=\lambda^k M(x,y)\text{ and } N(\lambda x,\lambda y)=\lambda^k N(x,y).$$

Prove these assertions yourselves.

Ex. 9. Solve Cauchy problem $xy' = y\ln\frac{y}{x},\ y(1) = 1$.

Let's divide both sides of the equation by x. We'll get an equation of the form (15) that is a homogeneous equation,

$$y' = \frac{y}{x}\ln\frac{y}{x},\ \varphi\left(\frac{y}{x}\right) = \frac{y}{x}\ln\frac{y}{x},$$

because of its right side is a function of the ratio y/x. Acting by the theory we have

$$\frac{y}{x} = u,\ y = xu,\ y' = xu' + u,\ xu' + u = u\ln u,\ xu' = u\ln u - u,\ x\frac{du}{dx} = u(\ln u - 1)\Bigg|\frac{dx}{xu(\ln u - 1)}$$

$$\frac{du}{u(\ln u - 1)} = \frac{dx}{x},\ \int\frac{du}{u(\ln u - 1)} = \int\frac{dx}{x} + \ln|C|,\ \ln|\ln u - 1| = \ln|Cx|,\ |\ln u - 1| = |Cx|,$$

$$\ln u - 1 = Cx,\ \ln\frac{y}{x} - 1 = Cx.$$

The initial condition gives

$$\ln\frac{1}{1} - 1 = C\cdot 1,\ C = -1.$$

The solution of Cauchy problem

$$\ln\frac{y}{x} = 1 - x \text{ or } y = xe^{1-x}.$$

Ex. 10. Find the general solution of a differential equation

$$\left(3y^2 + 3xy + x^2\right)dx = \left(x^2 + 2xy\right)dy.$$

The given equation is that homogeneous because of for any λ

$$3(\lambda y)^2 + 3(\lambda x)(\lambda y) + (\lambda x)^2 = \lambda^2\left(3y^2 + 3xy + x^2\right),\ (\lambda x)^2 + 2(\lambda x)(\lambda y) = \lambda^2\left(x^2 + 2xy\right).$$

To prove homogeneity of the equation directly we rewrite it in the next form

$$y' = \frac{dy}{dx} = \frac{3y^2 + 3xy + x^2}{x^2 + 2xy}$$

and then divide the numerator and denominator by x^2,

$$y' = \frac{3\left(\frac{y}{x}\right)^2 + 3\frac{y}{x} + 1}{1 + 2\frac{y}{x}}.$$

The right side of the equation is a function of the ratio y/x, and so the equation is homogeneous one. By the theory we have

$$\frac{y}{x} = u,\ y = xu,\ y' = xu' + u,\ xu' + u = \frac{3u^2 + 3u + 1}{1 + 2u},\ x\frac{du}{dx} = \frac{u^2 + 2u + 1}{1 + 2u},\ \frac{(1 + 2u)du}{u^2 + 2u + 1} = \frac{dx}{x},$$

$$\int \frac{(1 + 2u)du}{u^2 + 2u + 1} = \int \frac{dx}{x} + C,\ \int \frac{(1 + 2u)du}{(u + 1)^2} = \ln|x| + C,\ \int \frac{2(1 + u) - 1}{(u + 1)^2}du = 2\int \frac{du}{u + 1} - \int \frac{du}{(u + 1)^2} =$$

$$= 2\ln|u + 1| + \frac{1}{u + 1},\ 2\ln\left|\frac{y}{x} + 1\right| + \frac{1}{\frac{y}{x} + 1} = \ln|x| + C,\ 2\ln\left|\frac{x + y}{x}\right| + \frac{x}{x + y} = \ln|x| + C.$$

Ex. 11. Find a curve through a point $M_0(0; 1)$ if the subtangent at its arbitrary point $M(x; y)$ equals the sum of coordinates of this point.

It follows from the conditions of the problem that an unknown curve can't intersect the *Ox*-axis and so lies above of this axis.

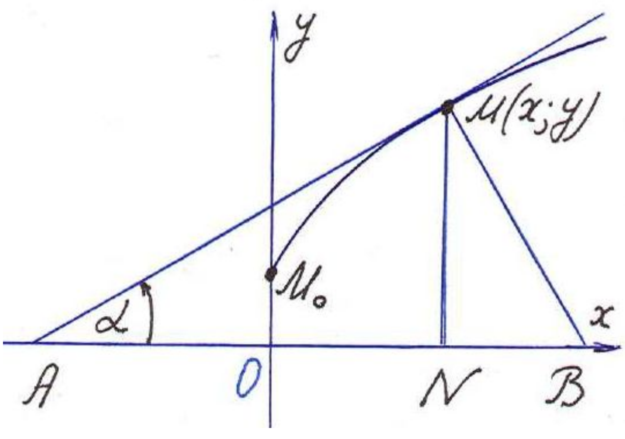


Fig. 2

Let $y = y(x),\ y(x) > 0,$ be an equation of a curve in question, MA and MB be the segments of the tangent and normal to the curve at its arbitrary point $M(x; y)$ respectively, and $MN \perp Ox$ (see fig. 2). Then directed segments AN and NB are called respectively the subtangent and subnormal to the curve at the point $M(x; y)$.

For the case $y(x) > 0,\ y'(x) > 0$ (see fig. 2, where the point *A* lies from the left of the point *N*)

$$NA = -\frac{y(x)}{y'(x)}.$$

We can prove this from the right triangle AMN, namely

$$NA = -NM \cdot \cot\alpha = -\frac{NM}{\tan\alpha} = -\frac{y}{y'};$$

the same value of NA we'll get proceeding from the equation of the tangent to the unknown curve at the point $M(x; y)$. Indeed, the equation of the tangent is

$$\eta = y(x) + y'(x)(\xi - x).$$

If we put $\eta = 0$ here, we'll obtain

$$\xi = OA = x - \frac{y(x)}{y'(x)},\ AN = ON - OA = \frac{y(x)}{y'(x)},\ NA = -\frac{y(x)}{y'(x)}.$$

The length of the subtangent equals

$$|NA| = \left|\frac{y(x)}{y'(x)}\right|,$$

and by the conditions of the problem we get the differential equation

$$\pm\frac{y}{y'} = x + y$$

with an initial condition

$$y(0) = 1$$

that is we have to solve Cauchy problem.

1. The first case

$$\frac{y}{y'} = x + y, \quad y(0) = 1.$$

To determine the type of the equation we write it as follows

$$y' = \frac{y}{x + y}$$

and divide the numerator and denominator of the fraction by x,

$$y' = \frac{\frac{y}{x}}{1 + \frac{y}{x}}.$$

We get a differential equation of the form (15), and therefore it's homogeneous one. Integrating it by corresponding method we have

$\frac{y}{x}=u,\ y=xu,\ y'=xu'+u,\ xu'+u=\frac{u}{1+u},\ xu'=\frac{u}{1+u}-u,\ x\frac{du}{dx}=-\frac{u^2}{1+u}\Big|\frac{(1+u)dx}{u^2},$

$$\frac{(1+u)du}{u^2}=-\frac{dx}{x},\ \int\frac{(1+u)du}{u^2}=-\int\frac{dx}{x}+C,\ -\frac{1}{u}+\ln|u|=-\ln|x|+C,\ -\frac{1}{u}+\ln|xu|=C,$$

$$-\frac{x}{y}+\ln|y|=C,\ -\frac{x}{y}+\ln y=C\quad(\textit{because of }\ y>0).$$

The value of C we find by virtue of the initial condition. Substituting 0 for x and 1 for y in the general solution we get

$$-\frac{0}{1}+\ln 1=C,\ C=0.$$

The curve in question has the next equation: $-\frac{x}{y}+\ln y=0$ or $y\ln y=x$.

2. The second case

$$-\frac{y}{y'}=x+y,\quad y(0)=1.$$

Integration of the differential equation (which is also homogeneous one) gives

$$-\frac{y'}{y}=\frac{1}{x+y},\ y'=-\frac{y}{x+y},\ y'=-\frac{y/x}{1+y/x},\ \frac{y}{x}=u,\ y=xu,\ y'=xu'+u,\ xu'+u=-\frac{u}{1+u},$$

$$x\frac{du}{dx}=-\frac{u}{1+u}-u,\ x\frac{du}{dx}=-\frac{2u+u^2}{1+u},\ \frac{(1+u)du}{2u+u^2}=-\frac{dx}{x},\ \frac{(1+u)du}{u(2+u)}=-\frac{dx}{x},$$

$$\frac{1}{2}\left(\frac{1}{u}+\frac{1}{u+2}\right)du=-\frac{dx}{x},\ \frac{1}{2}\left(\ln|u|+\ln|u+2|\right)=-\ln|x|+\ln|C|,\ \frac{1}{2}\ln|u(u+2)|+\ln|x|=\ln|C|,$$

$$\ln\left(|x|\sqrt{\left|\frac{y}{x}\left(\frac{y}{x}+2\right)\right|}\right)=\ln|C|,\ \ln\left(|x|\frac{1}{|x|}\sqrt{|y(y+2x)|}\right)=\ln|C|,\ln\sqrt{y|y+2x|}=\ln|C|,$$

$$\sqrt{y|y+2x|}=|C|,\ y|y+2x|=C^2,\ \begin{cases} y(y+2x)=C^2 \ \textit{for}\ y+2x\ge 0,\\ -y(y+2x)=C^2\ \textit{for}\ y+2x<0.\end{cases}$$

The initial condition is fulfiled only for $y+2x\ge 0$: $1\cdot(1+2\cdot 0)=C^2$, $C^2=1$, and therefore the curve in question is given by the equation $y(y+2x)=1$.

Answer. There are two solutions of the problem: $y\ln y=x$, $y(y+2x)=1$.

4. Linear DE-1

Def. 12. Linear differential equation is called an equation of the form

$$y' + P(x)y = Q(x), \qquad (19)$$

where a coefficient $P(x)$ and an absolute term $Q(x)$ are known functions.

Theorem 5. Integration of a linear differential equation (19) is reduced to sequential integration of two separable first order differential equations.

■Let's find a solution of the equation (19) in the form of a product of two unknown functions $u(x), v(x)$,

$$y = u(x)v(x). \qquad (20)$$

Differentiating and substituting the values of y, y' in the equation we have

$$y' = u'v + uv', u'v + uv' + P(x)uv = Q(x), u'v + u(v' + P(x)v) = Q(x).$$

We try to choose some function $v = v_0(x)$ to annulate the expression $v' + P(x)v = 0$. It leads us to two separable differential equations

$$v' + P(x)v = 0, \qquad (*)$$

$$u'v = Q(x). \qquad (**)$$

Integration of the first equation (*) gives

$$\frac{dv}{dx} + P(x)v = 0 \left| \frac{dx}{v}, \frac{dv}{v} + P(x)dx = 0, \ln|v| = -\int P(x)dx + \ln|C_1|, \ln\left|\frac{v}{C_1}\right| = -\int P(x)dx,\right.$$

$$v = C_1 e^{-\int P(x)dx}, v = v_0(x) = e^{-\int P(x)dx}.$$

We substitute the found value $v = v_0(x)$ in the equation (**) and find its general solution,

$$u'v_0(x) = Q(x), \frac{du}{dx}v_0(x) = Q(x) \left| \frac{dx}{v_0(x)}, du = \frac{Q(x)}{v_0(x)}dx, u = \int \frac{Q(x)}{v_0(x)}dx + C.\right.$$

Finally we get the general solution of the given equation (19), namely

$$y = uv = uv_0(x) = \left(\int \frac{Q(x)}{v_0(x)}dx + C\right)v_0(x). ■$$

Ex. 12. Integrate a differential equation $y' + y \cdot \tan x = \frac{1}{\cos x}$.

The differential equation is that linear with $P(x) = \tan x, Q(x) = \dfrac{1}{\cos x}$. Following the theory we put

$$y = uv = u(x)v(x)$$

and so

$$y' = u'v + uv', u'v + uv' + uv\tan x = \frac{1}{\cos x}, u'v + u(v' + v\tan x) = \frac{1}{\cos x}.$$

We have integrate successively the next two differential equations

$$v' + v\tan x = 0, \quad (*)$$

$$u'v = \frac{1}{\cos x}. \qquad (**)$$

As to (*) we have $\dfrac{dv}{dx} + v\tan x = 0 \ \left| \dfrac{dx}{v} \right., \dfrac{dv}{v} = -\dfrac{\sin x dx}{\cos x}, \int \dfrac{dv}{v} = -\int \dfrac{\sin x dx}{\cos x} + \ln|C|,$

$$\int \frac{dv}{v} = \int \frac{-\sin x dx}{\cos x} + \ln|C|, \ln|v| = \ln|\cos x| + \ln|C|, \ln|v| = \ln|C\cos x|, |v| = |C\cos x|,$$

By arbitrariness of C we can write

$$v = C\cos x, v = v_0(x) = \cos x.$$

Passing to the equation (**) we get

$$u'v_0 = \frac{1}{\cos x}, u'\cos x = \frac{1}{\cos x}, \frac{du}{dx}\cdot\cos x = \frac{1}{\cos x} \ \left| \frac{dx}{\cos x} \right., du = \frac{dx}{\cos^2 x}, u = \tan x + C.$$

Now the general solution of the given differential equation is

$$y = uv_0 = (\tan x + C)\cos x = \sin x + C\cos x, \ y = \sin x + C\cos x.$$

Ex. 13. Integrate the next differential equation $y' = \dfrac{y}{3x - y^2}$.

Let's rewrite the equation in the follows way:

$$\frac{dy}{dx} = \frac{y}{3x - y^2}, \quad \frac{dx}{dy} = \frac{3x - y^2}{y}, \quad \frac{dx}{dy} = \frac{3x}{y} - y, \quad \frac{dx}{dy} + (-\frac{3}{y})x = -y.$$

We see that the differential equation is that linear with respect to the unknown function $x = x(y)$. Therefore we do as follows

$$x = uv = u(y)v(y),\ x' = u'v + v'u,\ u'v + uv' - \frac{3}{y}uv = -y,\ u'v + u\left(v' - \frac{3}{y}v\right) = -y,$$

$$v' - \frac{3v}{y} = 0, \quad (*)$$

$$u'v = -y. \quad (**)$$

Solving the first equation (*) we have

$$\frac{dv}{dy} = 3\frac{v}{y} \left| \frac{dy}{v}, \frac{dv}{v} = 3\frac{dy}{y},\ \ln|v| = 3\ln|y| + \ln|C|,\ \ln|v| = \ln\left|Cy^3\right|,\ |v| = \left|Cy^3\right|,\ v = Cy^3. \right.$$

Putting $v = v_0 = y^3$, we integrate the equation (**), namely

$$u'v_0 = -y,\ u'y^3 = -y,\ \frac{du}{dy} = -\frac{1}{y^2},\ du = -\frac{dy}{y^2},\ u = -\int\frac{dy}{y^2} + C,\ u = \frac{1}{y} + C.$$

The general solution of the given differential equation

$$x = uv_0 = \left(\frac{1}{y} + C\right)y^3,\ x = y^2 + Cy^3.$$

Ex. 14 (funds flow [movement of funds]). Let $K(t)$ be amount of funds at a moment *t*. Retirement of funds during a time interval $[t, t + \Delta t]$ equals $\mu K(t)\Delta t$, where μ is some retirement coefficient. Growth of funds during the time interval $[t, t + \Delta t]$ equals $\rho I \Delta t$, where ρ is some coefficient (0< ρ <1 because of not all investments are putting in funds) and *I* is a known amount of investments during one year. Amount of funds at the moment of time $t + \Delta t$ equals

$$K(t + \Delta t) = K(t) - \mu K(t)\Delta t + \rho I \Delta t.$$

Rate of movement of funds at a moment *t* equals

$$\lim_{\Delta t \to 0}\frac{K(t + \Delta t) - K(t)}{\Delta t} = K'(t) = -\mu K(t) + \rho I.$$

We've got Cauchy problem for a differential equation

$$K'(t) = -\mu K(t) + \rho I$$

with an initial condition

$$K(0) = K_0.$$

Coefficients μ, ρ of the equation can be constant or known functions of *t*. The quantity *I* can be constant or a function of *t*. In these cases the differential equation is that linear. What is more, *I* can be a function of the *n*th power of *K*(*t*), and in this case the equation is that of Bernoulli (see lower).

