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ДОНЕЦЬКИЙ НАЦІОНАЛЬНИЙ ТЕХНІЧНИЙ УНІВЕРСИТЕТ

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INTRODUCTION IN MATHEMATICAL ANALYSIS
DIFFERENTIAL CALCULUS
ВСТУП ДО МАТЕМАТИЧНОГО АНАЛІЗУ
ДИФЕРЕНЦІАЛЬНЕ ЧИСЛЕННЯ

Методичний посібник по вивченню розділу курсу
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для студентів ДонНТУ
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Викладаються основні поняття теорії границь, теорії неперервності та диференціального числення функцій однієї та декількох змінних. Вивчаються застосування диференціального числення до дослідження функцій, в тому числі локальні та абсолютні екстремуми, а для функцій некількох змінних – умовні екстремуми. Докладно розглядаються приклади розв’язання типових задач. Вміщено англо-українсько-російський термінологічний словник. Дано завдання для самостійного розв’язання.

Велику допомогу в створенні посібника надали автору студенти факультету економіки і менеджменту ДонНТУ Мамічева В., Маринова К., Бородина Ю., Костюк О., Полєнок Т., Бердянська В., Фролофф Г. (впорядкування лекційних конспектів, редагування англомовного тексту, робота над термінологічним словником). Слід особливо відзначити роботу Галі Фролофф, яка ретельно перевірила всі математичні викладки, повторно розв”язала всі приклади і допомогла значно покращити текст посібника. Суттєвий внесок в написання посібника внесла старший викладач Слов”янського педагогічного університету Косолапова Н. В. (підготовка ілюстративного матеріалу, робота над англо-українсько-російським термінологічним словником). Всім своїм помічникам автор висловлює щиру подяку.

Для студентів і викладачів технічних вузів.

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MATHEMATICAL ANALYSIS

INTRODUCTION IN MATHEMATICAL ANALYSIS

LECTURE NO.12. LIMIT OF A FUNCTION

POINT 1.FUNCTION (ADDITIONAL REMARKS).

POINT 2. LIMIT. INFINITELY SMALL AND INFINITELY LARGE.

POINT 3. PROPERTIES OF LIMITS.

POINT 4. REMARKABLE [STANDARD] LIMITS.

POINT 5. INTERESTS IN INVESTMENTS.

POINT 1. FUNCTION (ADDITIONAL REMARKS).

Def.1 a) Set of all real numbers $\Re = \Re^1 = (-\infty, \infty)$ (Ox-axis), b) Ox_1x_2-plane ($\Re^2$), c) $Ox_1x_2x_3$ - space ($\Re^3$) are called correspondingly a) one-dimensional, b) two-dimensional, c) three-dimensional space.

Correspondingly points a) $x \in \Re^1$; b) $x = (x_1, x_2) \in \Re^2$; c) $x = (x_1, x_2, x_3) \in \Re^3$ are called a) one-dimensional, b) two-dimensional, c) three-dimensional points.

Def. 2. n-dimensional space $\Re^n$ is called the set of all n-dimensional points $x = (x_1, x_2, ..., x_n)$.

Def. 3. Distance between two points $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ of $\Re^n$ is called the next expression

$$\rho(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}.$$

Theorem 1. For any three points x, y, z of $\Re^n$

$$\rho(x, y) \le \rho(x, z) + \rho(z, y) \text{ (triangle inequality).}$$

Def. 4. Function *y=f(x)* with domain of definition $D(f) \in \Re^n$ and set of values $E(f) \subseteq \Re$ is called a mapping of *D(f)* onto *E(f)* that is some rule which puts in correspondence a certain (unique) number $y \in E(f) \subseteq \Re$ to every point $x \in D(f)$.

For n = 1, 2, 3, …, n we have a function of one, two, three, …, n variables $y = f(x)$, $y = f(x) = f(x_1, x_2)$, $y = f(x) = f(x_1, x_2, x_3)$, $y = f(x) = f(x_1, x_2, ..., x_n)$.

Def. 5. $f(x)$ is called the value of a function at a point x.

Ex. 1. **Number [numerical] sequence** (function of a natural argument). Let $D(f) = \aleph = \{1, 2, 3, ..., n, ...\}$ and $y_1 = f(1), y_2 = f(2), y_3 = f(3), ..., y_n = f(n), ...,$ or briefly $\{y_n = f(n)\}$. Values of function form a number sequence with general term $y_n = f(n)$.

Ways of definition of a function:

1. **Analytical way**: with the help of some formula

$$Ex.\ y = x^2,\quad y = x_1^2 + x_2^2,\quad y = x_1^2 + x_2^2 + x_3^2.$$

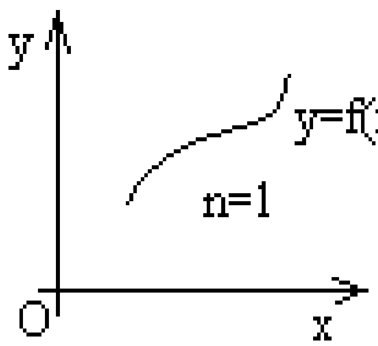


Fig. 1

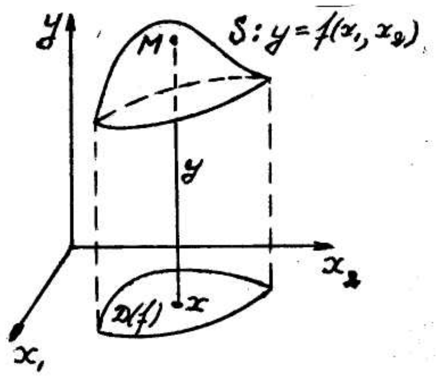


Fig. 2

2.**Graphical (geometrical) way** (for n = 1, 2): with the help of some graph(ic).

All is clear for n = 1 (see fig. 1).

Let n = 2 that is we deal with a function of two variables $y = f(x) = f(x_1, x_2)$. Then for any point $x = (x_1, x_2) \in D(f)$ we get a corresponding point $M(x_1, x_2, y)$, $y = f(x_1, x_2)$ of the space Ox_1x_2y. Set of all such the points M forms some surface S which is called the graph of the function (fig. 2).

A function of two variables $y = f(x) = f(x_1, x_2)$ can be geometrically represented with the help of so-called **level lines** [level curves, equiscalar lines] that is lines along which this function takes on constant values,

$$f(x_1, x_2) = C, C - const.$$

It's obvious that for every C a level line is the projection of the intersection line of the graph of the function $y = f(x) = f(x_1, x_2)$ and the plane $z = C$ onto the x_1Ox_2-plane.

Ex.2. Level lines of the function

$$y = f(x_1, x_2) = x_1^2 + x_2^2$$

are determined by the equation

$$x_1^2 + x_2^2 = C; C \geq 0.$$

For $C = 0$ we have $x_1 = x_2 = 0$ that is a point $O(0,0)$. If $C > 0$, the level lines are circles centered at the origin $O(0,0)$ with radii $R = \sqrt{C}$.

A function of three variables $y = f(x) = f(x_1, x_2, x_3)$ doesn't possess a graph but can be geometrically represented by **level surfaces** that is surfaces along every of them the function has a constant value, that is

$$f(x_1, x_2, x_3) = C, C - const.$$

Ex. 3. Level surfaces of the function

$$y = f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

are represented by the equation

$$x_1^2 + x_2^2 + x_3^2 = C; C \geq 0.$$

For $C = 0$ the level surface degenerates into a point $O(0,0,0)$, and for $C > 0$ the level surfaces are spheres centered at the origin $O(0,0,0)$ with radii $R = \sqrt{C}$.

3. **Tabular way** (for $n = 1, 2$): with the help of some table.

For $n = 1$ see for example tables of trigonometrical functions, of logarithms etc. There are double entry tables [two-input tables] for $n = 2$, three-entry tables [three-input tables] for $n = 3$ etc.

4. **Description way** (with the help of some description).

Ex. 4. Trigonometric functions of an arbitrary real argument were defined (see Lecture No. 2) by description with the help of trigonometric circle.

5. **Algorithmic way** (with the help of a program for a computer).

Def. 6. Basic elementary functions (of one variable) are called the next functions:

1) constant function $y = f(x) = C$, C -const;

2) power function

$$y = x^{\alpha}, \ \alpha \in \Re^1;$$

3) exponentional function

$$y = a^x, \ 0 < a \neq 1, \text{ in particular } y = e^x,$$

where , $e \approx 2.71828...$ is Euler number;

4) logarithmic function

$$y = \log_a x, \text{ in particular } y = \ln x = \log_e x;$$

5) trigonometrical functions

$$y = \sin x, \ y = \cos x, \ y = \tan x, \ y = \cot x;$$

6) inverse trigonometrical functions

$$y = \arcsin x, \ y = \arccos x, \ y = \arctan x, \ y = arc\cot x.$$

Def 7 (composite function). Let $y = f(u), \ u = \varphi(x)$ be two functions of one variable, and $E(\varphi) \subseteq D(f)$. A function $y = f(\varphi(x))$ is called a **composite** one [a function of a function, a **superposition** of functions *f* and φ].

Note. For functions of several variables a composite function can be defined analogously, for example a composite function of three variables

$$y = f(\varphi_1(x_1, x_2, x_3), \ \varphi_2(x_1, x_2, x_3))$$

where

$$y = f(u) = f(u_1, u_2), u_1 = \varphi_1(x) = \varphi_1(x_1, x_2, x_3), u_2 = \varphi_2(x) = \varphi_2(x_1, x_2, x_3),$$
$$u = (u_1, u_2) \in \Re^2, \ x = (x_1, x_2, x_3) \in \Re^3$$

Def. 8 (elementary function). A function $y = f(x)$ of one variable $x \in \Re^1$ is called that **elementary** if it is a basic elementary one or can be represented as result of finite number of arithmetical operations (addition, subtraction, multiplication, division) and superpositions on basic elementary functions.

Ex. 5. *n*-th degree polynomial (of one variable $x \in \Re^1$)

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n, \ a_n \neq 0$$

Ex. 6. A rational fraction (of $x \in \Re^1$) is called a ratio of two polynomials

$$R(x) = \frac{Q_m(x)}{P_n(x)}, \ Q_m(x) = b_0 + b_1 x + b_2 x^2 + ... + b_m x^m$$

The fraction is called proper one if $m < n$ and improper otherwise ($m \ge n$).

Def. 9. Let $a \in \Re^1$. A **neighbourhood** U_a of the point a is called every interval which contains this point. Specifically an interval $U_{a,\varepsilon} = (a-\varepsilon, a+\varepsilon)$, which is defined by the inequality $|x-a| < \varepsilon$, is called the ε - neighbourhood of the point a.

Def. 10. A **deleted neighbourhood** U'_a of the point $a \in \Re^1$ is called its neighbourhood U_a without this point: $U'_a = U_a \setminus \{a\}$. In particular the deleted ε-neighbourhood $U'_{a,\varepsilon}$ of the point a is the union of two intervals: $U'_{a,\varepsilon} = (a-\varepsilon, a) \bigcup (a, a+\varepsilon)$.

Analogous definitions can be stated in n-dimensional space for any n. We'll limit ourselves by the case $n = 2$ that is by the case $\Re^2$ (the plane x_1Ox_2).

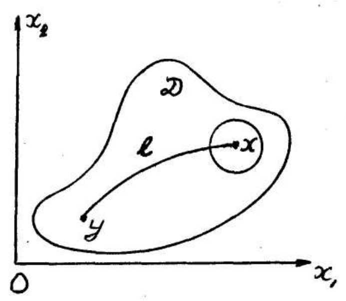


Fig. 3

Def. 11. A **domain** on the plane is called a point set $D \subseteq \Re^2$ satisfying two conditions: 1) every point $x = (x_1, x_2)$ of D belongs to D with some circle centered at $x = (x_1, x_2)$; 2) every two points $x = (x_1, x_2)$, $y = (y_1, y_2)$ of D can be joined by some line l which entirely lies in D ($l \subset D$) (fig. 3).

Ex. 7. Open circle $K(a, R)$ of radius R, centered at a point $a = (a_1, a_2)$ (a circle without its boundary that is the circumference $S(a, R)$).

By analogy with definitions 9, 10 we can state

Def. 12. A **neighbourhood** U_a of the point $a = (a_1, a_2) \in \Re^2$ is called every domain containing this point (for example an open circle $K(a, R)$).

Def. 13. A **deleted neighbourhood** U'_a of the point $a = (a_1, a_2) \in \Re^2$ is called its neighbourhood U_a without the point a that is the set $U'_a = U_a \setminus \{a\}$ (for example deleted circle $K'(a, R) = K(a, R) \setminus \{a\}$).

Many functions (of one and several variables) are studied in economics: production (виробнича) function, productive (продуктивна) function, profit [return] function (функція прибутку), cost function (функція витрат, функція вартості), demand function (функція попиту), supply function (функція пропозиції), payoff function (функція виграшу), utility function (функція корисності), loss function [expenditure function] (функція втрат), risk function (функція ризику), damage function (функція збитків), effectiveness function (функція ефективності), Cobb-Douglas function (функція Кобба-Дугласа), insolvency function (функція банк-рутства), loss-of-utility function (функція втрати корисності), preference function (функція переваги (предпочтения)), propositional function (пропозиційна функ-ція) etc.

POINT 2. LIMIT. INFINITELY SMALL AND INFINITELY LARGE

A. Limit of a function at a point.

We'll begin by the next example.

Ex. 8. Let there be given a function (fig. 4)

$$f(x) = \frac{x^2 - 9}{x - 3}$$

with domain of definition $D(f) = (-\infty, 3) \cup (3, \infty)$, and let x tend to the number 3 ($x \to 3$). We see (table 1) that the values of the function tend to 6, $f(x) \to 6$, as $x \to 3$. This fact is usually fixed by the next notations

$$\lim_{x \to 3} f(x) = 6, \quad f(x) \to 6 \ as\ x \to 3,$$

but it requires exact definition.

Table 1

x	2.94	2.96	2.98	3	3.02	3.04	3.06
$y = f(x)$	5.94	5.96	5.98	Doesn't exist	6.02	6.04	6.06
$\lvert f(x) - 6 \rvert$	0.06	0.04	0.02		0.02	0.04	0.06

Let $x \neq 3$ and ε be arbitrary number, which is positive and however small. We study the modulus of difference between values of the function and the number 6 and we have

$$|f(x)-6| = \left|\frac{x^2-9}{x-3}-6\right| = |(x+3)-6| = |x-3| < \varepsilon \ \ if \ -\varepsilon < x-3 < \varepsilon,$$

$$3-\varepsilon < x < 3+\varepsilon, \ x \in (3-\varepsilon, 3+\varepsilon), \ x \neq 3 \ or \ x \in (3-\varepsilon, 3) \cup (3, 3+\varepsilon).$$

For example $|f(x)-6| < 0.01 \, (\varepsilon = 0.01)$ if $x \in (2.99, 3) \cup (3, 3.01)$; $|f(x)-6| < 0.001$ $\varepsilon = 0.001)$ if $x \in (2.999, 3) \cup (3, 3.001)$ and $x \neq 3$.

Thus for any positive however small number ε there exists a neighbourhood of the point x = 3, that is the interval $U_3 = (3-\varepsilon, 3+\varepsilon)$ (on the fig. 4 $U_3 = (m, n)$), such that for any $x \in D(f)$, if x reaches deleted neighbourhood of the point x = 3, that is $U_3' = (3-\varepsilon, 3+\varepsilon) \setminus \{3\}$ $= (3-\varepsilon, 3) \cup (3, 3+\varepsilon) = (m, 3) \cup (3, n)$, then the inequality $|f(x)-6| < \varepsilon$ holds. Symbolically:

$$\forall \varepsilon > 0, \exists U_3 = (3-\varepsilon, 3+\varepsilon), \forall x \in D(f): (x \in U_3' = (3-\varepsilon, 3) \cup (3, 3+\varepsilon) \Rightarrow |f(x)-6| < \varepsilon)$$

It is exact definition of the fact that the **limit** of our function, as x tends to 3, equals 6 or, which is the same, that the function **tends** to 6 as its argument x tends to 3.

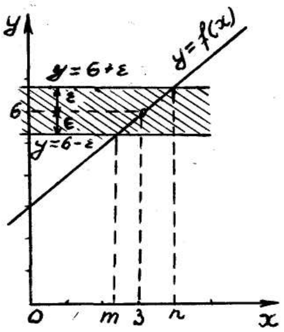


Fig. 4

The inequality $|f(x)-6| < \varepsilon$ is equivalent to the next one $6-\varepsilon < f(x) < 6+\varepsilon$, so we can state geometric sense of given defini-tion of the fact that $\lim\limits_{x \to 3} f(x) = 6$ (see fig. 4). Namely, if x belongs to the deleted neighbourhood $U_3' = (m, 3) \cup (3, n)$ of the point x = 3, then corresponding part of the graph of the function $f(x)$ lies in the hatched 2ε-strip bounded by the straight lines

$$y = 6-\varepsilon, \ y = 6+\varepsilon.$$

On the base of studied example we are able to state general definition of the li-mit of a function $y = f(x)$ as x tends [goes] to some point a (or the limit of the func-tion $y = f(x)$ at the point x = a). A function can be dependent as on one as on n vari-ables.

Def. 14. A number b is called the limit of a function $y = f(x)$ as $x \to a$ (the limit of the function at the point a), $\lim\limits_{x \to a} f(x) = b$ or $f(x) \to b$ as $x \to a$, if for any positive however small number ε there exists some neighbourhood U_a of the point a

such that for any value x from the domain of definition $D(f)$ of the function, if x belongs to deleted neighbourhood U'_a of the point a then the inequality

$$|f(x)-b|<\varepsilon,$$

or, which is the same, the double inequality

$$b-\varepsilon<f(x)<b+\varepsilon,$$

holds.

Symbolically,

$$\lim_{x\to a} f(x)=b$$

if

$$\forall\varepsilon>0, \exists U_a, \forall x\in D(f): \left(x\in U'_a \Rightarrow |f(x)-b|<\varepsilon \Leftrightarrow (\, b-\varepsilon<f(x)<b+\varepsilon)\right).$$

Remarks.

1) A point a can belong or not belong to the domain of definition $D(f)$ of a function $y=f(x)$. That is why deleted neighbourhood U'_a of the point a is introduced in the definition of limit. It can be substituted by U_a if $a\in D(f)$.

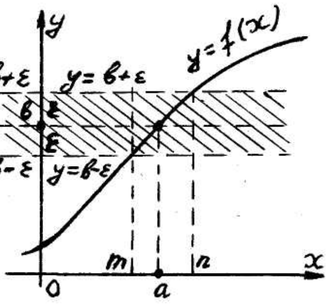


Fig. 5

2) In the case of function of several variables the definition of limit is stated under indispensable assumption that x can tend to a along arbitrary path which wholly lies in the domain of definition of a function.

3) In the case $n = 1$ that is for a function of one variable it's easy to give geometric sense of definition of the limit of the function at the point a (fig. 5). Namely for any $\varepsilon>0$ there exists a neighbourhood U_a of the point a (an interval (m, n) on the fig. 5) such that for all points $x\in D(f)$ of the deleted neighbourhood of the point a, namely $U'_a=(m, a)\cup(a, n)$, the corresponding part of the graph of the function lies in the hatched 2ε-strip between the lines $y=b-\varepsilon$ and $y=b+\varepsilon$.

Ex. 9. Prove that $\lim\limits_{x\to 2} x^2=4$.

Domain of definition of the function $y = f(x) = x^2$ is the set of all real numbers $\Re^1 = (-\infty, \infty)$. Behaviour of the function for $x \to 2$ is represented by a table 2.

Table 2

x	1.96	1.97	1.98	1.99	2.00	2.01	2.02	2.03	2.04
$y = x^2\ (\approx)$	3.84	3.88	3.92	3.96	4.00	4.04	4.08	4.12	4.16
$\lvert x^2 - 4\rvert$	0.16	0.12	0.08	0.04	0.00	0.04	0.08	0.12	0.16

Let $\varepsilon > 0$ be positive however small number. Then

$$|f(x) - 4| = |x^2 - 4| < \varepsilon \ \textit{if} \ -\varepsilon < x^2 - 4 < \varepsilon, 4 - \varepsilon < x^2 < 4 + \varepsilon, \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}.$$

Therefore for any $\varepsilon > 0$ it exists a neighbourhood of the point $x = 2$ namely $U_2 =$ $= (m, n) = \left(\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}\right)$ such that for all values of $x \in U_2$ the inequality $|x^2 - 4| < \varepsilon$ holds. By definition of limit and according to remark 1) we can write

$$\forall \varepsilon > 0, \exists U_2 = (\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon}), \forall x \in \Re : \left(x \in U_2 \Rightarrow |x^2 - 4| < \varepsilon\right), \ \lim_{x\to 2} x^2 = 4.$$

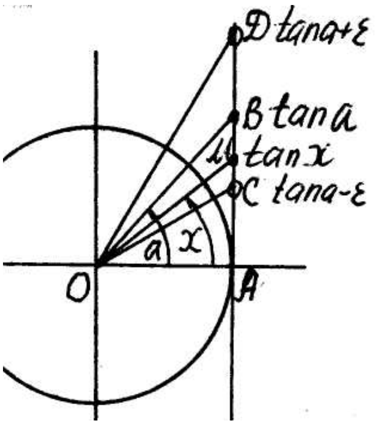


Fig. 6.

State yourselves geometric sense of studied passage to the limit.

Ex. 10. With the help of definition of tangent prove that $\lim_{x\to a} \tan x = \tan a$ for any $a \in (0, \pi/2)$.

■Let's mark three points $\tan a - \varepsilon, \tan a, \tan a + \varepsilon$ on the tangent line (points C, B, D correspondingly, fig. 6) and join these points with the centre O of the trigonometric circle. Let

$$\alpha = \angle AOC, \ a = \angle AOB, \beta = \angle AOD, x = \angle AOM \ \text{(fig. 6)}.$$

We get the next result (in symbolic form):

$$\forall \varepsilon > 0, \exists U_a = (\alpha, \beta), \forall x \in (0, \pi/2) : (x \in (\alpha, \beta) \Rightarrow \tan a - \varepsilon < \tan x < \tan a + \varepsilon$$
$$\textit{that is} \quad |\tan x - \tan a| < \varepsilon).$$

By definition of the limit $\lim_{x\to a} \tan x = \tan a$ ■

It's possible to extend this result on any $a \neq \pi/2, n = 0, \pm 1, \pm 2, \pm 3, \ldots$. Try to do it yourselves.

Prove yourselves with the help of definitions of sine, cosine, cotangent that

$$\lim_{x\to a}\sin x=\sin a;\ \lim_{x\to a}\cos x=\cos a;\ \lim_{x\to a}\cot x=\cot a\quad (a\neq \pi n,\ n=0,\pm 1,\pm 2,\pm 3,\dots).$$

Note. Examples 9, 10 and above-sited results as to *sin x, cos x, cot x* give us the first examples of functions possessing the property of the form

$$\lim_{x\to a} f(x)=f(a)$$

(limit of a function at a point *a* equals the value of the function at this point). There are very many functions of such kind (so-called **continuous** functions), for example all basic elementary and elementary functions. We'll especially study continuous functions in the next lecture, but here we'll apply continuity of elementary functions in simple cases.

Ex. 11. Prove that a function of two variables $f(x)=f(x_1, x_2)=\dfrac{x_1x_2}{x_1^2+x_2^2}$ doesn't possess limit at the origin $O(0, 0)$.

■It's sufficient to approach the origin along two different paths. Along the straight line $x_2=x_1$ one has

$$f(x_1, x_2)=f(x_1, x_1)=\frac{x_1x_1}{x_1^2+x_1^2}=\frac{1}{2}\ \text{and}\ \lim_{(x_1,x_2)\to(0,0)} f(x_1, x_2)=\lim_{x_1\to 0}\frac{x_1x_1}{x_1^2+x_1^2}=\frac{1}{2};$$

along the other straight line $x_2=2x_1$ one gets the other limit, for

$$f(x_1, x_2)=f(x_1, 2x_1)=\frac{x_1\cdot 2x_1}{x_1^2+(2x_1)^2}=\frac{2}{5}\ \text{and}\ \lim_{(x_1,x_2)\to(0,0)} f(x_1, x_2)=\lim_{x_1\to 0}\frac{x_1\cdot 2x_1}{x_1^2+(2x_1)^2}=\frac{2}{5}.$$

In accordance with remark 2) the limit of the function for $(x_1, x_2)\to(0,0)$ doesn't exist.■

We have defined the limit of a function $y=f(x)$ of one or several variables at a point *a*. There are some other types of passage to the limit. We'll briefly study them for a function of one variable $x\in D(f)\subseteq \Re^1$ (see **B, C, D**).

B. Unilateral limits of a function at a point

Let $x < a$ and $x \to a$. One says that *x* tends to *a* **from the left** and denotes this fact by the next way: $x \to a-0$. Corresponding limit b_1 of a function $y = f(x)$, if it exists, is called the **left limit** of the function at the point *a* and is denoted

$$b_1 = f(a-0) = \lim_{x \to a-0} f(x) \text{ (fig. 7).}$$

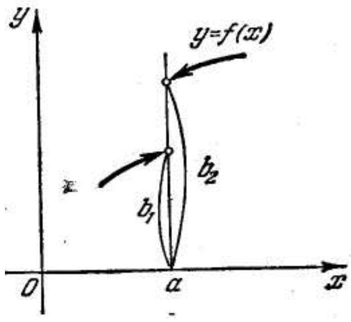


Fig. 7

Def. 15. A number b_1 is called the left limit of the function $y = f(x)$ at the point *a* (that is if *x* approaches *a* from the left) if (symbolically)

$$\forall \varepsilon > 0, \exists (m, a), \forall x \in D(f) : (x \in (m, a) \Rightarrow |f(x) - b_1| < \varepsilon).$$

In analogous way one says about tending of *x* to *a* **from the right** ($x > a$ and $x \to a$, $x \to a+0$) and the **right limit** b_2 of the function at the point *a*,

$$b_2 = f(a+0) = \lim_{x \to a+0} f(x) \text{ (fig. 7).}$$

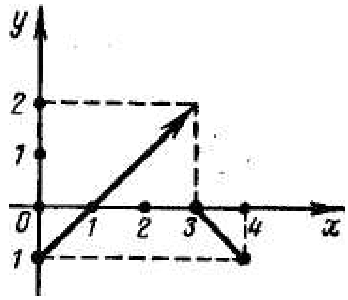


Fig. 8

Def. 16. A number b_2 is called the right limit of the function $y = f(x)$ at the point *a* (that is if *x* approaches *a* from the right) if

$$\forall \varepsilon > 0, \exists (a, n), \forall x \in D(f) : (x \in (a, n) \Rightarrow |f(x) - b_2| < \varepsilon).$$

Ex. 12. A function

$$f(x) = \begin{cases} x-1, & \text{if } 0 \le x < 3, \\ 3-x, & \text{if } 3 \le x \le 4 \end{cases}$$

has the left limit 2 and the right limit 0 at the point $x = 3$,

$$f(3-0) \lim_{x \to 3-0} f(x) = \lim_{x \to 3-0} (x-1) = 2, \ \lim_{x \to 3+0} f(x) = \lim_{x \to 3+0} (3-x) = 0 \text{ (see fig. 8).}$$

State yourselves geometric sense of right and left (unilateral) limits.

Theorem 2. Limit of a function of one variable at a point *a* exists if and only if left and right limits at this point exist and are equal,

$$\left(\exists \lim_{x \to a} f(x)\right) \Leftrightarrow \left(\exists f(a-0) = \lim_{x \to a-0} f(x), \exists f(a+0) = \lim_{x \to a+0} f(x), f(a-0) = f(a+0)\right)$$

■Validity of the theorem follows from the definitions 14 (for n = 1), 15, 16 ■

C. Limit of numerical sequence

Ex. 13. Let there be given number sequence

$$\left\{ x_n = \frac{2n+1}{3n-2} \right\}.$$

Its bihaviour is represented by a table 3

Table 3

n	10	10^2	10^3	10^6	10^9
x_n $(\approx)$	0.7500000	0.6744966	0.6674449	0.6666674	0.6666667
$\lvert x_n - 2/3 \rvert$ $(\approx)$	0.0833333	0.0078300	0.0007783	0.0000008	0.0000000

From the table we see that general term x_n of the sequence tends to 2/3 = 0.(6). We usually denote such the fact by the next way

$$\lim_{n\to\infty} \frac{2n+1}{3n-2} = \frac{2}{3}$$

and say that the sequence $\{x_n\}$ tends ("converges", "is convergent") to 2/3.

To express exactly this fact let's determine for which values of n the inequality

$$\lvert x_n - 2/3 \rvert = \left| \frac{2n+1}{3n-2} - \frac{2}{3} \right| < \varepsilon$$

holds for any however small number ε. We have

$$\lvert x_n - 2/3 \rvert = \left| \frac{2n+1}{3n-2} - \frac{2}{3} \right| = \left| \frac{7}{3(3n-2)} \right| = \{for\ n \geq 1\} = \frac{7}{3(3n-2)},$$

$$\frac{7}{3(3n-2)} < \varepsilon\ if\ 3(3n-2)\varepsilon > 7,\ 3n-2 > \frac{7}{3\varepsilon},\ n > \frac{1}{3}\left(\frac{7}{3\varepsilon} + 2 \right) = \frac{7+6\varepsilon}{9\varepsilon}.$$

Let the natural number $N = \left[\frac{7+6\varepsilon}{9\varepsilon} \right] \in \aleph$ is the integer part of the number $\frac{7+6\varepsilon}{9\varepsilon}$.

We have found that for however small positive number ε the inequality $\lvert x_n - 2/3 \rvert < \varepsilon$ holds for all natural numbers n which are greater than the found number N. Symbolically

$$\forall \varepsilon > 0, \exists N = \left[\frac{7+6\varepsilon}{9\varepsilon} \right], \forall n : \left(n > N \Rightarrow \lvert x_n - 2/3 \rvert = \left| \frac{2n+1}{3n-2} - \frac{2}{3} \right| < \varepsilon \right).$$

Generalizing the reasonings of the example we can state the definition of the limit of arbitrary number sequence $\{ y_n : y_1, y_2, ..., y_n, ...\}$.

Def. 17. A number *b* is called the limit of number sequence $\{ y_n \}$ if for however small positive number ε there exists a natural number *N* such that for any natural number *n*, which is greater than *N*, the inequality $|y_n - b| < \varepsilon$ holds.

One writes in this case

$$\lim_{n\to\infty} y_n = b$$

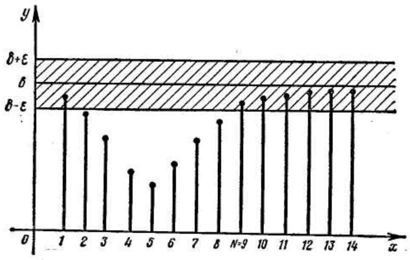


Fig. 9

and says that the sequence tends [converges, is convergent] to *b*.

Symbolic expression of the **Def. 17** is the next:

$$\lim_{n\to\infty} y_n = b$$

if

$$\forall \varepsilon > 0, \exists N \in \aleph, \forall n \in \aleph : \left(n > N \Rightarrow |y_n - b| < \varepsilon\right).$$

Geometric sense of the limit consists in the next: for *n* > *N* all terms of the sequence lie in hatched strip between straight lines $y = b - \varepsilon,\ y = b + \varepsilon$ (fig. 9).