5. Bernoulli DE-1

Def. 13. Bernoulli differential equation is called the next first order equation

$$y' + P(x)y = Q(x)y^n \tag{21}$$

where *n* is an arbitrary real number distinct from 0 and 1 ($n \neq 0, n \neq 1$).

For $n = 1$ the equation is a separable one and for $n = 0$ it is a linear one.

We can integrate Bernoulli equation as a linear one if we put

$$y = u(x)v(x).$$

The second method of integration consists in reducing of the equation (21) to that linear. Let's divide both its sides by y^n,

$$\frac{y'}{y^n} + \frac{P(x)}{y^{n-1}} = Q(x),\ y'y^{-n} + P(x)y^{1-n} = Q(x),$$

and then suppose

$$y^{1-n} = z\,.$$

We'll get

$$z' = (1-n)y^{-n}y',\ y'y^{-n} = \frac{z'}{1-n},\ \frac{z'}{1-n} + P(x)z = Q(x),\ z' + (1-n)P(x)z = (1-n)Q(x).$$

The latter equation is a linear one in *z* (*x*).

Ex. 15. Integrate a differential equation $y' - \dfrac{4}{x}y = x\sqrt{y}$.

The equation is Bernoulli one, $n = 1/2$. Finding its solution as a product

$$y = uv = u(x)v(x)$$

we have

$$y' = u'v + uv',\, u'v + uv' - \frac{4}{x}uv = x\sqrt{uv},\, u'v + u\left(v' - \frac{4}{x}v\right) = x\sqrt{uv},$$

$$v' - \frac{4}{x}v = 0, \qquad (*)$$

$$u'v = x\sqrt{uv}. \qquad (**)$$

$$\frac{dv}{dx} - \frac{4}{x}v = 0 \quad \left|\frac{dx}{v},\, \frac{dv}{v} - 4\frac{dx}{x} = 0,\, \ln|v| = 4\ln|x| + \ln|C_1|,\, \ln\left|\frac{v}{C_1}\right| = \ln x^4,\, v = C_1x^4,\, v = v_0 = x^4.\right.$$

$$\frac{du}{dx}v_0 = x\sqrt{uv_0},\, \frac{du}{dx}x^4 = x\sqrt{ux^4} \quad \left|\frac{dx}{x^4\sqrt{u}},\, \frac{du}{\sqrt{u}} = \frac{dx}{x},\, 2\sqrt{u} = \ln|x| + \ln|C|,\, 2\sqrt{u} = \ln|Cx|,\right.$$

$$\sqrt{u} = \frac{1}{2}\ln|Cx| = \ln\sqrt{|Cx|},\, u = \ln^2\sqrt{|Cx|},\, y = uv_0 = x^4\ln^2\sqrt{|Cx|}.$$

POINT 3. ORDER REDUCING SECOND ORDER DIFFERENTIAL EQUATIONS

1. Second order differential equation resolved with respect to the higher derivative if the right member of the equation depends only on the independent variable

$$y'' = f(x). \qquad (22)$$

One finds the general solution of this equation by twice integration, namely

$$y = \int dx \int f(x)dx + C_1x + C_2x \qquad (23)$$

$$\blacksquare\; y'' = (y')' = \frac{dy'}{dx} = f(x),\, dy' = f(x)dx,\, y' = \int f(x)dx + C_1,$$

$$\frac{dy}{dx} = \int f(x)dx + C_1,\, dy = \left(\int f(x)dx + C_1\right)dx,\, y = \int\left(\int f(x)dx + C_1\right)dx + C_2 =$$

$$= \int dx \int f(x)dx + \int C_1dx + C_2 = \int dx \int f(x)dx + C_1x + C_2 .\blacksquare$$

Note 7. If it is a matter of Cauchy problem for the equation (22) with initial conditions

$$y(x_0) = y_0,\, y'(x_0) = y_0'$$

then it is well to integrate over an interval $[x_0, x]$, namely

$$y' = \int_{x_0}^{x} f(x)dx + C_1,\ y = \int_{x_0}^{x} dx \int_{x_0}^{x} f(x)dx + C_1(x - x_0) + C_2 .$$

In this case we at once have

$$C_1 = y'(x_0) = y'_0,\ C_2 = y(x_0) = y_0 ,$$

and the solution of Cauchy problem takes on the next form

$$y = \int_{x_0}^{x} dx \int_{x_0}^{x} f(x)dx + y'_0(x - x_0) + y_0 .$$

Ex. 16. *S*olve the next Cauchy problem

$$\begin{cases} y'' = \cos x, \\ y(0) = 1,\ y'(0) = -2. \end{cases}$$

Twice integrating we at first have the general solution of the differential equation,

$$y' = \sin x + C_1,\ y = -\cos x + C_1 x + C_2 .$$

Taking into account the initial conditions we find C_1, C_2,

$$y(0) = 1 = -\cos 0 + C_1 \cdot 0 + C_2,\ C_2 = 2;\ y'(0) = -2 = \sin 0 + C_1,\ C_1 = -2 .$$

Therefore the solution of Cauchy problem

$$y = -\cos x - 2x + 2 .$$

2. Second order differential equation **not containing explicitly the unknown function**

$$F(x, y', y'') = 0 \tag{24}$$

reduces to the first order differential equation by introducing a new unknown function

$$y' = p(x) . \tag{25}$$

So long as

$$y'' = p'(x),$$

the equation (24) passes to the next one

$$F(x, p(x), p'(x)) = 0 . \tag{26}$$

We've got the first order differential equation in $p(x)$.

Suppose that we've found the general solution of the equation (26) of the form

$$p(x)=\varphi(x, C_1).$$

In this case we can finish integration of the given equation as follows

$$y'=\varphi(x, C_1), \frac{dy}{dx}=\varphi(x, C_1), dy=\varphi(x, C_1)dx, y=\int\varphi(x, C_1)dx+C_2.$$

Ex. 17. Find the general solution of the second order differential equation

$$x^2y''+xy'=1.$$

This equation doesn't contain explicitly the unknown function.

The first step. Putting $y'=p(x)$ we get the first order linear equation in $p(x)$,

$$y''=p', x^2p'+xp=1, p'+\frac{1}{x}p=\frac{1}{x^2}.$$

By corresponding theory we put $p(x)=uv$ and so

$$p'=u'v+uv', u'v+uv'+\frac{uv}{x}=\frac{1}{x^2}, u'v+u\left(v'+\frac{v}{x}\right)=\frac{1}{x^2},$$

$$a)\ v'+\frac{v}{x}=0, \frac{dv}{dx}+\frac{v}{x}=0, \frac{dv}{v}+\frac{dx}{x}=0, \ln|v|+\ln|x|=\ln|C_0|, vx=C_0, v=v_0(x)=\frac{1}{x};$$

$$b)\ u'v=\frac{1}{x^2}, u'v_0=\frac{1}{x^2}, \frac{du}{dx}\cdot\frac{1}{x}=\frac{1}{x^2}, du=\frac{dx}{x}, u=\ln|x|+C_1, p(x)=uv_0=\frac{\ln|x|+C_1}{x}.$$

The second step. Returning to y' we find the general solution in question,

$$y'=\frac{\ln|x|+C_1}{x}, \frac{dy}{dx}=\frac{\ln|x|+C_1}{x}, dy=\frac{\ln|x|}{x}dx+C_1\frac{dx}{x}, y=\frac{\ln^2|x|}{2}+C_1\ln|x|+C_2.$$

3. Second order differential equation **not containing explicitly the independent variable**

$$F(y, y', y'')=0 \qquad (27)$$

reduces to the first order differential equation by introducing an auxiliary unknown function

$$y'=p(y) \qquad (28)$$

Differentiating a composite function we obtain

$$y'' = p(y)p'(y), \quad (29)$$

and so

$$F(y, p(y), p(y)p'(y)) = 0. \quad (30)$$

We get the first order differential equation in $p(y)$.

If we can find the general solution of the equation (30) in the next form

$$p(y) = \phi(y, C_1)$$

then

$$y' = \phi(y, C_1), \frac{dy}{dx} = \phi(y, C_1) \quad \left| \frac{dx}{\phi(y, C_1)}, \frac{dy}{\phi(y, C_1)} = dx, \int \frac{dy}{\phi(y, C_1)} = x + C_2 \right..$$

Ex. 18. Solve Cauchy problem for an equation $y'' = 2\sin^3 y \cos y$ with initial conditions $y(1) = \frac{\pi}{2}$, $y'(1) = 1$.

The equation doesn't contain explicitly the independent variable, and so we put $y' = p(y)$.

The first step.

$$y' = p(y), y'' = pp', pp' = 2\sin^3 y \cos y, p\frac{dp}{dy} = 2\sin^3 y \cos y, pdp = 2\sin^3 y \cos y dy,$$

$$\int pdp = 2\int \sin^3 y \cos y dy + \frac{C_1}{2}, \frac{p^2}{2} = \frac{\sin^4 y}{2} + \frac{C_1}{2}, p^2 = \sin^4 y + C_1, p = \sqrt{\sin^4 y + C_1};$$

by initial conditions

$$p\left(\frac{\pi}{2}\right) = y'(1) = 1 = \sqrt{\sin^4 \frac{\pi}{2} + C_1}, 1 = \sqrt{1 + C_1}, C_1 = 0, \textit{and } p = \sqrt{\sin^4 y} = \sin^2 y.$$

The second step.

$$y' = \sin^2 y, \frac{dy}{dx} = \sin^2 y, \frac{dy}{\sin^2 y} = dx, -\cot y = x + C_2.$$

On the base of the first initial condition $-\cot\frac{\pi}{2} = 1 + C_2, 0 = 1 + C_2, C_2 = -1$, and the solution of Cauchy problem is given by the equality

$$\cot y = 1 - x.$$

Ex. 19. Solve Cauchy problem $yy''-(y')^2=y^4,\ y(0)=1,\ y'(0)=0$.

The first step. $y'=p(y),\ y''=pp',\ ypp'-p^2=y^4$.

By virtue of initial conditions $y\neq 0,\ p\neq 0$ (verify!) and so (after division by py)

$$p'-\frac{1}{y}\cdot p=y^3\cdot\frac{1}{p}.$$

This last differential equation is Bernoulli one, and we find its solution of the form

$$p=u(y)v(y),$$

whence

$$p'=u'v+uv',\ u'v+uv'-\frac{uv}{y}=\frac{y^3}{uv},\ u'v+u\left(v'-\frac{v}{y}\right)=\frac{y^3}{uv}.$$

$$v'-\frac{v}{y}=0,\ \frac{dv}{dy}=\frac{v}{y},\ \frac{dv}{v}=\frac{dy}{y},\ \ln|v|=\ln|x|+\ln|C_0|,\ \ln|v|=\ln|C_0x|,\ v=C_0x,\ v=v_0(y)=y;$$

$$u'v_0=\frac{y^3}{uv_0},\ u'y=\frac{y^3}{uy},\ udu=ydy,\ \frac{u^2}{2}=\frac{y^2}{2}+\frac{C_1}{2},\ u=\pm\sqrt{y^2+C_1},\ p=\pm y\sqrt{y^2+C_1};$$

from the initial conditions

$$p(y(0))=y'(0)=0=\pm y(0)\sqrt{y^2(0)+C_1}=\pm\sqrt{1+C_1},\ \sqrt{1+C_1}=0,\ C_1=-1,$$

$$p=\pm y\sqrt{y^2-1}.$$

The second step.

$$y'=\pm y\sqrt{y^2-1},\ \frac{dy}{dx}=\pm y\sqrt{y^2-1},\ \frac{dy}{y\sqrt{y^2-1}}=\pm dx,\ \int\frac{dy}{y\sqrt{y^2-1}}=\pm x+C_2;$$

$$\int\frac{dy}{y\sqrt{y^2-1}}=\left|\begin{array}{l} y=\dfrac{1}{\cos z},\\ dy=\dfrac{\sin z dz}{\cos^2 z}\end{array}\right|=\int\frac{\dfrac{\sin z dz}{\cos^2 z}}{\dfrac{1}{\cos z}\sqrt{\dfrac{1}{\cos^2 z}-1}}=\pm\int\frac{\dfrac{\sin z dz}{\cos^2 z}}{\dfrac{\sin z}{\cos^2 z}}=\pm\int dz=\pm z=$$

$$=\pm\arccos\frac{1}{y};\ \pm\arccos\frac{1}{y}=\pm x+C_2;\ \pm\arccos\frac{1}{y(0)}=\pm 0+C_2,\ \pm\arccos 1=C_2,\ C_2=0$$

The required solution of Cauchy problem

$$\pm\arccos\frac{1}{y}=\pm x\ \text{or}\ y=\frac{1}{\cos x}.$$

LECTURE NO.26. SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

POINT 1. GENERAL NOTIONS

POINT 2. LINEAR DEPENDENCE AND INDEPENDENCE

POINT 3. HOMOGENEOUS EQUATIONS

POINT 4. NONHOMOGENEOUS EQUATIONS

POINT 1. GENERAL NOTIONS

Def. 1. The second order linear differential equation (SO LDE, LDE) is called the next equation

$$y'' + a(x)y' + b(x)y = f(x). \qquad (1)$$

Coefficients $a(x), b(x)$ and the second [free, absolute] term $f(x)$ of equation are known functions, $y = y(x)$ is an unknown function.

Initial conditions for the equation (1) have a usual form

$$y(x_0) = y_0,\ y'(x_0) = y'_0 \qquad (2)$$

Equation (1) is called **nonhomogeneous** one (SO LNDE, LNDE) if its second term $f(x)$ doesn't equal zero identically.

If $f(x) \equiv 0$, the corresponding [associated] equation

$$y'' + a(x)y' + b(x)y = 0 \qquad (3)$$

is called **homogeneous** equation (SO LHDE, LHDE) which corresponds to the (nonhomogeneous) equation (1).

Theorem 1 (on one-valued solvability of Cauchy problem)**.** If coefficients $a(x), b(x)$ and a second term $f(x)$ of the equation (1) are continuous functions on some segment $[a, b]$ then Cauchy problem (1), (2) (in particular (3), (2)) has unique solution, and this solution is determined on the whole segment $[a, b]$.

Ex. 1. Cauchy's problem for homogeneous equation (3) with zero initial conditions

$$y(x_0)=0,\ y'(x_0)=0 \tag{4}$$

has unique trivial solution $y=0$.

Def. 2. The left side of the differential equations (1), (3) is called a **linear differential operator** and is denoted by $L[y]$,

$$L[y]=y''+a(x)y'+b(x)y. \tag{5}$$

The linear differential operator possesses evident **properties**

1. $L[y_1+y_2]=L[y_1]+L[y_2]$ (**additivity**).

■$L[y_1+y_2]=(y_1+y_2)''+a(x)(y_1+y_2)'+b(x)(y_1+y_2)=$

$=y_1''+a(x)y_1'+b(x)y_1+y_2''+a(x)y_2'+b(x)y_2=L[y_1]+L[y_2]$.■

2. $L[ky]=kL[y]$ for any constant k (**homogeneity**).

■$L[ky]=(ky)''+a(x)(ky)'+b(x)(ky)=k(y''+a(x)y'+b(x)y)=kL[y]$■

As the corollary we have

$$L[k_1y_1+k_2y_2]=k_1L[y_1]+k_2L[y_2]$$

for any constant k_1, k_2 (**linearity**).

With the help of the linear differential operator the equations (1), (3) can be written as follows

$$L[y]=f(x),$$
$$L[y]=0.$$

Properties of solutions of a linear homogeneous differential equation (3).

1. Sum of two solutions of the equation (3) is also a solution.

■Let y_1, y_2 be solutions of the equation (3) that is $L[y_1]\equiv 0,\ L[y_2]\equiv 0$. By the property 1 of a linear differential operator $L[y_1+y_2]=L[y_1]+L[y_2]\equiv 0$.■

2. Product of any solution of the equation (3) by a constant is also a solution.

■Let y be a solution of the equation (3), i.e. $L[y]\equiv 0$, and k is a constant. Then by the property 2 of a linear differential operator $L[ky]=kL[y]\equiv 0$.■

3. If a function $y=y_1+iy_2$ is a complex solution of the equation (3) (with real coefficients $a(x), b(x)$) then its real and imaginary parts y_1, y_2 are also the solutions.

POINT 2. LINEAR DEPENDENCE AND INDEPENDENCE

Def. 3. Two functions $\varphi_1(x), \varphi_2(x)$ are called **linearly dependent** on a segment $[a, b]$ if there exist two numbers λ_1, λ_2 not all zeros ($\lambda_1^2 + \lambda_2^2 > 0$) such that for any $x \in [a, b]$ an identity

$$\lambda_1\varphi_1(x) + \lambda_2\varphi_2(x) \equiv 0 \tag{6}$$

holds. If the identity holds only in the case $\lambda_1 = \lambda_2 = 0$, these functions are called **linearly independent**.

Theorem 2. Two functions $\varphi_1(x), \varphi_2(x)$ are linearly dependent on a segment $[a, b]$ if and only if their ratio identically equals a constant on $[a, b]$, that is for any $x \in [a, b]$

$$\frac{\varphi_1(x)}{\varphi_2(x)} \equiv C = const. \tag{7}$$

■1. Let functions $\varphi_1(x), \varphi_2(x)$ be linearly dependent on a segment $[a, b]$, and so the identity (6) holds for $\lambda_1^2 + \lambda_2^2 > 0$. If, for example, $\lambda_1 \neq 0$ then we get from (6)

$$\varphi_1 = -\frac{\lambda_2}{\lambda_1}\varphi_2 \Rightarrow \frac{\varphi_1}{\varphi_2} = -\frac{\lambda_2}{\lambda_1} = const$$

and a ratio of functions is an identical constant.

2. Now let $\varphi_1(x)/\varphi_2(x) \equiv C = const$ on $[a, b]$. Then

$$\varphi_1(x) \equiv C\varphi_2(x), 1\cdot\varphi_1(x) + (-C)\cdot\varphi_2(x) \equiv 0 \left(\lambda_1 = 1 \neq 0,\ \lambda_2 = -C,\ \lambda_1^2 + \lambda_2^2 > 0\right),$$

and $\varphi_1(x), \varphi_2(x)$ are linearly dependent by the definition of linear dependence.■

Ex. 2. Functions $\cos ax, \sin ax$ for $a \neq 0$ are linearly independent on $(-\infty, \infty)$ because of their ratio isn't a constant.

Def. 4. n functions $\varphi_1(x), \varphi_2(x), ..., \varphi_n(x)$ are called linearly dependent on a segment $[a, b]$ if there exist n numbers $\lambda_1, \lambda_2, ..., \lambda_n$ not all zeros ($\sum_{i=1}^{n}\lambda_i^2 > 0$) such that for any $x \in [a, b]$ an identity

$$\lambda_1\varphi_1(x) + \lambda_2\varphi_2(x) + ... + \lambda_n\varphi_n(x) \equiv 0 \tag{8}$$

holds. If the identity holds only in the case $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$, these functions are called **linearly independent** on $[a, b]$.

Ex. 3. Functions $1, x, x^2, x^3, \ldots, x^n$ are linearly independent on $\Re = (-\infty, \infty)$ because of a polynomial in x

$$\lambda_1 \cdot 1 + \lambda_2 \cdot x + \lambda_3 \cdot x^2 + \ldots + \lambda_{n+1} \cdot x^n \equiv 0$$

only if all its coefficients equal zero, i.e. $\lambda_1 = \lambda_2 = \lambda_3 = \ldots = \lambda_{n+1} \equiv 0$.

In what follows we'll deal with linear dependence or independence of solutions of differential equations. There is well mathematical tool to study corresponding questions namely the wronskian [Wronskian determinant, determinant of Wronski] of several functions.

Def. 5. Wronskian of functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$ is called the next determinant

$$W(x) = W[\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)] = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) & \varphi_3(x) & \ldots & \varphi_n(x) \\ \varphi_1'(x) & \varphi_2'(x) & \varphi_3'(x) & \ldots & \varphi_n'(x) \\ \varphi_1''(x) & \varphi_2''(x) & \varphi_3''(x) & \ldots & \varphi_n''(x) \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots & \ldots\ldots\ldots \\ \varphi_1^{(n-1)}(x) & \varphi_2^{(n-1)}(x) & \varphi_3^{(n-1)}(x) & \ldots & \varphi_n^{(n-1)}(x) \end{vmatrix} \quad (9)$$

Ex. 4. The wronskian of functions $\cos ax, \sin ax$ (see Ex. 2) is

$$W(x) = \begin{vmatrix} \cos ax & \sin ax \\ (\cos ax)' & (\sin ax)' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a\cos^2 ax + a\sin^2 ax = a\,.$$

Theorem 3. If functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$ are linearly dependent on a segment $[a, b]$ then their wronskian identically equals zero on $[a, b]$, $W(x) \equiv 0$.

■ For the sake of simplicity let two functions $\varphi_1(x), \varphi_2(x)$ be linearly dependent on $[a, b]$.

The *first mode* of proving. The ratio of the functions $\varphi_1(x), \varphi_2(x)$ is identical constant on $[a, b]$. For example let $\varphi_2(x)/\varphi_1(x) \equiv C = const$, $\varphi_2(x) \equiv C\varphi_1(x)$. Then the wronskian of these functions

$$W[\varphi_1(x), \varphi_2(x)] = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix} \equiv \begin{vmatrix} \varphi_1(x) & C\varphi_1(x) \\ \varphi_1'(x) & C\varphi_1'(x) \end{vmatrix} \equiv C \begin{vmatrix} \varphi_1(x) & \varphi_1(x) \\ \varphi_1'(x) & \varphi_1'(x) \end{vmatrix} \equiv 0 .$$

The *second mode* of proving (it can be easily extended on any number of functions). By the definition of linear dependence there are two numbers λ_1, λ_2 such that $\lambda_1^2 + \lambda_2^2 > 0$ and $\lambda_1\varphi_1(x) + \lambda_2\varphi_2(x) \equiv 0$ on $[a, b]$. We differentiate this identity and compile a system of linear homogeneous equations in λ_1, λ_2,

$$\begin{cases} \lambda_1\varphi_1(x) + \lambda_2\varphi_2(x) \equiv 0, \\ \lambda_1\varphi_1'(x) + \lambda_2\varphi_2'(x) \equiv 0. \end{cases}$$

This system has non-trivial solution, and therefore its principal determinant equals zero for any $x \in [a, b]$,

$$\Delta = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix} = W[\varphi_1(x), \varphi_2(x)] \equiv 0 .\blacksquare$$

Theorem 4. If functions $\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)$ are linearly independent solutions of the *n*th order linear homogeneous differential equation with coefficients continuous on a segment $[a, b]$, then the wronskian of these solutions doesn't equal zero at all the points of the segment.

■Let's prove the theorem for two linearly independent solutions $\varphi_1(x), \varphi_2(x)$ of the second order linear homogeneous differential equation (3). Suppose that the wronskian of these functions equals zero at a point $x_0 \in [a, b]$, $W(x_0) = 0$. We choose two numbers λ_1, λ_2 such that $\lambda_1^2 + \lambda_2^2 > 0$ and

$$\begin{cases} \lambda_1\varphi_1(x_0) + \lambda_2\varphi_2(x_0) \equiv 0, \\ \lambda_1\varphi_1'(x_0) + \lambda_2\varphi_2'(x_0) \equiv 0. \end{cases} \qquad (10)$$

It's possible because the system (10) in λ_1, λ_2 has zero principal determinant $W(x_0)$. Let's compile the next function

$$y = \lambda_1\varphi_1(x) + \lambda_2\varphi_2(x).$$

It is a solution of the equation (3) and satisfies zero initial conditions (10). Hence on the base of Ex. 1 $y = 0$, that is

$$\lambda_1\varphi_1(x)+\lambda_2\varphi_2(x)\equiv 0\,.$$

But it means that the functions $\varphi_1(x), \varphi_2(x)$ are linearly dependent (we remember that $\lambda_1^2+\lambda_2^2>0$).

We've arrived at contradiction which proves the theorem.■

Ex. 5. The functions $\cos ax, \sin ax\ (a\neq 0)$ (see Ex. 2) are linearly independent solutions of a differential equation $y''+a^2y=0$ on $\Re=(-\infty,\infty)$. Their wronskian equals a (see Ex. 4) and doesn't equal zero for any $x\in(-\infty,\infty)$.

POINT 3. HOMOGENEOUS EQUATIONS

Structure of the general solution of SO LHDE

Theorem 5. (structure of the general solution of the second order linear homogeneous differential equation (3)). If $y=y_1(x), y=y_2(x)$ be two linearly independent solutions of LHDE (3), then its general solution has the next form

$$y=C_1y_1(x)+C_2y_2(x) \qquad (11)$$

where C_1, C_2 are arbitrary constants.

■The function $y=C_1y_1(x)+C_2y_2(x)$ is a solution of the equation (3) for any values of C_1, C_2 because of linear property of solutions of LHDE. On the base of the definition 8 of the lecture No. 25 it's necessary to prove that for any initial conditions (2) one can find values of C_1, C_2 to satisfy these conditions. But

$$y(x_0)=C_1y_1(x_0)+C_2y_2(x_0),\ y'(x_0)=C_1y_1'(x_0)+C_2y_2'(x_0),$$

and so it is a matter of compatibility of the next system of linear equations in C_1, C_2

$$\begin{cases} C_1y_1(x_0)+C_2y_2(x_0)=y_0, \\ C_1y_1'(x_0)+C_2y_2'(x_0)=y_0'. \end{cases}$$

This system has unique solution because of its principal determinant

$$\Delta = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = W[y_1(x_0), y_2(x_0)] \neq 0$$

distinct from zero by virtue of linear independence of the solutions $y_1(x), y_2(x)$ ■

Ex. 6. A function $y = C_1 \cos ax + C_2 \sin ax$ is the general solution of the equation $y'' + a^2 y = 0$ (see Ex. 5).

SO LHDE with constant coefficients

Let there be given the second order linear differential equation

$$y'' + py' + qy = 0 \qquad (12)$$

with constant coefficients p, q. We'll seek its solutions of the form

$$y = e^{kx} \qquad (13)$$

where *k* is an unknown (real or complex) number. Finding the derivatives of the function (13)

$$y' = ke^{kx},\ y'' = k^2 e^{kx}$$

and substituting the values of y, y', y'' in the equation we get

$$k^2 e^{kx} + pke^{kx} + qe^{kx},\ e^{kx}\left(k^2 + pk + q\right) = 0,$$

$$k^2 + pk + q = 0. \qquad (14)$$

Equation (14) is called the **characteristic equation**. We have to study three cases.

1. Roots k_1, k_2 of the characteristic equation are **real distinct**. We get two linearly independent solutions of the equation (12) that is

$$y_1 = e^{k_1 x},\ y_2 = e^{k_2 x},\ y_1/y_2 = e^{k_1 x}/e^{k_2 x} = e^{(k_1 - k_2)x} \neq const$$

because of $k_1 \neq k_2$. Therefore the general solution of the equation (12) is

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{k_1 x} + C_2 e^{k_2 x}. \qquad (15)$$

Ex. 7. $y'' + 5y' + 6y = 0$.

The characteristic equation of the differential equation

$$k^2 + 5k + 6 = 0$$

has two distinct real roots $k_1 = -2, k_2 = -3$, so the differential equation has two linearly independent solutions

$$y_1 = e^{-2x},\ y_2 = e^{-3x}$$

and so its general solution is

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{-2x} + C_2 e^{-3x}.$$

2. Roots k_1, k_2 of the characteristic equation are **real equal** $k_1 = k_2 = k_0$.

We have one solution $y = e^{k_0 x}$, and we must find the second linear independent solution y_2 of the equation (12). By Vieta theorem

$$k_1 + k_2 = 2k_0 = -p,\ k_1 k_2 = k_0^2 = q,\ p = -2k_0,\ q = k_0^2,$$

so the equation (12) has in this case the form

$$y'' - 2k_0 y' + k_0^2 y = 0 \qquad (16)$$

and possesses the evident solution

$$y_2 = xy_1 = xe^{k_0 x}.$$

Indeed,

$$y_2' = e^{k_0 x} + xk_0 e^{k_0 x} = (1 + xk_0)e^{k_0 x},\ y_2'' = k_0 e^{k_0 x} + (1 + xk_0)k_0 e^{k_0 x} = (2k_0 + xk_0^2)e^{k_0 x}.$$

Substituting in the equation (16) we get

$$y_2'' - 2k_0 y_2' + k_0^2 y_2 = (2k_0 + xk_0^2)e^{k_0 x} - 2k_0(1 + xk_0)e^{k_0 x} + k_0^2 xe^{k_0 x} = 0.$$

Solutions y_1, y_2 are linearly independent, for

$$y_1 / y_2 = e^{k_0 x} / xe^{k_0 x} = 1/x \neq const.$$

Therefore the general solution of the equation (12) (of the equation (16) in this case) is

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{k_0 x} + C_2 xe^{k_0 x} = e^{k_0 x}(C_1 + xC_2). \qquad (17)$$

Ex. 8. $y'' - 10y' + 25y = 0$.