D. Limit of a function on plus or minus infinity

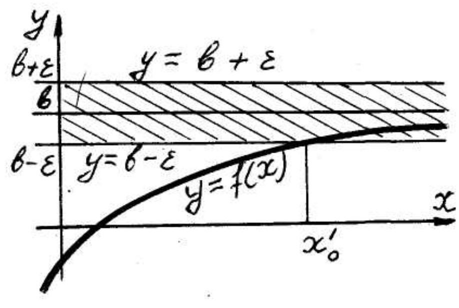


Fig. 10

Def. 18. A number b is called the limit of a function $y = f(x)$ as $x \to +\infty$, $\lim\limits_{x\to+\infty} f(x) = b$, if for any $\varepsilon > 0$ there exists a number x_0' such that for all values of $x \in D(f)$, which are greater than x_0', the inequality

$$|f(x) - b| < \varepsilon \ \left(b - \varepsilon < f(x) < b + \varepsilon\right)$$

holds. Symbolically

$$\lim_{x\to+\infty} f(x) = b \ \text{ if } \forall \varepsilon > 0, \forall x_0', \forall \in D(f) : \left(x > x_0' \Rightarrow |f(x) - b| < \varepsilon\right).$$

Geometrically (fig. 10): for $D(f) \ni x > x_0'$ the corresponding part of the graph of the function lies in hatched strip bounded by straight lines $y = b - \varepsilon,\ y = b + \varepsilon$.

Ex. 14. Prove that $\lim\limits_{x\to+\infty}\dfrac{3x+2}{4x-5}=\dfrac{3}{4}$.

$$\left|\frac{3x+2}{4x-5}-\frac{3}{4}\right|=\left|\frac{23}{4(4x-5)}\right|=\left\{if\ x>\frac{5}{4}\right\}=\frac{23}{4(4x-5)}<\varepsilon\ if\ 4\varepsilon(4x-5)>23,\ x>\frac{1}{4}\left(\frac{23}{4\varepsilon}+5\right).$$

$$\forall\varepsilon>0,\exists x_0'=\frac{1}{4}\left(\frac{23}{4\varepsilon}+5\right),\ \forall x\in D(f):\left(x>x_0'\Rightarrow\left|\frac{3x+2}{4x-5}-\frac{3}{4}\right|<\varepsilon\right)\Rightarrow\lim_{x\to+\infty}\frac{3x+2}{4x-5}=\frac{3}{4}$$

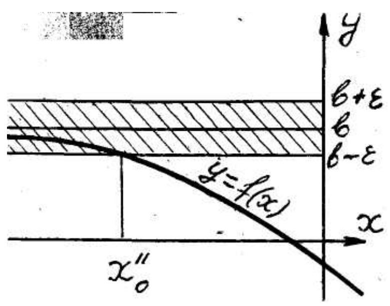


Fig. 11

Def. 19. A number b is called the limit of a function $y=f(x)$ as $x\to-\infty$, $\lim\limits_{x\to-\infty}f(x)=b$, if for any $\varepsilon>0$ there exists a number x_0'' such that for all values $x\in D(f)$, which are less than x_0'', the inequality

$$|f(x)-b|<\varepsilon\ (b-\varepsilon<f(x)<b+\varepsilon)$$

holds.

Symbolically

$$\lim_{x\to-\infty}f(x)=b\ \text{if}\ \forall\varepsilon>0,\forall x_0'',\ \forall\in D(f):\left(x<x_0''\Rightarrow|f(x)-b|<\varepsilon\right).$$

Geometrically (fig. 11): for $D(f)\ni x<x_0''$ corresponding part of the graph of the function lies in hatched strip bounded by the straight lines $y=b-\varepsilon,\ y=b+\varepsilon$.

Ex. 15. Prove that $\lim\limits_{x\to-\infty}\dfrac{3x+2}{4x-5}=\dfrac{3}{4}$.

Indeed,

$$\left|\frac{3x+2}{4x-5}-\frac{3}{4}\right|=\left|\frac{23}{4(4x-5)}\right|=\left\{if\ x<\frac{5}{4}\right\}=-\frac{23}{4(4x-5)}=\frac{23}{4(5-4x)}<\varepsilon\ if\ 4\varepsilon(5-4x)>23,$$

$$5-4x>\frac{23}{4\varepsilon},\ 4x<5-\frac{23}{4\varepsilon},\ x<\frac{1}{4}\left(5-\frac{23}{4\varepsilon}\right),$$

$$\forall\varepsilon>0,\exists x_0''=\frac{1}{4}\left(5-\frac{23}{4\varepsilon}\right),\ \forall x\in D(f):\left(x<x_0''\Rightarrow\left|\frac{3x+2}{4x-5}-\frac{3}{4}\right|<\varepsilon\right)\Rightarrow\lim_{x\to-\infty}\frac{3x+2}{4x-5}=\frac{3}{4}$$

E. Infinitely small

Def. 20. A function $y=f(x)$ is called infinitely small (*IS*) in some passage to the limit if its limit is equal to zero.

For the case of $x \to a$ one get the definition of *IS* from **Def. 14** for $b = 0$: function $y = f(x)$ is called *IS* in the case $x \to a$ if

$$\forall \varepsilon > 0, \exists U_a, \forall x \in D(f) : \left(x \in U'_a \Rightarrow |f(x) - 0| = |f(x)| < \varepsilon \Leftrightarrow (-\varepsilon < f(x) < \varepsilon)\right)$$

Ex. 16. A function $y = x^2$ is *IS* as $x \to 0$, $\lim\limits_{x\to 0} x^2 = 0$, because of $\left|x^2\right| = |x|^2 < \varepsilon$ if $|x| < \varepsilon, -\varepsilon < x < \varepsilon, x \in U_0 = (-\varepsilon, \varepsilon)$, $\forall \varepsilon > 0, \exists U_0 = (-\varepsilon, \varepsilon), \forall x : \left(x \in U_0 \Rightarrow \left|x^2\right| < \varepsilon\right)$.

Ex. 17. A function $y = 1/x$ is *IS* as $x \to \pm\infty$, $\lim\limits_{x\to\pm\infty} 1/x = 0$.

■ $\left|\dfrac{1}{x}\right| = \dfrac{1}{|x|} < \varepsilon$ if $|x| > \dfrac{1}{\varepsilon}$ that is if $x > x'_0 = \dfrac{1}{\varepsilon}$ or $x < x''_0 = -\dfrac{1}{\varepsilon}$ ■.

Theorem 3. All elementary functions are *IS* at their zeros.

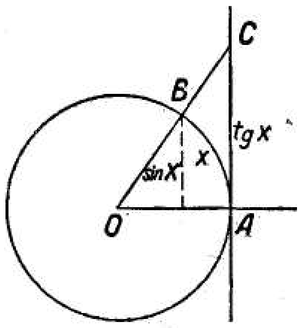


Fig. 12

Let's prove for example that $\sin x$ is *IS* at the point $x = 0$, that is $\lim\limits_{x\to 0} \sin x = 0$.

■From the trigonometric circle (fig.12) we see that sin$x$< x for $0 < x < \pi/2$ and $|\sin x| < |x|$ if $-\pi/2 < x < \pi/2$. So $|\sin x| < \varepsilon$ if $|x| < \varepsilon, -\varepsilon < x < \varepsilon$ or $x \in U_0 = (-\varepsilon, \varepsilon)$. Thus we we can write

$$\forall \varepsilon > 0, \exists U_0 = (-\varepsilon, \varepsilon), \forall x \in (-\pi/2, \pi/2) : (x \in U_0 \Rightarrow |\sin x| < \varepsilon) \Rightarrow \lim_{x\to 0} \sin x = 0 \blacksquare$$

Theorem 4. All next functions: a) $\dfrac{1}{x^n}, n \in \aleph,$ for $x \to \pm\infty$; b) a^x for $a > 1$ and $x \to -\infty$; c) a^x for $0 < a < 1$ and $x \to +\infty$ are *IS*.

One can remember these facts with the help of graphs of corresponding functions.

F. Infinitely large

Let, for example, be given a function $y = f(x)$ of one variable $x \in \Re^1$ and $x \to a - 0$.

Def. 21. The function $y = f(x)$ is called infinitely large (*IL*) as $x \to a-0$, $\lim\limits_{x\to a-0}|f(x)| = +\infty$, if for however large positive number *N* there exists an interval

(m, a) such that for any value of the argument *x*, if $x \in (m, a)$ then the inequality $|f(x)| > N$ holds, that is

$$\forall N > 0, \exists (m, a), \forall x \in D(f): (x \in (m, a) \Rightarrow |f(x)| > N).$$

Note. If a function $y = f(x)$ is *IL* for $x \to a-0$ and $f(x) > 0$ ($f(x) < 0$) from the left of the point *a* then one can say that $\lim\limits_{x\to a-0} f(x) = +\infty$ ($\lim\limits_{x\to a-0} f(x) = -\infty$).

Ex. 18. Function $1/x$ is *IL* if $x \to 0$. Namely $\lim\limits_{x\to 0-0}\frac{1}{x} = -\infty, \lim\limits_{x\to 0+0}\frac{1}{x} = +\infty$.

■Let, for example, $x \to 0-0$ ($x \to 0$ and $x < 0$). For however large positive number *N*

$$\left|\frac{1}{x}\right| = \frac{1}{|x|} = \frac{1}{-x} = -\frac{1}{x} > N, \frac{1}{x} < -N \text{ if } -1 < xN, x > -\frac{1}{N}, x \in \left(-\frac{1}{N}, 0\right).$$ Thus,

$$\forall N > 0, \exists\left(-\frac{1}{N}, 0\right), \forall x: \left(x \in \left(-\frac{1}{N}, 0\right) \Rightarrow \left|\frac{1}{x}\right| > N \text{ or } \frac{1}{x} < -\frac{1}{N}\right) \Rightarrow \lim_{x\to 0-0}\frac{1}{x} = -\infty \ ■$$

Ex. 19. With the help of trigonometric circle prove that

$$\lim_{x\to\pi/2-0}\tan x = +\infty.$$

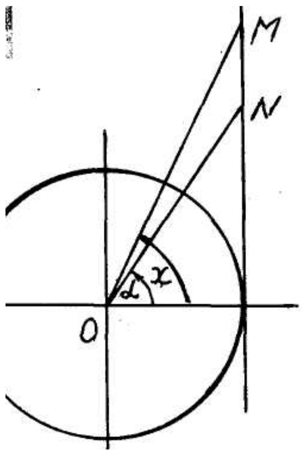


Fig. 13

Let *N* be however large positive number and $\alpha = \arctan N$ (see fig. 13). Then for any *x* from the interval

$$\left(m, \frac{\pi}{2}\right) = \left(arctan\, N, \frac{\pi}{2}\right), m = arctan\, N,$$

the inequality $0 \tan x > N$ holds, that is

$$\lim_{x\to\pi/2-0}\tan x = +\infty.$$

Finally, by definition of *IL* (for the case $f(x) = \tan x > 0$ on the interval $(0, \pi/2)$)

$$\forall M > 0, \exists\left(m, \frac{\pi}{2}\right), \forall x \in \left(0, \frac{\pi}{2}\right): \left(x \in \left(m, \frac{\pi}{2}\right) \Rightarrow tan\, x > M\right).$$

Analogous definitions can be stated for the other types of passage to the limit.

Ex. 20. x^3 is *IL* if $x \to \pm\infty$. Namely, $\lim\limits_{x\to+\infty} x^3 = +\infty,\ \lim\limits_{x\to-\infty} x^3 = -\infty.$

■If $x \to +\infty$ then (we can consider that *x* is positive) $x^3 > N$ for $x > x_0' = \sqrt[3]{N}$. If $x \to -\infty$ then (we can consider that *x* is negative) $x^3 < -N$ for $x < x_0'' = -\sqrt[3]{N}$ ■

Theorem 5. All the next functions: a) $x^n, n \in \aleph$ for $x \to \pm\infty$; b) a^x for $a > 1$ and $x \to +\infty$; c) a^x for $0 < a < 1$ and $x \to -\infty$; d) $\log_a x$ for $x \to +\infty$ or $x \to 0+0$; e) $\tan x$ for $x \to \pi/2 + \pi k$ from the left or from the right, $k \in \mathrm{Z}$; g) $\cot x$ for $x \to \pi k$ from the left or from the right, $k \in \mathrm{Z}$, are *IL*.

One can remember these facts with the help of graphs of corresponding functions.

POINT 3. PROPERTIES OF LIMITS

Def. 22. A function *f*(*x*) is called **bounded above** on some set $X \subseteq D(f)$ if there exists some number C_1 such that the inequality $f(x) \le C_1$ holds for any value of the argument *x* containing in the set *X*. Symbolically

$$\exists C_1, \forall x \in X : f(x) \le C_1.$$

A function *f*(*x*) is called **bounded below** on the set *X* if

$$\exists C_2, \forall x \in X : f(x) \ge C_2.$$

A function *f*(*x*) is called **bounded** one on *X* if it's bounded above and below.

Theorem 6. A function *f*(*x*) is bounded on X iff (if and only if)

$$\exists C, \forall x \in X : |f(x)| \le C.$$

Prove this theorem yourselves.

General properties of limits of functions

All this properties are true for any types of passage to limit. We'll state them for the case of the limite of a function at a point *a*.

1. If the limit

$$\lim_{x \to a} f(x) = A$$

exists then the function $f(x)$ is bounded in some neighbourhood of the point a.

■By definition of limit

$$\forall \varepsilon > 0, \exists U_a, \forall x \in D(f): \left(x \in U'_a \Rightarrow |f(x) - A| < \varepsilon \Rightarrow A - \varepsilon < f(x) < A + \varepsilon\right).$$

Thus in U'_a the function $f(x)$ is bounded above and below and so it is bounded one■

2. If

$$\lim_{x \to a} f(x) = A > 0,$$

then the function $f(x)$ is positive in some neighbourhood of the point a.

■Proving follows from that of preceding property if one takes ε such small that A - ε be positive. Then in U'_a one has $0 < A - \varepsilon < f(x), f(x) > 0$ ■

3 (corollary). If in some neighbourhood U_a of a point a one has $f(x) < 0$ (or $f(x) \le 0$) then $\lim_{x \to a} f(x) \le 0$.

■Prove this corollary yourselves by reduction to absurdity■

Ex. 21. $\frac{1}{n} > 0$ for any natural n, but $\lim_{n \to \infty} \frac{1}{n} = 0$.

4. **Theorem about two militiamen.** If in some neighbourhood $U_{a,1}$ of a point a a double inequality

$$g(x) < f(x) < h(x)$$

for three functions $g(x), f(x), h(x)$ holds and

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = A,$$

then there exists the limit of the function $f(x)$ at the point a and $\lim_{x \to a} f(x) = A$.

■ $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = A$ means that

$$\forall \varepsilon > 0, \exists U_{a,2}, \forall x \in D(f): \left(x \in U'_{a,2} \Rightarrow \begin{matrix} |g(x) - A| < \varepsilon,\ A - \varepsilon < g(x) < A + \varepsilon \\ |h(x) - A| < \varepsilon,\ A - \varepsilon < h(x) < A + \varepsilon \end{matrix} \right).$$

Let $U'_a = U'_{a,1} \cap U'_{a,2}$ is the common part of $U'_{a,1}$ and $U'_{a,2}$. In U'_a all the inequalities

$$A-\varepsilon < g(x) < f(x) < h(x) < A+\varepsilon$$

hold, therefore

$$A-\varepsilon < f(x) < A+\varepsilon, |f(x)-A| < \varepsilon .$$

Thus

$\forall\varepsilon > 0, \exists U_a, \forall x \in D(f): (x \in U'_a \Rightarrow |f(x)-A| < \varepsilon)$, that is $\lim_{x\to a} f(x) = A$ ■

5. If a numerical sequence $\{y_n\}$ is increasing and bounded above then it has the limit ("it converges", "it is convergent").

Properties of IS (of infinitely small)

1. Sum of two *IS* is *IS*.
2. Product of *IS* by bounded function is *IS*.
3 (corollary). Product of two *IS* is *IS*.

Note. One can say nothing about a quotient of two *IS*. It's undetermined expression of the type $\frac{0}{0}$.

4. If $\alpha(x)$ is *IS* then $f(x) = \dfrac{1}{\alpha(x)}$ is *IL* (symbolically: $\frac{1}{0} = \infty$).

5. If $f(x)$ is *IL* then $\alpha(x) = \dfrac{1}{f(x)}$ is *IS* (symbolically: $\frac{1}{\infty} = 0$).

6. A function $f(x)$ has a limit A at a point a (for $x \to a$) if and only if in some neighbourhood of this point the function can be represented in the form

$$f(x) = A + \alpha(x),$$

where $\alpha(x)$ is *IS* for $x \to a$.

■ a) If $\exists \lim_{x\to a} f(x) = A$, that is

$$\forall\varepsilon > 0, \exists U_a, \forall x \in D(f): (x \in U'_a \Rightarrow | f(x) - A | < \varepsilon),$$

then the function $\alpha(x) = f(x) - A$ is *IS* as $x \to a$ and $f(x) = A + \alpha(x)$ in U'_a.

b) Inversely let $f(x) = A + \alpha(x)$, where $\alpha(x)$ is *IS* for $x \to a$, that is

$$\forall\varepsilon > 0, \exists U_a, \forall x \in D(f): (x \in U'_a \Rightarrow |\alpha(x)| = | f(x) - A | < \varepsilon).$$

Hence if follows by definition of limit that $\lim_{x\to a} f(x)=A$ ■

“Arithmetical” properties of limits

1. Limit of sum, difference, product, quotient of two functions equals (correspondingly) sum, difference, product, quotient of limits of these functions, that is

$$\lim_{x\to a}(f(x)\pm g(x))=\lim_{x\to a} f(x)\pm \lim_{x\to a} g(x),$$
$$\lim_{x\to a}(f(x)\cdot g(x))=\lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x),$$
$$\lim_{x\to a}(f(x)/g(x))=\lim_{x\to a} f(x)\Big/\lim_{x\to a} g(x) \text{ provided } \lim_{x\to a} g(x)\neq 0.$$

■ Proof for the limit of a product. Let

$$\lim_{x\to a} f(x)=A, \lim_{x\to a} g(x)=B .$$

Then by the property 6 of properties of *IS*

$$f(x)=A+\alpha(x), g(x)=B+\beta(x),$$

where $\alpha(x), \beta(x)$ are IS for $x\to a$. Product of these functions equals:

$$f(x)\cdot g(x) = A\cdot B+\underbrace{\underbrace{A\beta(x)}_{IS}+\underbrace{B\alpha(x)}_{IS}+\underbrace{\alpha(x)\beta(x)}_{IS}}_{IS}$$

It means that $f(x)\cdot g(x)=A\cdot B+IS \Rightarrow \lim_{x\to a}(f(x)\cdot g(x))=A\cdot B=\lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$ ■

Note. It can be said nothing without special investigation about the limit of quotient of two functions

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

when

$$\lim_{x\to a} f(x)=0, \lim_{x\to a} g(x)=0$$

or when functions $f(x), g(x)$ are infinitely large as $x\to a$. In such the cases one says about indeterminate expressions [indeterminate forms, indeterminacies, indeterminatenesses, indeterminations, indeterminedness] of the types $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Corollaries. a) For any constant C

$$\lim_{x\to a}\left(C\cdot f(x)\right)=C\cdot\lim_{x\to a}f(x)$$

that is a constant factor can be taken outside the limit sign.

b) For any natural number n the limite of the nth power of a function is equal to the nth power of the limite of this function,

$$\lim_{x\to a}\left(f(x)\right)^n=\left(\lim_{x\to a}f(x)\right)^n$$

2 (limit of a composite function). Let there be given a composite function

$$y=f(\varphi(x))$$

(where $y=f(u), u=\varphi(x)$). If $\lim_{x\to a}\varphi(x)=b$ and $\lim_{u\to b}f(u)=A$ then there exists the limit of the composite function at the point a which equals

$$\lim_{x\to a}f(\varphi(x))=A.$$

Ex. 22. Let $u=\varphi(x)=\dfrac{1}{x-2}$, $y=f(u)=e^u$ that is $y=f(\varphi(x))=e^{\frac{1}{x-2}}$.

$$\lim_{x\to 2+0}\varphi(x)=\lim_{x\to 2+0}\frac{1}{x-2}=+\infty,\ \lim_{u\to+\infty}f(u)=\lim_{u\to+\infty}e^u=+\infty\Rightarrow\lim_{x\to 2+0}f(\varphi(x))=\lim_{x\to 2+0}e^{\frac{1}{x-2}}=+\infty$$

$$\lim_{x\to 2-0}\varphi(x)=\lim_{x\to 2-0}\frac{1}{x-2}=-\infty,\ \lim_{u\to-\infty}f(u)=\lim_{u\to-\infty}e^u=0\Rightarrow\lim_{x\to 2-0}f(\varphi(x))=\lim_{x\to 2-0}e^{\frac{1}{x-2}}=0;$$

$$\lim_{x\to 2}e^{\frac{1}{x-2}}=\begin{cases}+\infty, \text{ if } x\to 2+0,\\ 0, \text{ if } \quad x\to 2-0.\end{cases}$$

Def. 23. Two functions $f(x)$, $g(x)$ are called equivalent as $x\to a$ ($f(x)\sim g(x)$ or $f(x)\approx g(x)$ on computer) if limit of their ratio is equal to unity,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=1.$$

3. Finding limits we can substitute any factor by its equivalent one.

■ Let $f(x)\sim h(x)$, $g(x)\sim k(x)$ as $x\to a$ and it's necessary to find a limit

$$\lim_{x\to a}\frac{f(x)u(x)w(x)}{g(x)v(x)}.$$

Multiplying and dividing by $h(x)$ and $k(x)$ one gets

$$\lim_{x\to a}\frac{f(x)u(x)w(x)}{g(x)v(x)}=\lim_{x\to a}\frac{f(x)k(x)h(x)u(x)w(x)}{h(x)g(x)k(x)v(x)}=$$

$$= \lim_{x\to a}\frac{f(x)}{h(x)}\cdot\lim_{x\to a}\frac{k(x)}{g(x)}\cdot\lim_{x\to a}\frac{h(x)u(x)w(x)}{k(x)v(x)} = \lim_{x\to a}\frac{h(x)u(x)w(x)}{k(x)v(x)}.$$

Factors *f* (*x*), *g* (*x*) are substituted by *h* (*x*), *k* (*x*) without changing the limit ■

Properties of IL (of infinitely large)

1. If $f(x)\to\pm\infty, g(x)\to\pm\infty$ then $f(x)+g(x)\to\pm\infty$.

2. If $f(x)\to\pm\infty, g(x)\to\mp\infty$ then $f(x)-g(x)\to\pm\infty$

3. *IL*·*IL* = *IL*

4. If *f (x)*, *g* (*x*) be two functions, *f* (*x*) is *IL* for $x\to a$ and $g(a)=A\neq 0$ or $\lim_{x\to a} g(x)=A\neq 0$, where *A* is some finite number then the product *f* (*x*)·*g* (*x*) of these functions is *IL* for $x\to a$.

Ex. 23. *n*th degree polynomial

$$P_n(x)=a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}+a_nx^n,\ a_n\neq 0$$

is *IL* for *x* tending to infinity $(x\to\pm\infty)$. It's equivalent to its highest term [term with higher exponent] a_nx^n for $x\to\pm\infty$.

■ Taking x^n out the parentheses we get a product

$$P_n(x)=x^n\cdot\left(\frac{a_0}{x^n}+\frac{a_1}{x^{n-1}}+\frac{a_2}{x^{n-2}}+\ldots+\frac{a_{n-1}}{x}+a_n\right)$$

of *IL* x^n and a function having the finite limit $a_n\neq 0$. Therefore this product is *IL* as $x\to\infty$. Futher

$$\lim_{x\to\pm\infty}\frac{P_n(x)}{a_nx^n}=\lim_{x\to\pm\infty}\frac{x^n\cdot\left(\frac{a_0}{x^n}+\frac{a_1}{x^{n-1}}+\frac{a_2}{x^{n-2}}+\ldots+\frac{a_{n-1}}{x}+a_n\right)}{a_nx^n}=\frac{a_n}{a_n}=1\Rightarrow P_n(x)\sim a_nx^n.\ ■$$

Ex. 24. Using this last fact we find the next limit:

$$\lim_{x\to\pm\infty}\frac{4x^8-3x^5+2x-4}{5x^7+2x^4-3x^2+4}=\left(\frac{\infty}{\infty}\right)=\left|\begin{matrix}4x^8-3x^5+2x-4\sim 4x^8\\ 5x^7+2x^4-3x^2+4\sim 5x^7\end{matrix}\right|=$$

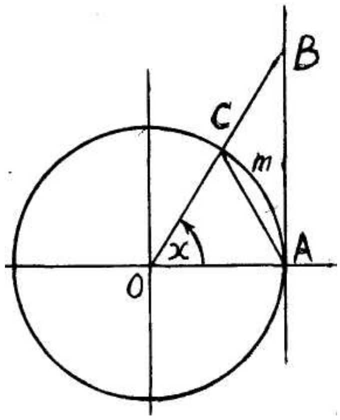
$$=\lim_{x\to\pm\infty}\frac{4x^8}{5x^7}=\frac{4}{5}\lim_{x\to\pm\infty}x=\begin{cases}+\infty\ if\ x\to+\infty,\\ -\infty\ if\ x\to-\infty.\end{cases}$$

POINT 4. REMARKABLE [STANDARD] LIMITS

The first remarkable limit

The first remarkable [standard] limit is called the next one

$$\lim_{x\to 0}\frac{\sin x}{x}=\left(\frac{0}{0}\right)=1 \qquad (1)$$



■Using the trigonometrical circle we'll study the case $0<x<\pi/2$ (fig. 14). Finding the areas $S_{\Delta AOC}$, $S_{\Delta AOB}$, S_{OAmC} of the triangles *AOC*, *AOB* and circular sector *OAmC* we see that

$$S_{\Delta AOC}<S_{OAmC}<S_{\Delta AOB}.$$

Hence

Fig. 14 $\frac{1}{2}OA\cdot OC\cdot\sin x<\frac{OA^2\cdot x}{2}<\frac{1}{2}OA\cdot AB,\ \frac{1}{2}\cdot 1\cdot 1\cdot\sin x<\frac{1^2\cdot x}{2}<\frac{1}{2}\cdot 1\cdot\tan x,$

$$\sin x<x<\tan x,\ 1<\frac{x}{\sin x}<\frac{1}{\cos x}$$

or better

$$\cos x<\frac{\sin x}{x}<1.$$

This last double inequality is valid and for the case $-\pi/2<x<0$.

As for as $\lim\limits_{x\to 0}\cos x=1$ we get the required result by virtue of the theorem about two militiamen.■

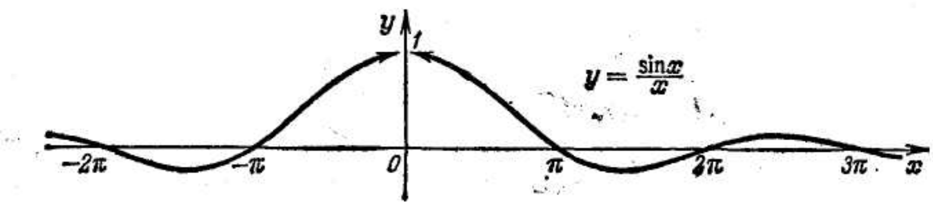


Fig. 15

The graph of the function

$$f(x)=\frac{\sin x}{x}$$

is representted on the figure 15.

Corollaries.

1. $\lim\limits_{x\to 0}\dfrac{\arcsin x}{x}=\left(\dfrac{0}{0}\right)=1;\ \lim\limits_{x\to 0}\dfrac{\tan x}{x}=\left(\dfrac{0}{0}\right)=1;\ \lim\limits_{x\to 0}\dfrac{\arctan x}{x}=\left(\dfrac{0}{0}\right)=1$

We'll prove the third of these limits. Prove the other yourselves.

$$\blacksquare \lim_{x\to 0}\frac{\arctan x}{x}=\left|\begin{array}{l}\arctan x=y,\\ y\to 0,\\ x=\tan y\end{array}\right|=\lim_{y\to 0}\frac{y}{\tan y}=\lim_{y\to 0}\frac{y\cos y}{\sin y}=\lim_{y\to 0}\frac{y}{\sin y}\cdot\lim_{y\to 0}\cos y=1\cdot 1=1\blacksquare$$

2. For $x\to 0$ functions $\sin x, \arcsin x, \tan x, \arctan x$ are *IS*, which are equivalent to their argument *x*: $\sin x\sim x,\ \arcsin x\sim x,\ \tan x\sim x,\ \arctan x\sim x$.

Ex. 25. With the help of the third property of "arithmetical" properties of limits

$$\lim_{x\to 0}\frac{\sin 4x\cdot\tan 7x}{\arcsin 3x\cdot\arctan 8x}=\left(\frac{0}{0}\right)=\left|\begin{array}{ll}\sin 4x\sim 4x & \tan 7x\sim 7x\\ \arcsin 3x\sim 3x & \arctan 8x\sim 8x\end{array}\right|=\lim_{x\to 0}\frac{4x\cdot 7x}{3x\cdot 8x}=\frac{7}{6}.$$

Ex. 26. Find the limit $A=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1-\sin^3 13x}{1-\sin^3 9x}$.

Let's remark that

$$\sin\frac{13\pi}{2}=\sin\left(6\pi+\frac{\pi}{2}\right)=\sin\frac{\pi}{2}=1,\ \sin\frac{9\pi}{2}=\sin\left(4\pi+\frac{\pi}{2}\right)=\sin\frac{\pi}{2}=1,$$

and so by "arithmetical" properties of limits

$$A=\lim_{x\to\frac{\pi}{2}}\frac{1-\sin^3 13x}{1-\sin^3 9x}=\left(\frac{0}{0}\right)=\lim_{x\to\frac{\pi}{2}}\frac{(1-\sin 13x)(1+\sin 13x+\sin^2 13x)}{(1-\sin 9x)(1+\sin 9x+\sin^2 9x)}=\lim_{x\to\frac{\pi}{2}}\frac{1-\sin 13x}{1-\sin 9x}$$

because of

$$\lim_{x\to\frac{\pi}{2}}(1+\sin 13x+\sin^2 13x)=1+1+1=3,\ \lim_{x\to\frac{\pi}{2}}(1+\sin 9x+\sin^2 9x)=1+1+1=3.$$

Now we introduce a substitution $\frac{\pi}{2}-x=y,\ y\to 0,$ whence it follows that

$$\sin 13x=\sin 13\left(\frac{\pi}{2}-y\right)=\sin\left(\frac{13\pi}{2}-13y\right)=\sin\left(6\pi+\frac{\pi}{2}-13y\right)=\sin\left(\frac{\pi}{2}-13y\right)=\cos 13y$$

$$\sin 9x=\sin 9\left(\frac{\pi}{2}-y\right)=\sin\left(\frac{9\pi}{2}-9y\right)=\sin\left(4\pi+\frac{\pi}{2}-9y\right)=\sin\left(\frac{\pi}{2}-9y\right)=\cos 9y\,.$$

Hence,

$$A=\lim_{y\to 0}\frac{1-\cos 13y}{1-\cos 9y}=\lim_{y\to 0}\frac{2\sin^2\frac{13y}{2}}{2\sin^2\frac{9y}{2}}=\left|\begin{matrix}\sin^2\frac{13y}{2}\sim\left(\frac{13y}{2}\right)^2\\ \sin^2\frac{9y}{2}\sim\left(\frac{9y}{2}\right)^2\end{matrix}\right|=\lim_{y\to 0}\frac{\left(\frac{13y}{2}\right)^2}{\left(\frac{9y}{2}\right)^2}=\left(\frac{13}{9}\right)^2=\frac{169}{81}$$

The second remarkable limit

Let's study the next number sequence

$$\left\{y_n=\left(1+\frac{1}{n}\right)^n\right\}.$$

Approximate values (to 3 decimals) of some terms of the sequence are given in the table 4

Table 4.

n	10	50	100	150	1000	2000	3000	10000
y_n	2.594	2.692	2.705	2.709	2.717	2.717	2.717	2.718

We come to conclusions (and there is a strict proving of these facts): a) the given sequence increaces; b) it is bounded above. Therefore (by virtue of property 5 of general properties of limits of functions) it possesses the limit which is denoted by a letter e (Euler's number; it's known that e = 2.718281828459045…). Thus we can write

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e.$$

More general result is true, namely

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e,$$

where x can tend as to $+\infty$, as to $-\infty$. This result can be represented in the next form:

$$\lim_{x\to 0}(1+x)^{1/x}=e.$$

We'll write all these formulae together and call them the **second standard limit**

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e;\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e;\ \lim_{x\to 0}(1+x)^{1/x}=e \qquad (2)$$

Corollaries.