The characteristic equation

$$k^2 - 10k + 25 = 0$$

has equal real roots $k_1 = k_2 = 5\ (k_0 = 5)$. Therefore the differential equation has two linearly independent solutions

$$y_1 = e^{5x},\ y_2 = xe^{5x},$$

and its general solution is

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{5x} + C_2 x e^{5x} = e^{5x}(C_1 + xC_2).$$

3. Roots of the characteristic equation (14) are **complex**, $k_{1,2} = \alpha \pm i\beta$.

We have two complex solution of the equation (12)

$$y_{1,2} = e^{(\alpha \pm i\beta)x},$$

but we can find two real solutions. By Euler formula

$$e^{ix} = \cos x + i \sin x$$

we do the next transformations

$$y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x + i\beta x} = e^{\alpha x} e^{i\beta x} = e^{\alpha x}(\cos \beta x + i \sin \beta x) = e^{\alpha x} \cos \beta x + i \cdot e^{\alpha x} \sin \beta x.$$

By virtue of the property 3 of solutions of LHDE (see Lecture No.26, P. 1) the next two real functions

$$y_1 = e^{\alpha x} \cos \beta x,\ y_2 = e^{\alpha x} \sin \beta x$$

are linearly independent solutions of the equation (12) $(y_1 / y_2 = \cot \beta x \neq const)$. Therefore its general solution is

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{\alpha x} \cos \beta x + + C_2 e^{\alpha x} \sin \beta x = e^{\alpha x}(C_1 \cos \beta x + + C_2 \sin \beta x).$$

Ex. 9. $y'' - 4y' + 18y = 0$.

The characteristic equation

$$k^2 - 4k + 18 = 0$$

has complex roots

$$k_{1,2} = \frac{2 \pm \sqrt{-56}}{2} = \frac{2 \pm i\sqrt{56}}{2} = \frac{2 \pm 2i\sqrt{14}}{2} = 1 \pm i\sqrt{14},\ \alpha = 1,\ \beta = \sqrt{14},$$

therefore the differential equation has two real linearly independent solutions

$$y_1 = e^{1 \cdot x} \cos \sqrt{14}x = e^x \cos \sqrt{14}x,\ y_2 = e^{1 \cdot x} \sin \sqrt{14}x = e^x \sin \sqrt{14}x$$

and the general solution

$$y = C_1 y_1 + C_2 y_2 = C_1 e^x \cos \sqrt{14}x + + C_2 e^x \sin \sqrt{14}x = e^x(C_1 \cos \sqrt{14}x + C_2 \sin \sqrt{14}x).$$

Ex. 10. For the known equation $y'' + a^2 y = 0$ (Ex. 5, 6) the characteristic equation $k^2 + a^2 = 0$ has complex roots $k_{1,2} = \pm\sqrt{-a^2} = \pm ai, \alpha = 0, \beta = a$. Therefore linearly independent solutions and the general solution of the differential equation are

$$y_1 = e^{0\cdot x}\cos ax = \cos ax,\ y_2 = e^{0\cdot x}\sin ax = \sin ax,$$
$$y = C_1 y_1 + C_2 y_2 = C_1\cos ax + C_2\sin ax.$$

POINT 4. NONHOMOGENEOUS EQUATIONS

Structure of the general solution of SO LNDE

Theorem 6 (structure of the general solution of the second order linear nonhomogeneous differential equation (1)**).** The general solution of SO LNDE equals the sum of the general solution y_{GH} of corresponding [associated] LHDE (3) and some particular solution y_{PN} of the given equation,

$$y = y_{GN} = y_{GH} + y_{PN}. \qquad (18)$$

■ Let $y_{GH} = C_1 y_1 + C_2 y_2$ be the general solution of LHDE (3), where y_1, y_2 are two linearly independent solutions of LHDE. A function

$$y = C_1 y_1 + C_2 y_2 + y_{PN}$$

is a solution of the equation (1) for any values of C_1, C_2 because of

$$L[y] = L[C_1 y_1 + C_2 y_2 + y_{PN}] = C_1 L[y_1] + C_2 L[y_2] + L[y_{PN}] \equiv 0 + 0 + f(x) = f(x).$$

We have only to prove that for any initial conditions (2) one can find values of C_1, C_2 to satisfy these conditions. Since

$$y(x_0) = C_1 y_1(x_0) + C_2 y_2(x_0) + y_{PN}(x_0),\ y'(x_0) = C_1 y_1'(x_0) + C_2 y_2'(x_0) + y_{PN}'(x_0),$$

we get a system of linear equations in C_1, C_2

$$\begin{cases} C_1 y_1(x_0) + C_2 y_2(x_0) + y_{PN}(x_0) = y_0, \\ C_1 y_1'(x_0) + C_2 y_2'(x_0) + y_{PN}'(x_0) = y_0' \end{cases}$$

which has unique solution because of its principal determinant is $W[y_1(x_0), y_2(x_0)]$ and doesn't equal zero on account of linear independence of y_1, y_2.■

Ex. 11. A function $y = C_1 \cos x + C_2 \sin x + 1/a^2$ is the general solution of a differential equation $y'' + a^2 y = 1$.

Indeed, $y = C_1 \cos x + C_2 \sin x$ is the general solution of the corresponding homogeneous equation $y'' + a^2 y = 0$ (see Ex. 6, 9), and a function $y = 1/a^2$ is a particular solution of the given equation.

Method of variation of arbitrary constants

Let's suppose that we must find the general solution of SO LNDE (1). We can do it with Lagrange in the next two steps.

1. We find the general solution

$$y_{GH} = C_1 y_1 + C_2 y_2$$

of the corresponding LHDE (3), where y_1, y_2 are its linearly independent solutions.

2. Now we find the general solution of SO LNDE (1) in the same form as y_{GH}, but we treat C_1, C_2 as unknown functions, namely

$$y = y_{GN} = C_1(x) y_1 + C_2(x) y_2 . \qquad (19)$$

We find the first derivative of y,

$$y' = C_1'(x) y_1 + C_2'(x) y_2 + C_1(x) y_1' + C_2(x) y_2',$$

and we suppose that

$$C_1'(x) y_1 + C_2'(x) y_2 = 0 .$$

Then

$$y' = C_1(x) y_1' + C_2(x) y_2', \; y'' = C_1'(x) y_1' + C_2'(x) y_2' + C_1(x) y_1'' + C_2(x) y_2'' .$$

Substituting the values of y, y', y'' in the equation (1), we have

$$C_1'(x) y_1' + C_2'(x) y_2' + C_1(x) y_1'' + C_2(x) y_2'' + a(x)\left(C_1(x) y_1' + C_2(x) y_2'\right) +$$
$$b(x)\left(C_1(x) y_1 + C_2(x) y_2\right) = f(x),$$

$$C_1'(x) y_1' + C_2'(x) y_2' + C_1(x)\left(y_1'' + a(x) y_1' + b(x) y_1\right) + C_2(x)\left(y_2'' + a(x) y_2' + b(x) y_2\right) = f(x),$$

$$C_1'(x) y_1' + C_2'(x) y_2' + C_1(x) \cdot 0 + C_2(x) \cdot 0 = f(x), \; C_1'(x) y_1' + C_2'(x) y_2' = f(x).$$

We get the next system of linear equations in C_1', C_2'

$$\begin{cases} C_1'(x)y_1 + C_2'(x)y_2 = 0, \\ C_1'(x)y_1' + C_2'(x)y_2' = f(x). \end{cases} \qquad (20)$$

Solving the system (20) we get

$$C_1'(x) = \varphi_1(x), C_2'(x) = \varphi_2(x),$$

where $\varphi_1(x), \varphi_2(x)$ be some functions. Integrating, we have finally

$$C_1(x) = \int \varphi_1(x)dx + \widetilde{C}_1, C_2(x) = \int \varphi_2(x)dx + \widetilde{C}_2 \qquad (21)$$

where $\widetilde{C}_1$, $\widetilde{C}_2$ are arbitrary constants. The general solution of the equation (1) is

$$y = y_{GN} = C_1(x)y_1 + C_2(x)y_2 = \left(\int \varphi_1(x)dx + \widetilde{C}_1\right)y_1 + \left(\int \varphi_2(x)dx + \widetilde{C}_2\right)y_2 . \qquad (22)$$

Note 1. We can represent the general solution (22) in the next form:

$$y = y_{GN} = \widetilde{C}_1 y_1 + \widetilde{C}_2 y_2 + y_1 \int \varphi_1(x)dx + y_2 \int \varphi_2(x)dx,$$

and we see that

$$\widetilde{C}_1 y_1 + \widetilde{C}_2 y_2$$

is the general solution of the homogeneous equation (3) and

$$y_1 \int \varphi_1(x)dx + y_2 \int \varphi_2(x)dx$$

is the particular solution of the nonhomogeneous equation (1).

Ex. 12. Find the general solution of an equation $y'' - 2y' + y = \dfrac{e^x}{x^2+1}$.

1. Corresponding LHDE is $y'' - 2y' + y = 0$, its characteristic equation

$$k^2 - 2k + 1 = 0$$

has real equal roots $k_1 = k_2 = 1$ $(k_0 = 1)$, and LHDE has linearly independent particular solutions

$$y_1 = e^x, y_2 = xe^x$$

and the general solution

$$y_{GH} = C_1 y_1 + C_2 y_2 = C_1 e^x + C_2 x e^x.$$

2. Now we seek the general solution of the given equation in the form

$$y = y_{GN} = C_1(x)y_1 + C_2(x)y_2 = C_1(x)e^x + C_2(x)xe^x.$$

By virtue of the formula (20) the system of linear equations in C_1', C_2' and its solution are

$$\begin{cases} C_1'(x)y_1 + C_2'(x)y_2 = 0, \\ C_1'(x)y_1' + C_2'(x)y_2' = \dfrac{e^x}{x^2+1}; \end{cases} \begin{cases} C_1'(x)e^x + C_2'(x)xe^x = 0, \\ C_1'(x)e^x + C_2'(x)(1+x)e^x = \dfrac{e^x}{x^2+1}; \end{cases}$$

$$\begin{cases} C_1'(x) + C_2'(x)x = 0, \\ C_1'(x) + C_2'(x)(1+x) = \dfrac{1}{x^2+1}; \end{cases} \Delta = \begin{vmatrix} 1 & x \\ 1 & 1+x \end{vmatrix} = 1,\ \Delta_1 = \begin{vmatrix} 0 & x \\ \dfrac{1}{x^2+1} & 1+x \end{vmatrix} = -\frac{x}{x^2+1},$$

$$\Delta_2 = \begin{vmatrix} 1 & 0 \\ 1 & \dfrac{1}{x^2+1} \end{vmatrix} = \frac{1}{x^2+1},\quad C_1'(x) = \frac{\Delta_1}{\Delta} = -\frac{x}{x^2+1},\ C_2'(x) = \frac{\Delta_2}{\Delta} = \frac{1}{x^2+1}.$$

After integration

$$C_1(x) = -\int \frac{xdx}{x^2+1} + \widetilde{C}_1 = -\frac{1}{2}\int \frac{2xdx}{x^2+1} + \widetilde{C}_1 = -\frac{1}{2}\ln(x^2+1) + \widetilde{C}_1,$$

$$C_2(x) = \int \frac{dx}{x^2+1} + \widetilde{C}_2 = \arctan x + \widetilde{C}_2,$$

and the general solution of the given differential equation is

$$y = y_{GN} = C_1(x)e^x + C_2(x)xe^x = \left(-\frac{1}{2}\ln(x^2+1) + \widetilde{C}_1\right)e^x + \left(\arctan x + \widetilde{C}_2\right)xe^x.$$

Ex. 13. Solve Cauchy problem $y'' - 3y' + 2y = \dfrac{e^x}{1+e^{-x}}$, $\quad y(0) = y'(0) = 0$.

1. For corresponding [associated] homogeneous equation $y'' - 3y' + 2y = 0$ linearly independent particular solutions $y_1 = e^x$, $y_2 = e^{2x}$ and the general solution

$$y = y_{GH} = C_1e^x + C_2e^{2x}.$$

2. We seek the general solution of the given equation in the form

$$y = y_{GN} = C_1(x)e^x + C_2(x)e^{2x},$$

and by virtue of (20) we get the system of equations in C_1', C_2'

$$\begin{cases} C_1'(x)e^x + C_2'(x)e^{2x} = 0, \\ C_1'(x)e^x + 2C_2'(x)e^{2x} = \dfrac{e^x}{1+e^{-x}} \end{cases} \text{or} \begin{cases} C_1'(x) + C_2'(x)e^x = 0, \\ C_1'(x) + 2C_2'(x)e^x = \dfrac{1}{1+e^{-x}}. \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & e^x \\ 1 & e^{2x} \end{vmatrix} = e^x,\ \Delta_1 = \begin{vmatrix} 0 & e^x \\ \dfrac{1}{1+e^{-x}} & 2e^{2x} \end{vmatrix} = -\frac{e^x}{1+e^{-x}},\ \Delta_2 = \begin{vmatrix} 1 & 0 \\ 1 & \dfrac{1}{1+e^{-x}} \end{vmatrix} = \frac{1}{1+e^{-x}},$$

$$C_1'(x) = \frac{\Delta_1}{\Delta} = -\frac{1}{1+e^{-x}},\ C_2'(x) = \frac{\Delta_2}{\Delta} = \frac{1}{\left(1+e^{-x}\right)e^x}.$$

$$C_1(x) = -\int \frac{1}{1+e^{-x}}dx + \widetilde{C}_1 = -\int \frac{e^x}{e^x+1}dx + \widetilde{C}_1 =$$

$$= -\int \frac{\left(e^x+1\right)'}{e^x+1}dx + \widetilde{C}_1 = -\ln\left(e^x+1\right) + \widetilde{C}_1,$$

$$C_2(x) = \int \frac{1}{\left(1+e^{-x}\right)e^x}dx + \widetilde{C}_2 = \int \frac{e^{-x}}{1+e^{-x}}dx + \widetilde{C}_2 =$$

$$= -\int \frac{\left(1+e^{-x}\right)'}{1+e^{-x}}dx + \widetilde{C}_2 = -\ln\left(1+e^{-x}\right) + \widetilde{C}_2.$$

The general solution of the given differential equation is

$$y = y_{GN} = \left(-\ln\left(e^x+1\right) + \widetilde{C}_1\right)e^x + \left(-\ln\left(1+e^{-x}\right) + \widetilde{C}_2\right)e^{2x}.$$

3. Determination of values of $\widetilde{C}_1, \widetilde{C}_2$ with the help of the initial conditions. Let's find at first the derivative of the unknown function,

$$y' = y'_{GN} = -\frac{e^x \cdot e^x}{e^x+1} + \left(-\ln\left(e^x+1\right) + \widetilde{C}_1\right)e^x - \frac{-e^{-x} \cdot e^{2x}}{1+e^{-x}} + \left(-\ln\left(1+e^{-x}\right) + \widetilde{C}_2\right)2e^{2x},$$

and the values of the functions y, y' at the point $x = 0$,

$$y(0) = \widetilde{C}_1 - \ln 2 + \widetilde{C}_2 - \ln 2 = \widetilde{C}_1 + \widetilde{C}_2 - 2\ln 2,\ y'(0) = \widetilde{C}_1 + 2\widetilde{C}_2 - 3\ln 2.$$

By the initial conditions we must have $y(0) = 0,\ y'(0) = 0$, whence we get and solve the system of equations in $\widetilde{C}_1, \widetilde{C}_2$

$$\begin{cases} \widetilde{C}_1 + \widetilde{C}_2 - 2\ln 2 = 0, \\ \widetilde{C}_1 + 2\widetilde{C}_2 - 3\ln 2 = 0, \end{cases} \quad \begin{cases} \widetilde{C}_1 + \widetilde{C}_2 = 2\ln 2, \\ \widetilde{C}_1 + 2\widetilde{C}_2 = 3\ln 2, \end{cases} \quad \begin{matrix} \widetilde{C}_1 = \ln 2, \\ \widetilde{C}_2 = \ln 2. \end{matrix}$$

The solution of Cauchy problem in question is

$$y = \left(-\ln\left(e^x+1\right) + \ln 2\right)e^x + \left(-\ln\left(1+e^{-x}\right) + \ln 2\right)e^{2x} = e^x \ln\frac{2}{e^x+1} + e^{2x}\ln\frac{2}{1+e^{-x}}.$$

Method of undetermined coefficients for SO LNDE with constant coefficients

Let there be given a linear nonhomogeneous differential equation with constant coefficients. If its second term has a specific form, it's possible to find a particular solution of the equation with the help of the method of undetermined coefficients, without integration.

Let's dwell on the second order linear nonhomogeneous differential equation

$$y'' + py' + qy = f(x) \qquad (23)$$

with constant coefficients p, q.

1. Let the second term of the eqation (23) be a so-called quasipolynomial that is a product of an exponential function $e^{\alpha x}$ (α is a constant) and some *n*th-order polynomial,

$$f(x) = e^{\alpha x}\left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0\right). \qquad (24)$$

In this case we find a particular solution of the eqation (23) in the form

$$y = y_{PN} = x^r e^{\alpha x}\left(A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0\right), \qquad (25)$$

where

$$A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0$$

is a polynomial of the same power as in (24) but with undetermined coefficients $A_n, A_{n-1}, \ldots, A_1, A_0$, and a number *r* is defined by the next conditions:

а) $r = 0$, if α isn't a root of the characteristic equation,

б) $r = 1$, if α is a simple root of the characteristic equation, (26)

в) $r = 2$, if α a double root of the characteristic equation.

If in particular

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \qquad (27)$$

that is $f(x)$ be some *n*th-order polynomial (the case of $\alpha = 0$), then we find a particular solution in the form

$$y = y_{PN} = x^r\left(A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0\right), \qquad (28)$$

where

$$A_n x^n + A_{n-1}x^{n-1} + ... + A_1 x + A_0$$

is a polynomial of the same power as in (27) with undetermined coefficients $A_n, A_{n-1}, ..., A_1, A_0$, and is defined by the next conditions:

а) $r = 0$, if 0 isn't a root of the characteristic equation,

б) $r = 1$, if 0 is a simple root of the characteristic equation, (29)

в) $r = 2$, if 0 a double root of the characteristic equation.

2. Now let the second term of the equation (23) be of the form

$$f(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x), \quad (30)$$

(with constant α, β, A, B). We seek a particular solution of the form

$$y = y_{PN} = x^r e^{\alpha x}(M\cos\beta x + N\sin\beta x), \quad (31)$$

where M, N are undetermined coefficients and

$$r = \begin{cases} 0, \text{ if } \alpha + i\beta \text{ isn't a root of the characteristic equation,} \\ 1, \text{ if } \alpha + i\beta \text{ is a root of the characteristic equation.} \end{cases} \quad (32)$$

If in particular

$$f(x) = A\cos\beta x + B\sin\beta x \quad (33)$$

(the case of $\alpha = 0$), then we find a particular solution in the form

$$y = y_{PN} = x^r(M\cos\beta x + N\sin\beta x), \quad (34)$$

$$r = \begin{cases} 0, \text{ if } i\beta \text{ isn't a root of the characteristic equation,} \\ 1, \text{ if } i\beta \text{ is a root of the characteristic equation} \end{cases} \quad (35)$$

Ex. 14. $y'' + 5y' + 6y = 2xe^{-3x}$.

1. The corresponding homogeneous equation $y'' + 5y' + 6y = 0$ (see Ex. 7). Roots of its characteristic equation $k^2 + 5k + 6 = 0$ are – 2, - 3, general solution is

$$y_{GH} = C_1 e^{-2x} + C_2 e^{-3x}.$$

2. We find a particular solution of the given equation in correspondence with the formulas (24), (25), (26) ($a_1 = 2, a_0 = 0$, $\alpha = -3$ is a simple root of the characteristic equation, $r = 1$)

$$y = y_{PN} = x^1 e^{-3x}(A_1 x + A_0) = e^{-3x}(A_1 x^2 + A_0 x).$$

Finding the derivatives of $y = y_{PN}$,

$$y' = e^{-3x}\left(-3A_1x^2 - 3A_0x + 2A_1x + A_0\right),$$

$$y'' = e^{-3x}\left(9A_1x^2 - 12A_1x + 9A_0x + 2A_1 - 6A_0\right),$$

we substitute the values of y, y', y'' in the given equation, and after reducing similar terms we get

$$\left(-2A_1x + 2A_1 - A_0\right)e^{-3x} = 2xe^{-3x}, -2A_1x + 2A_1 - A_0 = 2x.$$

Equating coefficients of the same powers of x leads to a system of linear equations in A_1, A_0, namely

$$\begin{array}{c|l} x^1 & \left\{\begin{array}{ll} -2A_1 = 2, & A_1 = -1, \\ 2A_1 - A_0 = 0; & A_0 = -2. \end{array}\right. \\ x^0 & \end{array}$$

Therefore

$$y = y_{PN} = e^{-3x}\left(-x^2 - 2x\right),$$

and the general solution of the given differential equation is

$$y = y_{GN} = y_{GH} + y_{PN} = C_1e^{-2x} + C_2e^{-3x} - e^{-3x}\left(x^2 + 2x\right).$$

Ex. 15. $y'' - 10y' + 25y = 3e^{5x}$.

1. The corresponding homogeneous equation $y'' - 10y' + 25y = 0$ (see Ex. 8). Its characteristic equation $k^2 - 10k + 25 = 0$ has real equal roots $k_1 = k_2 = 5$, and

$$y_{GH} = C_1e^{5x} + C_2xe^{5x}.$$

2. By the same formulas (24), (25), (26) ($a_0 = 3, \alpha = 5$ is a double root of the characteristic equation, $r = 2$) we seek a particular solution of the given equation in the form

$$y = y_{PN} = x^2e^{5x}A_0,$$

and so

$$y' = \left(2x + 5x^2\right)e^{5x}A_0, \; y'' = \left(2 + 20x + 25x^2\right)e^{5x}A_0.$$

Substitution of y, y', y'' in the given equation gives

$$2e^{5x}A_0 = 3e^{5x},\ 2A_0 = 3,\ A_0 = \frac{3}{2},\ \text{and so}\ y = y_{PN} = \frac{3}{2}x^2e^{5x},$$

$$y = y_{GN} = y_{GH} + y_{PN} = C_1e^{5x} + C_2xe^{5x} + \frac{3}{2}x^2e^{5x}.$$

Ex. 16. $y'' - 4y' + 18y = 3\cos 2x + 5\sin 2x$.

1. Corresponding homogeneous equation $y'' - 4y' + 18y = 0$ was studied in the Ex. 9. Its characteristic equation has complex roots $1 \pm i\sqrt{14}$, and the general solution

$$y_{GH} = e^x\left(C_1\cos\sqrt{14}x + +C_2\sin\sqrt{14}x\right).$$

2. To find a particular solution of the given equation we use the formulas (33), (34), (35) ($A = 3,\ B = 5,\ \beta = 2,\ r = 0$ because $i\beta = 2i$ isn't a root of the characteristic equation). We find y_{PN} in the form

$$y = y_{PN} = x^0\left(M\cos 2x + N\sin 2x\right) = M\cos 2x + N\sin 2x.$$

Since

$$y' = -2M\sin 2x + 2N\cos 2x,\ y'' = -4M\cos 2x - 4N\sin 2x,$$

substitution of values of $y,\ y',\ y''$ in the given equation gives

$$(14M - 8N)\cos 2x + (8M + 14N)\sin 2x = 3\cos 2x + 5\sin 2x.$$

Equating coefficients of $\cos 2x,\ \sin 2x$ we obtain a system of equations in $M,\ N$

$$\begin{cases} 14M - 8N = 3, \\ 8M + 14N = 5, \end{cases} \Rightarrow \begin{matrix} M = 41/130, \\ N = 23/130, \end{matrix} \Rightarrow y_{PN} = \frac{41}{130}\cos 2x + \frac{23}{130}\sin 2x \Rightarrow$$

$$y = y_{GN} = y_{GH} + y_{PN} = e^x\left(C_1\cos\sqrt{14}x + +C_2\sin\sqrt{14}x\right) + \frac{41}{130}\cos 2x + \frac{23}{130}\sin 2x.$$

Ex. 17. Solve Cauchy problem

$$y'' - 4y' + 13y = 4e^{2x}\sin 3x, \quad y(0) = 1,\ y'(0) = 0.$$

1. For corresponding homogeneous equation $y'' - 4y' + 13y = 0$ we have the characteristic equation $k^2 - 4k + 13 = 0$ with complex roots $2 \pm 3i$, and so

$$y_1 = e^{2x}\cos 3x,\ y_2 = e^{2x}\sin 3x, \quad y_{GH} = C_1e^{2x}\cos 3x + C_2e^{2x}\sin 3x.$$

2. To get a particular solution of the given equation we take into account the formulas (30), (31), (32) ($A = 0, B = 4, \alpha = 2, \beta = 3, \alpha + i\beta = 2 + 3i$ is the root of the characteristic equation, and so $r = 1$), and we put

$$y = y_{PN} = xe^{2x}\left(M\cos 3x + N\sin 3x\right)$$

with undetermined coefficients M, N. Finding

$$y'_{PN} = e^{2x}\left(\left(M + 2xM + 3xN\right)\cos 3x + \left(N + 2xN - 3xM\right)\sin 3x\right),$$

$$y''_{PN} = e^{2x}\left(\left(4M + 6N + 4xM + 12xN - 9xM\right)\cos 3x + \right.$$

$$\left. + \left(4N - 6M + 4xN - 12xM - 9xN\right)\sin 3x\right)$$

and substituting the values of y_{PN}, y'_{PN}, y''_{PN} in the given equation we obtain the equality

$$e^{2x}\left(6N\cos 3x - 6M\sin 3x\right) = 4e^{2x}\sin 3x,\ 6N\cos 3x - 6M\sin 3x = 4\sin 3x.$$

Equating coefficients of $\cos 3x, \sin 3x$ we get the system of equations in M, N

$$\begin{array}{r|l} \cos 3x & 6N = 0, \quad\quad N = 0, \\ \sin 3x & -6M = 4,\ M = -2/3, \end{array} \quad y_{PN} = -\frac{2}{3}xe^{2x}\cos 3x,$$

and therefore

$$y_{GN} = y_{GH} + y_{PN} = C_1e^{2x}\cos 3x + C_2e^{2x}\sin 3x - \frac{2}{3}xe^{2x}\cos 3x.$$

3. Taking into account the initial conditions.

$$y'_{GN} = 2e^{2x}\left(\left(C_1 - \frac{2}{3}x\right)\cos 3x + C_2\sin 3x\right) + e^{2x}\left(\left(-\frac{2}{3} + 3C_2\right)\cos 3x + \left(2x - 3C_1\right)\sin 3x\right)$$

$$y_{GN}(0) = C_1 = 1,\ y'_{GN}(0) = 2C_1 - 2/3 + 3C_2 = 0,\ C_2 = -4/9.$$

Answer. The solution of Cauchy problem for the given equation is

$$y = e^{2x}\cos 3x - \frac{4}{9}e^{2x}\sin 3x - \frac{2}{3}xe^{2x}\cos 3x.$$

Remark (superposition principle). We often meet with situations when a second term of a differential equation (23) is a sum of several different summands of a specific form. For example let

$$y'' + py' + qy = f_1(x) + f_2(x). \quad (36)$$

A particular solution y_{PN} of the equation (36) equals the sum of particular solutions y_{PN1}, y_{PN2} of the next equations

$$y'' + py' + qy = f_1(x), \quad y'' + py' + qy = f_2(x).$$

■Let $L[y] = y'' + py' + qy$. Then $L[y_{PN1}] \equiv f_1(x)$, $L[y_{PN2}] \equiv f_2(x)$, and therefore

$$L[y_{PN1} + y_{PN2}] = L[y_{PN1}] + L[y_{PN2}] \equiv f_1(x) + f_2(x).$$

It means that the sum $y_{PN1} + y_{PN2}$ is a particular solution of the equation (36), that is $y_{PN1} + y_{PN2} = y_{PN}$.■

In practice one can find $y_{PN1} + y_{PN2} = y_{PN}$ with the help of one procedure.

Ex. 18. Find the general solution of a differential equation

$$y'' - 4y' + 4y = 8\left(x^2 + e^{2x} + \sin 2x\right).$$

1. The general solution of the associated homogeneous equation

$$y'' - 4y' + 4y = 0$$

is

$$y_{GH} = C_1 e^{2x} + C_2 x e^{2x}$$

because of the characteristic equation $k^2 - 4k + 4 = 0$ has two equal real roots $k_1 = k_2 = 2$.

2. The second term of the given differential equation is the sum of three summands

$$f_1(x) = 8x^2 = e^{0x} \cdot 8x^2,\ f_2(x) = 8e^{2x} = e^{2x} \cdot 8,\ f_3(x) = 8\sin 2x = 0 \cdot \cos 2x + 8 \cdot \sin 2x.$$

On the base of the superposition principle and the formulas (27) - (28), (25) - (26), (33) - (34) we can sequentially seek three corresponding particular solutions

$$y'' - 4y' + 4y = 8x^2, \quad y_{PN1} = x^0 \cdot e^{0x}\left(A_0 + A_1 x + A_2 x^2\right);$$

$$y'' - 4y' + 4y = 8e^{2x}, \quad y_{PN2} = x^2 \cdot e^{2x} B_0;$$

$$y'' - 4y' + 4y = 8\sin 2x, \quad y_{PN3} = x^0 \cdot \left(M\cos 2x + N\sin 2x\right)$$

and then take their sum. But it's better to find at once the particular solution of the given equation as the sum of such solutions

$$y_{PN} = y_{PN1} + y_{PN2} + y_{PN3} = A_0 + A_1 x + A_2 x^2 + B_0 x^2 e^{2x} + M\cos 2x + N\sin 2x.$$

Since

$$y'_{PN} = A_1 + 2A_2x + 2B_0xe^{2x} + 2B_0x^2e^{2x} - 2M\sin 2x + 2N\cos 2x,$$

$$y''_{PN} = 2A_2 + 2B_0e^{2x} + 8B_0xe^{2x} + 4B_0x^2e^{2x} - 4M\cos 2x - 4N\sin 2x,$$

the substitution of the values of y_{PN}, y'_{PN}, y''_{PN} in the given equation gives

$$y''_{PN} - 4y'_{PN} + 4y_{PN} =$$

$$2A_2 + 2B_0e^{2x} + 8B_0xe^{2x} + 4B_0x^2e^{2x} - 4M\cos 2x - 4N\sin 2x -$$

$$-4\left(A_1 + 2A_2x + 2B_0xe^{2x} + 2B_0x^2e^{2x} - 2M\sin 2x + 2N\cos 2x\right) +$$

$$+4\left(A_0 + A_1x + A_2x^2 + B_0x^2e^{2x} + M\cos 2x + N\sin 2x\right) = 8x^2 + 8e^{2x} + 8\sin 2x.$$