1. $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$ the **third remarkable limit** (3)

$$\blacksquare \lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\frac{1}{x}\ln(1+x)=\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=\ln\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\ln e=1.\blacksquare$$

Legitimacy of the passage to the limit under logarithm sign will be proved later.

2. $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$ the **fourth remarkable limit** (4)

$$\blacksquare \lim_{x\to 0}\frac{e^x-1}{x}=\left|\begin{array}{c} e^x-1=y, \\ y\to 0, \\ x=\ln(1+y)\end{array}\right|=\lim_{y\to 0}\frac{y}{\ln(1+y)}=1\blacksquare$$

3. From the formulas (3), (4) it follows that for $x\to 0$ the functions $\ln(1+x)$, e^x-1 are *IS* which are equivalent to their argument x, $\ln(1+x)\sim x$, $e^x-1\sim x$.

4. $\log_a(1+x)=\dfrac{\ln(1+x)}{\ln a}\sim\dfrac{1}{\ln\alpha}\cdot x$.

5. $a^x-1=\left(e^{\ln a}\right)^x-1=e^{x\ln a}-1\sim x\cdot\ln a$.

6. $(1+x)^\alpha-1=e^{\alpha\ln(1+x)}-1\sim\alpha\ln(1+x)\sim\alpha x$.

As the final consequence we form the **table of equivalent IS**

$$\left.\begin{array}{l}\sin x\\ tgx\\ \arcsin x\\ arctgx\\ \ln(1+x)\\ e^x-1\end{array}\right\}\sim x \text{ as } x\to 0 \qquad\qquad \left.\begin{array}{c}\log_a(1+x)\sim\dfrac{x}{\ln a}\\ a^x-1\sim x\ln a\\ (1+x)^\alpha-1\sim\alpha x\end{array}\right\} \text{ as } x\to 0$$

Ex. 27.

$$\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{x-1}=\left|\begin{array}{c}\dfrac{2x-3}{2x+5}\to 1,\quad 1^\infty\\ x-1\to\infty\end{array}\right|=\lim_{x\to\infty}\left(1+\frac{2x-3}{2x+5}-1\right)^{x-1}=\lim_{x\to\infty}\left(1+\frac{-8}{2x+5}\right)^{x-1}=$$

$$=\lim_{x\to\infty}\left(\left(1+\frac{-8}{2x+5}\right)^{\frac{2x+5}{-8}}\right)^{\frac{-8}{2x+5}(x-1)}=e^{\lim\limits_{x\to\infty}\frac{-8}{2x+5}(x-1)}=e^{\lim\limits_{x\to\infty}\frac{-8(x-1)}{2x+5}}=e^{\lim\limits_{x\to\infty}\frac{-8x}{2x}}=e^{-4}=\frac{1}{e^4}.$$

Ex. 28.

$$\lim_{x\to\infty}(3x-2)(\ln(4x+3)-\ln(4x-7))=(\infty\cdot(\infty-\infty))=\lim_{x\to\infty}(3x-2)\ln\frac{4x+3}{4x-7}=$$

$$=\lim_{x\to\infty}(3x-2)\ln\left(1+\frac{4x+3}{4x-7}-1\right)=\lim_{x\to\infty}(3x-2)\underbrace{\ln\left(1+\frac{10}{4x-7}\right)}_{\sim\frac{10}{4x-7}}=\lim_{x\to\infty}\frac{(3x-2)\cdot 10}{4x-7}=\frac{15}{2}$$

Ex. 29.

$$\lim_{x\to\infty}\frac{\overbrace{\left(\sqrt[5]{1+\tan 4x}-1\right)}^{\sim\frac{1}{5}\tan 4x}\cdot\overbrace{\log_8(1-\arcsin 3x)}^{\sim-\frac{\arcsin 3x}{\ln 8}}}{\underbrace{\left(6^{\sin 3x}-1\right)}_{\sim\sin 3x}\underbrace{\arctan 5x}_{\sim 5x}}=\lim_{x\to 0}\frac{\frac{1}{5}\tan 4x\cdot\left(-\frac{\arcsin 3x}{\ln 8}\right)}{\sin 3x\ln 6\cdot 5x}=$$

$$=\lim_{x\to 0}\frac{-\overbrace{\tan 4x}^{\sim 4x}\cdot\overbrace{\arcsin 3x}^{\sim 3x}}{25\ln 8\ln 6\underbrace{\sin 3x}_{\sim 3x}\cdot x}=-\frac{1}{25\ln 8\cdot\ln 6\cdot}\lim_{x\to 0}\frac{4x\cdot 3x}{3x\cdot x}=-\frac{4}{375\cdot\ln 8\cdot\ln 6}$$

Ex. 30.

$$\lim_{x\to 0}\frac{4^{3x}-5^{8x}}{\underbrace{\sqrt{1-\sin 3x}-1}_{\sim\frac{1}{2}(-\sin 3x)}}=\lim_{x\to 0}\frac{e^{3x\ln 4}-e^{8x\ln 5}}{-\frac{1}{2}\underbrace{\sin 3x}_{\approx\sim 3x}}=\lim_{x\to 0}\frac{\overbrace{e^{8x\ln 5}}^{\to 1}\overbrace{\left(e^{x(3\ln 4-8\ln 5)}-1\right)}^{\sim x(3\ln 4-8\ln 5)}}{-\frac{3}{2}x}=$$

$$=\lim_{x\to 0}\frac{2x(3\ln 4-8\ln 5)}{-3x}=\frac{-2(3\ln 4-8\ln 5)}{3}=-\frac{2}{3}\ln\frac{4^3}{5^8}.$$

POINT 5. INTERESTS IN INVESTMENTS

Let

$P(t)$ is a principal (that is amount of money) invested to a time moment *t*,

$I(t)$ is an interest (прибуток) to a time moment *t*,

$B(t) = P(t) + I(t)$ is the balance to a time moment *t* that is the general amount of money because of investments and an interest (прибуток),

$B(0) = P(0) + I(0) = P(0) = P$ is the opening capital at the time moment $t = 0$ (початковий капітал в момент часу $t = 0$),

α is the per cent of the interest per unite of time (відсоток прибутку на одиницю часу).

Let we do an investment of our opening capital $B(0) = P$ to a time moment *T*. At this moment *T* we have the next general amount of money (the sum of the opening capital *P* and the interest $I(T)$ to the moment *T*)

$$B(T) = P + I(T) = P + \frac{\alpha}{100}TP = P(1 + \frac{\alpha}{100}T). \qquad (5)$$

It's a formula of **simple interests** (формула простих відсотків).

Let we fulfil *n* investments of all our money during time interval *T* (in the time moments *0*, $\frac{T}{n}$, $\frac{2T}{n}$,…, $\frac{(n-1)T}{n}$)

To the time moment $\frac{T}{n}$ we'll have (the sum of the opening capital *P* and the interest $I\left(\frac{T}{n}\right)$ to the moment $\frac{T}{n}$)

$$B\left(\frac{T}{n}\right) = P + I\left(\frac{T}{n}\right) = P + P\cdot\frac{\alpha}{100}\cdot\frac{T}{n} = P\left(1 + \frac{\alpha}{100}\cdot\frac{T}{n}\right).$$

To the time moment $\frac{2T}{n}$ we'll have (the sum of the invested capital $B\left(\frac{T}{n}\right)$ and the interest $I\left(\frac{2T}{n}\right)$ from the moment $\frac{T}{n}$ to the moment $\frac{2T}{n}$)

$$B\left(\frac{2T}{n}\right)=B\left(\frac{T}{n}\right)+B\left(\frac{T}{n}\right)\frac{\alpha}{100}\cdot\frac{T}{n}=B\left(\frac{T}{n}\right)\cdot\left(1+\frac{\alpha}{100}\cdot\frac{T}{n}\right)=$$

$$=P\left(1+\frac{\alpha}{100}\cdot\frac{T}{n}\right)\left(1+\frac{\alpha}{100}\cdot\frac{T}{n}\right)=P\left(1+\frac{\alpha}{100}\cdot\frac{T}{n}\right)^2.$$

To the time moment $\frac{3T}{n}$ we'll have

$$B\left(\frac{3T}{n}\right)=P\left(1+\frac{\alpha}{100}\cdot\frac{T}{n}\right)^2\left(1+\frac{\alpha}{100}\cdot\frac{T}{n}\right)=P\left(1+\frac{\alpha}{100}\cdot\frac{T}{n}\right)^3$$

and so on.

At the time moment $T=\frac{nT}{n}$ we'll have final amount of money

$$B(T)=B\left(\frac{nT}{n}\right)=P\left(1+\frac{\alpha}{100}\cdot\frac{T}{n}\right)^n \qquad (6)$$

The formula (6) is that of **compound interests** (формула складних відсотків)

Let the number of investments $n\to+\infty$ during time *T*. In this case final amount of money $B^*(T)$ at the time moment *T* by virtue of the second remarkable limit will be equal

$$B^*(T)=\lim_{n\to\infty}B(T)=\lim_{n\to\infty}P(1+\frac{\alpha}{100}\cdot\frac{T}{n})^n=P\lim_{n\to\infty}(1+\frac{\alpha}{100}\cdot\frac{T}{n})^n=$$

$$=P\lim_{n\to\infty}((1+\frac{\alpha T}{100n})^{\frac{100n}{\alpha T}})^{\frac{\alpha T}{100}}=Pe^{\frac{\alpha}{100}T},$$

$$B^*(T)=P\cdot e^{\frac{\alpha}{100}\cdot T} \qquad (7)$$

The formula (7) is that of **continuous interests** (формула неперервних відсотків). It gives final amount of money at the time moment *T* by condition that we fulfil investments continuously.

LECTURE NO.13. CONTINUITY OF FUNCTIONS

POINT 1. CONTINUITY OF A FUNCTION AT A POINT

POINT 2. DISCONTINUITY POINTS

POINT 3. PROPERTIES OF FUNCTIONS WHICH ARE CONTINUOUS ON A SEGMENT OR IN A CLOSED BOUNDED DOMAIN.

POINT 4. INTERVAL METHOD AND ITS EXTENSION

POINT 1. CONTINUITY OF A FUNCTION AT A POINT

Def. 1. A function $y = f(x)$ of one variable or of n variables is called continuous one at a point a $\left(a \in \Re^1 \text{ or } a = (a_1, a_2, ..., a_n) \in \Re^n\right)$ if:

1) the function is defined at the point a and in some its neighbourhood;

2) there exists the limit $\lim\limits_{x \to a} f(x)$ at the point a;

3) this limit equals the value of the function at the point a,

$$\lim_{x \to a} f(x) = f(\lim_{x \to a} x) = f(a). \qquad (1)$$

On the language of the limit theory this definition means:

$$\forall \varepsilon > 0, \exists U_a, \forall x \in D(f) : \left(x \in U_a \Rightarrow |f(x) - f(a)| < \varepsilon\right).$$

Theorem 1. A function of one variable $x \in \Re^1$ is continuous at a point $a \in \Re^1$ if and only if [iff]: a) there exist the left and right limits

$$f(a-0) = \lim_{x \to a-0} f(x), \quad f(a+0) = \lim_{x \to a+0} f(x)$$

of the function at the point *a*; b) these limits are equal to the value of the function at this point,

$$f(a-0) = f(a+0) = f(a). \qquad (2)$$

■ Validity of the theorem follows from the theorem 2 of preceding lecture.■

Def. 2. A function of one variable x is called continuous at the point a **from the left** if it's defined in some interval (m, a) and $f(a-0) = f(a)$. It is called continuous at the point a **from the right** if it's defined in some interval (a, n) and $f(a+0) = f(a)$.

Therefore a function of one variable is continuous at a point iff (if and only if) it is continuous at this point from the left and from the right.

Def. 3. For a function $y = f(x)$ of one variable a difference

$$\Delta x = x - a$$

is called the **increment of the argument** x and a difference

$$\Delta y = \Delta f(a) = f(x) - f(a) = f(a + \Delta x) - f(a) \qquad (3)$$

is called the **increment of the function** at the point a.

It's evident that $x \to a$ iff $\Delta x \to 0$, $(x \to a) \Leftrightarrow (\Delta x \to 0)$.

Def. 4. For a **function of *n* variables** the next differences

$$\Delta x_1 = x_1 - a_1, \Delta x_2 = x_2 - a_2, ..., \Delta x_n = x_n - a_n,$$

are called **increments of its arguments**, n – dimension vector

$$\Delta x = (\Delta x_1, \Delta x_2, ..., \Delta x_n) = (x_1 - a_1, x_2 - a_2, ..., x_n - a_n)$$

is called the **increment** of its (n-dimensional) argument and the difference

$$\begin{aligned}\Delta y = \Delta f(a) = f(x) - f(a) = f(x_1, x_2, ..., x_n) - f(a_1, a_2, ..., a_n) = \\ = f(a + \Delta x) - f(a) = f(a_1 + \Delta x_1, a_2 + \Delta x_2, ..., a_n + \Delta x_n) - f(a_1, a_2, ..., a_n).\end{aligned} \qquad (4)$$

is called the (total) **increment of the function** at the point $a = (a_1, a_2, ..., a_n)$.

It's evident that $x \to a$ iff $\Delta x \to 0$, $(x \to a) \Leftrightarrow (\Delta x \to 0)$.

Theorem 2. A function $y = f(x)$ is continuous at the point a if and only if from tending to zero of the increment Δx of the argument it follows tending to zero of the increment $\Delta y = \Delta f(a) = f(x) - f(a)$ of the function at this point, that is iff *IS* increment of the function at the point a corresponds to *IS* increment of the argument.

■ Theorem 2 follow from the theory of limits if one supposes $b = f(a)$ ■

Def. 5. A function $y = f(x)$ is called continuous on some set if it is continuous at any point of this set. In particular a function of one variable is continuous on the segment $[a, b]$ if : 1) it's continuous at all points of the interval (a, b), 2) at the point a it's continuous from the right ($\lim_{x \to a+0} f(x) = f(a)$), 3) at the point b it's continuous from the left ($\lim_{x \to b-0} f(x) = f(b)$).

Properties of continuous functions.

1 **(**continuity of arithmetic operations on continuous functions**)**. The sum, difference, product of two continuous at a point a functions $f(x), g(x)$ are continuous at this point. The ratio $f(x)/g(x)$ of these functions is continuous if $g(a)\neq 0$.

■(for a product). Let $F(x)=f(x)\cdot g(x)$. By virtue of the property 1 of "Arithmetical properties of limits"

$$\lim_{x\to a}F(x)=\lim_{x\to a}f((x)\cdot g(x))=\lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x)=f(a)\cdot g(a)=F(a)$$

that is the function $F(x)=f(x)\cdot g(x)$ is continuous one at the point a ■

2 **(**continuity of a superposition of functions**)**. If a function $u=\varphi(x)$ is continuous at a point a and a function $y=f(u)$ is continuous at the corresponding point $b=\varphi(a)$ then the composite function $y=f(\varphi(x))$ is continuous at the point a.

It means that if $\lim\limits_{x\to a}\varphi(x)=b=\varphi(a)$ and $\lim\limits_{u\to b}f(u)=f(b)=f(\varphi(a))$, then

$$\lim_{x\to a}f(\varphi(x))=f(\lim_{x\to a}\varphi(x))=f\left(\varphi\left(\lim_{x\to a}x\right)\right)=f(\varphi(a)).$$

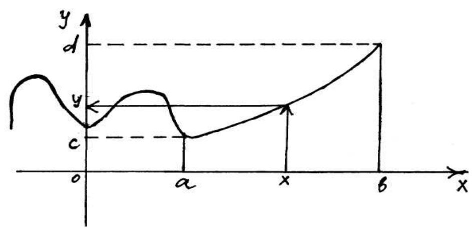


Fig. 1.

3 (continuity of an inverse function). If a function of one variable $y=f(x)$ is continuous and increasing (decreasing) in some interval (a,b), then its inverse function $x=g(y)$ is continuous and increasing (decreasing) in the interval $(c,d)=(f(a),f(b))$.

Ex. 1. A function $y=f(x)=x^2$ with the domain of definition $D(f)=[0,+\infty)$ and the set of values $E(f)=[0,+\infty)$ is continuous at any point of $D(f)=[0,+\infty)$ and increases. Therefore its inverse function $x=g(y)=\sqrt{y}$ is continuous at any point of $E(f)=[0,+\infty)$ and increases.

■It's sufficient to prove continuity of the function $y=f(x)=x^2$. for example let a be any positive number. We must prove that $\lim\limits_{x\to a}f(x)=\lim\limits_{x\to a}x^2=a^2=f(a)$.

Let $\varepsilon>0$ be such small that $a^2-\varepsilon>0$. We have $|f(x)-f(a)|=|x^2-a^2|<\varepsilon$ if

$$-\varepsilon < x^2 - a^2 < \varepsilon, 0 < a^2 - \varepsilon < x^2 < a^2 + \varepsilon, \sqrt{a^2 - \varepsilon} < x < \sqrt{a^2 + \varepsilon}, x \in U_a = $$
$$= (\sqrt{a^2 - \varepsilon}, \sqrt{a^2 + \varepsilon}). \text{Therefore } \lim_{x \to a} x^2 = a^2. \blacksquare$$

Ex. 2. A function $y = f(x) = \sin x \in [-1, 1]$ is continuous at any point a in particular on the segment $[-\pi/2, \pi/2]$ on which it increases. Therefore its inverse function $x = g(y) = \arcsin y$ is continuous one and increases on the segment $[-1, 1]$.

■ It's sufficient to prove continuity of the sine that is to prove, that

$$\lim_{x \to a} \sin x = \sin a.$$

But for any $\varepsilon > 0$

$$|\sin x - \sin a| = \left|2\cos\frac{x+a}{2}\sin\frac{x-a}{2}\right| = 2\left|\cos\frac{x+a}{2}\right|\left|\sin\frac{x-a}{2}\right| \le 2\left|\sin\frac{x-a}{2}\right| \le$$
$$\le 2\left|\frac{x-a}{2}\right| = |x-a|. \text{Therefore } |\sin x - \sin a| < \varepsilon \text{ if } |x-a| < \varepsilon, x \in U_a = (a-\varepsilon, a+\varepsilon).$$

It means that $\lim_{x \to a} \sin x = \sin a$ ■

Prove yourselves continuity of functions $y = f(x) = \cos x, x = g(y) = \arccos y$.

Ex. 3. Continuity of $\tan x$ at any point $a \ne \pi/2 + \pi k, k \in Z,$and of $\cot x$ at any point $a \ne \pi k, k \in Z,$ follows from property 1 and continuity of $\sin x, \cos x$. Prove yourselves continuity of $\arctan x, arc\cot x$.

Ex. 4. Continuity of a power function $y = x^\alpha, \alpha \in \Re^1,$ and an exponential function $y = a^x, a \in \Re^1, 0 < a \ne 1,$ is laid in the strict definition of these functions (on the base of the strict theory of real numbers).

Continuity of a logarithmic function $y = \log_a x, a \in \Re^1, 0 < a \ne 1,$ follows from continuity and (strict) monotonicity of the exponential function.

From properties 1 – 3 and examples 1 – 4 it follows the next theorem.

Theorem 3. All elementary functions are continuous in their domains of definition.

Ex. 5. Proving the third remarkable limit in point 4 of the preceding lecture we have used continuity of the logarithmic function.

Remark. Finding limits we make use of continuity of elementary functions.

POINT 2. DISCONTINUITY POINTS

Def. 6. Let a function $y = f(x)$ be continuous in some deleted neighbourhood $U'_{x_0} = U_{x_0} \setminus \{x_0\}$ of a point x_0 excluding this point. In this case the point x_0 is called a discontinuity point of the function.

In the case of a function of one variable $y = f(x), x \in \Re^1$, we can do classification of discontinuity points in terms of the left and right limits $f(x_0 \pm 0) = \lim\limits_{x \to x_0 \pm 0} f(x)$ of the function at the point x_0 (see figures 2, 3).

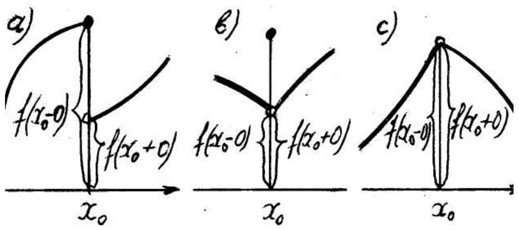


Fig. 2

1. Discontinuity point **of the first type** is a point x_0 for which there exist both (finite) the left and right limits (fig. 2). Three cases can occur for discontinuity point of the first type:

a) $f(x_0 - 0) \neq f(x_0 + 0)$ (see fig. 2a); in this case the difference

$$h = f(x_0 + 0) - f(x_0 - 0)$$

is called a (finite) jump of the function at its discontinuity point x_0;

b) $f(x_0 - 0) = f(x_0 + 0)$ and the value of the function at the point x_0 exists (see fig. 2b);

c) $f(x_0 - 0) = f(x_0 + 0)$ and the value of the function at the point x_0 doesn't exist (fig. 2c). In the cases b), c) the point x_0 is called the **point of a removable discontinuity**.

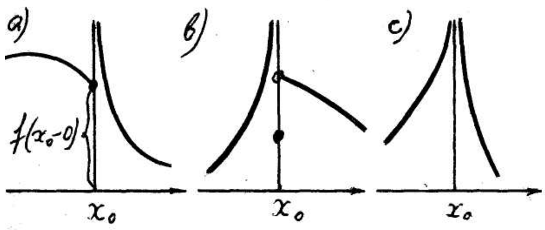


Fig. 3

2. Discontinuity point **of the second type** is a point x_0 for which at least one of the limits $f(x_0 \pm 0)$ is infinite or doesn't exist (fig. 3).

Corollary. The graph of a function $y = f(x)$ of one variable which is continuous on some

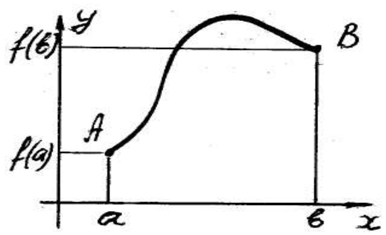


Fig. 4

interval (a,b) is some continuous line (fig. 4).

Ex. 6. The discontinuity points of functions $\tan x$, $\cot x$ ($\pi/2 + k\pi, k\pi$ correspondingly, $k \in Z$) are those of the second type.

Ex. 7. In Ex. 22 of preceding lecture we have found

$$\lim_{x\to 2} e^{\frac{1}{x-2}} = \begin{cases} +\infty, \text{if } x \to 2+0, \\ 0, \text{if } \quad x \to 2-0. \end{cases}$$

Therefore a discontinuity point $x = 2$ of the function $f(x) = e^{\frac{1}{x-2}}$ is that of the second type.

Ex. 8. A discontinuity point $x = 2$ of the function

$$f(x) = \frac{3 - 5 \cdot e^{\frac{1}{x-2}}}{4 + 2 \cdot e^{\frac{1}{x-2}}}$$

is that of the first type.

■Let

$$y = e^{\frac{1}{x-2}} \text{ and } f(x) = \frac{3 - 5 \cdot y}{4 + 2 \cdot y}.$$

If $x \to 2-0$ then (by virtue of Ex. 7) $y \to 0$ and $f(x) \to \frac{3}{4}$. If $x \to 2+0$ then (by virtue of the same Ex.) $y \to +\infty$ and

$$\lim_{x\to 2+0} f(x) = \lim_{y\to+\infty} \frac{3 - 5 \cdot y}{4 + 2 \cdot y} = \lim_{y\to+\infty} \frac{-5 \cdot y}{2 \cdot y} = -\frac{5}{2}.$$

Thus $f(2-0) = 3/4$, $f(2+0) = -5/2$, $f(2-0) \neq f(2+0)$, and the point $x = 2$ is a discontinuity point of the first type. The function suffers a jump

$$h = f(2+0) - f(2-0) = -13/4$$

at this point■

Ex. 9. Let $f(x) = \begin{cases} 3x - 1 & \text{if } x \le 1, \\ 2 + ax^2 & \text{if } x > 1. \end{cases}$

For what a will the function $f(x)$ be contiuous?

$$f(1-0)=\lim_{x\to 1-0} f(x)=\lim_{x\to 1-0}(3x-1)=2;\ f(1+0)=\lim_{x\to 1+0} f(x)=\lim_{x\to 1+0}(2+ax^2)=2+a,$$

$f(1-0)=f(1+0)$ *if* $2=2+a$ that is if $a = 0$.

Ex. 10. Discontinuity points of a function $f(x,y)=(3x-4y+5)/(x-y)$ generate the straight line $x = y$. This example demonstrates that a set of discontinuity points of a function of several variable can be extremely complicated.

POINT 3. PROPERTIES OF A FUNCTION WHICH IS CONTINUOUS ON A SEGMENT OR IN A CLOSED BOUNDED DOMAIN

Theorem 4. If a function of one variable is continuous on a segment $[a, b]$ then (see fig. 5):

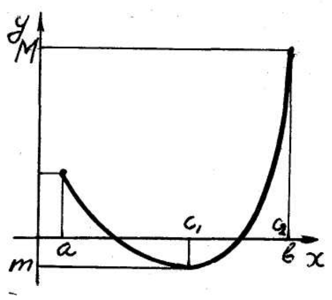


Fig. 5

1) it takes on the greatest M and the least m values on $[a, b]$: there are points $c_1 \in [a,b], c_2 \in [a,b]$ such that

$$f(c_2)=M= =\max_{[a,b]} f(x),\quad f(c_1)=m=\min_{[a,b]} f(x)$$

(**Weierstrass**[1] **theorem**);

2) it takes on all values containing between m and M (**Bolzano**[2]**-Cauchy**[3] **theorem**);

3) if it has values of different signs in two points of the segment then it has at least one zero between these points.

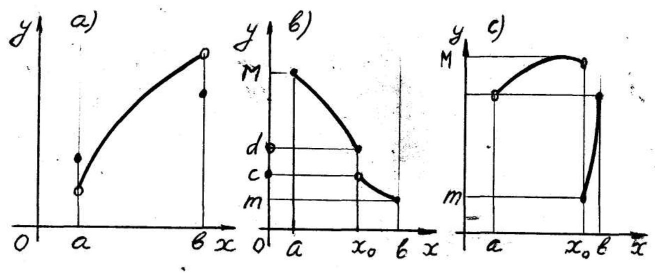


Fig. 6

Remark. Conclusions of the theorem can not fulfil (but sometimes can fulfil) if a function has at least one discontinuity point. For example a function on the fig. 6a with discontinuity points a and b hasn't the greatest

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[1] Weierstrass, K.Th.W. (1815 - 1897), a German mathematician
[2] Bolzano, B. (1781 - 1848), a Czech mathematician, philosopher, and logician
[3] Cauchy, A.L. (1780 - 1859), an eminent French mathematician

and the least values. A function on the fig. 6 b with one discontinuity point x_0 possesses the greatest *M* and the least *m* values but doesn't take on values which belong to an interval $[c, d)$. Finally a function on the fig. 6 c has two discontinuity points *a* and x_0, but the conclusions 1) and 2) of the theorem are fulfilled.

Analogous theorem is valid for a function of several variables.

Def. 7. Union $\overline{D}$ of a domain D and its boundary ∂D is called **closed domain**, $\overline{D} = D \bigcup \partial D$.

Def. 8. A domain is called **bounded** one if it's contained in some circle centered at the origin.

Theorem 5. If a function of several variables is continuous in a closed bounded domain $\overline{D}$ then:

1) It takes on the greatest *M* and the least *m* values in $\overline{D}$.

2) It takes on all values containing between *m* and *M*.

3) If it has values of different signs in two points of the domain then it has at least one zero in $\overline{D}$.

POINT 4. INTERVAL METHOD AND ITS EXTENSION

The third conclusion of the theorem 4 often applies in so-called interval method for solving inequalities or definition of signs of functions.

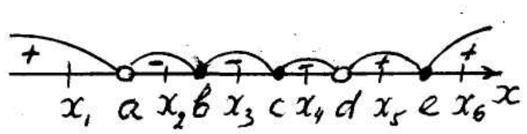

Fig. 7

Let for example a function $y = f(x)$ of one variable has three zeros *b*, *c*, *e* and two discontinuity points *a*, *d* on $\Re^1 = (-\infty, \infty)$ (fig. 7). The points *a*, *b*, *c*, *d*, *e* generate six intervals

$$(-\infty, a), (a, b), (b, c), (c, d), (d, e), (e, +\infty)$$

on every of which the function, by virtue of the third conclusion of the theorem 4, has a constant sign. To determine this sign it's sufficient to find it at arbitrary point of an interval. On fig. 7 points $x_1, x_2, x_3, x_4, x_5, x_6$ are taken and a possible distribution of signs of the function on the intervals $(-\infty, a)$, $(a, b), (b, c), (c, d), (d, e), (e, +\infty)$ is shown.

Analogous method is applicable for functions of two variables.

Ex. 11. Solve the inequality $x^2 > a^2$ for $a > 0$.

Solution. A function $f(x) = x^2 - a^2$ is contininous one on $\Re^1$ and has two zeroes $\pm a$ which generate three intervals $(-\infty, -a), (-a, a), (a, +\infty)$. For the points $x = -2a \in (-\infty, -a)$ and $x = 2a \in (a, +\infty)$ we have $f(-2a) > 0, f(2a) > 0$. For the point $x = 0 \in (-a, a)$ $f(0) < 0$. Therefore the inequality is true if

$$x \in (-\infty, -a) \cup (a, +\infty), \text{ or if } |x| > a.$$

Ex. 12. Find the domain of definition of a function of two variable *x*, *y*

$$Z = \sqrt{\frac{x^2 + y^2 - 16}{x^2 + y^2 - 4}}.$$

Solution. Let

$$f(x, y) = \frac{x^2 + y^2 - 16}{x^2 + y^2 - 4}.$$

The domain of definition of the function *Z* is the set of points (x, y) of the *x*O*y*-plane for which the inequality

$$f(x, y) \equiv \frac{x^2 + y^2 - 16}{x^2 + y^2 - 4} \geq 0$$

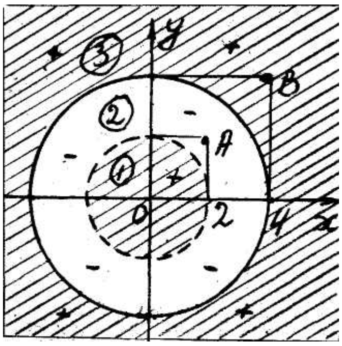

Fig. 8

holds. It's necessary to solve this inequality. The function $f(x, y)$ equals zero on the circle $x^2 + y^2 - 16 = 0$ and doesn't exixt on the circle $x^2 + y^2 - 4 = 0$. These circles divide *xOy*-plane into 3 parts 1, 2, 3 (see fig. 8) in every of which the function, by virtue of the third conclusion of the theorem 5, has constant sign. To find this sign we calculate

$$f(0) = f(0, 0) = 4 > 0, f(A) = f(2, 2) = -2 < 0, f(B) = f(4, 4) = 4/7 > 0.$$

Therefore the function $f(x, y)$ is positive in the parts 1 and 3 of the *xOy*-plane.