After convenient grouping of the summands

$$(4A_0 - 4A_1 + 2A_2) + (4A_1 - 8A_2)x + 4A_2x^2 + 2B_0e^{2x} - 8N\cos 2x + 8M\sin 2x =$$

$$= 8x^2 + 8e^{2x} + 8\sin 2x$$

we get

$$(4A_0 - 4A_1 + 2A_2) + (4A_1 - 8A_2)x + 4A_2x^2 = 8x^2 \Rightarrow$$

$$\begin{array}{c|l} x^2 & \\ x^1 & \\ x^0 & \end{array}\left\{\begin{array}{rl} 4A_2 = 8, & A_2 = 2, \\ 4A_1 - 8A_2 = 0, & A_1 = 4, \\ 4A_0 - 4A_1 + 2A_2 = 0, & A_0 = 3. \end{array}\right.$$

$$2B_0e^{2x} = 8e^{2x} \Rightarrow 2B_0 = 8,\ B_0 = 4,$$

$$-8N\cos 2x + 8M\sin 2x = 8\sin 2x \Rightarrow -8N = 0,\ 8M = 8 \Rightarrow M = 1,\ N = 0.$$

Therefore

$$y_{PN} = 3 + 4x + 2x^2 + 4x^2e^{2x} + \cos 2x,$$

and the general solution of the differential equation is

$$y_{GN} = y_{GH} + y_{PN} = C_1e^{2x} + C_2xe^{2x} + 3 + 4x + 2x^2 + 4x^2e^{2x} + \cos 2x.$$

LECTURE NO. 27. SYSTEMS OF DIFFERENTIAL EQUATIONS. APPROXIMATE INTEGRATION OF DIFFERENTIAL EQUATIONS

POINT 1. NORMAL SYSTEMS OF DIFFERENTIAL EQUATIONS

POINT 2. APPROXIMATE INTEGRATION OF DIFFERENTIAL EQUATIONS

POINT 1. NORMAL SYSTEMS OF DIFFERENTIAL EQUATIONS

Def. 1. Normal system of differential equations with n unknown functions $y_1(x), y_1(x), ..., y_1(x)$ is called the next system

$$\begin{cases} y_1' = f_1(x, y_1, y_2, ..., y_n), \\ y_2' = f_2(x, y_1, y_2, ..., y_n), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ y_n' = f_n(x, y_1, y_2, ..., y_n). \end{cases} \qquad (1)$$

Def. 2. Solution of a normal system (1) is called an ordered set of n functions $y = \varphi_1(x), y = \varphi_2(x), ..., y = \varphi_n(x)$ which satisfies every equation of the system.

Def. 3. Cauchy problem for a normal system (1) is called a problem of finding solutions of the system which satisfy the next initial conditions

$$y_1(x_0) = y_{10},\ y_2(x_0) = y_{20},\ ...,\ y_n(x_0) = y_{n0} \qquad (2)$$

Theorem 1. Let the functions $f_i(x, y_1, y_2, ..., y_n),\ i = \overline{1, n},$ and all their first order partial derivatives with respect to $y_1, y_2, ..., y_n$ be continuous in some domain D of the $(n+1)$-dimensional space $x, y_1, y_2, ..., y_n$. Then for any point

$$M_0(x_0, y_{10}, y_{20}, ..., y_{n0}) \in D$$

Cauchy problem (1), (2) has unique solution.

Def. 4. The **general solution** of a normal system (1) (in the domain D of the theorem 1) is called an ordered set of n functions $y = \varphi_i(x, C_1, C_2, ..., C_n),\ i = \overline{1, n}$, containing n arbitrary constants, which satisfies two conditions: a) this set is a solution of the system for any values of $C_1, C_2, ..., C_n$; b) for any initial conditions (2) (if $M_0(x_0, y_{10}, y_{20}, ..., y_{n0}) \in D$) it is possible to find values $C_{10}, C_{20}, ..., C_{n0}$ of arbitrary constants to satisfy these conditions.

It can be proved that any system of differential equations, in particular every nth-order differential equation can be reduced to a normal system.

■For example, let there be given the third order differential equation

$$y''' = f(x, y, y', y'').$$

Putting $y = y_1, y' = y_1' = y_2, y'' = y_2' = y_3$ we'll get

$$\begin{cases} y_1' = y_2, \\ y_2' = y_3, \\ y_3' = f(x, y_1, y_2, y_3), \end{cases}$$

that is a normal system of equations in three unknown functions y_1, y_2, y_3.■

A normal system (1) can often be reduced to one nth-order differential equation with the help of so called **elimination method**.

Let's confine ourselves by a normal system of two linear equations with unknown functions $y(x)$, $z(x)$ and constant coefficients

$$\begin{cases} y'(x) = a_1 y(x) + b_1 z(x) + f_1(x), \\ z'(x) = a_2 y(x) + b_2 z(x) + f_2(x). \end{cases} \quad (3)$$

We differentiate the first equation and then substitute the first derivatives of $y(x)$, $z(x)$ by right sides of the equations of the system,

$$y''(x) = a_1 y'(x) + b_1 z'(x) + f_1'(x),$$

$$y''(x) = a_1(a_1 y(x) + b_1 z(x) + f_1(x)) + b_1(a_2 y(x) + b_2 z(x) + f_2(x)) + f_1'(x),$$

$$y''(x) = a_3 y(x) + b_3 z(x) + f_3(x), \quad (4)$$

where

$$a_3 = a_1^2 + a_2 b_1, b_3 = a_1 b_1 + b_1 b_2, f_3(x) = a_1 f_1(x) + b_1 f_2(x) + f_1'(x).$$

From the first equation of the system we find $z(x)$ (if this is possible) and substitute in the equation (4),

$$z(x)=\frac{y'(x)-a_1y(x)-f_1(x)}{b_1}, \tag{5}$$

$$y''(x)=a_3y(x)+b_3\frac{y'(x)-a_1y(x)-f_1(x)}{b_1}+f_3(x),$$

$$y''(x)=ay'(x)+by(x)+f_4(x), \tag{6}$$

where

$$a=\frac{b_3}{b_1},\ b=a_3-\frac{a_1b_3}{b_1},\ f_4(x)=f_3(x)-\frac{b_3}{b_1}f_1(x).$$

Therefore the system of equations (3) is reduced to the second order differential equation (6). Let

$$y(x)=\varphi_1(x,C_1,C_2)$$

is its general solution. Finding $z(x)$ from (5),

$$z(x)=\frac{\dfrac{\partial\varphi_1(x,C_1,C_2)}{\partial x}-a_1\varphi_1(x,C_1,C_2)-f_1(x)}{b_1}=\varphi_2(x,C_1,C_2),$$

we get the general solution of the system (3)

$$y(x)=\varphi_1(x,C_1,C_2),\ z(x)=\varphi_2(x,C_1,C_2).$$

Ex. 1. Solve Cauchy problem for a system

$$\begin{cases} y'=2y+z,\\ z'=3y+4z \end{cases}$$

with initial conditions $y(0)=1,\ z(0)=-2$.

Using the theory we have

$$y''=2y'+z'=2(2y+z)+(3y+4z)=7y+6z=7y+6(y'-2y)=6y'-5y,$$

$$y''=6y'-5y,\ z=y'-2y.$$

$$y''=6y'-5y,\ y''-6y'+5y=0,\ y=C_1e^x+C_2e^{5x},\ z=y'-2y=-C_1e^x+3C_2e^{5x}.$$

The general solution of the system

$$y=C_1e^x+C_2e^{5x},\ z=-C_1e^x+3C_2e^{5x}.$$

Taking into account the initial conditions we get

$$\begin{cases} y(0) = C_1 + C_2 = 1, \\ z(0) = -C_1 + 3C_2 = -2; \end{cases} \begin{cases} C_1 + C_2 = 1, \\ -C_1 + 3C_2 = -2; \end{cases} C_1 = \frac{5}{4}, C_2 = -\frac{1}{4}.$$

Answer. The solution of Cauchy problem

$$y = \frac{5}{4}e^x - \frac{1}{4}e^{5x}, \; z = -\frac{5}{4}e^x - \frac{3}{4}C_2 e^{5x}.$$

Ex. 2. Find the general solution of the system

$$\begin{cases} y' = 2y + z - 15e^{-2x}, \\ z' = y + 2z. \end{cases}$$

By the theory

$$z = y' - 2y + 15e^{-2x}, \; y'' = 2y' + z' + 30e^{-2x} = 2\left(2y + z - 15e^{-2x}\right) + (y + 2z) + 30e^{-2x} =$$
$$= 5y + 4z = 5y + 4\left(y' - 2y + 15e^{-2x}\right) = 4y' - 3y + 60e^{-2x}, \; y'' = 4y' - 3y + 60e^{-2x},$$

$$y'' - 4y' + 3y = 60e^{-2x},$$

a) $y'' - 4y' + 3y = 0, \; y_{GH} = C_1 e^x + C_2 e^{3x}$;

b) $$y = y_{PN} = Ae^{-2x}, \; y' = -2Ae^{-2x}, \; y'' = 4Ae^{-2x},$$
$$4Ae^{-2x} + 8Ae^{-2x} + 3Ae^{-2x} = 60e^{-2x}, 15Ae^{-2x} = 60e^{-2x}, \; A = 4, \; y_{PN} = 4e^{-2x}$$

$$y = y_{GH} + y_{PN} = C_1 e^x + C_2 e^{3x} + 4e^{-2x},$$

$$z = y' - 2y + 15e^{-2x} = -C_1 e^x + C_2 e^{3x} + 15e^{-2x}.$$

The general solution of the system

$$y = C_1 e^x + C_2 e^{3x} + 4e^{-2x}, \; z = -C_1 e^x + C_2 e^{3x} + 15e^{-2x}.$$

POINT 2. APPROXIMATE INTEGRATION OF DIFFERENTIAL EQUATIONS

Successive approximations method

Suppose that it's necessary to solve Cauchy problem

$$y' = f(x, y), \quad (7)$$

$$y(x_0) = y_0. \quad (8)$$

Theorem 2. Cauchy problem (7), (8) is equivalent to the next integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))dx \qquad (9)$$

■a) If a function $y = y(x)$ is a solution of Cauchy problem, then

$$y'(x) \equiv f(x, y(x)),\; y(x_0) = y_0 .$$

Integrating the identity we have

$$y(x) \equiv \int_{x_0}^{x} f(x, y(x))dx + C,\; y(x_0) = y_0 = \int_{x_0}^{x_0} f(x, y(x))dx + C,\; C = y_0,$$

$$y(x) \equiv y_0 + \int_{x_0}^{x} f(x, y(x))dx,$$

and so the function $y = y(x)$ is also a solution of the integral equation.

b) Now let a function $y = y(x)$ be a solution of the integral equation, that is

$$y(x) \equiv y_0 + \int_{x_0}^{x} f(x, y(x))dx .$$

Then $y(x_0) = y_0$, and after differentiation

$$y'(x) \equiv f(x, y(x)).$$

Therefore this function is a solution of Cauchy problem.■

Let's put

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)dx ,$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_1(x))dx ,$$

$$y_3(x) = y_0 + \int_{x_0}^{x} f(x, y_2(x))dx ,$$

………………………………..

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1}(x))dx , \qquad (10)$$

It can be proved that if a function $f(x,y)$ and its partial derivative $f'_y(x,y)$ are continuous in some domain *D* of the xOy-plane, then there is a function $y(x)$ such that for any *x* from a certain interval $[a,b]$

$$\lim_{n\to\infty} y_n(x) = y(x).$$

Passing to the limit as $n\to\infty$ in the equality (10) we see that

$$y(x) = y_0 + \int_{x_0}^{x} f(x, y(x))dx$$

and therefore the function $y(x)$ is a solution of the integral equation (9) and of Cauchy problem.

On this base the formula (10) represents an approximate value of a solution of Cauchy problem.

Note. A function y_n represented by the formula (10) is called the *n*th approximation to an unknown solution of Cauchy problem.

Ex. 3. Find first three approximations to a solution of Cauchy problem

$$y' = x^2 + y^2,\ y(0) = 0..$$

Here $x_0 = y_0 = 0$, and Cauchy problem is equivalent to the integral equation

$$y(x) = \int_0^x \left(x^2 + y^2(x)\right)dx.$$

Therefore

$$y_1(x) = \int_0^x \left(x^2 + 0^2(x)\right)dx = \frac{x^3}{3},$$

$$y_2(x) = \int_0^x \left(x^2 + \left(\frac{x^3}{3}\right)^2\right)dx = \frac{x^3}{3} + \frac{x^7}{63},$$

$$y_3(x) = \int_0^x \left(x^2 + \left(\frac{x^3}{3} + \frac{x^7}{63}\right)^2\right)dx = \frac{x^3}{3} + \frac{x^7}{63} + \frac{2x^{11}}{2079} + \frac{x^{15}}{59535}.$$

We can find the value of a solution of Cauchy problem with arbitrary accuracy.

Euler method

Let's we find solution of Cauchy problem on a segment $[x_0, b]$. We divide the segment into n equal parts of the length

$$h = \frac{b - x_0}{n}$$

by division points

$$x_0 < x_1 < x_2 < \ldots < x_n = b,\ x_1 = x_0 + h,\ x_2 = x_1 + h, \ldots, x_n = x_{n-1} + h\,.$$

Further we substitute the derivative of the unknown function

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

by the divided difference

$$y'(x) \approx \frac{y(x+h) - y(x)}{h}$$

and the differential equation (7) by the difference equation

$$\frac{y(x+h) - y(x)}{h} = f(x, y(x)),$$

from which

$$y(x+h) = y(x) + hf(x, y(x)). \qquad (11)$$

With the help of (11) we have

$$y(x_1) = y(x_0 + h) = y(x_0) + hf(x_0, y(x_0)),\ y(x_1) = y_0 + hf(x_0, y_0), \qquad (12.1)$$

$$y(x_2) = y(x_1) + hf(x_1, y(x_1)) \qquad (12.2)$$

$$y(x_3) = y(x_2) + hf(x_2, y(x_2)) \qquad (12.3)$$

…………………………………..

$$y(x_n) = y(b) = y(x_{n-1}) + hf(x_{n-1}, y(x_{n-1})). \qquad (12.n)$$

If we join points $M_0(x_0, y_0), M_1(x_1, y(x_1)), M_2(x_2, y(x_2)), \ldots, M_n(x_n, y(x_n))$, we'll represent an approximate graph of the solution of Cauchy problem.

Ex. 4. Find an approximate solution of Cauchy problem $y' = xy,\ y(0) = 1$ on a segment $[0, 1]$.