Answer. Domain of definition of the function Z is hatched union of the disk

$x^2 + y^2 < 4$ without the boundary $x^2 + y^2 = 4$ and the outer part of the big circle $x^2 + y^2 = 16$ including this circle.

Ex. 13. Investigate a function

$$y = f(x) = \frac{x^3}{8 - x}$$

and graph it.

Investigation is fulfilled in the next order.

1) Domain of definition of the function is $D(f) = (-\infty, 8) \cup (8, +\infty)$. The graph of the function doesn't intersect the straight line $x = 8$ which is perpendicular to the Ox-axis.

2) Intervals of constant sign of the function. Points $x = 0$ (zero of the function) and $x = 8$ (discontinuity point) generate three intervals $(-\infty, 0), (0, 8), (8, +\infty)$. On the interval $(0, 8)$ the function is positive so its graph lies above the Ox-axis. On the intervals $(-\infty, 0), (8, +\infty)$ the function is negative and its graph lies below the Ox-axis.

3) Knowing the sign of the function we easy find its right and left limits at the discontinuity point $x = 8$ namely

$$f(8-0) = \lim_{x \to 8-0} \frac{x^3}{8-x} = \left(\frac{1}{0} = \infty\right) = +\infty, \; f(8+0) = \lim_{x \to 8+0} \frac{x^3}{8-x} = \left(\frac{1}{0} = \infty\right) = -\infty.$$

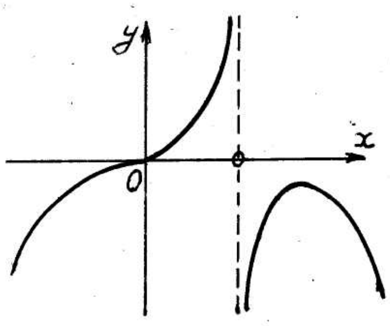

Fig. 9

Graph of the function goes up if $x \to 8-0$ and goes down if $x \to 8+0$.

4) Limit of the function as $x \to \pm\infty$

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^3}{8-x} = \lim_{x \to \pm\infty} \frac{x^3}{-x} = -\lim_{x \to \pm\infty} x^2 = -\infty.$$

Graph of the function goes down as $x \to \pm\infty$.

5) Intersection points of the graph of the function with Ox-, Oy-axes.

Oy: $x = 0 \Rightarrow y = 0 \Rightarrow$ O(0;0);

Ox: $y = 0 \Rightarrow x = 0 \Rightarrow$ O(0;0).

Taking into account obtained results we graph the function (fig. 9).

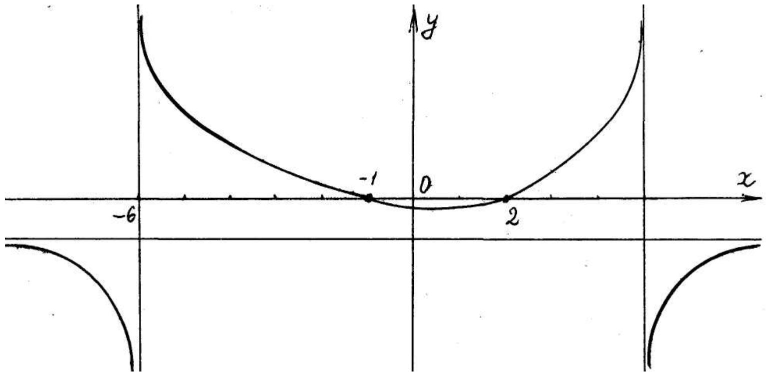

Fig. 10

Ex. 14. Investigate and plot yourselves the graph of the function

$$f(x)=\frac{(x+1)(2-x)}{(x-5)(x+6)}$$

The key.

1) $D(f)=(-\infty,-6)\cup(-6,5)\cup(5,+\infty)$.

2) $f(x)>0$ on $(-6,5)$; $f(x)<0$ on $(-\infty,-6)\cup(-1,2)\cup(5,+\infty)$.

3) $f(-6-0)=-\infty$, $f(-6+0)=+\infty$; $f(5-0)=+\infty$, $f(5+0)=-\infty$.

4) $\lim\limits_{x\to\pm\infty} f(x)=-1$.

5) $(-1;0)\in Ox$, $(2;0)\in Ox$, $(0;-1/15)\in Oy$ (see fig. 10).

INTRODUCTION IN MATHEMATICAL ANALYSIS:

basic terminology

1. Appróach [tend to, go to] a númber *a* (from the left/right) [the plus or mínus infinity] (about an árgument, a nùmérical séquence, a fúnction)	Прямувати до числа *a* (зліва, справа) [до плюс чи мінус нескінченності] (про арґумент, числову послідовність, функцію)	Стремиться к числу *a* (слева, справа) [к плюс или минус бесконечности] (об аргументе, числовой последовательности, функции)
2. Ball [globe] (of/with a rádius *R* céntered at a point *A*)	Куля (радіуса *R* з центром (в точці) *A*)	Шар (радиуса *R* с центром (в точке) *A*)
3. Bóundary [fróntier] of a domáin [a région], of a set	Границя області, множини	Граница области, множества
4. Bóundary [fróntier] póint of a set	Гранична точка множини	Граничная точка множества
5. Bóunded	Обмежений	Ограниченный
6. Bóunded [límited] domáin [région]	Обмежена область	Ограниченная область
7. Bóunded [límited] set	Обмежена множина	Ограниченное множество
8. Bounded above	Обмежений зверху	Ограниченный сверху
9. Bóunded belów	Обмежений знизу	Ограниченный снизу
10. Bóunded fúnction [nùmérical séquence]	Обмежена функція [числова послідовність]	Ограниченная функция [числовая последовательность]
11. Cháracter/náture of discòntinúity (póint)	Характер (точки) розриву	Характер (точки) разрыва
12. Chóose an árbitrary póint in/on each/évery ínterval	Вибрати довільну точку на кожному інтервалі	Выбрать произвольную точку на каждом интервале
13. Circle (with a rádius *R* and with a céntre (at a póint) *A*; circle céntered [the céntre of which is] at (a póint) *A*	Круг (радіуса *R* з центром (в точці) *A*)	Круг (радиуса *R* с центром (в точке) *A*)
14. Círcular néighbourhood of a póint	Круговий окіл точки	Круговая окрестность точки
15. Clósed bóunded domain/région	Замкнена обмежена область	Замкнутая ограниченная область
16. Clósed domáin [région]	Замкнена область	Замкнутая область
17. Cómposite fúnction,	Складена функція, функ-	Сложная функция, функ-

fúnction of a fúnction, superposítion [còmposítion] of fúnctions	ція від функції, суперпозиція функцій	ция от функции, суперпозиция функций
18.Connécted [tie] set	Зв"язна множина	Связное множество
19.Consérve a constant/fixed sign [do not chánge a sign] in/on/over an ínterval	Зберігати сталий/фіксований знак [не змінювати знак] на інтервалі	Сохранять постоянный/фиксированный знак (не изменять знак) на интервале
20.Continuity of a fúnction at a póint *a*	Неперервність функції в точці *a*	Непрерывность функции в точке *a*
21.Continúity of a fúnction on/in/over a(n) ínterval/segment	Неперервність функції на інтервалі/відрізку	Непрерывность функции на интервале/отрезке
22.Continúity on the left [on the right], [left/right continúity] of a fúnction at the póint *a*	Неперервність функції зліва/справа в точці *a*	Непрерывность функции слева/справа в точке *a*
23.Contínuous curve	Неперервна крива	Непрерывная кривая
24.Contínuous fúnction	Неперервна функція	Непрерывная функция
25.Contínuous fúnction (on the left [on the right] [(left/right) contínuous fúnction] at the póint *a*	Функція, неперервна в точці *a* (зліва/справа)	Функция, непрерывная в точке *a* (слева/справа)
26.Contínuous fúnction on/in/over a(n) interval/ ségment	Функція, неперервна на інтервалі/відрізку	Функция, непрерывная на интервале/отрезке
27.Convérge (to a númber *a*)	Збігатися (до числа *a*)	Стремиться (к числу *a*)
28.Convérgence of a nùmérical séquence (to a númber *a*)	Збіжність числової послідовності (до числа *a*)	Сходимость числовой последовательности (к числу *a*)
29.Convérgent (to the númber *a*) nùmérical séquence	Збіжна (до числа *a*) числова послідов-ність	Сходящаяся (к числу *a*) числовая последовательность
30.Decréase (strictly, nónstríctly)	Спадати (строго, нестрого)	Убывать (строго, нестрого)
31.Décrease (strict, nónstríct)	Спадання (строге, нестроге)	Убывание (строгое, нестрогое)
32.Decréasing (strictly, nónstríctly)	Спадний (строго, нестрого)	Убывающий (строго, нестрого)
33.Deléted ε-néighbourhood of a póint *a*	Проколений ε-окіл точки *a*	Проколотая ε-окрестность точки *a*
34.Deléted néighbourhood of a póint	Проколений окіл, окіл з виколеною точкою	Проколотая окрестность, окрестность с выколотой

		точкой
35.Discontinuity (of the first/second kind) of a fúnction at the póint *a*	Розрив функції (першого /другого роду) в точці *a*	Разрыв функции (первого/второго рода) в точке *a*
36.Discontinuity of a fúnction at the póint *a* (fínite, ínfinite, remóvable)	Розрив функції (скінченний/нескінченний, усувний) в точці *a*	Разрыв функции (конечный/бесконечный, устранимый) в точке *a*
37.Discontinuity póint [póint of discòntinúity] of the first/second kind	Точка розриву першого/ другого роду	Точка разрыва первого/ второго рода
38.Dìscontínuous fúnction at the póint *a*	Функція, розривна в точці *a*	Функция, разрывная в точке *a*
39.Dìstribútion of signs of a fúnction on the intervals	Розподіл знаків функції на интервалах	Распределение знаков функции на интервалах
40.Divíde/partition/dècompóse an ínterval ínto parts by noughts/zéros and discòntinúity póints of a fúnction	Ділити/поділити інтервал на частини нулями й точками розриву функції	Делить/разделить интервал на части нулями и точками разрыва функции
41.Domáin [région]	Область	Область
42.Domáin of dèfinítion of a fúnction	Область визначення функции	Область определения функции
43.Equívalence, equívalency	Еквівалентність	Эквивалентность
44.Equívalent infíntely larges	Еквівалентні нескінченно великі	Эквивалентные бесконечно большие
45.Equívalent infíntely smalls [equívalent ìnfinitésimals]	Еквівалентні нескінченно малі	Эквивалентные бесконечно малые
46.Evàluáte (find the válue of) an ìndetérminate form/expréssion [an ìndetérminacy/ìndetérminateness/indetermination/ìndetérminedness]	Розкрити невизначеність	Раскрыть неопределённость
47.Extérior [óutside] póint of a set	Зовнішня точка множини	Внешняя точка области
48.Find the íntervals of cónstant/fixed/invàriable sign of a fúnction by the méthod of íntervals, by the ínterval méthode	Знайти інтервали знакосталості функції методом інтервалів	Найти интервалы знакопостоянства методом интервалов
49.Find the límit (of a	Знайти границю (функ-	Найти предел (функции,

fúnction, of a nùmérical sequence)	ції, числової послідовності)	числовой последовательности)
50.Find/detérmine the sign of a fúnction at the chósen póint, in/on each/évery ínterval	Знайти/визначити знак функції у вибраній точці, на (кожному) інтервалі	Найти/определить знак функции в выбранной точке, на (каждом) интервале
51.Fínite discòntinúity of a fúnction at the póint *a*	Скінченний розрив функції в точці *a*	Конечный предел функции в точке *a*
52.Fínite jump of a fúnction at the póint of its discòntinúity [at its discontinuity (póint)]	Скінченний стрибок функції в точці її розриву	Конечный прыжок функции в точке её разрыва
53.Fínite límit	Скінченна границя	Конечный предел
54.Fúnction of a nátural árgument	Функція натурального арґументу	Функция натурального аргумента
55.Fúnction of one [two, three, *n*, séveral] váriables	Функція однієї [двох, трьох, *n*, декількох] змінних	Функция одной [двух, трёх, *n*, нескольких] переменных
56.Géneral term/élement of a nùmérical séquence	Загальий член/елемент числової послідовності	Общий член/элемент числовой последовательности
57.Graph of a fúnction of two váriables	Графік функції двох змінних	График функции двух переменных
58.Graph of a fúnction contínuous on/in/over a ségment	Графік функції, неперервної на відрізку	График функции, непрерывной на отрезке
59.Graph of a fúnction having póints of discòntinúity [discòntinúities, discòntinúity póints]	Графік функції, яка має точки розриву	График функции, которая имеет точки разрыва
60.Gréatest válue (*M*) of a fúnction contínuous on/in/over a ségment	Найбільше (*M*) значення функції, неперервної на відрізку	Наибольшее (*M*) значение функции, непрерывной на отрезке
61.Have a discòntinúity, jump at the póint *a* (about a fúnction)	Мати/зазнавати/терпіти розрив, скачок в точці *a* (про функцію)	Иметь/претерпевать разрыв, скачок в точке *a* (о функции)
62.Have/possés a límit	Мати границю	Иметь предел
63.Hóle in a graph of a fúnction (at the póint of its remóvable discòntinúity)	Дірка в графіку функції (в точці її усувного розриву)	Дыра в графике функции (в точке ее устранимого разрыва)
64.Íncrease [-s] (strict, nónstríct)	Зростання (строге, нестроге)	Возрастание (строгое, нестрогое)
65.Incréase [-s] (strictly, nónstríctly)	Зростати (строго, нестрого)	Возрастать (строго, нестрого)

66.Incréasing [-s-] (strictly, nónstríctly)	Зростаючий (строго, нестрого)	Возрастающий (строго, нестрого)
67.Íncrement of a fúnction at a póint a	Приріст функції в точці a	Приращение функции в точке a
68.Íncrement of an árgument	Приріст арґументу	Приращение аргумента
69.Ìndetérminate form/expéssion [ìndetérminacy, ìndetérminateness, ìndeterminátion, ìndetérminedness] of the type/form	Невизначеність вигляду	Неопределённость вида
70.Ínfinit límit	Нескінченна границя	Бесконечный предел
71.Ínfinite discòntinúity of a fúnction at the póint a	Нескінченний розрив функції в точці a	Бесконечный разрыв функции в точке a
72.Infínitely large	Нескінченно велика (величина)	Бесконечно большая (величина)
73.Infínitely large function, nùmérical séquence	Нескінченно велика функція, числова послідовність	Бесконечно большая функция, числовая последовательность
74.Infínitely small, ìnfinitésimal	Нескінченно мала (величина)	Бесконечно малая (величина)
75.Infíntely small [ìnfinitésimal] function, nùmérical séquence; ìnfinitésimal	Нескінченно мала функція, числова послідовність	Бесконечно малая функция, числовая последовательность
76.Intérior [ínner] póint of a set	Внутрішня точка множини	Внутренняя точка множества
77.Ínterval of cónstant/fíxed/inváriable sign of a function	Інтервал знакосталості функції	Интервал знакопостоянства функции
78.Intíre/únbróken curve	Неперервна/суцільна крива	Непрерывная/сплошная кривая
79.Invéstigate (a fúnction onto/upon/for a continúity, cháracter/náture of a póint of discòntinúity [discòntinúity póint])	Дослідити (функцію на неперервність, на характер точки розриву)	Исследовать (функцию на непрерывность, на характер точки разрыва)
80.Jump (fínite, ínfinite) of a graph of a fúnction at the póint of its discòntinúity [at its discòntinúity (póint)]	Стрибок (скінченний/нескінченний) графіка функції в точці її розриву	Прыжок (конечный/бесконечный) графика функции в точке её разрыва
81.Léast (m) válue of a	Найменше (m) значення	Наименьшее (m) значе-

fúnction contínuous on/in-/over a ségment	функції, неперервної на відрізку	ние функции, непрерывной на отрезке
82.Left-hand límit [límit on the left] of a fúnction at the póint *a*	Ліва [лівобічна, лівостороння] границя функції в точці *a*	Левый [левосторонний] предел функции в точке *a*
83.Lével line/curve of a fúnction of two váriables	Лінія рівня функції двох змінних	Линия уровня функции двух переменных
84.Lével súrface of a fúnction of three váriables	Поверхня рівня функції трьох змінних	Поверхность уровня функции трёх переменных
85.Límit (of a fúnction, of nùmérical séquence)	Границя (функції, числової послідовності)	Предел (функции, числовой последовательности)
86.Límit of a fúnction at the plus or mínus infínity, if/as/when/while *x* appróaches [tends to, goes to] the plus or mínus infínity	Границя функції на плюс чи мінус нескінченності	Предел функции на плюс или минус бесконечности
87.Limit of a fúnction at the point *a* (biláteral/two-sided/doublesided, unilateral/one-sided)	Границя функції в точці *a* (двобічна/двостороння, однобічна/одностороння)	Предел функции в точке *a* (двусторонняя, односторонняя)
88.Límit of a fúnction at the póint *a* (from the left [from the right])	Границя функції в точці *a* (зліва, справа)	Предел функции в точке *a* (слева, справа)
89.Límit of a fúnction $f(x)$ if/as/when/while *x* appróaches [tends to, goes to] … (by/for ténding/téndency of *x* to…, for *x* appróaching [tending to]…)	Границя функції $f(x)$, якщо *x* прямує до… (при прямуванні *x* до…, при *x* прямуючому до…)	Предел функциии $f(x)$, если *x* стремится к … (при стремлении *x* к …; при *x*, стремящемся к…)
90.Máp(ping)	Відображення	Отображение
91.Mápping of a set *X* **into /onto** a set *Y*	Відображення множини *X* **в/на** множину *Y*	Отображение множества *X* **в/на** множество *Y*
92.Méthod of íntervals, ínterval méthode (for solútion of an ìnequálity, for detèrminátion a fúnction sign)	Метод інтервалів (для розв"язання нерівності, для визначення знака функції)	Метод интервалов (для решения неравенства, для определения знака функции)
93.Nátural domáin of dèfinítion of a fúnction	Природна [натуральна] область визначення функції	Естественная [натуральная] область определения функции
94.*n*-diménsional space	Ен-вимірний (*n*-вимірний) простір	Эн-мерное (*n*-мерное) пространство

95.Néighbourhood of a póint	Окіл точки	Окрестность точки
96.Nóte [mark (off), trace, óutline] póints on the áxis and get [obtain, receive, deríve] séveral/some intervals	Відкласти [відмітити, нанести] точки на осі й отримати декілька інтервалів	Отложить (отметить, нанести) точки на оси и получить несколько интервалов
97.Nùmérical séquence	Числова послідовність	Числовая последовательность
98.One-diménsional space	Одновимірний простір	Одномерное пространство
99.Open set	Відкрита множина	Открытое множество
100. Pássage to the límit	Граничний перехід, перехід до границі	Предельный переход, переход к пределу
101. Póint of discòntinúity [discontinuity póint] of the first/second kind	Точка розриву першого/ другого роду	Точка разрыва первого/ второго рода
102. Póint of remóvable discòntinúity	Точка усувного розриву	Точка устранимого разрыва
103. Póint set, set of póints, púnctual set	Множина точок, точкова множина	Множество точек, точечное множество
104. Próperty (*pl* properties) (of límit)	Властивість (границі)	Свойство (предела)
105. Remárcable/stándard/stándardized límit	Стандартна границя	Замечательный предел
106. Remóvable discontinuity of a fúnction at the póint *a*	Усувний розрив функції в точці *a*	Устранимый разрыв функции в точке *a*
107. Right-hand límit [límit on the right] of a fúnction at the póint *a*	Права/правобічна/правостороння границя функції в точці *a*	Правый/правосторонний предел функции в точке *a*
108. Right-hand/right-side continúity of a fúnction at the póint *a*	Правобічна/правостороння неперервність функції в точці *a*	Правая/правосторонняя непрерывность функции в точке *a*
109. Solve the ìnequálity by the méthod of íntervals, by the ínterval méthode	Розв"язати нерівність методом інтервалів	Решить неравенство методом интервалов
110. Sphére (of/with a rádius *R* céntered at a point *A*)	Сфера (радіуса *R* з центром (в точці) *A*)	Сфера (радиуса *R* с центром (в точке) *A*)
111. Sphérical [glóbular] néighbourhood of a póint	Кульовий окіл точки	Шаровая окрестность точки
112. Sphérical néighbourhood of a póint	Сферичний окіл точки	Сферическая окрестность точки

113. Table/list of equívalent infíntly smalls [of equívalent ìnfinitésimals]	Таблиця еквівалентних нескінченно малих	Таблица эквивалентных бесконечно малых
114. Take on válues of dif-ferrent signs	Набувати значення різних знаків	Принимать значения разных знаков
115. Ténding/téndency (of an árgument, of a fúnction) to the númber *a* (from the left [from the right]), to the plus or mínus infínity	Прямування (аргумента, функції) до числа *а* (зліва, справа), до плюс чи мінус нескінченності	Стремление (аргумента, функции) к числу *а* (слева, справа), к плюс или минус бесконечности
116. Term/élement of a nùmérical séquence	Член/елемент числової послідовності	Член/элемент числовой последовательности
117. Three-diménsional space	Тривимірний простір	Трёхмерное пространство
118. Tótal íncrement of a function	Повний приріст функції	Полное приращение функции
119. Turn/chánge/transfórm into [redúce, go to, becóme] zéro/nought; vánish	Обертатися на нуль, перетворюватися на/в нуль, анулюватися	Обращаться в нуль, аннулироваться
120. Two-diménsional space	Двовимірний простір	Двумерное пространство
121. Ùnconnécted set	Незв"язна множина	Несвязное множество
122. Úniláteral/one-sided continúity of a fúnction at the póint *a*	Однобічна [одностороння] неперервність функції в точці *а*	Односторонняя непрерывность функции в точке *а*
123. Úniláteral/one-sided límit of a fúnction at the póint *a*	Однобічна [одностороння] границя функції в точці *а*	Односторонний предел функции в точке *а*
124. Uníqueness of the límit	Єдиність границі	Единственность предела
125. Unlímited set	Необмежена множина	Неограниченное множество
126. Zéro/nought [róot] (of an equátion, of a númerator/denominator, of a function)	Нуль [корінь] (рівняння, чисельника, знаменника, функції)	Нуль [корень] (уравнения, числителя, знаменателя, функции)

DIFFERENTIAL CALCULUS
LECTURE NO. 14. DERIVATIVE

POINT 1. PROBLEMS LEADING TO THE CONCEPT OF THE DERIVATIVE
POINT 2. DERIVATIVES AND PARTIAL DERIVATIVES
POINT 3. DERIVATIVES OF SOME BASIC ELEMENTARY FUNCTIONS
POINT 4. DIFFERENTIABILITY AND CONTINUITY
POINT 5. DERIVATIVES OF THE SUM, DIFFERENCE, PRODUCT AND QUOTIENT OF FUNCTIONS

POINT 1. PROBLEMS LEADING TO THE CONCEPT OF THE DERIVATIVE

1. The rate of changing of a function

Let $y = f(x)$ be a function of one variable and $x = x_0$ is some point. If the argument *x* receives an increment $\Delta x = x - x_0$ then the function receives an increment

$$\Delta y = \Delta f(x_0) = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0)$$

which is a changing of the function on the interval $[x_0, x_0 + \Delta x] \equiv [x_0, x]$. The ratio

$$V_{av} = \frac{\Delta y}{\Delta x} = \frac{\Delta f(x_0)}{\Delta x} = \frac{f(x) - f(x_0)}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

is called the average rate [the mean rate] of changing of the function on the interval $[x_0, x_0 + \Delta x] \equiv [x_0, x]$. Let $\Delta x \to 0$ that is $x \to x_0$. The limit

$$V(x_0) = \lim_{\Delta x \to 0} V_{av} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \qquad (1)$$

is called the **rate of changing** of the function at the point x_0.

2. The labour productivity

Let $U(t)$ is a produced quantity of some factory during a time *t* (that is during a time interval from 0 to *t*). Then the increment of the function $U(t)$ at a point t_0,

$$\Delta U(t_0) = U(t_0 + \Delta t) - U(t_0),$$

is the produced quantity during the time interval from t_0 to $t_0 + \Delta t$. The ratio

$$f_{av} = \frac{\Delta U(t_0)}{\Delta t} = \frac{U(t_0 + \Delta t) - U(t_0)}{\Delta t}$$

is the average [the mean] labour productivity during this time interval. Limit of the average labour productivity as $\Delta t \to 0$,

$$f(t_0) = \lim_{\Delta t \to 0} f_{av} = \lim_{\Delta t \to 0} \frac{\Delta U(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{U(t_0 + \Delta t) - U(t_0)}{\Delta t}, \qquad (2)$$

is called the labour productivity of the factory at the time moment t_0.

2. The tangent to a curve

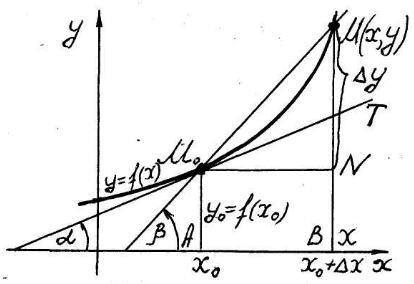

Fig. 1

Let be given a curve $y = f(x)$ and $M_0(x_0, y_0)$, $y_0 = f(x_0)$ is its fixed point, $M(x, y)$ is its arbitrary point. Straight line M_0M is called a **secant** of the curve $y = f(x)$. Let $M \to M_0$ along the curve. If there exists the **limiting position** M_0T of the secant M_0M as $M \to M_0$ (from the right and from the left), then the straight line M_0T is called the **tangent** (or the tangent line) to the curve $y = f(x)$ at the point $M(x_0, y_0)$. Its slope (angular coefficient) equals

$$k_{tg} = tg\alpha = \lim_{\substack{\beta \to \alpha \\ M \to M_0 \\ \Delta x \to 0}} tg\beta = \lim_{\Delta x \to 0} \frac{NM}{M_0N} = \lim_{\Delta x \to 0} \frac{BM - BN}{AB} = \lim_{\Delta x \to 0} \frac{f(x) - f(x_0)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \qquad (3)$$

POINT 2. DERIVATIVES AND PARTIAL DERIVATIVES

The derivative of a functions of one variable

Let be given a function of one variable $y = f(x)$. Giving arbitrary increment $\Delta x = x - x_0$ to the argument x and finding corresponding increment of the function at the point x_0

$$\Delta y = \Delta f(x_0) = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0) \qquad (4)$$

we find their ratio

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f(x_0)}{\Delta x} = \frac{f(x) - f(x_0)}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

and pass to the limit as $\Delta x = x - x_0 \to 0$.

Def.1. The limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x) - f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}, \qquad (5)$$

that is the limit of the ratio of the increment of the function $y = f(x)$ at the point x_0 to the corresponding increment of the argument Δx as this latter tends to zero is called the derivative of the function at the point x_0. We denote the derivative by one of the next notations

$$y' = y'(x_0) = f'(x_0) = \frac{dy}{dx} = \frac{df(x_0)}{dx}$$

and so

$$y' = f'(x_0) = \frac{dy}{dx} = \frac{df(x_0)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \qquad (6)$$

Above-stated examples allow to establish some **senses of the derivative**.

1. From the formula (1) it follows that the **rate of changing of the function** $y = f(x)$ at the point x_0 is the derivative of the function at this point

$$V(x_0) = f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \qquad (7)$$

2. From the formula (2) it follows that the **labour productivity** of the factory at the time moment t_0 is the derivative of the function $U(t)$, that is the derivative of the produced quantity of the factory, at this moment,

$$f(t_0) = U'(t_0) = \lim_{\Delta t \to 0} \frac{\Delta U(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{U(t_0 + \Delta t) - U(t_0)}{\Delta t} \qquad (8)$$

3. From the formula (3) it follows the **geometric sense of the derivative:**

The **slope** k_{tg} **of the tangent** M_0T to the graph of the function $y = f(x)$ at its point $M(x_0, y_0)$, $y_0 = f(x_0)$ (fig. 1) is the derivative of the function at the point x_0,

$$k_{tg} = tg\alpha = f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f(x_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \qquad (9)$$

Equation of the tangent M_0T (with the slope $k_{tg} = tg\alpha = f'(x_0)$) is

$$y = y_0 + f'(x_0)(x - x_0),\; y_0 = f(x_0). \qquad (10)$$

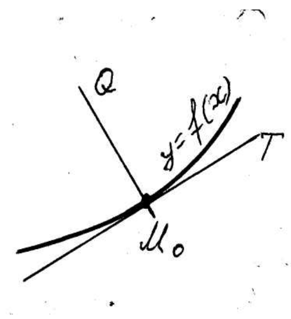

Fig. 2

The normal $M_0Q \perp M_0T$ to the graph of the function $y = f(x)$ at the point $M(x_0, y_0)$, $y_0 = f(x_0)$ (fig. 2) has the slope

$$k_{norm} = -\frac{1}{k_{tg}} = -\frac{1}{f'(x_0)}$$

and the next equation

$$y - y_0 = -\frac{1}{f'(x_0)} \cdot (x - x_0) \qquad (11)$$

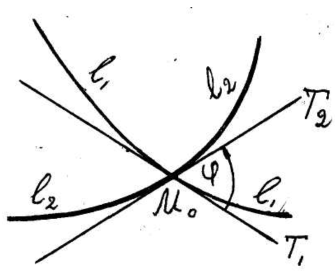

Fig. 3

It's interesting the next problem. Find the angle φ at which two curves $L_1 : y = f_1(x)$ and $L_2 : y = f_2(x)$ intersect (fig. 3).

Solving. Let $M_0(x_0, y_0)$ be intersection point of the curves L_1 and L_2 and M_0T_1, M_0T_2 are the tangents to L_1, L_2 at the point M_0. Their slopes are $k_{M_0T_1} = f_1'(x_0)$, $k_{M_0T_2} = f_2'(x_0)$ therefore

$$\tan\varphi = \frac{k_{M_0T_2} - k_{M_0T_1}}{1 + k_{M_0T_1} \cdot k_{M_0T_2}} = \frac{f_2'(x_0) - f_1'(x_0)}{1 + f_1'(x_0) \cdot f_2'(x_0)} \qquad (12)$$

Partial derivatives of a function of several variables

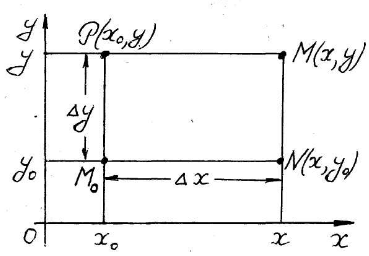

Fig. 4

For a functions of several variables we introduce the concept of partial derivatives. for example let there be given a function of two variables x, y

$$z = f(M) = f(x, y),\; M(x, y).$$

We introduce four points $M_0(x_0, y_0)$, $M(x, y)$, $N(x, y_0)$,

$P(x_0, y)$ and we imply $\Delta x = x - x_0, \Delta y = y - y_0$ whence it follows that $x = x_0 + \Delta x$, $y = y_0 + \Delta y$ (fig. 4).

Def. 2. Difference (for fixed $y = y_0$)

$$\Delta_x z = \Delta_x f(M_0) = f(N) - f(M_0) = f(x, y_0) - f(x_0, y_0) = f(x_0 + \Delta x, y_0) - f(x_0, y_0)$$

is called the **partial increment** of the function $z = f(M) = f(x, y)$ with respect to x at the point $M_0(x_0, y_0)$. Difference (for fixed $x = x_0$)

$$\Delta_y z = \Delta_y f(M_0) = f(P) - f(M_0) = f(x_0, y) - f(x_0, y_0) = f(x_0, y_0 + \Delta y) - f(x_0, y_0)$$

is called the partial increment of the function with respect to y at this point.

Def. 3. Partial derivatives of the function $z = f(M) = f(x, y)$ with respect to x, y at the point $M_0(x_0, y_0)$ are called (and denoted) correspondingly the next limits

$$\begin{aligned} z'_x &= f'_x(M_0) = f'_x(x_0, y_0) = \frac{\partial f(M_0)}{\partial x} = \frac{\partial f(x_0, y_0)}{\partial x} = \lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta_x f(M_0)}{\Delta x} = \\ &= \lim_{\Delta x \to 0} \frac{f(x, y_0) - f(x_0, y_0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \\ z'_y &= f'_y(M_0) = f'_y(x_0, y_0) = \frac{\partial f(M_0)}{\partial y} = \frac{\partial f(x_0, y_0)}{\partial y} = \lim_{\Delta y \to 0} \frac{\Delta_y z}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta_y f(M_0)}{\Delta y} = \\ &= \lim_{\Delta y \to 0} \frac{f(x_0, y) - f(x_0, y_0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y} \end{aligned} \qquad (13)$$

POINT 3. DERIVATIVES OF SOME BASIC ELEMENTARY FUNCTIONS

Derivatives of many basic elementary functions can be find on the base of definition of the derivative.

1. $C' = 0, C - const$.

■Let $y = f(x) = C$. Then

$$f(x + \Delta x) = C, \Delta y = f(x + \Delta x) - f(x) = C - C = 0, \frac{\Delta y}{\Delta x} = 0, y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0 \blacksquare$$

2. $x' = 1$.

■ Let $y = f(x) = x$. Then

$$f(x+\Delta x) = x + \Delta x,\ \Delta y = f(x+\Delta x) - f(x) = \Delta x,\ \frac{\Delta y}{\Delta x} = 1,\ y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 1,\ x' = 1 \blacksquare$$

3. $\left(x^{\alpha}\right)' = \alpha x^{\alpha-1},\ \alpha \in \Re$. In particular $\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}},\ \left(\sqrt[3]{x}\right)' = \frac{1}{3\sqrt[3]{x^2}}$.

■ Let $y = f(x) = x^{\alpha}$. Then

$$f(x+\Delta x) = (x+\Delta x)^{\alpha},\ \Delta y = f(x+\Delta x) - f(x) = (x+\Delta x)^{\alpha} - x^{\alpha} = \left(x\left(1+\frac{\Delta x}{x}\right)\right)^{\alpha} - x^{\alpha} =$$

$$= x^{\alpha}\left(\left(1+\frac{\Delta x}{x}\right)^{\alpha} - 1\right) \sim\ = x^{\alpha} \cdot \alpha \frac{\Delta x}{x} = \alpha x^{\alpha-1}\Delta x \Rightarrow \frac{\Delta y}{\Delta x} = \alpha x^{\alpha-1},\ y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \alpha x^{\alpha-1} \blacksquare$$

4. $\left(a^{x}\right)' = a^{x} \ln a^{x}$. In particular $\left(e^{x}\right)' = e^{x}$.

■Let $y = f(x) = a^{x}$. Then

$$f(x+\Delta x) = a^{x+\Delta x},\ \Delta y = f(x+\Delta x) - f(x) = a^{x+\Delta x} - a^{x} = a^{x}\left(a^{\Delta x} - 1\right) \sim\ a^{x} \cdot \Delta x \cdot \ln a$$

$$\frac{\Delta y}{\Delta x} = a^{x} \cdot \ln a \Rightarrow y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = a^{x} \cdot \ln a \blacksquare$$

5. $\left(\log_a x\right)' = \frac{1}{x \ln a}$. In particular $\left(\ln x\right)' = \frac{1}{x}$.

■Let $y = f(x) = \log_a x$. Then

$$f(x+\Delta x) = \log_a(x+\Delta x),\ \Delta y = f(x+\Delta x) - f(x) = \log_a(x+\Delta x) - \log_a x =$$

$$= \log_a \frac{x+\Delta x}{x} = \log_a\left(1+\frac{\Delta x}{x}\right) \sim \frac{\Delta x}{x \ln a} \Rightarrow \frac{\Delta y}{\Delta x} = \frac{1}{x \ln a},\ y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{1}{x \ln a} \blacksquare$$

6. $\left(\sin x\right)' = \cos x, \quad \left(\cos x\right)' = -\sin x$.

■Let for example $y = f(x) = \sin x$. Then

$$f(x+\Delta x) = \sin(x+\Delta x),\ \Delta y = f(x+\Delta x) - f(x) = \sin(x+\Delta x) - \sin x =$$

$$= 2\cos\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2} \sim 2\cos\left(x+\frac{\Delta x}{2}\right) \cdot \frac{\Delta x}{2} = \cos\left(x+\frac{\Delta x}{2}\right) \cdot \Delta x \Rightarrow \frac{\Delta y}{\Delta x} = \cos\left(x+\frac{\Delta x}{2}\right)$$

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \cos\left(x+\frac{\Delta x}{2}\right) = \cos x \blacksquare$$

Ex. 1. Find the angle between two intersecting lines $f_1(x) = \sin x,\ f_2(x) = \cos x$.

Solving. Intersection points of the lines are determined by the equation

$$sin\, x = cos\, x \Rightarrow tan\, x = 1,\ x_0 = \pi/4 + \pi n,\ n \in Z$$

$$f_1'(x_0) = \cos x_0 = \cos(\pi/4 + \pi n),\ f_2'(x_0) = -\sin x_0 = -\sin(\pi/4 + \pi n)$$

and by virtue of the formula (12)

$$tg\varphi = \frac{-\sin(\pi/4 + n\pi) - \cos(\pi/4 + n\pi)}{1 + \cos(\pi/4 + n\pi)\cdot(-\sin(\pi/4 + n\pi))} = -\frac{\sqrt{2}\sin(\pi/4 + n\pi)}{1 - \frac{1}{2}\sin(\pi/4 + 2n\pi)} =$$

$$= -\frac{\sqrt{2}\cos n\pi}{1/2} = -2\sqrt{2}\cos n\pi = -2\sqrt{2}(-1)^n = 2\sqrt{2}(-1)^{n+1}.$$

Ex. 2. Compile equations of the tangent and the normal to the curve $y = \sin x$ at the point with abscissa $x_0 = \frac{\pi}{6}$.

Solution. Let $y = f(x) = \sin x$. We have $y_0 = f(x_0) = \sin x_0 = \sin\frac{\pi}{6} = \frac{1}{2}$; $f'(x) = \cos x,\ f'(x_0) = \cos x_0 = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. Making use of the formulas (10), (11) we compile the equation of the tangent

$$y = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$$

and the equation of the normal

$$y = \frac{1}{2} - \frac{2}{\sqrt{3}}\left(x - \frac{\pi}{6}\right)$$

POINT 4. DIFFERENTIABILITY AND CONTINUITY

Def. 4. Function of one variable $y = f(x)$ $(x \in \Re)$ is called differentiable at the point x_0 if it has derivative $f'(x_0)$ at this point.

Let a function $y = f(x)$ is differentiable at the point x_0. On the base of definition of the derivative and the theory of limits

$$f'(x_0) = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} \Rightarrow \frac{\Delta y}{\Delta x} = f'(x_0) + \alpha,\ \Delta y = f'(x_0)\Delta x + \alpha\cdot\Delta x$$

where $\alpha = \alpha(\Delta x)$ is *IS* (infinitely small) as $\Delta x \to 0$. Therefore the increment of a function which is differentiable at the point x_0 can be represented in the next form

$$\Delta y = \Delta f(x_0) = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0) = A \cdot \Delta x + \alpha(\Delta x) \cdot \Delta x \qquad (14)$$

where $A = f'(x_0)$ and $\alpha = \alpha(\Delta x)$ is *IS* for $\Delta x \to 0$.

Definition of differentiable function of several variables is more delicate and is connected with generalization of the formula (14). For the sake of simplicity we'll say about function of two variables.

Def. 5. Function of two variables $z = f(M) = f(x, y)$ is called differentiable one at a point $M_0(x_0, y_0)$ if its total increment at this point

$$\Delta z = \Delta f(M_0) = f(M) - f(M_0) = f(x, y) - f(x_0, y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

(see Def. 4 in the Lecture 11 and fig. 4 in this Lecture) can be represented in the next form

$$\Delta z = f(M) - f(M_0) = A \cdot \Delta x + B \cdot \Delta y + \alpha \cdot \Delta x + \beta \cdot \Delta y \qquad (15)$$

where A, B are some numbers and α, β are *IS* as $\Delta x \to 0, \Delta y \to 0$. It's easy to prove that $A = f'_x(M_0) = f'_x(x_0, y_0), B = f'_y(M_0) = f'_y(x_0, y_0)$ and therefore

$$\begin{aligned} \Delta z = f(M) - f(M_0) = f'_x(M_0) \cdot \Delta x + f'_y(M_0) \cdot \Delta y + \alpha \cdot \Delta x + \beta \cdot \Delta y = \\ = f'_x(x_0, y_0) \cdot \Delta x + f'_y(x_0, y_0) \cdot \Delta y + \alpha \cdot \Delta x + \beta \cdot \Delta y \end{aligned} \qquad (16)$$

Theorem 1 (sufficient condition for differentiability)**.** If a function $z = f(M) = f(x, y)$ has partial derivatives in some neighbourhood of the point $M_0(x_0, y_0)$ and these derivatives are continuous at this point itself, then the function is differentiable one at this point.

We'll prove this theorem later.

As can be illustrated by examples it isn't sufficiently for a function to possess the partial derivatives at the point $M_0(x_0, y_0)$ for to be differentiable at this point.

Theorem 2 (necessary but not sufficient condition for differentiability)**.** If a function is differentiable at a point then it's continuous at this point (but not vice versa!).

■Let for example $y = f(x)$ is function of one variable which is differentiable at a point x_0 and let $\Delta x = x - x_0 \to 0$. It follows from (14) that increment of the function at the point x_0 goes to zero, $\Delta y = f(x_0 + \Delta x) - f(x_0) \to 0$, which means continuity of the function at the point x_0 ■

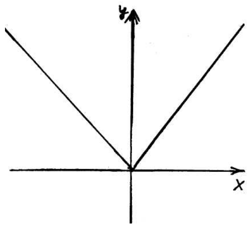

Fig. 5

Note. There're continuous functions which haven't derivative at least at one point.

Ex. 3. Function $y = |x|$ (see fig. 5) is continuous one at all points $x \in \Re$ but its derivative doesn't exist an the point $x = 0$.

■We've $x_0 = 0,\ f(x_0) = f(0) = |0| = 0,\ f(x_0 + \Delta x) = |\Delta x|$

$$\Delta y = f(x_0 + \Delta x) - f(x_0) = |\Delta x|;\ \frac{\Delta y}{\Delta x} = 1 \text{ for } \Delta x > 0,\ \frac{\Delta y}{\Delta x} = -1 \text{ for } \Delta x < 0$$

and so $y' = \lim\limits_{\Delta x \to 0} \Delta y / \Delta x$ doesn't exist.

POINT 5. DERIVATIVES OF THE SUM, DIFFERENCE, PRODUCT AND QUOTIENT OF FUNCTIONS

Let $u = u(x),\ v = v(x)$ be two differentiable functions of one variable x. The next rules are valid

1. $(u \pm v)' = u' \pm v'$ **2.** $(u \cdot v)' = u' \cdot v + u \cdot v'$ **3.** $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u \cdot v'}{v^2}$

■(for the product). Let's remark that

$\Delta u = u(x + \Delta x) - u(x),\ u(x + \Delta x) = u(x) + \Delta u = u + \Delta u;\ \Delta u = u + \Delta u$ *and* $\Delta v = v + \Delta v$.

If $\Delta x \to 0$ then $\Delta u \to 0,\ \Delta v \to 0$ because of differentiable functions $u = u(x),\ v = v(x)$ are those continuous. Therefore

$$(u \cdot v)' = \lim_{\Delta x \to 0} \frac{u(x + \Delta x) \cdot v(x + \Delta x) - u(x) \cdot v(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(u + \Delta u) \cdot (v + \Delta v) - u \cdot v}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{u \cdot v + v \cdot \Delta u + u \cdot \Delta v + \Delta u \cdot \Delta v - u \cdot v}{\Delta x} = \lim_{\Delta x \to 0} \frac{v \cdot \Delta u + u \cdot \Delta v + \Delta u \cdot \Delta v}{\Delta x} =$$

$$= \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x} \cdot v + u \cdot \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta x} \cdot \Delta v \right) = u' \cdot v + u \cdot v' + 0 = u' \cdot v + u \cdot v' \ \blacksquare$$

Particular cases.

a) $(C \cdot u) = C \cdot u'$ $(C - const)$ (constant factor can be taken outside the sign of differentiation) because of $(C \cdot u)' = C' \cdot u + C \cdot u' = 0 \cdot u + C \cdot u' = C \cdot u'$.

b) $\left(\frac{1}{v}\right)' = -\frac{v'}{v^2}$ by virtue of $\left(\frac{1}{v}\right)' = \frac{1' \cdot v - 1 \cdot v'}{v^2} = \frac{0 \cdot v - 1 \cdot v'}{v^2} = -\frac{v'}{v^2}$

Ex. 4. Derivatives of functions $\tan x, \cot x$.

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$(\cot x)' = \left(\frac{1}{\tan x}\right)' = -\frac{(\tan x)'}{\tan^2 x} = -\frac{1}{\tan^2 x \cdot \cos^2 x} = -\frac{1}{\sin^2 x}$$

Ex. 5. Find partial derivatives of the function $z = \ln x \cdot 5^y - \arctan x \cdot y^5$ with respect to *x* and *y*.

Remark. Finding partial derivative with respect to *x* (*y*) we consider the other variable *y* (correspondingly *x*) as fixed (or constant) one.

$$\frac{\partial z}{\partial x} = z'_x = \left(\ln x \cdot 5^y\right)'_x - \left(\arctan x \cdot y^5\right)'_x = (\ln x)'_x \cdot 5^y - (\arctan x)'_x \cdot y^5 =$$

$$= \frac{1}{x} \cdot 5^y - \frac{1}{1+x^2} \cdot y^5;$$

$$\frac{\partial z}{\partial y} = z'_y = \left(\ln x \cdot 5^y\right)'_y - \left(\arctan x \cdot y^5\right)'_y = \ln x \cdot (5^y)'_y - \arctan x \cdot (y^5)'_y =$$

$$= \ln x \cdot 5^y \ln 5 - \arctan x \cdot 5y^4.$$

Ex. 6. Differentiate the next function

$$y = \arcsin x \cdot \sqrt[3]{x}.$$

$$y' = \left(\arcsin x \cdot \sqrt[3]{x}\right)' = (\arcsin x)' \cdot \sqrt[3]{x} + \arcsin x \cdot \left(\sqrt[3]{x}\right)' = \frac{\sqrt[3]{x}}{\sqrt{1-x^2}} + \frac{\arcsin x}{3\sqrt[3]{x^2}}.$$

Ex. 7. Prove the formula for derivative of a product of three factors

$$(u \cdot v \cdot w)' = u' \cdot v \cdot w + u \cdot v' \cdot w + u \cdot v \cdot w'$$

Hint: consider the product $u \cdot v \cdot w$ as $u \cdot (v \cdot w)$.

LECTURE NO. 15. TECHNIQUE OF DIFFERENTIATION

POINT 1. THE DERIVATIVE OF A COMPOSITE FUNCTION
POINT 2. DIFFERENTIATION OF IMPLICIT, INVERSE AND PARAMETRICALLY REPRESENTED FUNCTIONS
POINT 3. THE HIGHER ORDER DERIVATIVES
POINT 4. THE DIFFERENTIAL
POINT 5. THE DIRECTIONAL DERIVATIVE. THE GRADIENT
POINT 6. DERIVATIVES IN ECONOMICS. THE ELASTICITY

POINT 1. THE DERIVATIVE OF A COMPOSITE FUNCTION

Theorem 1. If functions of one variable $y = f(u), u = \varphi(x)$ are those differrentiable, then the composite function $y = f(\varphi(x))$ possesses the derivative which is calculated by the next rule

$$y' = f'(\varphi(x))\cdot\varphi'(x) \text{ or for short } y'_x = y'_u \cdot u'_x \qquad (1)$$

■ From the theorem 2 of preceding lecture it follows that the differentiable functions $y = f(u), u = \varphi(x)$ are those continuous. So if the increment of the argument x tends to zero, $\Delta x \to 0$, then $\Delta u = \varphi(x + \Delta x) - \varphi(x) \to 0$ and therefore the increment of the function tends to zero, $\Delta y = f(u + \Delta u) - f(u) \to 0$.

On the base of the formula (14) of the Lecture No. 14 we can write

$$\Delta y = f'(u)\Delta u + \alpha \cdot \Delta u$$

where α is *IS* for $\Delta u \to 0$ (and so for $\Delta x \to 0$). Dividing both sides of the equality by Δx and passing to the limit for $\Delta x \to 0$ we get

$$\frac{\Delta y}{\Delta x} = f'(u)\frac{\Delta u}{\Delta x} + \alpha \cdot \frac{\Delta u}{\Delta x},\ \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = f'(u)\cdot \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} + \alpha \cdot \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} = f'(u)\cdot u' + 0\cdot u'$$

whence the formula (1) follows■

Note. Function $u = \varphi(x)$ is often called an **intermediate argument** or an **inner function**. We can state the nest rule: derivative of a composite function equals the product of its derivative with respect to the intermediate argument [to the inner function] and the derivative of the intermediate argument [of the inner function].

Applying the theorem and all preceding formulas for differentiation of the basic elementary functions (see Lecture No. 12) we can compile the next table in which $u = u(x)$ means some function.

Table of derivatives

1. $\left(u^{\alpha}\right)' = \alpha u^{\alpha-1} \cdot u'$ a) $\left(\sqrt{u}\right)' = \dfrac{1}{2\sqrt{u}} \cdot u'$ b) $\left(\sqrt[3]{u}\right)' = \dfrac{1}{3\sqrt[3]{u^2}} \cdot u'$ c) $\left(\dfrac{1}{u}\right)' = -\dfrac{1}{u^2} \cdot u'$
2. $\left(a^u\right)' = a^u \ln a \cdot u'$ a) $\left(e^u\right)' = e^u \cdot u'$
3. $\left(\log_a u\right)' = \dfrac{1}{u}\log_a e \cdot u' = \dfrac{1}{u \ln a} \cdot u'$ a) $\left(\ln u\right)' = \dfrac{1}{u} \cdot u'$
4. $(\sin u)' = \cos u \cdot u'$
5. $(\cos u)' = -\sin u \cdot u'$
6. $(\tan u)' = \dfrac{1}{\cos^2 u} \cdot u' = \sec^2 u \cdot u'$ $\left(\sec x = \dfrac{1}{\cos x}\right)$
7. $(\cot u)' = -\dfrac{1}{\sin^2 u} \cdot u' = -\operatorname{cosec}^2 u \cdot u'$ $\left(\operatorname{cosec} x = \dfrac{1}{\sin x}\right)$
8. $(\sec u)' = \left(\dfrac{1}{\cos u}\right)' = \sec u \cdot \tan u \cdot u'$
9. $(\operatorname{cosec} u)' = \left(\dfrac{1}{\sin u}\right)' = -\operatorname{cosec} u \cdot \cot u \cdot u'$
10. $(\arcsin u)' = \dfrac{1}{\sqrt{1-u^2}} \cdot u'$
11. $(\arccos u)' = -\dfrac{1}{\sqrt{1-u^2}} \cdot u'$
12. $(\arctan u)' = \dfrac{1}{1+u^2} \cdot u'$
13. $(\operatorname{arccot} u)' = -\dfrac{1}{1+u^2} \cdot u'$

Ex. 1. $\left(\sin^6 x\right)' = \left((\sin x)^6\right)' = 6(\sin x)^5 \cdot (\sin x)' = 6\sin^5 x \cdot \cos x$

Ex. 2. $\left(\sqrt[3]{arc\cot x}\right)' = \dfrac{1}{3\sqrt[3]{(arc\cot x)^2}} \cdot (arc\cot x)' = -\dfrac{1}{3\sqrt[3]{(arc\cot x)^2}\left(1+x^2\right)}$

Ex. 3. Find partial derivatives of the function $z=\sqrt{x^2+y^2}$ with respect to x and y. If the variable y is fixed then

$$\frac{\partial z}{\partial x}=\left(\sqrt{x^2+y^2}\right)'_x=\frac{1}{2\sqrt{x^2+y^2}}\cdot\left(x^2+y^2\right)'_x=\frac{1}{2\sqrt{x^2+y^2}}\cdot(2x+0)=\frac{x}{\sqrt{x^2+y^2}}$$

For fixed x

$$\frac{\partial z}{\partial y}=\left(\sqrt{x^2+y^2}\right)'_y=\frac{1}{2\sqrt{x^2+y^2}}\cdot\left(x^2+y^2\right)'_y=\frac{1}{2\sqrt{x^2+y^2}}\cdot(0+2y)=\frac{y}{\sqrt{x^2+y^2}}$$

Ex. 4 (logarithmic differentiation). Let there be given a function

$$y=(\varphi(x))^{\phi(x)} \qquad (2)$$

Let's take logarithm both of the left and right sides of the equality and then differentiate termwise:

$$\ln y=\ln(\varphi(x))^{\phi(x)},\ \ln y=\phi(x)\cdot\ln\varphi(x),\ (\ln y)'=(\phi(x)\cdot\ln\varphi(x))',$$

$$\frac{1}{y}\cdot y'=(\phi(x))'\cdot\ln\varphi(x)+\phi(x)\cdot(\ln\varphi(x))'=\phi'(x)\cdot\ln\varphi(x)+\frac{\phi(x)}{\varphi(x)}\cdot\varphi'(x).$$

Multiplying both members of this last equality by $y=(\varphi(x))^{\phi(x)}$ we find y',

$$y'=(\varphi(x))^{\phi(x)}\left(\phi'(x)\cdot\ln\varphi(x)+\frac{\phi(x)}{\varphi(x)}\cdot\varphi'(x)\right)..$$

Ex. 5. Let's applicate this method to differentiate the function $y=x^{\tan x}$.

$$\ln y=\ln x^{\tan x},\ \ln y=\tan x\cdot\ln x,\ (\ln y)'=(\tan x\cdot\ln x)',\ \frac{1}{y}\cdot y'=(\tan x)'\cdot\ln x+$$

$$+\tan x\cdot(\ln x)'=\frac{\ln x}{\cos^2 x}+\frac{\tan x}{x}\Rightarrow y'=y\cdot\left(\frac{\ln x}{\cos^2 x}+\frac{\tan x}{x}\right)=x^{\tan x}\cdot\left(\frac{\ln x}{\cos^2 x}+\frac{\tan x}{x}\right)$$

For functions of several variables we can get a lot of analogous formulae. One of them is given by the next theorem.

Theorem 2. If functions $y=f(u,v),\ u=u(x),\ v=v(x)$ are differentiable, then there exists the derivative of the composite function $y=f(u(x),v(x))$ which equals

$$y'=\frac{\partial f}{\partial u}\cdot u'+\frac{\partial f}{\partial v}\cdot v' \qquad (3)$$

Prove the theorem yourselves with the help of the formula (16) and the next scheme of proof.

$$\Delta y = f'_u(u,v)\cdot\Delta u + f'_v(u,v)\cdot\Delta v + \alpha\cdot\Delta u + \beta\cdot\Delta v,$$
$$\frac{\Delta y}{\Delta x} = f'_u(u,v)\cdot\frac{\Delta u}{\Delta x} + f'_v(u,v)\cdot\frac{\Delta v}{\Delta x} + \alpha\cdot\frac{\Delta u}{\Delta x} + \beta\cdot\frac{\Delta v}{\Delta x},$$
$$y' = \lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x} = f'_u(u,v)\cdot u' + f'_v(u,v)\cdot v'.$$

Ex. 6. Find the derivative of the function (2) using the formula (3).

Let $u=\varphi(x), v=\phi(x)$. We get $y=u^v$, $\frac{\partial y}{\partial u} = v\cdot u^{v-1}$, $\frac{\partial y}{\partial v} = u^v \ln u$ and therefore

$$y' = \frac{\partial y}{\partial u}\cdot u' + \frac{\partial y}{\partial v}\cdot v' = v\cdot u^{v-1}\cdot u' + u^v \ln u\cdot v' =$$
$$= \phi(x)(\varphi(x))^{\phi(x)-1}\cdot\varphi'(x) + (\varphi(x))^{\phi(x)}\ln\varphi(x)\cdot\phi'(x)$$

Ex. 7. Calculate the derivative of a function $y=(\cos x)^{\sin x}$.

$$u=\cos x, v=\sin x,\ y=u^v,\ y'_u = vu^{v-1},\ y'_v = u^v\ln u,\ y' = y'_u\cdot u' + y'_v\cdot v'$$
$$y' = \sin x(\cos x)^{\sin x-1}\cdot(-\sin x) + (\cos x)^{\sin x}\ln\cos x\cdot\cos x$$

Ex. 8. Find formulas for differentiation of functions

$$y = f(x,\varphi(x),\phi(x)),\ z = F(u(x),v(x),w(x)) \qquad (4)$$

Answer.

$$y' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial\varphi}\cdot\varphi' + \frac{\partial f}{\partial\phi}\cdot\phi',\ z' = F'_u\cdot u' + F'_v\cdot v' + F'_w\cdot w'. \qquad (5)$$

Ex. 9. Let's find the derivative $(u\cdot v\cdot w)'$. Denoting $F=u\cdot v\cdot w$ we get $F'_u = v\cdot w$, $F'_v = u\cdot w$ $F'_w = u\cdot v$. Now with the help of the second formula (5) of preceding example we get the same result as in Ex. 7 of the 12-th lecture,

$$(u\cdot v\cdot w)' = F'_u\cdot u' + F'_v\cdot v' + F'_w\cdot w' = u'\cdot v\cdot w + u\cdot v'\cdot w + u\cdot v\cdot w'$$

POINT 2. DIFFERENTIATION OF IMPLICIT, INVERSE AND PARAMETRICALLY REPRESENTED FUNCTIONS

The case of an implicit function

Def. 1. A function $y = f(x)$ of one variable $x \in \Re$ is called implicit one (or defined implicitly) if it's defined by an equation of the form

$$F(x, y) = 0 \qquad (6)$$

which isn't resolved with respect to *y*.

If one can find $y = y(x)$ from the equation (6) then the function $y(x)$ turns the equation into identity ($F(x, y(x)) \equiv 0$).

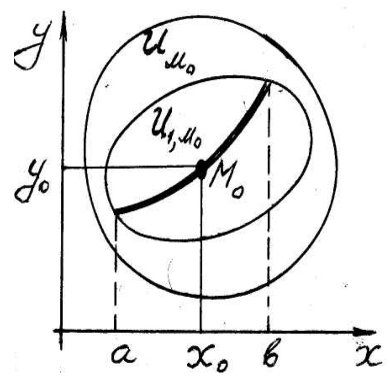

Fig. 1

Ex. 10. Equation $x^2 + y^2 = 1$ defines two implicit func-tions $y = \pm\sqrt{1 - x^2}$. Their substitution in the equation gives identity $x^2 + \left(\pm\sqrt{1 - x^2}\right)^2 \equiv 1$.

Theorem 3. Let:

1) a function $F(x, y)$ and its partial derivatives F'_x, F'_y are defined and continuous is some neighbourhood U_{M_0} of a point $M_0(x_0, y_0)$;

2) $F(M_0) = F(x_0, y_0) = 0$ but $F'_y(M_0) = F'_y(x_0, y_0) \neq 0$.

Then the equation (6) defines the unique implicit function $y = f(x)$ in some neighbourhood $U_{1,M_0} \subseteq U_{M_0}$ of the point M_0. This function is continuous and differentiable in some interval $(a, b) \subseteq \Re^1$, containing the point x_0, and satisfies the condition $f(x_0) = y_0$ (fig. 1).

To find the derivative of implicit function $y = f(x)$ we consider the equation (6) as identity $(F(x, f(x)) \equiv 0 \text{ for } x \in (a, b))$ and differentiate it with respect to *x*:

$$(F(x, y))'_x = 0,\ F'_x \cdot x'_x + F'_y \cdot y'_x = 0,\ F'_x \cdot 1 + F'_y \cdot y'_x = 0,\ F'_x \cdot 1 + F'_y \cdot y'_x = 0,$$

$$y' = y'_x = -\frac{F'_x}{F'_y} \qquad (7)$$

In what follows we can apply both the formula (7) and the method of its development.

Ex. 10. Find the derivative of a function defined implicitly by an equation

$$x^2 + y^2 = 7xy + 6.$$

Solution. The first way.

$$F(x, y) = x^2 + y^2 - 7xy - 6;\ F'_x = 2x - 7y,\ F'_y = 2y - 7x,$$

and by the formula (7)

$$y' = -\frac{F'_x}{F'_y} = -\frac{2x - 7y}{2y - 7x} = \frac{7y - 2x}{2y - 7x}.$$

The second way. Let's, in accordance with the method of deduction of the formula (7), differentiate both members of the given equality with respect to *x* taking into account that *y* is the function of *x*. We'll have

$$\left(x^2 + y^2\right)'_x = (7xy + 6)'_x,\ 2x + 2y \cdot y' = 7(x' \cdot y + x \cdot y'),\ 2x + 2y \cdot y' = 7y + 7x \cdot y'.$$

We've got the first degree equation in y'. Solving it we get y':

$$2y \cdot y' - 7x \cdot y' = 7y - 2x,\ (2y - 7x) \cdot y' = 7y - 2x,\ y' = \frac{7y - 2x}{2y - 7x}.$$

Ex. 11. Write an equation of the tangent to a circle $x^2 + y^2 = 16$ through a point *A* (0; 5).

Solving. a) Let's find a desired equation in the form $y - 5 = k(x - 0)$, and it's necessary to find a slope *k*.

b) We differentiate both members of the equation of the circle, $2x + 2yy' = 0$, finding a slope of a tangent to the circle at any its point $y' = -x / y$.

c) It must be in the tangent point (x, y):

$$\begin{cases} y - 5 = kx, \\ x^2 + y^2 = 16, \\ k = -x / y. \end{cases}$$

d) It's sufficient to find only *k* from this system of equations. We do in the next way:

$$\begin{cases} x=-ky, \\ k^2y^2+y^2=16, \\ y-5=k(-ky); \end{cases} \begin{cases} y^2(k^2+1)=16, \\ y(k^2+1)=5; \end{cases} \frac{y^2(k^2+1)}{y^2(k^2+1)^2}=\frac{16}{25}; 25=16\cdot(k^2+1); k^2=\frac{3}{4}; k=\pm\frac{\sqrt{3}}{2}.$$

There are two values of k and so two tangents to the circle of the equations

$$y=5\pm\frac{\sqrt{3}}{2}x.$$

Ex. 12. Find the angle between two intersecting lines $x^2+y^2=8, y^2=2x.$

Solution. a) At firxt we find intersection points of the lines (of a circle and a parabola) solving a system of equations

$$\begin{cases} x^2+y^2=8, \\ y^2=2x\ (x\ge 0); \end{cases} \begin{cases} y^2=2x \\ x^2+2x-8=0; \end{cases} \begin{cases} x=2 \\ y=\pm 2 \end{cases} M_{01}(2;2), M_{02}(2;-2)$$

b) Secondly we find the slopes of the tangents to the curves at arbitrary their points as the derivatives of the implicit functions,

$$a)\ x^2+y^2=8,\ 2x+2yy'=0,\ y'=y_1'=-\frac{x}{y};\quad b)\ y^2=2x,\ 2yy'=2,\ y'=y_2'=\frac{1}{y}.$$

c) For the point $M_{01}(2;2)\ (x_0=2, y_0=2)$ the slopes of the tangents to the curves are equal

$$k_1=y_1'(x_0)=-\frac{x_0}{y_0}=-\frac{2}{2}=-1,\quad k_2=y_2'(x_0)=\frac{1}{y_0}=\frac{1}{2},$$

and on the base of the formula (12) of the lecture 14 the angle between the curves at this point is defined by the equality

$$\tan\varphi_1=\frac{k_2-k_1}{1+k_1k_2}=\frac{1/2-(-1)}{1+1/2\cdot(-1)}=3.$$

d) For the point $M_{02}(2;-2)$ we get by the same way $\tan\varphi_2=-3$. Verify!