Here

$x_0 = 0,\ y_0 = 1,\ f(x, y(x)) = xy(x),\ y(x_{i+1}) = y(x_i) + hx_i y(x_i) = y(x_i)(1 + hx_i),\ i = 1, 2, \ldots, n$

Dividing the segment $[0, 1]$ into 10 equal parts of the length $h = 0.1$ we get the calculating formula

$$y(x_{i+1}) = y(x_i)(1 + 0.1x_i),\ i = 1, 2, \ldots, 10.$$

We represent all calculations with the help of the next table

i	x_i	$y(x_i)$	$1 + 0.1 \cdot x_i$	$y(x_{i+1})$
0	0.0	1.00	1.00	1.00
1	0.1	1.00	1.01	1.01
2	0.2	1.01	1.02	1.03
3	0.3	1.03	1.03	1.06
4	0.4	1.06	1.04	1.10
5	0.5	1.10	1.05	1.16
6	0.6	1.16	1.06	1.23
7	0.7	1.23	1.07	1.31
8	0.8	1.31	1.08	1.42
9	0.9	1.42	1.09	1.55

Corresponding points of the graph of the approximate solution of Cauchy problem

$$M_0(0.0; 1.00), M_1(0.1; 1.00), M_2(0.2; 1.01), M_3(0.3; 1.03), M_4(0.4; 1.06), M_5(0.5; 1.10),$$
$$M_6(0.6; 1.16),\ M_7(0.7; 1.23),\ M_8(0.8; 1.31),\ M_9(0.9; 1.42),\ M_{10}(1.0; 1.55)$$

DIFFERENTIAL EQUATIONS: Basic Terminology RUE

1. быть разделёнными	бути відокремленими	be séparated
2. быть разрешённым относительно (производной, старшей производной)	бути розв"язаним відносно (похідної, старшої похідної)	be (re)sólved for [with respect to] the (hígher) derívative [be nórmalized]
3. введение новой функции	введення нової функції	ìntrodúction a new fúnction
4. ввести новую функцию	ввести нову функцію	ìntrodúce a new fúnction
5. вещественная часть	дійсна частина	réal part
6. вещественные и равные корни характеристичес-кого уравнения	дійсні й рівні корені характеристичного рівняння	réal and équal roots of the characteristic equátion
7. вещественные и различные корни характеристического уравнения	дійсні й різні корені характеристичного рівняння	réal and distínct roots of the chàracterístic equá-tion
8. вронскиан (определитель вронского)	вронскіан (визначник вронського)	wrónskian
9. геометрический смысл	геометричний сенс	geométric(al) sense [méaning]
10. график решения	графік розв"язку	graph of a solútion
11. двукратный корень	двократний корінь	double [repéated] root
12. дифферециальное уравнение	диференціальне рівняння	dìfferéntial equátion
13. дифферециальное уравнение относительно искомой функции $y(x)$	диференціальне рівняння відносно шуканої функції $y(x)$	dìfferéntial equátion in an únknówn fúnction $y(x)$
14. дифферециальное уравнение первого, второго, n-го, высшего порядка	диференціальне рівняння першого, другого, n-го, вищого порядку	dìfferéntial equátion of the first/second/n-th/hígher order
15. дифферециальное уравнение с разделёнными переменными	диференціальне рівняння з відокремленими змінними	dìfferéntial equátion in/with séparated váriables [séparated (dìfferéntial) equátion]
16. дифферециальное уравнение с разделяемыми переменными	диференціальне рівняння з відокремлюваними змінними	dìfferéntial equátion in/with séparable váriables [séparable (dìfferéntial) equátion]
17. дифферециальное уравнение, допускающее	диференціальне рівняння, яку допускає знижен-	órder redúcing diffe-réntial equátion

понижение порядка	ня порядку	
18.диффереациальное уравнение, не содержащее явно искомую функцию [независимую переменную]	диференціальне рівняння, яке не містить явно шукану функцію [незалежну змінну]	dìfferéntial equátion not contáining explícitly an únknówn fúnction [an ìndepéndent váriable]
19.диффереациальное уравнение, разрешённое относительно (старшей) производной	диференціальне рівняння, розв”язане відносно (старшої) похідної	dìfferéntial equátion (re)sólved for [with respect to] the (hígher) derívative
20.дополнительное условие	додаткова умова	addítional condítion
21.допускать понижение порядка	допускати зниження порядку	admít/allów/permít redúction of the órder
22.единственный	єдиний	uníque [(one and) only one]
23.зависеть от …	залежати від ...	depénd on/upon …
24.задача коши	задача коші	cauchy´s próblem
25.интегральная кривая	інтегральна крива	íntegral curve
26.интегрирование дифференциального уравнения (в квадратурах)	інтегрування диференціального рівняння (в квадратурах)	ìntegrátion of a dìfferéntial equátion (in quádratures)
27.интегрировать дифференциальное уравнение	інтегрувати диференціальне рівняння	íntegrate a dìfferéntial equátion
28.интегрируемый тип	інтеґровний тип	íntegrable type
29.искать решение в форме	шукати розв”язок у вигляді	find [search, look for, look up, seek] a solútion in the form
30.искомая функция	шукана функція	únknówn [desíred, sought, requíred] fúnction
31.квадратное уравнение	квадратне рівняння	quadrátic equátion; quadrátic
32.квадратура	квадратура	quádrature
33.квазиполином	квазіполіном	quási-pòlynómial
34.комплексный корень характеристического уравнения	комплексний корінь характеристичного рівняння	cómplex root of a chàracterístic equátion
35.корень характеристического уравнения	корінь характеристичного рівняння	root of a characte-rístic equátion
36.коэффициент	коефіцієнт	còeffícient
37.коэффициенты при одинаковых степенях x	коефіцієнти при однакових степенях x	còeffícients of équal [of the same] degrées/pó-wers (in like pówers) of x
38.краевое [граничное]	крайова [гранична] умо-	bóundary condítion

условие	ва	
39.кратность корня характеристического уравнения	кратність кореня характеристичного рівняння	mùltiplícity of the root of the chàracterístic equá-tion
40.кратный корень	кратний корінь	múltiple [repéated] root
41.линейная зависимость	лінійна залежність	línear depéndence
42.линейная независимость	лінійна незалежність	línear ìndepéndence
43.линейно зависимые (независимые) функции, ре-шения	лінійно залежні (незалежні) функції, розв"язки	línearly (ìn)depéndent functions, solutions
44.линейное дифференциальное уравнение (лду)	лінійне диференціальне рівняння (лдр)	línear dìfferéntial equátion (lde)
45.линейное дифференциальное уравнение с постоянными коэффициентами	лінійне диференціальне рівняння з сталими коефіцієнтами	linear dìfferéntial equátion with cónstant cò-effícients
46.линейное неоднородное дифференциальное уравнение (лнду)	лінійне неоднорідне диференціальне рівняння (лндр)	línear nónhòmogéne-ous dìfferéntial equátion, línear dìfferéntial equátion with a right-hand mémber (lnde)
47.линейное однородное дифференциальное уравнение (лоду)	лінійне однорідне диференціальне рівняння (лодр)	linear hòmogéneous dìfferéntial equátion, línear dìfferéntial equátion withóut a right-hand member (lhde)
48.линейный дифференциальный оператор	лінійний диференціальний оператор	linear dìfferéntial ópe-rator
49.метод бернулли	метод бернуллі	bernoulli(΄s) méthod
50.метод вариации произвольных постоянных	метод варіації довільних сталих	méthod of vàriátion of of árbitrary constants
51.метод исключения	метод виключення	méthod of elìminátion [elìminátion méthod]
52.метод неопределённых коэффициентов	метод невизначених коефіцієнтів	méthod of úndetérmined còeffícients
53.мнимая часть	уявна частина	imáginary part
54.многочлен	многочлен	pòlynómial
55.многочлен той же степени, что и …, с неопределёнными коэффициентами	многочлен того ж степеня, що й ..., з невизначеними коефіцієнтами	pòlynómial of the same degrée as … with úndetérmined còeffícients
56.наивысший порядок производных искомой функции, которые со-	найвищий порядок похідних шуканої функції, які містяться в диферен-	híghest órder of the derívatives of an únknówn [desíred, sought, requíred]

держатся в дифференциальном уравнении	ціальному рівнянні	fúnction contáining in a dìfferéntial equátion
57.найти (общее решение, решение задачи коши)	знайти (загальний розв”язок, розв”язок задачі коші)	find (the géneral solú-tion, the solútion of cau-chy´s próblem)
58.начальная задача [задача коши]	початкова задача [задача коші]	inítial válue próblem [cauchy(´s) próblem]
59.начальное условие [условие коши]	початкова умова [умова коші]	inítial condítion [cauchy(´s) condítion]
60.не быть единственным	не бути єдиним	not to be unique
61.не содержать явно (независимую переменную, искомую функцию)	не містити явно (незалежну змінну, шукану функцію)	do not contáin explícit-ly (an ìndepéndent váriable, a(n) desired/únknó-wn/sought fúnction)
62.не требовать интегрирования	не вимагати інтегрування	not to demánd/requíre ìntegrátion
63.независимая переменная	незалежна змінна	ìndepéndent váriable
64.необходимое и достаточ-ное условие чего	необхідна й достатня умова чогось	nécessary and suffici-ent condítion for …
65.нормальная система дифференциальных уравнений	нормальна система диференціальних рівнянь	nórmal sýstem of dìfferéntial equátions
66.область	область	domáin [région]
67.обращать [превращать] дифференциальное уравнение в тождество	перетворювати диференціальне рівняння у тотожність	convért [transfórm, turn] a dìfferéntial equá-tion into idéntity
68.обращаться в тождество	перетворюватися в тотожність	turn [transfórm, revért] into idéntity
69.общее решение	загальний розв”язок	géneral solútion
70.общий вид	загальний вигляд	géneral form
71.однородное дифференциальное уравнение	однорідне диференціальне рівняння	hòmogéneous differential equátion
72.определитель вронского	визначник вронського	wronski´s detérmi-nant
73.определить тип дифференциального уравнения	визначити тип диференціального рівняння	detérmine the type of the dìfferéntial equátion
74.отношение (переменных, двух функций, двух решений)	відношення (змінних, двох функцій, двох розв”язків)	rátio (of váriables, of two fúnctions, of two solútions)
75.отношение *чего-либо* (не) является постоянной величиной	відношення *чогось* (не) є сталою величиною	rátio of *smth* is(n´t) a cónstant quántity

76.первообразная	первісна	ántiderívative/prímitive
77.подставить *что-либо* в уравнение	підставити *щось* в рівняння	súbstitute *smth* into an equátion
78.понижение порядка диф-ференциального уравнения	зниження порядку диференціального рівняння	redúction of the órder of a dìfferéntial equátion
79.понизить порядок дифференциального уравнения	знизити порядок диференціального рівняння	redúce/lówer the órder of a dìfferéntial equátion
80.порядок (дифференциального уравнения, производной)	порядок (диференціального рівняння, похідної)	órder (of a dìfferéntial equátion, of a derívative)
81.последовательное интегрирование	послідовне інтегрування	succéssive [consé-cutive] ìntegrátion
82.принимать вид	набувати вигляду	take the form
83.приравнивать, отождествлять коэффициенты при одинаковых/равных степенях *x*	прирівнювати, ототожнювати коефіцієнти при однакових/рівних степенях *x*	equáte/set équal/idéntify the còeffícients of équal (of the same, in like) degrées/pówers of *x*
84.продифференцировать	(про)диференціювати	dìfferéntiate
85.произвольная постоянная	довільна стала	árbitrary cónstant
86.простой корень	простий корінь	símple [single] root
87.проходить через данную точку	проходити через дану точку	pass thróugh a gíven póint
88.равняться [быть равным] нулю тождественно	дорівнювати нулю тотожньо	be idéntically [be équal idéntically, be équal idéntically to] zéro
89.разделить переменные	відокремити змінні	séparate váriables
90.разрешить дифференциальное уравнение относительно (старшей) производной	розв”язати диференціальне рівняння відносно (старшої) похідної	resólve/sólve a dìfferéntial equátion for [with respect to] the (hígher) derívative
91.рассматривать, трактовать, интерпретировать *c* как неизвестную функцию (c_1, c_2, … как неизвестные функции)	розглядати, трактувати, інтерпретувати *c* як невідому функцію (c_1, c_2, … як невідомі функції)	consíder [tréat, intérp-ret] *c* as an ùnknówn fúnction (c_1, c_2, …as ùnknówn fúnctions)
92.решение в неявном виде	розв”язок в неявному вигляді	implícit solútion
93.решение дифферен-циа-льного уравнения	розв”язок диференціального рівняння	solútion of a differ-réntial equátion
94.решить *что-либо*	розв”язати *щось*	sólve/resólve *smth*
95.свести дифференци-	звести диференціальне	redúce a dìfferéntial

альное уравнение к нор-ма-льной системе	рівняння до нормальної системи	equátion to a nórmal sýs-tem
96.свести нормальную систему к одному диф-ференциальному уравне-нию	звести нормальну систе-му до одного диференці-ального рівняння	redúce a nórmal sýs-tem to one dìfferéntial equátion
97.свестись к *чему-л.*	звестись до *чогось*	be redúced to *smth*,
98.свободный член	вільний член	right-hand mémber [ábso-lute term, free term]
99.свободный член спе-циального вида (специ-альной формы)	вільний член спеціально-го вигляду (спеціальної форми)	right-hand mémber of a spécial/specífic form
100. свойство решений	властивість розв"язків	próperty of solútions
101. семейство инте-гральных кривых, зави-сящее от од-ного пара-метра, от двух парамет-ров	сім"я інтегральних кри-вих, яка залежить від од-ного параметра, від двох параметрів	fámily of íntegral cur-ves depénding on a single parámeter, on two paráme-ters
102. система дифферен-циа-льных уравнений	система диференціаль-них рівнянь	sýstem of dìfferén-tial equátions
103. система линейных уравнений относительно неопределённых коэф-фициентов	система лінійних рівнянь відносно невизначених коефіцієнтів	sýstem of linear equátions in [to detérmine] úndetér-mined còeffícients
104. содержать произ-водную или дифферен-циал искомой функции	містити похідну чи ди-ференціал шуканої фун-кції	contáin a derívative or a dìfferéntial of a desíred/ únknówn/sought fúnction
105. соответствующее [присоединённое] линей-ное однородное диффе-ренциальное уравнение	відповідне [приєднане] лінійне однорідне дифе-ренціальне рівняння	còrrespónding/adjoint/ assóciated hòmogéneous línear dìfferéntial equátion [línear dìfferéntial equátion withóut a right-hand member]
106. специальный вид свободного члена	спеціальний вигляд віль-ного члену	spécial/specífic form of a right-hand mém-ber
107. старшая производ-ная	старша походна	hígher derívative
108. структура общего решения	структура загального розв"язку	strúcture of the géneral solútion
109. существование	існування	exístence
110. существовать	існувати	exíst
111. считать c_1, c_2 неиз-вестными функциями	вважати c_1, c_2 невідоми-ми функціями	suppóse [think] that c_1, c_2 are/be ùnknówn fúnctions
112. теорема существо-	теорема існування і єди-	théorem on the exí-stence

вания и единственности решения (задачи коши)	ності розв"язку (задачі коші)	and uníqueness/úni-city [on the uníque exís-tence] of the solútion [thé-orem on the one-válued sòl-vabílity] (of cauchy´s próblem)
113. тип дифференци-ального уравнения	тип диференціального рівняння	type of a differen-tial equátion
114. угловой коэффици-ент касательной	кутовий коефіцієнт до-тичної	ángular còeffícient [slópe] of a tángent
115. удовлетворять *че-му-либо*	задовольняти *щось*	sátisfy *smth*
116. уравнение бернул-ли	рівняння бернуллі	bernoulli(´s) equátion
117. учесть *что-л.*	врахувати *щось*	take *smth* ínto con-siderátion [ínto accóunt]
118. функция от отно-шения переменных	функція від відношення змінних	fúnction of the rátio of váriables
119. функция, завися-щая от ...	функція, яка залежить від ...	fúnction depéning on ..
120. характеристическое ура-внение	характеристичне рівнян-ня	chàracterístic equátion
121. частная производ-ная по ...	частинна походна по...	pártial derívative with re-spéct to ...
122. частное значение произвольной постоян-ной	частинне значення дові-льної сталої	partícular value of an árbi-trary cónstant
123. частное решение	частинний розв"язок	partícular solútion
124. частное решение, линей-но независимое от (найденного решения)	частинний розв"язок, лі-нійно незалежний від (знайденого розв"язку)	partícular solútion which is línearly ìndepén-dent of (the found solútion)
125. эн-кратный (*n*-кратный) корень	ен-кратний (*n*-кратний) корінь	*n*-th root, *n*-tuple root