A differentiable implicit function $z=f(x,y)$ of two variables x, y can be determined by an equation of the form

$$F(x,y,z)=0. \tag{8}$$

In this case its partial derivatives with respect to x and y can be calculated with the help of the next formulas

$$z'_x = -\frac{F'_x(x,y,z)}{F'_z(x,y,z)}, \quad z'_y = -\frac{F'_y(x,y,z)}{F'_z(x,y,z)} \text{ if } F'_z(x,y,z) \neq 0 \qquad (9)$$

Prove these formulas yourselves!

Instructions. $(F(x,y,z))'_x = 0,\ F'_x \cdot x'_x + F'_y \cdot y'_x + F'_z \cdot z'_x = 0,$

$$F'_x \cdot 1 + F'_y \cdot 0 + F'_z \cdot z'_x = 0,\ F'_x + F'_z \cdot z'_x = 0,\ z'_x = -F'_x/F'_z,$$

and by the same way for z'_y.

Ex. 13. Find partial derivatives of an implicit function $z = f(x,y)$ determined by an equation

$$\cos(x^2 + y^3 + z^4) = e^{xyz}.$$

Solving. In accordance with the formula (9)

$$F(x,y,z) = e^{xyz} - \cos(x^2 + y^3 + z^4), \quad F'_x = yze^{xyz} + 2x\sin(x^2 + y^3 + z^4),$$

$$F'_y = xze^{xyz} + 3y^2\sin(x^2 + y^3 + z^4), \quad F'_z = xye^{xyz} + 4z^3\sin(x^2 + y^3 + z^4),$$

$$\frac{\partial z}{\partial x} = -\frac{yze^{xyz} + 2x\sin(x^2 + y^3 + z^4)}{xye^{xyz} + 4z^3\sin(x^2 + y^3 + z^4)}, \frac{\partial z}{\partial y} = -\frac{xze^{xyz} + 3y^2\sin(x^2 + y^3 + z^4)}{xye^{xyz} + 4z^3\sin(x^2 + y^3 + z^4)}$$

Differentiable implicit functions can be determined by a system of equations.

The case of an inverse function

Theorem 4. Let a function $y = f(x)$ of one variable satisfy conditions of the third property of continuous functions (see point 1 of the lecture No. 13) and is differentiable one. In this case the inverse function $x = g(y)$ is too differentiable and its derivative can be found by the next formula

$$x'_y = g'(y) = \frac{1}{f'(x)} = \frac{1}{y'_x} \qquad (10)$$

■Both functions $y = f(x),\ x = g(y)$ are continuous and so if $\Delta x \to 0$ then $\Delta y \to 0$ and conversely if $\Delta y \to 0$ then $\Delta x \to 0$. Besides $\Delta y \neq 0$ if $\Delta x \neq 0$ and vice versa. Therefore

$$x' = x'_y = \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = \lim_{\Delta y \to 0} \frac{1}{\frac{\Delta y}{\Delta x}} = \frac{1}{\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}} = \frac{1}{y'_x} \ ■$$

Ex. 14. Derivatives of inverse trigonometric function

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}, (\arctan x)' = \frac{1}{1+x^2}, (arc\cot x)' = -\frac{1}{1+x^2}.$$

■(for arctan*x*). Let $y = f(x) = \arctan x$ and $x = g(y) = \tan y$. By virtue of the formula (10)

$$(\arctan x)' = y'_x = \frac{1}{x'_y} = \frac{1}{(\tan y)'_y} = \frac{1}{\frac{1}{\cos^2 y}} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} \blacksquare$$

The case of a parametrically represented function

Function $y = f(x)$ of one variable *x* can be determined with the help of certain pair of equations

$$\begin{cases} x = x(t), \\ y = y(t), \end{cases} \tag{11}$$

containing some auxiliary variable (parameter) t.

Ex. 15. The equations $x = a\cos t,\ y = b\sin t$ for $0 \le t \le \pi$ determine a function with the upper part of the ellipse (of semiaxes *a*, *b*) as the graph; for $\pi \le t \le 2\pi$ they determine a function with the graph which is the lower part of the same ellipse.

Ex. 16. Equations $x = a(t - \sin t),\ y = a(1 - \cos t)$ determine a function whose graph is the cycloid.

Parametrically represented function can be given in the form of direct dependence between *x* and *y* if in parametric equations (11) the function $x = x(t)$ possesses an inverse function $t = t(x)$. In this case we can write $y = y(t(x))$ what means that *y* is defined directly as a function of the argument *x*.

To evaluate the derivative of a function which is represented parametrically it isn't necessary to express *t* in terms of *x*.

Theorem 5. If the functions $x = x(t),\ y = y(t)$ in the parametric representation (11) of a function $y = f(x)$ are differentiable and the function $x = x(t)$ has an inverse one then the function $y = f(x)$ has the derivative which is given by the next formula

$$y' = y'_x = \frac{y'_t}{x'_t}. \qquad (12)$$

■Using the rules of differentiation of composite and inverse functions we do as follows

$$y' = y'_x = y'_t \cdot t'_x = y'_t \cdot \frac{1}{x'_t} = \frac{y'_t}{x'_t} \blacksquare$$

Ex. 17. Write equations of a tangent and a normal to an ellipse $x = a \cos t$, $y=$ $= b \sin t$ at a point for which $t = t_0 = \frac{\pi}{3}$.

Solving. We find the equations of the tangent and normal in the next form

$$y - y_0 = y'(x_0)(x - x_0), \quad y - y_0 = -\frac{1}{y'(x_0)}(x - x_0).$$

But $x_0 = \cos t_0 = \cos\frac{\pi}{3} = \frac{1}{2}$, $y_0 = \sin t_0 = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$,

$$y'(x) = \frac{y'_t}{x'_t} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t,\ y'(x_0) = -\frac{b}{a}\cot t_0 = -\frac{b}{a}\cot\frac{\pi}{3} = -\frac{b}{a}\frac{\sqrt{3}}{3}$$

and therefore the equations in question are

$$y - \frac{\sqrt{3}}{2} = -\frac{b}{a}\frac{\sqrt{3}}{3}\cdot\left(x - \frac{1}{2}\right),\ y - \frac{\sqrt{3}}{2} = \frac{3a}{b\sqrt{3}}\left(x - \frac{1}{2}\right).$$

POINT 3. THE HIGHER ORDER DERIVATIVES

Let $y = f(x)$ be a function of one independent variable x and $y' = f'(x)$ be its derivative. It's a function of x and we can differentiate it. Such the procedure leads us to the concepts of derivatives of the second, third, … orders (second order, third order, … derivatives).

Def. 2. The derivative of a derivative of a function of one variable is called a derivative of the second order (the second order derivative, the second derivative) of this function and is denoted as follows

$$y'' = y''_{x^2} = \frac{d^2y}{dx^2} = f''(x) = f''_{x^2}(x) = \frac{d^2f(x)}{dx^2} = (y')' = (f'(x))' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(f'(x)).$$

By analogous way are defined the third, fourth, … nth order derivatives,

$$y''' = (y'')' = f'''(x),\ y^{IV} = y^{(4)} = (y''')' = f^{IV}(x) = f^{(4)}(x), \ldots, y^{(n)} = \left(y^{(n-1)}\right)' = f^{(n)}(x).$$

Ex. 18. Let $y = a^x$. Then

$$y' = a^x \ln a,\ y'' = a^x \ln^2 a,\ y''' = a^x \ln^3 a,\ y^{IV} = y^{(4)} = a^x \ln^4 a, \ldots, y^{(n)} = a^x \ln^n a.$$

Ex. 19. Let $y = \sin x$. Then

$$y' = \cos x = \sin\left(x + 1\cdot\frac{\pi}{2}\right),\ y'' = -\sin x = \sin\left(x + 2\cdot\frac{\pi}{2}\right),\ y''' = -\cos x = \sin\left(x + 3\cdot\frac{\pi}{2}\right),$$

and in general

$$y^{(n)} = (\sin x)^{(n)} = \sin\left(x + n\cdot\frac{\pi}{2}\right).$$

For the function $y = \cos x$ we can by analogy deduce that

$$y^{(n)} = (\cos x)^{(n)} = \cos\left(x + n\cdot\frac{\pi}{2}\right).$$

Ex. 20. Find the second derivative of an implicit function given by an equation $x^2 + y^3 = 4$.

Solving.

$$2x + 3y^2 y' = 0,\ y' = -\frac{2x}{3y^2},\ y'' = \frac{2}{3}\cdot\frac{(x)'\cdot y^2 - x\cdot(y^2)'}{y^4} = \frac{2}{3}\cdot\frac{1\cdot y^2 - x\cdot 2yy'}{y^4} =$$

$$= \frac{2}{3}\cdot\frac{y - 2xy'}{y^3} = \frac{2}{3}\cdot\frac{y - 2x\cdot\left(-\frac{2x}{3y^2}\right)}{y^3} = \frac{2}{3}\cdot\frac{3y^3 + 4x^2}{y^5} = \frac{2}{3}\cdot\frac{x^2 + 3(x^2 + y^3)}{y^5} = \frac{2}{3}\cdot\frac{x^2 + 12}{y^5}.$$

Ex. 21. Second derivative of a function which is represented parametrically.

Let $x = x(t), y = y(t)$. By double application of the formula (12) we get

$$y' = y'_x = \frac{y'_t}{x'_t},\ y'' = y''_{x^2} = (y'_x)'_x = \frac{(y'_x)'_t}{x'_t} = \frac{y''_{t^2} x'_t - x''_{t^2} y'_t}{(x'_t)^3}.$$

Thus

$$y'' = y''_{x^2} = \frac{(y'_x)'_t}{x'_t} = \frac{y''_{t^2} x'_t - x''_{t^2} y'_t}{(x'_t)^3}. \qquad (13)$$

Ex. 22. Let $x = a\cos t,\ y = b\sin t$. Then

$$y' = y'_x = \frac{y'_t}{x'_t} = \frac{(b\sin t)'_t}{(a\cos t)'_t} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t,\ y'' = y''_{x^2} = \frac{\left(-\frac{b}{a}\cot t\right)'_t}{(a\cos t)'_t} = -\frac{b}{a^2}\cdot\frac{1}{\sin^3 t}.$$

For functions of several variables one introduces the second, third, … partial derivatives.

Let for example $z = f(x, y)$ be a function of two variables. Then the second order partial derivatives of the function are

$$z''_{x^2} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f(x,y)}{\partial x^2} = (z'_x)'_x,\ z''_{xy} = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 f(x,y)}{\partial x\partial y} = (z'_x)'_y,$$
$$z''_{yx} = \frac{\partial^2 z}{\partial y\partial x} = \frac{\partial^2 f(x,y)}{\partial y\partial x} = (z'_y)'_x,\ z''_{y^2} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f(x,y)}{\partial y^2} = (z'_y)'_y..$$

The partial derivatives z''_{xy}, z''_{yx} are called those mixed.

Ex. 23. Let $z = f(x, y) = x^4y^6 + x^2y^5$. Then

$$z'_x = 4x^4y^6 + 2xy^5,\ z'_y = 6x^4y^5 + 5x^2y^4,$$
$$z''_{x^2} = (4x^4y^6 + 2xy^5)'_x = 16x^3y^6 + 2y^5,\ z''_{xy} = (4x^4y^6 + 2xy^5)'_y = 24x^4y^5 + 10xy^4,$$
$$z''_{yx} = (6x^4y^5 + 5x^2y^4)'_x = 24x^3y^5 + 10xy^4,\ z''_{y^2} = (6x^4y^5 + 5x^2y^4)'_y = 30x^4y^4 + 20x^2y^3$$

In the example the mixed partial derivatives are equal, $z''_{xy} = z''_{yx}$, and it's the general fact. Namely the next theorem holds.

Theorem 6. Mixed partial derivatives z''_{xy}, z''_{yx} are equal at any point at which they are continuous.

POINT 4. THE DIFFERENTIAL

Def. 3. Let a function $y = f(x)$ of one variable x be differentiable one at a point x_0 and therefore its increment at this point can be given by a formula (see the formula (14) in the lecture 14)

$$\Delta y = \Delta f(x_0) = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0) = f'(x_0) \cdot \Delta x + \alpha(\Delta x) \cdot \Delta x \quad (14)$$

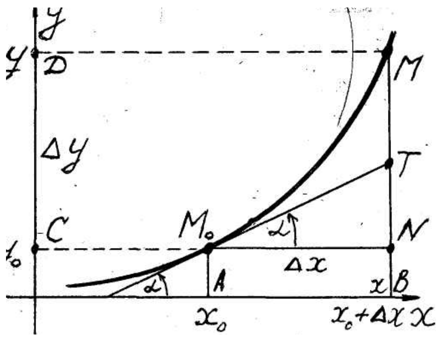

Fig. 1

where $\alpha = \alpha(\Delta x)$ is *IS* as $\Delta x \to 0$. Expression

$$f'(x_0) \cdot \Delta x \quad (15)$$

is called the differential of the function $y = f(x)$ at the point x_0. It is denoted by $dy = df(x_0)$ and therefore

$$dy = df(x_0) = f'(x_0) \cdot \Delta x \quad (16)$$

For example let $y = f(x) = x$. Then

$$dy = dx = x' \cdot \Delta x = 1 \cdot \Delta x = \Delta x,$$

$$dx = \Delta x.$$

This result means that the differential of an independent variable equals its increment. Now we can represent the differential of the function in its usual form

$$dy = df(x_0) = f'(x_0)dx. \quad (17)$$

Geometric sense of the differential we can see from the fig. 1:

$$dy = f'(x_0) \cdot \Delta x = \tan \alpha \cdot M_0 N = NT$$

that is the differential is the increment of the ordinate of the tangent to the graph of the function at the point $M_0(x_0, y_0)$ where $y_0 = f(x_0)$.

The concept of the differential is also introduced for functions of several variables.

Def. 4. Let $z = f(x, y)$ be a function of two variables which is differentiable one at a point $M_0(x_0, y_0)$ that is its total increment at this point has the next form (see the formula (16) of the lecture No. 14)

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f'_x(x_0, y_0) \cdot \Delta x + f'_y(x_0, y_0) \cdot \Delta y + \alpha \cdot \Delta x + \beta \cdot \Delta y.$$

Here α, β are *IS* as $\Delta x \to 0, \Delta y \to 0$. Differential $dz = df(x_0, y_0)$ of the function $z = f(x, y)$ at the point $M_0(x_0, y_0)$ is called the next expression

$$dz = df(x_0, y_0) = f'_x(x_0, y_0) \cdot \Delta x + f'_y(x_0, y_0) \cdot \Delta y. \quad (18)$$

If we put $z = x$, then $dz = x'_x \cdot \Delta x + x'_y \cdot \Delta y = 1 \cdot \Delta x + 0 \cdot \Delta y = \Delta x$, $\Delta x = dx$. By analogy if $z = y$, then $\Delta y = dy$, and so the differential (18) can be written in the form

$$dz = df(x_0, y_0) = f'_x(x_0, y_0) \cdot dx + f'_y(x_0, y_0) \cdot dy \qquad (19)$$

Properties of differentials. If u, v be two differentiable functions then

1. $d(u \pm v) = du \pm dv$. 2. $d(u \cdot v) = u \cdot dv + v \cdot du$. 3. $d\left(\frac{u}{v}\right) = \frac{v \cdot du - u \cdot dv}{v^2}$

and therefore $d(Cu) = C \cdot du$ $(C - const)$, $d\left(\frac{1}{v}\right) = -\frac{dv}{v^2}$..

■If for example $u = u(x), v = v(x)$ be two differentiable functions of one variable, then $d(u \cdot v) = (u \cdot v)' dx = vu'dx + uv'dx = v \cdot du + u \cdot dv$■

4 (differential of composite function of one or several variables).

a) If $y = f(x)$, $x = \varphi(t)$ then

$$dy = f'(x)dx.$$

b) If for example $z = f(x, y)$, $x = x(t)$, $y = y(t)$ then

$$dz = f'_x(x, y)dx + f'_y(x, y)dy.$$

These results mean that the differential has the same form no matter if arguments of a function are independent variables or functions (**invariance** of the differential form).

■a) $dy = y'_t \cdot dt = y'_x \cdot x'(t)dt = f'(x)dx$;

b) $dz = z'_t \cdot dt = (z'_x \cdot x' + z'_y \cdot y') \cdot dt = z'_x \cdot x'dt + z'_y \cdot y'dt = z'_x \cdot dx + z'_y \cdot dy$■

Differentials can be used in approximate calculations.

A) On the one hand we can use the next approximate formulas

$$f(x) = f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \cdot \Delta x \qquad (20)$$

$$f(x, y) = f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f'_x(x_0, y_0) \cdot \Delta x + f'_y(x_0, y_0) \cdot \Delta y \qquad (21)$$

B) On the other hand we can put for a function of one variable

$$f(x) = f(x_0 + \Delta x) \approx f(x_0) \qquad (22)$$

with the absolute error

$$|f(x_0 + \Delta x) - f(x_0)| \approx |f'(x_0) \cdot \Delta x| = |f'(x_0)||\Delta x|;$$

for a function of two variables we can put

$$f(x, y) = f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) \qquad (23)$$

with the absolute error

$$\left|f(x,y)-f(x_0,y_0)\right| \approx \left|f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta y\right| \le \left|f'_x(x_0,y_0)\right|\left|\Delta x\right| + \left|f'_y(x_0,y_0)\right|\left|\Delta y\right|.$$

Ex. 24. Let's find an approximate value of $\sqrt[3]{8.003}$.

A) Taking into account the formula (20) we'll have

$$f(x)=\sqrt[3]{x},\ f'(x)=\frac{1}{3\sqrt[3]{x^2}},x_0=8.000,\ \Delta x=0.003, f(x_0+\Delta x)=\sqrt[3]{x_0+\Delta x}=$$

$$=\sqrt[3]{2.000+0.003} \approx f(x_0)+f'(x_0)\Delta x=\sqrt[3]{8.000}+\frac{1}{3\sqrt[3]{8.000^2}}\cdot 0.003=2+\frac{0.003}{12}\approx 2.000$$

B) Taking into account the formula (22) we'll have

$$f(x_0+\Delta x)=\sqrt[3]{x_0+\Delta x}=\sqrt[3]{2.000+0.003}\approx f(x_0)=\sqrt[2]{8.000}\approx 2.000$$

with the absolute error

$$\left|f'(x_0)\right|\left|\Delta x\right|=\frac{1}{3\sqrt[3]{8.000^2}}\cdot 0.003=0.00025.$$

Compare this result with more precise value of the root:

$$\sqrt[3]{8.003}\approx 2.000249969\ldots .$$

Ex. 25 Find approximate value $1.03^{1.98}$.

A) Using the formula (21) we have

$$f(x,y)=x^y,\ f'_x(x,y)=yx^{y-1},\ f'_y(x,y)=x^y\ln x,\ x_0=1,\ y_0=2,\ \Delta x=0.03,\ \Delta y=-0.02,$$

$$f(x_0+\Delta x, y_0+\Delta y)=(x_0+\Delta x)^{y_0+\Delta y}=(1+0.03)^{2+(-0.02)}\approx f(x_0,y_0)+f'_x(x_0,y_0)\Delta x+$$

$$+f'_y(x_0,y_0)\Delta y=1^2+2\cdot 1^{2-1}\cdot 0.03+1^2\cdot\ln 1\cdot(-0.02)=1+0.06=1.06.$$

B) Using the formula (23) we have

$$f(x_0+\Delta x, y_0+\Delta y)=(1+0.03)^{2+(-0.02)}\approx f(x_0,y_0)=1^2=1.0$$

with absolute error not greater than

$$\left|f'_x(x_0,y_0)\right|\left|\Delta x\right|+\left|f'_y(x_0,y_0)\right|\left|\Delta y\right|\le 2\cdot 1^{2-1}\cdot 0.03+1^2\cdot\ln 1\cdot(-0.02)=0.06<0.1.$$

Def. 5. Differential of the second, third, ..., nth order of a function is called the differential of the differential of the first, second, ... (n - 1)-th order,

$$d^2f=d(df),\ d^3f=d(d^2f), \ldots,\ d^nf=d(d^{n-1}f) \qquad (24)$$

If $y = f(x)$ is a function of one independent variable x, then $dx = \Delta x$ is an arbitrary increment of the argument and so it is an arbitrary constant. Therefore

$$d^2 f(x) = d(df(x)) = d(f'(x)dx) = dx \cdot d(f'(x)) = dx \cdot f''(x)dx = f''(x)dx^2 .$$

Similary we can prove that

$$d^3 f(x) = f'''(x)dx^3, ..., d^n f(x) = f^{(n)}(x)dx^n . \qquad (25)$$

If $z = f(x, y)$ is a function of two independent variables x, y then $dx = \Delta x$, $dy = \Delta y$ are arbitrary increments of the arguments and so they are arbitrary constants. Assuming continuity of the second order partial derivatives (and therefore equality of the mixed partial derivatives) we'll have

$$d^2 f(x, y) = d(df(x, y)) = d(f'_x dx + f'_y dy) = dx \cdot df'_x + dy \cdot df'_y =$$
$$= dx \cdot (f''_{x^2} dx + f''_{xy} dy) + dy \cdot (f''_{yx} dx + f''_{y^2} dy) = f''_{x^2} dx^2 + 2 f''_{xy} dxdy + f''_{y^2} dy^2$$

$$d^2 f(x, y) = f''_{x^2} dx^2 + 2 f''_{xy} dxdy + f''_{y^2} dy^2 = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^2 f(x, y) \qquad (26)$$

Analogously

$$d^3 f(x, y) = f'''_{x^3} dx^3 + f'''_{x^2 y} dx^2 dy + f'''_{xy^2} dxdy^2 + f'''_{x^3} dx^3 = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^3 f(x, y)$$

$$d^n f(x, y) = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^n f(x, y) \qquad (27)$$

Formula (26) indicates that the second order differential of a function $f(x, y)$ of two independent variables is the quadratic form with the matrix

$$H = \begin{pmatrix} f''_{x^2} & f''_{xy} \\ f''_{yx} & f''_{y^2} \end{pmatrix} \qquad (28)$$

Ex. 26. Find the second order differential of the function $z = x^5 y^8$.

The partial derivatives of the function are equal

$z'_x = 5x^4 y^8$, $z'_y = 8x^5 y^7$, $z''_{x^2} = 20x^3 y^8$, $z''_{xy} = z''_{yx} = 40x^4 y^7$, $z''_{y^2} = 56x^5 y^6$,

and by virtue of the formula (26) we'll get

$$d^2z = 20x^3y^8dx^2 + 80x^4y^7dxdy + 56x^5y^6dy^2 .$$

POINT 5. THE DIRECTIONAL DERIVATIVE. THE GRADIENT

Let some direction l on the plane xOy be determined by the unit vector

$$\overline{l^\circ} = (\cos\alpha,\, \cos\beta), \qquad (29)$$

and $M_0(x_0,\, y_0),\, M(x,\, y)$ be two points such that $\overline{M_0M} \uparrow\uparrow \overline{l^\circ}$ (see fig. 2).

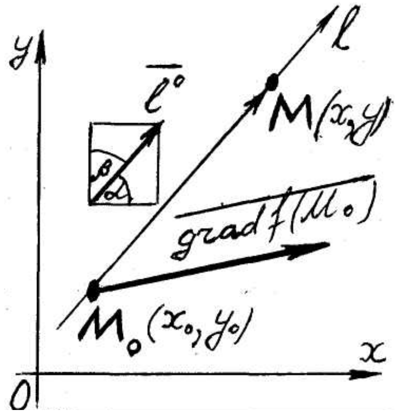

Fig. 2

Def. 6. The derivative of a function of two variables $z = f(M) = f(x,y)$ in the direction l (the direction derivative) at the point $M_0(x_0,\, y_0)$ is called (and is denoted) the next limit

$$\frac{\partial f(M_0)}{\partial l} = \frac{\partial f(x_0, y_0)}{\partial l} = \lim_{M \to M_0} \frac{f(M) - f(M_0)}{\left|\overline{M_0M}\right|} \qquad (30)$$

Def. 7. The gradient of the function of two variables $z = f(M) = f(x,y)$ at the point $M_0(x_0,\, y_0)$ is called (and is denoted) the next vector

$$\overline{grad\ f(M_0)} = \overline{grad\ f(x_0, y_0)} = \left(\frac{\partial f(M_0)}{\partial x}, \frac{\partial f(M_0)}{\partial y}\right) = \left(\frac{\partial f(x_0, y_0)}{\partial x}, \frac{\partial f(x_0, y_0)}{\partial y}\right) \qquad (31)$$

Theorem 7. The derivative of the function $z = f(M) = f(x,y)$ in the direction l at the point $M_0(x_0,\, y_0)$ equals the scalar product of the gradient $\overline{grad\, f(M_0)}$ and the unit vector $\overline{l^\circ}$ of the direction l,

$$\frac{\partial f(M_0)}{\partial l} = \frac{\partial f(x_0, y_0)}{\partial l} = \overline{grad\ f(M_0)} \cdot \overline{l^\circ} = \frac{\partial f(M_0)}{\partial x}\cos\alpha + \frac{\partial f(M_0)}{\partial y}\cos\beta \qquad (32)$$

■Let $\left|\overline{M_0M}\right| = t$ and so

$$\overline{M_0M} = t \cdot \overline{l^\circ} = (t\cos\alpha, t\cos\beta) = (x - x_0, y - y_0);$$

$$x - x_0 = t\cos\alpha,\ y - y_0 = t\cos\beta;\ x = x_0 + t\cos\alpha,\ y = y_0 + t\cos\beta .$$

The given function $f(M) = f(x,y)$ can be considered as a function $\varphi(t)$ of one variable t namely

$$f(M)=f(x,y)=f(x_0+t\cos\alpha,\ y=y_0+t\cos\beta)=\varphi(t)\Rightarrow f(M_0)=f(x_0,y_0)=\varphi(0).$$

The formula (30) yields that

$$\frac{\partial f(M_0)}{\partial l}=\lim_{M\to M_0}\frac{f(M)-f(M_0)}{\left|\overline{M_0M}\right|}=\lim_{t\to 0}\frac{\varphi(t)-\varphi(0)}{t}=\varphi'(0)$$

and so we must find $\varphi'(0)$. But by virtue of the formula (3)

$$\varphi'(t)=f'_x(x,y)\cdot x'_t+f'_y(x,y)\cdot y'_t=f'_x(x,y)\cos\alpha+f'_y(x,y)\cos\beta=$$
$$=f'_x(x_0+t\cos\alpha,y_0+t\cos\beta)\cos\alpha+f'_y(x_0+t\cos\alpha,y_0+t\cos\beta)\cos\beta$$

and therefore

$$\varphi'(0)=\frac{\partial f(M_0)}{\partial l}=\frac{\partial f(x_0,y_0)}{\partial l}=f'_x(x_0,y_0)\cos\alpha+f'_y(x_0,y_0)\cos\beta\ \blacksquare$$

It follows from the definition of a scalar product that the direction derivative (32) equals

$$\frac{\partial f(M_0)}{\partial l}=\frac{\partial f(x_0,y_0)}{\partial l}=\left|\overline{grad\, f(M_0)}\right|\cdot\cos\left(\overline{grad\, f(M_0)}\,\hat{,}\,\overline{l^\circ}\right). \qquad (33)$$

So it possesses the greatest value if $\overline{l^\circ}\uparrow\uparrow\overline{grad\, f(M_0)}$ that is if the derivative of the function $z=f(M)=f(x,y)$ at the point $M_0(x_0,\ y_0)$ is taken in the direction of the gradient of this function at the same point. It can be written as follows

$$\max\frac{\partial f(M_0)}{\partial l}=\frac{\partial f(x_0,y_0)}{\partial grad\, f(M_0)}=\left|\overline{grad\, f(M_0)}\right|. \qquad (34)$$

One can say that the gradient of the function $z=f(M)=f(x,y)$ at the point $M_0(x_0,\ y_0)$ is the vector which in magnitude and in sense represents the greatest rate of growth of this function at this point.

Ex. 27. Partial derivatives of a function $z=f(M)=f(x,y)$ with respect to *x* or *y* are its derivatives in the directions of the *Ox*- and *Oy*-axes respectively.

Ex. 28. Find the derivatives of the function $z=x^2+y^2$ at the point $M_0(1;-2)$ in the direction of: a) a given vector $\overline{a}=(-3;\,4)$; b) the gradient of the function at the same point $M_0(1;-2)$; c) the gradient of the function at the point $N(2;\,3)$ distinct from the point $M_0(1;-2)$.

Solving. $\overline{grad\, z(M)} = \overline{grad\, z(x,y)} = \left(z'_x(x,y);\, z'_y(x,y)\right) = (2x;2y)$ and so

$$\overline{grad\, z(M_0)} = \left(z'_x(1;-2);\, z'_y(1;-2)\right) = (2;-4),\ \overline{grad\, z(N)} = \left(z'_x(2;\,3);\, z'_y(2;\,3)\right) = (4;\,6).$$

Unit vectors of the vector $\overline{a} = (-3;\,4)$ and the gradient of the function $z = x^2 + y^2$ at the point $N(2;\,3)$ are equal correspondingly

$$\overline{a^\circ} = \frac{\overline{a}}{|\overline{a}|} = \left(-\frac{3}{5};\frac{4}{5}\right),\ \overline{grad\, z(N)}^\circ = \left(\frac{4}{\sqrt{52}};\frac{6}{\sqrt{52}}\right) = \left(\frac{2}{\sqrt{13}};\frac{3}{\sqrt{13}}\right).$$

Therefore on the base of the formulae (32), (34)

$$\frac{\partial f(M_0)}{\partial a} = \overline{grad\, f(M_0)} \cdot \overline{a^\circ} = 2\cdot\left(-\frac{3}{5}\right) + (-4)\cdot\frac{4}{5} = -\frac{22}{5},$$

$$\frac{\partial f(M_0)}{\partial grad\, f(N)} = \overline{grad\, f(M_0)} \cdot \overline{grad\, f(N)}^\circ = 2\cdot\frac{2}{\sqrt{13}} + (-4)\cdot\frac{3}{\sqrt{13}} = -\frac{8}{\sqrt{13}},$$

$$\frac{\partial f(x_0, y_0)}{\partial grad\, f(M_0)} = \left|\overline{grad\, f(M_0)}\right| = \sqrt{2^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}.$$

Theorem 8. The gradient $\overline{grad\, f(M_0)}$ is perpendicular to the level line of the function $z = f(M) = f(x,y)$ which lies in the *xOy*-plane and passes through the point $M_0(x_0, y_0)$.