DIFFERENTIAL EQUATIONS: Basic Terminology ERU

1. addítional condítion	дополнительное условие	додаткова умова
2. admít/allów/permít redúction of the órder	допускать понижение порядка	допускати зниження порядку
3. ángular còeffícient [slópe] of a tángent	угловой коэффициент касательной	кутовий коефіцієнт дотичної
4. ántiderívative/prímitive	первообразная	первісна
5. árbitrary cónstant	произвольная постоянная	довільна стала
6. be (re)sólved for [with respect to] the (hígher) derívative [be nórmalized]	быть разрешённым относительно (производной, старшей производной)	бути розв”язаним відносно (похідної, старшої похідної)
7. be idéntically [be équal idéntically, be équal idéntically to] zéro	равняться [быть равным] нулю тождественно	дорівнювати нулю тотожньо
8. be redúced to *smth*,	свестись к *чему-л.*	звестись до *чогось*
9. be séparated	быть разделёнными	бути відокремленими
10.bernoulli(´s) equátion	уравнение бернулли	рівняння бернуллі
11.bernoulli(´s) méthod	метод бернулли	метод бернуллі
12.bóundary condítion	краевое [граничное] условие	крайова [гранична] умова
13.cauchy´s próblem	задача коши	задача коші
14.chàracterístic equátion	характеристическое уравнение	характеристичне рівняння
15.còeffícient	коэффициент	коефіцієнт
16.còeffícients of équal [of the same] degrées/pó-wers (in like pówers) of x	коэффициенты при одинаковых степенях x	коефіцієнти при однакових степенях x
17.consíder [tréat, intérpret] c as an ùnknówn fúnction (c_1, c_2, …as ùnknówn fúnctions)	рассматривать, трактовать, интерпретировать c как неизвестную функцию (c_1, c_2, … как неизвестные функции)	розглядати, трактувати, інтерпретувати c як невідому функцію (c_1, c_2, … як невідомі функції)
18.contáin a derívative or a dìfferéntial of a desíred/ únknówn/sought fúnction	содержать производную или дифференциал искомой функции	містити похідну чи диференціал шуканої функції
19.convért [transfórm, turn] a dìfferéntial equátion into idéntity	обращать [превращать] дифференциальное уравнение в тождество	перетворювати диференціальне рівняння у тотожність
20.còrrespónding/adjoint/ assóciated hòmogéneous línear dìfferéntial equátion	соответствующее [присоединённое] линейное однородное дифферен-	відповідне [приєднане] лінійне однорідне диференціальне рівняння

[línear dìfferéntial equátion withóut a right-hand member]	циальное уравнение	
21.depénd on/upon …	зависеть от …	залежати від ...
22.detérmine the type of the dìfferéntial equátion	определить тип дифференциального уравнения	визначити тип диференціального рівняння
23.dìfferéntial equátion	диффереციальное уравнение	диференціальне рівняння
24.dìfferéntial equátion (re)sólved for [with respect to] the (hígher) derívative	диффереciальное уравнение, разрешённое относительно (старшей) производной	диференціальне рівняння, розв”язане відносно (старшої) похідної
25.dìfferéntial equátion in an únknówn fúnction $y(x)$	диффереciальное уравнение относительно искомой функции $y(x)$	диференціальне рівняння відносно шуканої функції $y(x)$
26.dìfferéntial equátion in/with séparable váriables [séparable (dìfferéntial) equátion]	диффереciальное уравнение с разделяемыми переменными	диференціальне рівняння з відокремлюваними змінними
27.dìfferéntial equátion in/with séparated váriables [séparated (dìfferéntial) equátion]	диффереciальное уравнение с разделёнными переменными	диференціальне рівняння з відокремленими змінними
28.dìfferéntial equátion not contáining explícitly an únknówn fúnction [an ìndepéndent váriable]	диффереciальное уравнение, не содержащее явно искомую функцию [независимую переменную]	диференціальне рівняння, яке не містить явно шукану функцію [незалежну змінну]
29.dìfferéntial equátion of the first/second/*n*-th/hígher order	диффереciальное уравнение первого, второго, *n*-го, высшего порядка	диференціальне рівняння першого, другого, *n*-го, вищого порядку
30.dìfferéntiate	(про)дифференцировать	(про)диференціювати
31.do not contáin explícitly (an ìndepéndent váriable, a(n) desired/únknówn/sought fúnction)	не содержать явно (независимую переменную, искомую функцию)	не містити явно (незалежну змінну, шукану функцію)
32.domáin [région]	область	область
33.double [repéated] root	двукратный корень	двократний корінь
34.equáte/set équal/idéntify the còeffícients of équal (of the same, in like) degrées/pówers of x	приравнивать, отождествлять коэффициенты при одинаковых/равных степенях x	прирівнювати, ототожнювати коефіцієнти при однакових/рівних степенях x
35.exíst	существовать	існувати
36.exístence	существование	існування

37. fámily of íntegral curves depénding on a single parámeter, on two parámeters	семейство интегральных кривых, зависящее от одного параметра, от двух параметров	сім”я інтегральних кривих, яка залежить від одного параметра, від двох параметрів
38. find (the géneral solútion, the solútion of cauchy´s próblem)	найти (общее решение, решение задачи коши)	знайти (загальний розв”язок, розв”язок задачі коші)
39. find [search, look for, look up, seek] a solútion in the form	искать решение в форме	шукати розв”язок у вигляді
40. fúnction depéning on ..	функция, зависящая от ...	функція, яка залежить від ...
41. fúnction of the rátio of váriables	функция от отношения переменных	функція від відношення змінних
42. géneral form	общий вид	загальний вигляд
43. géneral solútion	общее решение	загальний розв”язок
44. geométric(al) sense [méaning]	геометрический смысл	геометричний сенс
45. graph of a solútion	график решения	графік розв”язку
46. hígher derívative	старшая производная	старша походна
47. híghest órder of the derívatives of an únknówn [desíred, sought, requíred] fúnction contáining in a dìfferéntial equátion	наивысший порядок производных искомой функции, которые содержатся в дифференциальном уравнении	найвищий порядок похідних шуканої функції, які містяться в диференціальному рівнянні
48. hòmogéneous differential equátion	однородное дифференциальное уравнение	однорідне диференціальне рівняння
49. imáginary part	мнимая часть	уявна частина
50. implícit solútion	решение в неявном виде	розв”язок в неявному вигляді
51. ìndepéndent váriable	независимая переменная	незалежна змінна
52. inítial condítion [cauchy(´s) condítion]	начальное условие [условие коши]	початкова умова [умова коші]
53. inítial válue próblem [cauchy(´s) próblem]	начальная задача [задача коши]	початкова задача [задача коші]
54. íntegrable type	интегрируемый тип	інтеґровний тип
55. íntegral curve	интегральная кривая	інтегральна крива
56. íntegrate a dìfferéntial equátion	интегрировать дифференциальное уравнение	інтегрувати диференціальне рівняння
57. ìntegrátion of a dìfferéntial equátion (in quádratures)	интегрирование дифференциального уравнения (в квадратурах)	інтегрування диференціального рівняння (в квадратурах)
58. ìntrodúce a new fúnction	ввести новую функцию	ввести нову функцію

59.ìntrodúction a new fúnction	введение новой функции	введення нової функції
60.linear hòmogéneous dìfferéntial equátion, línear dìfferéntial equátion withóut a right-hand member (lhde)	линейное однородное дифференциальное уравнение (лоду)	лінійне однорідне диференціальне рівняння (лодр)
61.línear depéndence	линейная зависимость	лінійна залежність
62.línear dìfferéntial equátion (lde)	линейное дифференциальное уравнение (лду)	лінійне диференціальне рівняння (лдр)
63.linear dìfferéntial equátion with cónstant còeffícients	линейное дифференциальное уравнение с постоянными коэффициентами	лінійне диференціальне рівняння з сталими коефіцієнтами
64.linear dìfferéntial óperator	линейный дифференциальный оператор	лінійний диференціальний оператор
65.línear ìndepéndence	линейная независимость	лінійна незалежність
66.línear nónhòmogéneous dìfferéntial equátion, línear dìfferéntial equátion with a right-hand mémber (lnde)	линейное неоднородное дифференциальное уравнение (лнду)	лінійне неоднорідне диференціальне рівняння (лндр)
67.línearly (ìn)depéndent functions, solutions	линейно зависимые (независимые) функции, решения	лінійно залежні (незалежні) функції, розв”язки
68.méthod of elìminátion [elìminátion méthod]	метод исключения	метод виключення
69.méthod of úndetérmined còeffícients	метод неопределённых коэффициентов	метод невизначених коефіцієнтів
70.méthod of vàriátion of of árbitrary constants	метод вариации произвольных постоянных	метод варіації довільних сталих
71.múltiple [repéated] root	кратный корень	кратний корінь
72.mùltiplícity of the root of the chàracterístic equátion	кратность корня характеристического уравнения	кратність кореня характеристичного рівняння
73.nécessary and sufficient condítion for …	необходимое и достаточное условие чего	необхідна й достатня умова чогось
74.nórmal sýstem of dìfferéntial equátions	нормальная система дифференциальных уравнений	нормальна система диференціальних рівнянь
75.not to be unique	не быть единственным	не бути єдиним
76.not to demánd/requíre ìntegrátion	не требовать интегрирования	не вимагати інтегрування
77.n-th root, n-tuple root	эн-кратный (n-кратный)	ен-кратний (n-кратний)

	корень	корінь
78.órder (of a dìfferéntial equátion, of a derívative)	порядок (дифференциального уравнения, производной)	порядок (диференціального рівняння, похідної)
79.órder redúcing differéntial equátion	диффereциальное уравнение, допускающее понижение порядка	диференціальне рівняння, яку допускає зниження порядку
80.pártial derívative with respéct to ...	частная производная по ...	частинна походна по...
81.partícular solútion	частное решение	частинний розв"язок
82.partícular solútion which is línearly ìndepéndent of (the found solútion)	частное решение, линейно независимое от (найденного решения)	частинний розв"язок, лінійно незалежний від (знайденого розв"язку)
83.partícular value of an árbitrary cónstant	частное значение произвольной постоянной	частинне значення довільної сталої
84.pass thróugh a gíven póint	проходить через данную точку	проходити через дану точку
85.pòlynómial	многочлен	многочлен
86.pòlynómial of the same degrée as … with úndetérmined còeffícients	многочлен той же степени, что и …, с неопределёнными коэффициентами	многочлен того ж степеня, що й ..., з невизначеними коефіцієнтами
87.próperty of solútions	свойство решений	властивість розв"язків
88.quadrátic equátion; quadrátic	квадратное уравнение	квадратне рівняння
89.quádrature	квадратура	квадратура
90.quási-pòlynómial	квазиполином	квазіполіном
91.rátio of *smth* is(n´t) a cónstant quántity	отношение *чего-либо* (не) является постоянной величиной	відношення *чогось* (не) є сталою величиною
92.rátio (of váriables, of two fúnctions, of two solútions)	отношение (переменных, двух функций, двух решений)	відношення (змінних, двох функцій, двох розв"язків)
93.réal and distínct roots of the chàracterístic equá-tion	вещественные и различные корни характеристического уравнения	дійсні й різні корені характеристичного рівняння
94.réal and équal roots of the characteristic equátion	вещественные и равные корни характеристического уравнения	дійсні й рівні корені характеристичного рівняння
95.réal part	вещественная часть	дійсна частина
96.redúce a dìfferéntial equátion to a nórmal sýstem	свести дифференциальное уравнение к нормальной системе	звести диференціальне рівняння до нормальної системи
97.redúce a nórmal sýs-	свести нормальную сис-	звести нормальну систе-

tem to one dìfferéntial equátion	тему к одному дифференциальному уравнению	му до одного диференціального рівняння
*98.*redúce/lówer the órder of a dìfferéntial equátion	понизить порядок дифференциального уравнения	знизити порядок диференціального рівняння
99.redúction of the órder of a dìfferéntial equátion	понижение порядка дифференциального уравнения	зниження порядку диференціального рівняння
100. resólve/sólve a dìfferéntial equátion for [with respect to] the (hígher) derívative	разрешить дифференциальное уравнение относительно (старшей) производной	розв’’язати диференціальне рівняння відносно (старшої) похідної
101. right-hand mémber [ábsolute term, free term]	свободный член	вільний член
102. right-hand mémber of a spécial/specífic form	свободный член специального вида (специальной формы)	вільний член спеціального вигляду (спеціальної форми)
103. root of a characterístic equátion	корень характеристического уравнения	корінь характеристичного рівняння
104. sátisfy *smth*	удовлетворять *чему-либо*	задовольняти *щось*
105. séparate váriables	разделить переменные	відокремити змінні
106. símple [single] root	простой корень	простий корінь
107. solútion of a differéntial equátion	решение дифференциального уравнения	розв’’язок диференціального рівняння
108. sólve/resólve *smth*	решить *что-либо*	розв’’язати *щось*
109. spécial/specífic form of a right-hand mém-ber	специальный вид свободного члена	спеціальний вигляд вільного члену
110. strúcture of the géneral solútion	структура общего решения	структура загального розв’’язку
111. súbstitute *smth* into an equátion	подставить *что-либо* в уравнение	підставити *щось* в рівняння
112. succéssive [consécutive] ìntegrátion	последовательное интегрирование	послідовне інтегрування
113. suppóse [think] that c_1, c_2 are/be ùnknówn fúnctions	считать c_1, c_2 неизвестными функциями	вважати c_1, c_2 невідомими функціями
114. sýstem of dìfferéntial equátions	система дифференциальных уравнений	система диференціальних рівнянь
115. sýstem of linear equátions in [to detérmine] úndetérmined còefficients	система линейных уравнений относительно неопределённых коэффициентов	система лінійних рівнянь відносно невизначених коефіцієнтів
116. take *smth* ínto con-	учесть *что-л.*	врахувати *щось*

sideration [ínto accóunt]		
117. take the form	принимать вид	набувати вигляду
118. théorem on the exístence and uníqueness/únicity [on the uníque exístence] of the solútion [théorem on the one-válued sòlvabílity] (of cauchy´s próblem)	теорема существования и единственности решения (задачи коши)	теорема існування і єдиності розв”язку (задачі коші)
119. turn [transfórm, revért] into idéntity	обращаться в тождество	перетворюватися в тотожність
120. type of a differential equátion	тип дифференциального уравнения	тип диференціального рівняння
121. uníque [(one and) only one]	единственный	єдиний
122. únknówn [desíred, sought, requíred] fúnction	искомая функция	шукана функція
123. wronski´s detérminant	определитель вронского	визначник вронського
124. wrónskian	вронскиан (определитель вронского)	вронскіан (визначник вронського)
125. cómplex root of a chàracterístic equátion	комплексный корень характеристического уравнения	комплексний корінь характеристичного рівняння

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