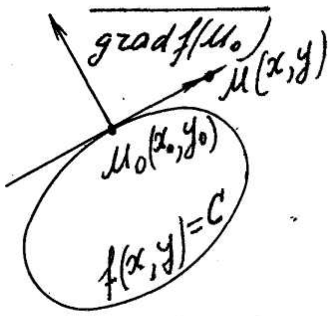

Fig. 3

■Let the level line $l: f(x,\, y) = C$ (for certain value of *C*) passes through the point $M_0(x_0, y_0)$ (fig. 3). The slope of the tangent to the line at the point $M_0(x_0, y_0)$ equals

$$y'(x_0) = -f'_x(x_0, y_0)/f'_y(x_0, y_0),$$

and the equation of the tangent is

$$y - y_0 = -f'_x(x_0, y_0)/f'_y(x_0, y_0)\cdot(x - x_0)$$

or

$$f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0) = 0.$$

It follows that $\overline{gradf(M_0)} = \left(f'_x(x_0, y_0), f'_y(x_0, y_0)\right)$ is perpendicular to the level line *l* because of it is perpendicular to the vector of the tangent $\overline{M_0M} = (x - x_0, y - y_0)$■

Analogous definitions and facts are valid in the 3-dimension space for a function of three variables $u = f(M) = f(x,y,z)$, namely:

$$\frac{\partial f(M_0)}{\partial l}=\frac{\partial f(x_0,y_0,z_0)}{\partial l}=\lim_{M\to M_0}\frac{f(M)-f(M_0)}{\left|\overline{M_0M}\right|}\ ,$$

$$\overline{l^\circ}=(\cos\alpha,\ \cos\beta,\cos\gamma),$$

$$\overline{grad\ f(M_0)}=\overline{grad\ f(x_0,y_0,z_0)}=\left(\frac{\partial f(x_0,y_0,z_0)}{\partial x},\frac{\partial f(x_0,y_0,z_0)}{\partial y},\frac{\partial f(x_0,y_0,z_0)}{\partial z}\right),$$

$$\frac{\partial f(M_0)}{\partial l}=\frac{\partial f(x_0,y_0,z_0)}{\partial l}=\left|\overline{grad\ f(M_0)}\right|\cdot\cos\left(\overline{grad\ f(M_0)}\ \hat{,}\ \overline{l^\circ}\right),$$

$$\max\frac{\partial f(M_0)}{\partial l}=\frac{\partial f(x_0,y_0,z_0)}{\partial grad\ f(M_0)}=\left|\overline{grad\ f(M_0)}\right|.$$

Theorem 9. The gradient $\overline{grad\ f(M_0)}=\overline{grad\ f(x_0,\ y_0,z_0)}$ is perpendicular to the level surface $f(x,\ y,z)=C$ of the function $z=f(M)=f(x,y,z)$ which passes through the point $M_0(x_0,\ y_0,z_0)$.

POINT 6. DERIVATIVES IN ECONOMICS. THE ELASTICITY

Tempo of changing of a function

Relative rate of changing [**tempo of changing**, rate of changing, speed of changing, pace of changing] of a function[1] $y=f(x)$ it's its logarithmic derivative

$$T_{f(x)}=(ln\ f(x))'=\frac{f'(x)}{f(x)}. \qquad (39)$$

Limiting quantities

Economics deals with lots of so-called limiting quantities which are based on the notion of the derivative: marginal costs of production [marginal production (manufacturing) costs, marginal expences of production][2], marginal gain [return, prode-

[1] Относительная скорость изменения [темп изменения] функции
[2] Предельные издержки производства

eds, receipts, takins, profit][1], marginal income [marginal revenue, marginal return, marginal yield][2], marginal product[3], marginal utility[4] and so on.

We'll dwall upon the notion of the marginal costs of production. The rest of quantities are introduced analogously.

Let's consider the costs of production as a function $y = f(x)$ of a quantity x of the output. If Δx is an increment of the output, then the increment of the function

$$\Delta y = \Delta f(x) = f(x + \Delta x) - f(x)$$

is the increment of the costs of production, and

$$\frac{\Delta y}{\Delta x} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is the average increment of the costs of production per unite of production. The derivative

$$y' = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

is the marginal costs of production. It characterizes approximately additional costs for making the unite of additional production.

Limiting [marginal] quantities don't characterize a condition [position, state, status], but a process, a changing of some economic(al) ofject. Therefore the derivative is the rate of changing of this economical ofject (that is the rate of a process) with respect to a time or to some factor to be studied.

Elasticity of a function

Def. 8. Relative increment δz of a given positive quantity $z > 0$ is called the ratio of a usual increment Δz and the initial value z of this quantity,

$$\delta z = \frac{\Delta z}{z}.$$

Let a function $y = f(x)$ and its argument be positive: $x > 0, f(x) > 0$.

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[1] Предельная выручка
[2] Предельный доход
[3] Предельный продукт
[4] Предельная полезность

By definition their relative increments be

$$\delta x = \frac{\Delta x}{x},\ \delta f(x) = \frac{\Delta f(x)}{f(x)} \tag{40}$$

Def. 9. Elasticity $E_x(f)$ of given (positive) function $y = f(x)$ with (positive) argument x is called the limit of the ratio of the relative increment of the function to the relative increment of its argument if this latter goes to zero,

$$E_x(f) = \lim_{\delta x \to 0} \frac{\delta f(x)}{\delta x} \tag{41}$$

The elasticity determines the percentage increment of a function per one persent of the increment of its argument.

Theorem 10 (elasticity and derivative).

$$E_x(f) = f'(x) \cdot \frac{x}{f(x)} \tag{42}$$

$$\blacksquare\, E_x(f) = \lim_{\Delta x \to 0} \frac{\Delta f(x)/f(x)}{\Delta x/x} = \lim_{\Delta x \to 0} \frac{\Delta f(x) \cdot x}{\Delta x \cdot f(x)} = \frac{x}{f(x)} \cdot \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = f'(x) \cdot \frac{x}{f(x)} \blacksquare$$

Corollary (elasticity and tempo of changing). The elasticity of a function equals the product of its argument and the tempo of changing,

$$E_x(f) = xT_{f(x)} \tag{43}$$

Ex. 29. $f(x) = x^2,\ E_x(f) = 2x \cdot \frac{x}{x^2} = 2;\quad g(x) = x^5,\ E_x(g) = 5x^4 \cdot \frac{x}{x^5} = 5$

Ex. 30. Let $f(x) = Ax^a$ where A, a be arbitrary real numbers. Then

$$E_x(f) = E_x\left(Ax^a\right) = a \tag{44}$$

because of by virtue of the formula (37)

$$E_x(f) = E_x\left(Ax^a\right) = \left(Ax^a\right)' \cdot \frac{x}{Ax^a} = Aa \cdot x^{a-1} \cdot \frac{x}{Ax^a} = a$$

Ex. 31. Let $f(x) = e^{ax}$ where a be an arbitrary real number. Then

$$E_x(f) = E_x\left(e^{ax}\right) = ax \tag{45}$$

$$\blacksquare\, E_x(f) = E_x\left(e^{ax}\right) = \left(e^{ax}\right)' \cdot \frac{x}{e^{ax}} = ae^{ax} \cdot \frac{x}{e^{ax}} = ax \blacksquare$$

Properties of the elasticity

1. $\operatorname{sign} E_x(f) = \operatorname{sign} f'(x)$ (because of $x > 0, f(x) > 0$).

2. Elasticity is dimensionless function that is its dimension $[E_x(f)] = 1$.

$$\blacksquare\ [E_x(f)] = \left[\lim_{\delta x \to 0} \frac{\delta f(x)}{\delta x}\right] = \left[\frac{\delta f(x)}{\delta x}\right] = \left[\frac{\Delta f(x)/f(x)}{\Delta x/x}\right] = \frac{[\Delta f(x)]\cdot[x]}{[\Delta x]\cdot[f(x)]} = 1\ \blacksquare$$

3. $E_x(f) = \dfrac{d(\ln f(x))}{d(\ln x)}$ or $E_x(f) = \dfrac{d(\log_a f(x))}{d(\log_a x)}$ for any a, $0 < a \neq 1$

$$\blacksquare\ \frac{d(\ln f(x))}{d(\ln x)} = \frac{(\ln f(x))' dx}{(\ln x)' dx} = \frac{\frac{1}{f(x)}\cdot f'(x)}{\frac{1}{x}} = f'(x)\cdot\frac{x}{f(x)} = E_x(f)\ \blacksquare$$

4. Elasticity of a product (of a quotient) of two functions equals the sum (corr. the difference) of their elasticities,

$$E_x(fg) = E_x(f) + E_x(g), \quad E_x(f/g) = E_x(f) - E_x(g).$$

$$\blacksquare\ E_x(fg) = (fg)'\cdot\frac{x}{fg} = f'g\cdot\frac{x}{fg} + fg'\cdot\frac{x}{fg} = f'\cdot\frac{x}{f} + g'\cdot\frac{x}{g} = E_x(f) + E_x(g),$$

$$E_x(f/g) = (f/g)'\cdot\frac{x}{f/g} = \frac{f'g}{g^2}\cdot\frac{x}{f/g} - \frac{fg'}{g^2}\cdot\frac{x}{f/g} = f'\cdot\frac{x}{f} - g'\cdot\frac{x}{g} = E_x(f) - E_x(g)\ \blacksquare$$

LECTURE NO.16. MAIN THEOREMS ON DIFFERENTIAL CALCULUS OF FUNCTIONS OF ONE VARIABLE

POINT 1. FERMAT AND ROLLE THEOREMS

POINT 2. LAGRANGE THEOREM. CAUCHY THEOREM

POINT 3. BERNOULLI - L'HOSPITALE RULE FOR REMOVAL INDETERMINACIES

POINT 4. TAYLOR AND MACLAURIN FORMULAS

POINT 1. FERMAT AND ROLLE THEOREMS

Theorem 1 (Fermat[1]). If a function $y = f(x)$ is defined in interval (a, b) and takes on the greatest or the least value at some (inner) point x_0 of this interval then the derivative of the function at this point equals zero $f'(x_0) = 0$ if it exists.

■ To fix the idea let the function $y = f(x)$ take on the greatest value at the point $x_0 \in (a, b)$ and so its increment at this point is negative, $\Delta y = f(x) - f(x_0) < 0$.

a) Let $\Delta x > 0$ and Δx be so small that $x = x_0 + \Delta x < b$. Then $\frac{\Delta y}{\Delta x} < 0$ and by virtue of the theory of limits

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \le 0.$$

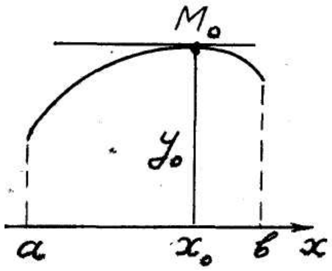

Fig. 1

b) Now let $\Delta x < 0$ and Δx be so small that $x = x_0 + \Delta x > a$. Then $\frac{\Delta y}{\Delta x} > 0$ and by the same reason $f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \ge 0$.

We've got $f'(x_0) \le 0$ and $f'(x_0) \ge 0$ whence it follows that $f'(x_0) = 0$ ■

Geometric sense of Fermat theorem: tangent to the graph of the function at the point $M_0(x_0, f(x_0))$, which is highest or lowest point of the graph over the interval

[1] Fermat, P. (1601 - 1665), a famous French mathematician

(a, b), is parallel to the Ox - axis (fig. 1).

Theorem 2 (Rolle[1]). If a function $y = f(x)$:

a) is continuous one on the segment (closed bounded interval) $[a, b]$;

b) has a derivative on the interval (open bounded interval) (a, b);

c) takes on equal values at the end points of the segment $[a, b]$,

then there exists at least one point x_0 in the interval (a, b) at which the derivative of the function takes on zero value, $f'(x_0) = 0$.

■We can suppose that $f(x) \neq const$ on (a, b) (otherwise $f'(x) \equiv 0$ at all points of (a, b)). By virtue of the condition a) the function $f(x)$ takes on its greatest and its least values in some two points of the segment $[a, b]$. According to the condition c) at least one of these points lies inside the segment. If x_0 is such inner point then by Fermat theorem and by the condition b) $f'(x_0) = 0$■

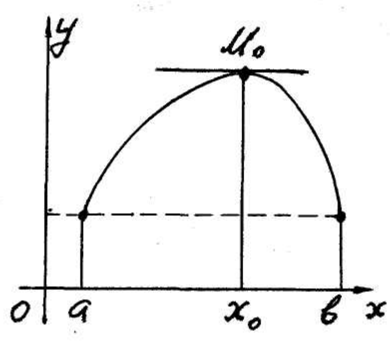

Fig. 2

Geometric sense of Rolle theorem is analogous to that of Fermat theorem: if the graph of the function $y = f(x)$ is continuous curve with equidistant from the Ox-axis points $A(a, f(a)), B(b, f(b)) \,(f(a) = f(b))$ and possesses the tangent at every its point over the interval (a, b) then there exists at least one point $M_0(x_0, f(x_0))$ of the graph at which the tangent to the graph is parallel to the Ox - axis (fig. 2).

Ex. 1. Prove that the derivative of the function

$$f(x) = x^4 - 2x^3 - 8x^2 + 18x - 9$$

has at least one root in the interval $(-3, 3)$.

Solution. The function $f(x)$ is continuous and differentiable one for any x and the points $x = \pm 3$ are its zeros. By Rolle theorem for the segment $[-3, 3]$ there exists

[1] Rolle, M. (1652 - 1719), a French mathematician

at least one root of the derivative $f'(x)$ in the interval $(-3, 3)$.

Verification. The derivative $f'(x)$ equals

$$f'(x)=4x^3-6x^2-16x+18$$

and has for example a root $x=1$ which belongs to the interval $(-3, 3)$.

Ex. 2. Prove and test yourselves that the derivative of the function

$$f(x)=x^4+3x^3+x^2-3x-2$$

has at least one root in the interval $(-2, 1)$.

Note 1. It follows from Rolle theorem that between two zeros x_1, x_2 of a function which is continuous on any segment $[a, b]\supseteq[x_1, x_2]$ and differentiable on corresponding interval $(a, b)\supseteq(x_1, x_2)$ lies at least one root of its derivative.

Ex. 3. The function $f(x)=9^{\sqrt{\cos x}}$ is continuous and differentiable one in every segment $[-\pi/2+2\pi n, \pi/2+2\pi n]\,(n\in Z)$, $f(-\pi/2)=f(\pi/2)=1$. By Rolle theorem the derivative $f'(x)$ at least once vanishes in the interval $(-\pi/2, \pi/2)$.

Testing. $f'(x)=-9^{\sqrt{\cos x}}\cdot\dfrac{\sin x}{2\sqrt{\cos x}}\cdot\ln 9$ and $f'(x)=0$ if $x=0\in(-\pi/2, \pi/2)$.

Ex. 4. Check the validity of Rolle theorem for the function $y=\sqrt{12-x-x^2}$ in the segment $[-4, 3]$

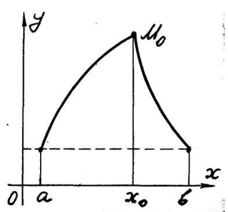

Fig. 3

Note 2. All the conditions of Rolle theorem are essential for its validity that is for existence of the tangent to the graph of the function $y=f(x)$ which is parallel to the Ox- axis. On the other hand they are those sufficient but not necessary for existing of such the tangent.

Ex. 5. There is no tangent which is parallel to the Ox- axis for the function represented on the fig. 3. This function satisfies the conditions a) c) of Rolle theorem but not the condition b) being nondifferentiable one at unique point x_0 of the interval (a, b).

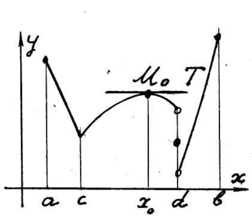

Fig. 4

Ex. 6. None of conditions of Rolle theorem are satisfied for a function represented by the fig. 4 (namely: a) it has discontinuity point $d \in [a, b]$; b) it isn't differentiable at the point $c \in (a, b)$; c) $f(a) \neq f(b)$) but there is a point $x_0 \in (a, b)$ for which the tangent $M_0T \| Ox$.

POINT 2. LAGRANGE THEOREM. CAUCHY THEOREM

THEOREM 3 (LAGRANGE[1]). If a function $y = f(x)$: a) is continuous on the
segment $[a,b]$; b) has the derivative in the interval (a, b), then there exists at least one point $c \in (a, b)$ for which the next equality holds

$$\frac{f(b) - f(a)}{b - a} = f'(c), \qquad (1)$$

or

$$f(b) - f(a) = f'(c)(b - a) \qquad (2)$$

■Let 's denote

$$\frac{f(b) - f(a)}{b - a} = Q$$

and so

$$f(b) - f(a) = Q(b - a), \; f(b) - f(a) - Q(b - a) = 0. \qquad (3)$$

Substituting *b* by *x* in (3) we introduce auxiliary function

$$F(x) = f(x) - f(a) - Q(x - a). \qquad (4)$$

It satisfies all the conditions of Rolle theorem: it's continuous on the segment $[a, b]$, possesses the derivative

$$F'(x) = f'(x) - Q$$

in the interval (a, b), because of properties of the function $f(x)$, and takes on equal zero values at the points *a*, *b* ($F(a) = 0$ by (4), $F(b) = 0$ by (3)). Therefore by virtue of Rolle's theorem there exists a point $c \in (a, b)$ at which $F'(c) = 0$ that is

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[1] Lagrange, J.L. (1736 - 1813), an outstanding French mathematician and astronomer

$$F'(c)=f'(c)-Q=0,\ f'(c)=Q,\ f'(c)=\frac{f(b)-f(a)}{b-a},\ \frac{f(b)-f(a)}{b-a}=f'(c)\ \blacksquare$$

Geometric sense of Lagrange's theorem consists in the following (fig. 3): if the graph of the function $y=f(x)$ is continuous curve and possesses the tangent at every its point over the interval (a,b) then there exists at least one point $C(c,\ f(c))$ of the graph at which the tangent to the graph is parallel to the segment AB joining end-points $A(a,\ f(a)),\ B(b,\ f(b))$ of the graph.

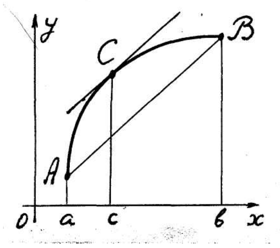

Fig. 5

Corollary. If in conditions of Lagrange's theorem the derivative of the function $f(x)$ equals zero, $f'(x)=0$, than the function is constant one on the segment $[a,b]$.

$\blacksquare$For any $x\in[a,b]$ there exists a point $c\in(a,x)$ such that by virtue of the formula (2) one has

$$f(x)-f(a)=f'(c)(x-a)=0\cdot(x-a)=0\Rightarrow f(x)=f(a)=const\ \blacksquare$$

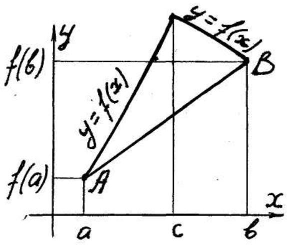

Fig. 6

Note 3. Both conditions of Lagrange theorem are essential for its validity that is for existence of the tangent to the graph of the function $y=f(x)$ which is parallel to the segment AB. On the other hand they are those sufficient but not necessary for existing of such the tangent.

Ex. 7. There is no tangent which is parallel to the segment AB for the function represented on the fig. 6. This function being continuous one on the segment $[a,b]$ doesn't possesses the derivative at the point $c\in(a,b)$.

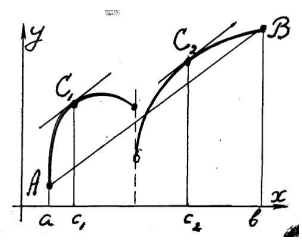

Fig. 7

Ex. 8. A function determined by the fig. 7 doesn't satisfy the conditions of Lagrange theorem but its graph has two tan-gents parallel to the segment AB.

Ex. 9. With the help of Lagrange theorem prove that for any a, b such that $0\le a<b$ the next inequality holds

$$\frac{b-a}{1+b^2} < \arctan b - \arctan a < \frac{b-a}{1+a^2}.$$

■The function $f(x) = \arctan x$ satisfies conditions of Lagrange theorem for any segment $[a, b]$ $(0 \le a < b)$ and so there exists a point $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b-a), \arctan b - \arctan a = \frac{(b-a)}{1+c^2}. \qquad (*)$$

After the next chain of estimates

$$0 \le a < c < b,\ a^2 < c^2 < b^2, 1+a^2 < 1+c^2 < 1+b^2, \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

we get the inequality

$$\frac{b-a}{1+b^2} < \frac{b-a}{1+c^2} < \frac{b-a}{1+a^2}$$

which by (*) is equivalent to the inequality in question■

Ex. 10. Prove the inequalities

a) $\dfrac{b-a}{b} \le \ln\dfrac{b}{a} \le \dfrac{b-a}{a}$ for $0 < a \le b$;

b) $\dfrac{\beta-\alpha}{\cos^2\alpha} \le \tan\beta - \tan\alpha \le \dfrac{\beta-\alpha}{\cos^2\beta}$ for $0 \le \alpha \le \beta < \pi/2$;

c) $\dfrac{b-a}{\sqrt{1-a^2}} \le \arcsin b - \arcsin a \le \dfrac{b-a}{\sqrt{1-b^2}}$ for $0 \le a \le b < 1$.

Ex. 11. Using Lagrange theorem form double-ended estimate and find approximate value of the number $\sqrt[4]{82}$.

Solution. Let $f(x) = \sqrt[4]{x}$, $a = 81, b = 82$. By Lagrange theorem there exists a point $c \in (81, 82)$ such that

$$f(82) - f(81) = f'(c)(82-81), \sqrt[4]{82} - \sqrt[4]{81} = \sqrt[4]{82} - 3 = \frac{1}{4\sqrt[4]{c^3}}.$$

The next estimates yield

$$3^4 < 81 < c < 82 < 3.01^4 < 82.085, 3^4 < c < 3.01^4, 3^{12} < c^3 < 3.01^{12}, 3^3 < \sqrt[4]{c^3} < 3.01^3,$$

$$\frac{1}{3.01^3} < \frac{1}{\sqrt[4]{c^3}} < \frac{1}{3^3}, \frac{1}{4\cdot 3.01^3} < \frac{1}{4\cdot\sqrt[4]{c^3}} < \frac{1}{4\cdot 3^3}, 0.00916 < \frac{1}{4\cdot\sqrt[4]{c^3}} < 0.00926$$

and therefore $0.00916 < \sqrt[4]{82} - 3 < 0.00926, 3.00916 < \sqrt[4]{82} < 3.00926$, $\sqrt[4]{82} \approx 3.009$. All the digits are correct.

Ex. 12. Find approximate value of the number $\sqrt{4.02}$.

Remark. Lagrange theorem permits to prove sufficient condition of differentiability of a function of several independent variables. We'll give the proving of the Theorem 1 of the lecture No. 12 (Point 4, formula (16)) concerning a function of two variables.

Let in accordance with the theorem a function $z = f(M) = f(x, y)$ has partial derivatives in some neighbourhood of a point $M_0(x_0, y_0)$ which are continuous at this point. Representing total increment of the function at the point $M_0(x_0, y_0)$, that is the expression

$$\Delta z = f(M) - f(M_0) = f(x, y) - f(x_0, y_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0),$$

in the next form

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) + f(x_0, y_0 + \Delta y) - f(x_0, y_0) =$$
$$= \left(f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)\right) + \left(f(x_0, y_0 + \Delta y) - f(x_0, y_0)\right),$$

we apply Lagrange theorem to two expressions in the parentheses in the second row namely

$$\Delta z = f'_x(c_1, y_0 + \Delta y)\Delta x + f'_y(x_0, c_2)\Delta y,\ c_1 \in (x_0, x_0 + \Delta x),\ c_2 \in (y_0, y_0 + \Delta y).$$

On the base of continuity of partial derivatives of the function at the point $M_0(x_0, y_0)$ we can write

$$f'_x(c_1, y_0 + \Delta y) = f'_x(x_0, y_0) + \alpha,\ f'_y(x_0, c_2) = f'_y(x_0, y_0) + \beta$$

where $\alpha = \alpha(\Delta x, \Delta y), \beta = \beta(\Delta x, \Delta y)$ are *IS* as $\Delta x \to 0, \Delta y \to 0.$ Therefore

$$\Delta z = \left(f'_x(x_0, y_0) + \alpha\right)\Delta x + \left(f'_y(x_0, y_0) + \beta\right)\Delta y = f'_x(x_0, y_0)\Delta x + f'_y(x_0, y_0) + \alpha\Delta x + \beta\Delta y$$

what it was required to be proved.

THEOREM 4 (Cauchy[1]). If functions $f(x), g(x)$

1. are continuous on the segment $[a, b]$;

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[1] Cauchy, A.L. (1780 - 1859), a famous French mathematician

2. have the derivatives in the interval (a, b);

3. $g(a) \neq g(b)$ $(\Rightarrow g'(x) \neq 0$ on $(a, b))$;

then there exists a point $c \in (a, b)$ for which the next equality holds

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \tag{5}$$

Prove the theorem yourselves putting

$$\frac{f(b)-f(a)}{g(b)-g(a)} = Q$$

and introducing auxiliary function $F(x) = f(x) - f(a) - Q(g(x) - g(a))$.

POINT 3. BERNOULLI - L'HOSPITALE RULE FOR REMOVAL INDETERMINACIES

Finding limits we dealt with various types of indeterminacies [indeterminate forms, indeterminate expressions]:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, 1^{\infty}, 0^{\infty}, \infty^{0}, 0^{0}, (\pm\infty)-(\pm\infty), (\pm\infty)+(\mp\infty).$$

Differential calculus gives some methods to remove them.

Indeterminacies of the types $0/0, \infty/\infty$

THEOREM 5 (Bernoulli[1] - L'Hospitale[2] rule). Limit of a ratio of two *IS* or *IL* (in every type of passage to limit) equals the limit of the ratio of their derivatives if this latter exists. Schematically

$$\lim \frac{f(x)}{g(x)} = \left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right) = \lim \frac{f'(x)}{g'(x)}.$$

■We'll study the simplest case namely if $x \to a+0$, $f(a) = g(a) = 0$ and functions $f(x)$, $g(x)$ satisfy the conditions of Cauchy theorem. Let there exist the limit

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[1] Bernoulli, Johann (1667 - 1748), the famous Swiss mathematician

[2] L'Hospital, J.F.A. (1661 - 1704), a French mathematician

$$K = \lim_{x\to a+0}\frac{f'(x)}{g'(x)}.$$

Then by Cauchy theorem

$$\lim_{x\to a+0}\frac{f(x)}{g(x)} = \left(\frac{0}{0}\right) = \lim_{x\to a+0}\frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a+0}\frac{f'(c)}{g'(c)} = K,\ \lim_{x\to a+0}\frac{f(x)}{g(x)} = K = \lim_{x\to a+0}\frac{f'(x)}{g'(x)}$$

because of $c \in (a, x)$ and $c \to a$ if $x \to a+0$ ■

Note 4. Bernoulli - L'Hospitale rule can be combined with other methods of evaluation the limits. For example we can use the table of equivalent *IS*.

Ex. 13.

$$\lim_{x\to 0}\frac{1-\cos 16x}{\sin 9x tg 24x} = \left(\frac{0}{0}\right) = \begin{vmatrix}\sin 9x \sim 9x \\ tg 24x \sim 24x\end{vmatrix} = \lim_{x\to 0}\frac{1-\cos 16x}{9x\cdot 24x} = \frac{1}{216}\lim_{x\to 0}\frac{1-\cos 16x}{x^2} =$$

$$= \frac{1}{216}\lim_{x\to 0}\frac{(1-\cos 16x)'}{(x^2)'} = \frac{1}{216}\lim_{x\to 0}\frac{16\sin 16x}{2x} = \frac{1}{27}\lim_{x\to 0}\frac{\sin 16x}{x} = |\sin 16x \sim 16x| = \frac{16}{27}$$

Ex. 14. $\lim_{x\to\infty}\frac{\ln 5x}{2^x} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty}\frac{(\ln 5x)'}{(2^x)'} = \lim_{x\to\infty}\frac{\frac{1}{5x}\cdot 5}{2^x \ln 2} = \frac{1}{\ln 2}\lim_{x\to\infty}\frac{1}{x\cdot 2^x} = \left(\frac{1}{\infty}\right) = 0$

Remark. Bernoulli - L'Hospitale rule can be applied several times (repeatedly) by necessity.

Ex. 15. For any natural n

$$\lim_{x\to\infty}\frac{x^n}{5^x} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to\infty}\frac{(x^n)'}{(5^x)'} = \frac{n}{\ln 5}\lim_{x\to\infty}\frac{x^{n-1}}{5^x} = \left(\frac{\infty}{\infty}\right) = \frac{n}{\ln 5}\lim_{x\to\infty}\frac{(x^{n-1})'}{(5^x)'} = \frac{n(n-1)}{(\ln 5)^2}\lim_{x\to\infty}\frac{x^{n-2}}{5^x} =$$

$$= \frac{n(n-1)}{(\ln 5)^2}\lim_{x\to\infty}\frac{(x^{n-2})'}{(5^x)'} = \ldots = \frac{n!}{(\ln 5)^n}\lim_{x\to\infty}\frac{1}{5^x} = \left(\frac{1}{\infty}\right) = 0$$

Some other types of indeterminacies

are reduced by various transformations to two first types. We'll regard some particular examples.

Ex. 16.

$$\lim_{x\to 0} x\ln x = (0\cdot\infty) = \lim_{x\to 0}\frac{\ln x}{x^{-1}} = \left(\frac{\infty}{\infty}\right) = \lim_{x\to 0}\frac{(\ln x)'}{(x^{-1})'} = \lim_{x\to 0}\frac{1/x}{-x^{-2}} = -\lim_{x\to 0} x = 0$$

Ex. 17. Using the result of preceding example we get

$$\lim_{x\to 0+0} x^x = \left(0^0\right) = \lim_{x\to 0+0}\left(e^{\ln x}\right)^x = \lim_{x\to 0+0} e^{x\ln x} = e^{\lim\limits_{x\to 0+0} x\ln x} = \left(e^{0\cdot\infty}\right) = e^0 = 1$$

Ex. 18. $\lim\limits_{x\to 0}\left(\frac{1}{\sin 2x} - \frac{1}{2\sin x}\right) = (\infty - \infty) = \lim\limits_{x\to 0}\left(\frac{1}{2\sin x\cos x} - \frac{1}{2\sin x}\right) =$

$$= \lim_{x\to 0}\frac{1-\cos x}{\sin 2x} = \left(\frac{0}{0}\right) = \lim_{x\to 0}\frac{1-\cos x}{2x} = \frac{1}{2}\lim_{x\to 0}\frac{(1-\cos x)'}{x'} = \frac{1}{2}\lim_{x\to 0}\sin x = 0$$

POINT 4. TAYLOR AND MACLAURIN FORMULAS

Maclaurin and Taylor formulas for polynomial

Let be given nth degree polynomial

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n. \qquad (6)$$

Differentiating it n times we get

$$\left.\begin{array}{c} P'(x) = 1\cdot a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \ldots + na_nx^{n-1}, \\ P''(x) = 1\cdot 2a_2 + 2\cdot 3a_3x + 3\cdot 4a_4x^2 + 4\cdot 5x^3 + \ldots + (n-1)na_nx^{n-2}, \\ P'''(x) = 1\cdot 2\cdot 3a_3 + 2\cdot 3\cdot 4a_4x + 3\cdot 4\cdot 5x^2 + \ldots + (n-2)(n-1)na_nx^{n-3}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ P^{(n)}(x) = 1\cdot 2\cdot 3\cdot\ldots\cdot na_n. \end{array}\right\} \qquad (7)$$

Putting $x = 0$ in the formulas (6), (7) we can express the coefficients of the polynomial in term of its value and the values of its derivatives at the point $x = 0$, namely

$$\left.\begin{array}{c} P(0) = a_0 = 1\cdot a_0 = 0!\cdot a_0 \text{ (by definition } 0! = 1\text{, zero - factorial)}, \\ P'(0) = a_1 = 1\cdot a_1 = 1!a_1 \text{ (by definition } 1! = 1\text{, one - factorial)}, \\ P''(0) = 1\cdot 2\cdot a_2 = 2!\cdot a_2 \,(2! = 1\cdot 2\text{, two - factorial}), \\ P'''(0) = 1\cdot 2\cdot 3\cdot a_3 = 3!\cdot a_3 \,(3! = 1\cdot 2\cdot 3\text{, three - factorial}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ P^{(n)}(0) = 1\cdot 2\cdot 3\cdot\ldots\cdot n\cdot a_n = n!\cdot a_n \,(\text{n}! = 1\cdot 2\cdot 3\cdot\ldots\cdot n\text{, n - factorial}) \end{array}\right\}$$

$$a_0 = P(0) = \frac{P^{(0)}(0)}{0!},\ a_1 = P'(0) = \frac{P'(0)}{1!},\ a_2 = \frac{P''(0)}{2!},\ a_3 = \frac{P'''(0)}{3!}, \ldots, a_n = \frac{P^{(n)}(0)}{n!}, \qquad (8)$$

$$P(x) = P(0) + P'(0)x + \frac{P''(0)}{2!}x^2 + \frac{P'''(0)}{3!}x^3 + \ldots + \frac{P^{(n)}(0)}{n!}x^n = \sum_{k=0}^{n}\frac{P^{(k)}(0)}{k!}x^k. \qquad (9)$$

Def. 1. Formula (9) is called Maclaurin (or Taylor[1] - Maclaurin[2]) formula for the polynomial (6). We've proved the next theorem.

Theorem 6. Every polynomial of the form (6) can be represented by Maclaurin (Taylor - Maclaurin) formula (9) (with coefficients (8)).

If nth degree polynomial is written as development with respect to powers of a difference $x - x_0$, namely

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \ldots + a_n(x - x_0)^n, \qquad (10)$$

then by the same way one can get

$$a_0 = P(x_0) = \frac{P^{(0)}(x_0)}{0!},\ a_1 = P'(x_0) = \frac{P'(x_0)}{1!},\ a_2 = \frac{P''(x_0)}{2!},\ \ldots,\ a_n = \frac{P^{(n)}(x_0)}{n!}, \qquad (11)$$

$$P(x) = P(x_0) + P'(x_0)(x - x_0) + \frac{P''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{P^{(n)}(x_0)}{n!}(x - x_0)^n, \qquad (12)$$

$$P(x) = \sum_{k=0}^{n} \frac{P^{(k)}(x_0)}{k!}(x - x_0)^k .$$

Def. 2. Formula (12) is called Taylor formula for the polynomial (10).

Theorem 7. Every polynomial of the form (10) can be represented by Taylor formula (12) (with coefficients (11)).

Note 5. Maclaurin (Taylor - Maclaurin) formula (9) is particular case of Taylor formula (12) for $x_0 = 0$.

Binomial expansion

Def. 3. Newton binomial is called the next expression

$$P(x) = (a + x)^n \qquad (13)$$

We'll expand Newton binomial (13) with the help of the formulas (6), (9).

$$P'(x) = n(a + x)^{n-1}, P''(x) = n(n-1)(a + x)^{n-2}, P'''(x) = n(n-1)(n-2)(a + x)^{n-3}, \ldots,$$

$$P^{(k)}(x) = n(n-1)(n-2)\ldots(n-(k-1))(a + x)^{n-k}, \ldots, P^{(n)}(x) = n!$$

---

[1] Taylor, B. (1685 - 1731), an English mathematician

[2] Maclaurin, C. (1698 - 1746), a Scotch mathematician

$$P(0)=a^n, P'(0)=na^{n-1}, P''(0)=n(n-1)a^{n-2}, P'''(0)=n(n-1)(n-2)a^{n-3},...,$$

$$P^{(k)}(0)=n(n-1)(n-2)...(n-(k-1))a^{n-k},..., P^{(n)}(0)=n!$$

$$\begin{aligned}(a+x)^n &= a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + ... + \\ &+ \frac{n(n-1)(n-2)...(n-(k-1))}{k!}a^{n-k}x^k + ... + \frac{n(n-1)}{2!}a^2x^{n-2} + nax^{n-1} + x^n.\end{aligned} \qquad (14)$$

Coefficients of the expansion (14) (b**inomial coefficients**) are denoted as follows

$$C_n^0 = 1, C_n^1 = n, C_n^2 = \frac{n(n-1)}{2!},..., C_n^{n-2} = C_n^2 = \frac{n(n-1)}{2!}, C_n^{n-1} = C_n^1 = n, C_n^n = C_n^0 = 1.$$

In general

$$C_n^k = \frac{n(n-1)(n-2)\cdot...\cdot(n-(k-1))}{k!},\ k = 0, 1, 2,..., n-2, n-1, n \qquad (15)$$

Note 6. Coefficients (15) possess the next property (prove it yourselves)

$$C_n^k = C_n^{n-k} \qquad (16)$$

Note 7. Binomial coefficients can be easy calculated with the help of so-called **Pascal[1] triangle**

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

....

Ex. 19.

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3,$$
$$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4,$$
$$(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5.$$

---

[1] Pascal, B. (1623 - 1662), a French mathematician, physicist, and philosopher

Taylor formula for arbitrary function of one variable

Let be given arbitrary function $y = f(x)$.

Def. 4. nth degree Taylor polynomial corresponding to the function $y = f(x)$ is called the next polynomial

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \quad (17)$$

Def. 5. Difference of n times differentiable function $y = f(x)$ and its nth degree Taylor polynomial $T_n(x)$ is called a **remainder** [remainder term, residual member] and is denoted by $R_n(x)$,

$$R_n(x) = f(x) - T_n(x) \quad (18)$$

It follows from the formulas (17), (18) that

$$R_n(x_0) = R_n'(x_0) = R_n''(x_0) = \ldots = R_n^{(n)}(x_0) = 0 \text{ and } R_n^{(k)}(x) = f^{(k)}(x) \text{ for } k > n \quad (19)$$

Theorem 8. For (n + 1)-times differentiable function $y = f(x)$ the remainder can be represented in the next form (**Lagrange form**)

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1} \quad (20)$$

where c is some point between x and x_0.

■Let for definiteness $x_0 < x$ and

$$\varphi(x) = (x - x_0)^{n+1} \quad (21)$$

be auxiliary function which satisfies conditions

$$\varphi(x_0) = \varphi'(x_0) = \varphi''(x_0) = \ldots = \varphi^{(n)}(x_0) = 0,\ \varphi^{(n+1)}(x) = (n+1)!. \quad (22)$$

Applying n times Cauchy theorem (with consecutive appearance of points $c_1, c_2, \ldots,$ c_n, c such that $x_0 < c < c_n < \ldots < c_2 < c_1 < x$) we get

$$\frac{R_n(x)}{\varphi(x)} = \frac{R_n(x) - R_n(x_0)}{\varphi(x) - \varphi(x_0)} = \frac{R_n'(c_1)}{\varphi'(c_1)} = \frac{R_n'(c_1) - R_n'(x_0)}{\varphi'(c_1) - \varphi'(x_0)} = \frac{R_n''(c_2)}{\varphi''(c_2)} = \ldots = \frac{R_n^{(n)}(c_n)}{\varphi^{(n)}(c_n)} =$$

$$= \frac{R_n^{(n)}(c_n) - R_n^{(n)}(x_0)}{\varphi^{(n)}(c_n) - \varphi^{(n)}(x_0)} = \frac{R_n^{(n+1)}(c)}{\varphi^{(n+1)}(c)} = \frac{f^{(n+1)}(c)}{(n+1)!} \Rightarrow R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\varphi(x),$$

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}\ \blacksquare$$

Note 8. Depending upon the way of reasoning there are many other forms of remainder (for example in the form of Cauchy, of Peano[1] *etc*.).

Knowing the remainder $R_n(x)$ we can represent the function $y=f(x)$ in the next form

$$f(x)=T_n(x)+R_n(x) \qquad (23)$$

Def. 6. Formula (23) which represents the function $y=f(x)$ through its Taylor polynomial $T_n(x)$ and the remainder $R_n(x)$ is called Taylor formula for this function. In particular case $x_0=0$ it is called Maclaurin formula.

Let's write Taylor and Maclaurin formulas with Lagrange's form of remainder:

$$\begin{aligned}f(x)&=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\\&+\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}x^k+\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1},\ c\in(x_0,x),\end{aligned} \qquad (24)$$

$$\begin{aligned}f(x)&=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}=\\&=\sum_{k=0}^{n}\frac{P^{(k)}(0)}{k!}x^k+\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1},\ c\in(0,x)\end{aligned} \qquad (25)$$

Ex. 20. Expand a function $f(x)=e^x$ by Maclaurin formula.

Solution. $f(x)=f'(x)=f''(x)=f'''(x)=\ldots=f^{(n)}(x)=f^{(n+1)}(x)=e^x$, $f(0)=$ $=f'(0)=f''(0)=f'''(0)=\ldots=f^{(n)}(0)=1, f^{(n+1)}(c)=e^c$ and so by (25)

$$e^x=T_n(x)+R_n(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^n}{n!}+\frac{x^{n+1}}{(n+1)!}e^c,\ c\in(0,x). \qquad (26)$$

If we'll put

$$e^x\approx T_n(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^n}{n!} \qquad (27)$$

[1] Peano, G. (1858 - 1932), an Italian mathematician

we'll find for approximate value of e^x with absolute error

$$\alpha = \left|e^x - T_n(x)\right| = \left|R_n(x)\right| = \frac{|x|^{n+1}}{(n+1)!}e^c. \qquad (28)$$

Ex. 21. Let's find approximate value of e putting $x=1$ and $n=8$ in the formulas (27), (28). We have

$$\alpha = \left|e - T_8(1)\right| = \left|R_8(1)\right| = \frac{1}{9!}e^c,\ 0<c<1,\ e^c<3,\ \alpha < \frac{3}{9!} < 0.000008 = 8\cdot 10^{-6},$$

$$T_8(1) = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} \approx 2.718278,$$

$$T_8(1) - 8\cdot 10^{-6} < e < T_8(1) + 8\cdot 10^{-6},\ 2.718278 - 0.000008 < e < 2.718278 + 0.000008,$$

$$2.718270 < e < 2.718286,\ e \approx 2.7182 \text{ and all digits are exact.}$$

Ex. 22. Expand functions $f(x) = \sin x$, $f(x) = \cos x$ by Maclaurin formula.

Let for example $f(x) = \sin x$. Derivatives of the function are

$$f'(x) = \cos x,\ f''(x) = -\sin x,\ f'''(x) = -\cos x,\ f^{(4)}(x) = \sin x,\ f^{(5)}(x) = \cos x,$$

$$f^{(6)}(x) = -\sin x,\ f^{(7)}(x) = -\cos x,\ f^{(8)}(x) = \sin x, \ldots,$$

in general (see lecture No. 15, Ex. 19)

$$f^{(n)}(x) = \sin\left(x + n\cdot\frac{\pi}{2}\right).$$

Values of the function and its derivatives at the point $x=0$ are equal to

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1,\ f^{(4)}(0) = 0,\ f^{(5)}(0) = 1,$$

$$f^{(6)}(0) = 0,\ f^{(7)}(0) = -1,\ f^{(8)}(0) = 0, \ldots,$$

$$f^{(2n-1)}(0) = \sin\left((2n-1)\cdot\frac{\pi}{2}\right) = \sin\left(\pi n - \frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - \pi n\right) = -\cos\pi n = (-1)^{n-1},$$

$$f^{(2n)}(0) = \sin\left(2n\cdot\frac{\pi}{2}\right) = \sin n\pi = 0,$$

The value of the $(2n+1)$-th derivative at a point c

$$f^{(2n+1)}(c) = \sin\left(c + (2n+1)\cdot\frac{\pi}{2}\right).$$

Now on the base of the formula (25) we'll get

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^{n+1}\frac{x^{2n+1}}{(2n+1)!}\sin\left(c + \frac{\pi}{2}(2n+1)\right). \quad (27)$$

By the same way we can obtain the expanding of the cosine in Maclaurin formula (do it yourselves)

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + (-1)^{n+1}\frac{x^{2n+2}}{(2n+2)!}\cos\left(c + \frac{\pi}{2}(2n+2)\right). (28)$$

Ex. 23. It follows from (27) that $\sin x \approx T_1(x) = x$ with absolute error

$$\alpha = |\sin x - T_1(x)| = |R_1(x)| = \left|(-1)^2\frac{x^3}{3!}\sin\left(c + 3\cdot\frac{\pi}{2}\right)\right| \le \frac{|x|^3}{3!} \le 0.001 \text{ if } |x| \le \sqrt[3]{0.006} < 0.18$$

Therefore with an accuracy to 0.001 $\sin x \approx x$ if $|x| < 0.18$ or $|x^\circ| < 10^\circ$.

Note 9. Putting $dx = \Delta x = x - x_0$ we can write Taylor formula (24) in terms of differentials (see lecture 13, Point 4, (25))

$$\Delta f(x_0) = f(x) - f(x_0) = df(x_0) + \frac{1}{2!}d^2 f(x_0) + \ldots + \frac{1}{n!}d^n f(x_0) + \frac{1}{(n+1)!}d^{n+1} f(c) \quad (29)$$

Taylor formula for a function of several variables

Taylor formula (29) for a function of one variable can be easy extended on the case of several variables.

Let $x = (x_1, x_2, \ldots, x_n) \in \Re^n$, $x_0 = (x_{10}, x_{20}, \ldots, x_{n0}) \in \Re^n$ and a function $y = f(x) = f(x_1, x_2, \ldots, x_n)$ be $(k+1)$-fold continuously differentiable function of n independent variables. Its total increment at the point $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})$ can be represented in the next form (**Taylor formula** with remainder in **Lagrange form**)

$$\Delta y = \Delta f(x_0) = f(x) - f(x_0) = f(x_1, x_2, \ldots, x_n) - f(x_{10}, x_{20}, \ldots, x_{n0}) =$$

$$= df(x_0) + \frac{1}{2!}d^2 f(x_0) + \ldots + \frac{1}{k!}d^k f(x_0) + \frac{1}{(k+1)!}d^{k+1} f(c),\ c = (c_1, c_2, \ldots, c_n). \quad (30)$$

For the case of two independent variables $z = f(x, y)$ **Taylor formula** is written in the next form

$$\Delta z = f(x,y) - f(x_0,y_0) = df(x_0,y_0) + \frac{1}{2!}d^2 f(x_0,y_0) + \frac{1}{3!}d^3 f(x_0,y_0) + \ldots + \\ + \frac{1}{k!}d^k f(x_0,y_0) + \frac{1}{(k+1)!}d^{k+1} f(c_1,c_2), \tag{31}$$

where

$$df(x_0,y_0) = f'_x(x_0,y_0)dx + f'_y(x_0,y_0)dy = \overline{gradf(x_0,y_0)} \cdot (dx,dy),$$

$$d^2 f(x_0,y_0) = f''_{xx}(x_0,y_0)dx^2 + 2f''_{xy}(x_0,y_0)dxdy + f''_{yy}(x_0,y_0)dy^2, \tag{32}$$

$$d^k f(x_0,y_0) = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^k f(x_0,y_0),\ k = 1, 2, 3, \ldots \tag{33}$$

The second order differential of a function of two variables $z = f(x,y)$ is a quadratic form with symmetric matrix (so-called Hesse[1] matrix) of the second order,

$$H(f,(x_0,y_0)) = \begin{pmatrix} f''_{xx}(x_0,y_0) & f''_{xy}(x_0,y_0) \\ f''_{yx}(x_0,y_0) & f''_{yy}(x_0,y_0) \end{pmatrix},\ f''_{xy}(x_0,y_0) = f''_{yx}(x_0,y_0). \tag{34}$$

In the case on *n* independent variables the second order differential of a function $y = f(x) = f(x_1, x_2, \ldots, x_n)$ is quadratic form

$$d^2 f(x_0) = d^2 f(x_{10}, x_{20}, \ldots, x_{n0}) = \sum_{i,j=1}^{n} f''_{x_i x_j}(x_0)dx_i dx_j \tag{35}$$

with *n*th order symmetric matrix (**Hesse matrix**)

$$H(f,x_0) = \begin{pmatrix} f''_{x_1x_1}(x_0) & f''_{x_1x_2}(x_0) & f''_{x_1x_3}(x_0) & \ldots & f''_{x_1x_n}(x_0) \\ f''_{x_2x_1}(x_0) & f''_{x_2x_2}(x_0) & f''_{x_2x_3}(x_0) & \ldots & f''_{x_2x_n}(x_0) \\ f''_{x_3x_1}(x_0) & f''_{x_3x_2}(x_0) & f''_{x_3x_3}(x_0) & \ldots & f''_{x_3x_n}(x_0) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ f''_{x_nx_1}(x_0) & f''_{x_nx_2}(x_0) & f''_{x_nx_3}(x_0) & \ldots & f''_{x_nx_n}(x_0) \end{pmatrix}, \tag{36}$$

$$f''_{x_i x_j}(x_0) = f''_{x_j x_i}(x_0),\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n.$$

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[1] Hesse, L.O. (1811 - 1874), a German mathematician

DIFFERENTIAL CALCULUS: basic terminology

1. Áccuracy of appróximate cálculation/evàluátion	Точність наближеного обчислення	Точность приближённого вычисления
2. Ángle between two intersecting curves	Кут між двома кривими, що перетинаються	Угол между двумя пересекающимися кривыми
3. Ángular póint of a graph	Кутова точка графіка	Угловая точка графика
4. Appróach [tend to, go to] *smth* (about a point of a graph/curve)	Наближатися до *чогось* (про точку кривої, графіка)	Приближаться к *чему-то* (о точке кривой, графика)
5. Appróximate calculation/evàluátion of a mágnitude/quántity	Наближене обчислення величини	Приближённое вычисление величины
6. Appróximate válue	Наближене значення	Приближённое значение
7. Auxíliary fúnction	Допоміжна функція	Вспомогательная функция
8. Be spécified [represented, defíned, detérmined] explícitly, by an explícit equátion; be explícit(ly) rèpresénted	Бути заданим [заданою] явно, явним рівнянням	Быть заданным [заданной] явно, явным уравнением
9. Be spécified [represented, defíned, detérmined] implícitly, by an implícit equátion; be implícit(ly) rèpresénted	Бути заданим [заданою] неявно, неявним рівнянням	Быть заданным [заданной] неявно, неявным уравнением
10.Be spécified [represented, defíned, detérmined] pàramétrically, by parámetric equations, be parametriccally rèpresénted	Бути заданим [заданою] параметрично, параметричними рівняннями	Быть заданным [заданной] параметрически, параметрическими уравнениями
11.Cálculate *smth* to the third décimal place, to withín 0,001, up to 0,001	Обчислити *щось* з точністю до 0.001	Вычислить что-либо с точностью до 0,001
12.Cómposite fúnction, fúnction of a fúnction, superposítion [còmposítion] of fúnctions	Складена функція, функція від функції, суперпозиція функцій	Сложная функция, функция от функции, суперпозиция функций
13.Compúte/eváluate/cálculate/find a derívative/ dìfferéntial	Знайти/відшуката/обчислити похідну/диференціал	Найти/отыскать/вычислить роизводную/дифференциал
14.Compúting [eváluating,	Знаходження/відшукан-	Нахожде-

evàluátion, càlculátion, fínding] a derívative/differrenttial	ня/обчислення похідної/ диференціала	ние/отыскание/вычисление производной/ дифференциала
15.Derívative of a fúnction in a given diréction; diréctional derívative of a fúnction	Похідна функції у даному напрямку [за даним напрямком]	Производная функции в данном направлении
16.Derivative of the first/ second/third/hígher order; first/second/third/hígher órder derívative	Похідна першого/другого/третього/вищого порядку	Производная первого/ второго/третьего/высшего порядка
17.Derívative (of fúnction at the point a)	Похідна (функції в точці)	Производная (функции в точке)
18.Derívative in a diréction; diréctional derívative	Похідна у напрямку [за напрямком]	Производная по направлению
19.Derívative of a composite fúnction	Похідна складеної функції	Производная сложной функции
20.Derívative of an implícit fúnction	Похідна неявної функції	Производная неявной функции
21.Derívative on the right, right [right-hand] derívative	Права похідна	Правая производная
22.Derívative with respéct to x	Похідна по x	Производная по x
23.Dìffentiátion	Диференціювання	Дифференцирование
24.Dìfferentiabílity of a fúnction	Диференційовність функції	Дифференцируемость функции
25.Dìfferéntiable fúnction	Диференційовна функція	Дифференцируемая функция
26.Dìfferéntial	Диференціал	Дифференциал
27.Dìfferéntial of the first [second, third, hígher] órder; first-[second-, third-, hígher] órder dìfferéntial; first [second, third, hígher] dìfferéntial	Диференціал першого [другого, третього, вищого] порядку	Дифференциал первого [второго, третьего, высшего] порядка
28.Dìfferéntial cálculus	Диференціальне числення	Дифференциальное исчисление
29.Dìfferéntiate (with respéct to x)	Диференціювати/продиференціювати (по x)	Дифференцировать/ продифференцировать (по x)
30.Dìfferentiátion rúle, rúle of differentiation	Правило диференціювання	Правило дифференцирования
31.Diréction	Напрям, напрямок	Направление
32.Diréction defined by	Напрям [напрямок], ви-	Направление, определён-

two given póints (from the póint *A* to the póint *B*)	значений двома даними точками (від точки *A* до точки *B*)	ное двумя данными точками (от точки *A* до точки *B*)
33.Diréction of a given véctor	Напрям [напрямок] даного вектора	Направление данного вектора
34.Draw a sécant through a póint	Провести січну через точку	Провести секущую через точку
35.Elasticity	Еластичність	Эластичность
36.Equátion of the nórmal [nórmal line] to a curve [to a súrface] (at its given póint)	Рівняння нормалі до кривої [до поверхні] (в даній її точці)	Уравнение нормали к кривой [к поверхности] (в данной её точке)
37.Equátion of the tangent [tangent line] to a curve (at its given póint)	Рівняння дотичної до кривої (в даній її точці)	Уравнение касательной к кривой (в данной её точке)
38.Equátion of the tángent plane to a súrface (at its given póint)	Рівняння дотичної площини до поверхні (в даній її точці)	Уравнение касательной плоскости к поверхности (в данной её точке)
39.Érror	Помилка, похибка	Ошибка, погрешность
40.Expánsion [development, expánding] of a fúnction by (méans of) Maclaurin(‘s) fórmula	Розвинення функції за допомогою формули Маклорена	Разложение функции с помощью формулы Маклорена
41.Explícit fúnction	Явна функція	Явная функция
42.Fínite derívative	Скінченна похідна	Конечная производная
43.For a fixed *x* (*y*) with *y* (*x*) as a váriable	При фіксованому *x* (*y*) і *y* (*x*) як змінному	При фиксированном *x* (*y*) и *y* (*x*) в качестве переменной
44.Gèométric(al) sense/méaning/signíficance	Геометричний сенс	Геометрический смысл
45.Get/obtáin an íncrement	Отримувати приріст	Получать приращение
46.Give an íncrement to an árgument	Давати арґументу приріст	Давать аргументу приращение
47.Grádient	Ґрадієнт	Градиент
48.Héssian mátrix	Матриця Гессе	Матрица Гессе
49.Implícit fúnction	Неявна функція	Неявная функция
50.Implícit fúnction defíned/detérmined [fúnction defíned/detérmined implicitly]: a) by an equátion; b) by a sýstem of equátions	Неявна функція, задана/визначена [функція, задана/визначена неявно]: а) одним рівнянням; б) системою рівнянь	Неявная функція, заданная/определённая [функція, заданная/определённая неявно]: а) одним уравнением; б) системой уравнений
51.Íncrement of an árgu-	Приріст арґументу/функ-	Приращение аргумента/

ment [of a fúnction] at a póint	ції (в точці)	функции (в точке)
52.Ínfinite derívative	Нескінченна похідна	Бесконечная производная
53.Intérior/ínner fúnction	Внутрішня функція	Внутренняя функция
54.Ìntermédiate árgument	Проміжний арґумент	Промежуточный аргумент
55.Ìnterséct [cut, cross] *smth*	Перетинати *щось*	Пересекать *что-то*
56.Ìnterséct [cut, intercross, meet] at a póint	Перетинатися в точці	Пересекаться в точке
57.Ìnterséct [cut, intercross, meet] with *smth*	Перетинатися з *чимсь*	Пересекаться с *чем-то*
58.Ìnterséction [intersecttion, concúrrence, cross, crossing, íntercept, meet] *of smth with smth*	Перетин *чогось з чимсь*	Пересечение *чего-то с чем-то*
59.Ìnterséction/cross póint, póint of intersecttion	Точка перетину	Точка пересечения
60.Inváriant próperty, próperty of inváriance of the form of a dìfferéntial	Властивість інваріантності форми диференціала	Свойство инвариантности формы дифферен-циала
61.Ínvérse fúnction	Обернена функція	Обратная функция
62.Left [left-hand] derívative, derívative on/from the left	Ліва похідна	Левая производная
63.Left [left-hand] tángent (líne)	Ліва дотична	Левая касательная
64.Límit of the rátio of the íncrement of the fúnction to còrrespónding íncrement of the árgument when/as the látter tends to [appróaches] nought/zéro	Границя відношення приросту функції до відповідного приросту арґументу при прямуванні останнього до нуля	Предел отношения приращения функции к соответствующему приращению аргумента при стремлении последнего к нулю
65.Límiting posítion of the sécant, tángent (line)	Граничне положення січної, дотична	Предельное положение секущей, касательная
66.Lògaríthmic derívative	Логарифмічна похідна	Логарифмическая производная
67.Lògaríthmic dìffentiátion, dìffentiátion by méans of táking the logarithm	Диференціювання за допомогою диференціювання	Дифференцирование при помощи логарифмирования
68.Maclaurin(‘s) fórmula	Формула Маклорена	Формула Маклорена

69.Mechánical sense/méaning/signíficance (of a derívative/dìfferéntial)	Механічний сенс (похідної/диференціала)	Механический смысл (производной/дифференциала)
70.Mixed pártial derívative	Мішана частинна похідна	Смешанная частная производная
71.Nórmal (line) to a cúrve at a gíven póint	Нормаль до кривої в даній точці	Нормаль к кривой в данной точке
72.Note [mark (off), trace, óutline] (crítical) póints on the áxis and get [obtáin, deríve] séveral/some íntervals	Відкласти, відмітити, нанести (критичні) точки на осі й отримати декілька інтервалів	Отложить, отметить, нанести (критические) точки на оси и получить несколько интервалов
73.*n*-th (order) derívative, derívative of the *n*-th order	Похідна *n*-го порядку	Производная *n*-го порядка
74.*n*-th (order) differenttial, dìferéntial of the *n*-th order	Диференціал *n*-го порядку	Дифференциал *n*-го порядка
75.*n*-th (order) pártial derívative, pártial derívative of the *n*-th order	Частинна похідна *n*-го порядку	Частная производная *n*-го порядка
76.Parámeter	Параметр	Параметр
77.Partial derívative of the first [second, third, hígher] órder; first-[second-, third-, hígher] order pártial derívative; first [second, third, hígher] pártial derívative	Частинна похідна першого [другого, третього, вищого] порядку	Частная производная первого [второго, третьего, высшего] порядка
78.Pártial derívative with respect to x, y, …	Частинна похідна по x, y, …	Частная производная по x, y, …
79.Pártial dìfferéntial with respect to x, y, …	Частинний диференціал по x, y, …	Частный дифференциал по x, y, …
80.Pártial íncrement with respect to x, y, …	Частинний приріст по x, y, …	Частное приращение по x, y, …
81.Pass through the point	Проходити через точку	Проходить через точку
82.Phýsical sense/méaning /signíficance	Фізичний сенс	Физический смысл
83.Póint of tángency/cóntact, tángency/cóntact/adhérent point	Точка дотику	Точка касания
84.Príncipal/dóminant línear part of the íncrement of a fúnction	Головна лінійна частина приросту функції	Главная линейная часть приращения функции
85.Rélative érror	Відносна похибка	Относительная погреш-

		ность
86.Relative increment	Відносний приріст	Относительное приращение
87.Remáinder (term)	Залишковий член	Остаточный член
88.Rèpresént (*for exámple* a cúrve)	Зображати/зобразити (*напр.* криву)	Изображать/изобразить (*напр.* кривую)
89.Rèpresentátion (*for exámple* of a cúrve)	Зображення (*напр.* кривої)	Изображение (*напр.*, кривой)
90.Right [right-hand] tángent	Права дотична	Правая касательная
91.Sécant	Січна	Секущая
92.Spécify [rèpresént, defíne, detérmine] (a fúnction/curve) explícitly, by an explícite equátion, implícitly, by an implícite equátion, pàramétrically, by paramétric equations, in pólar coórdinates, by a pólar equátion	Задавати/задати (функцію, криву) явно, явним рівнянням, неявно, неявним рівняням, параметрично, параметричними рівняннями, рівняням в полярних координатах, полярним рівнянням	Задавать/задать (функцию, кривую) явно, явным уравнением, неявно, неявным уравнением, параметрически, параметрическими уравнениями, уравнением в полярных координатах, полярным уравнением
93.Súbnórmal	Піднормаль	Поднормаль
94.Sùbtángent	Піддотична	Подкасательная
95.Table of the derívatives	Таблиця похідних	Таблица производных
96.Tángency/cóntact	Дотик	Касание
97.Tángent [cóntact, be tángent to, tóuch] *smth*	Дотикатися *чогось*	Касаться *чего-то*
98.Tángent (line)	Дотична	Касательная
99.Tángent (line) to a cúrve at a gíven póint	Дотична до кривої в даній точці	Касательная к кривой в данной точке
100. Taylor(‘s) fórmula	Формула Тейлора	Формула Тейлора
101. To be approximately équal [to appróximate] (to)	Наближено дорівнювати	Приближённо равняться
102. Tótal [exáct, órdinary, pérfect] dìfferéntial	Повний диференціал	Полный дифференциал
103. Tótal derívative of a cómposite fúnction	Повна похідна складеної функції	Полная производная сложной функции
104. Tótal íncrement	Повний приріст	Полное приращение

APPLICATIONS OF DIFFERENTIAL CALCULUS

LECTURE NO.17. INVESTIGATION OF FUNCTIONS OF ONE VARIABLE

POINT 1. CONDITIONS OF INCREASE AND DECREASE

Theorem 1 (necessary condition of íncrease of a function). If a differentiable function of one variable $y = f(x)$ increases on an interval (*a, b*), then its derivative is nonnegative one on this interval.

■Let a function $y = f(x)$ increases on the interval (*a, b*), *x* is an arbitrary point of the interval and an increment Δx of the argument *x* is so small that a point $x + \Delta x$ lies on (*a, b*) (fig. 1). If the increment $\Delta x > 0$, that is $x < x + \Delta x < b$, then the increment of the function at the point *x* is positive,

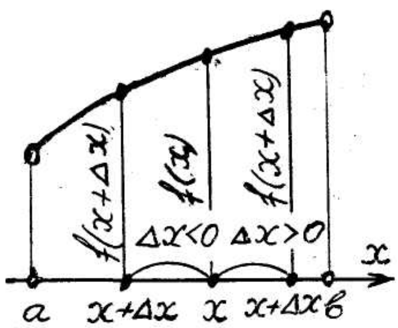


Fig. 1

$$\Delta f(x) = f(x + \Delta x) - f(x) > 0,$$

and so $\Delta f(x)/\Delta x > 0$. If $\Delta x < 0$, that is $a < x + \Delta x < x$, then the increment of the function at the point *x* is negative,

$$\Delta f(x) = f(x + \Delta x) - f(x) < 0,$$

and so $\Delta f(x)/\Delta x > 0$. Thus in both cases ($\Delta x > 0$ and $\Delta x < 0$) the ratio $\Delta f(x)/\Delta x$ is positive. By virtue of the limit theory the derivative of the function at the point *x* is non-negative, that is

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta f(x)}{\Delta x}\geq 0\ \blacksquare$$

Note 1. By analogy the inequality $f'(x)\leq 0$ on the interval (*a, b*) is the necessary condition for décrease of a function $y=f(x)$ on (*a, b*).

Theorem 2 (sufficient condition of íncrease of a function). If $f'(x)>0$ on an interval (*a, b*) then the function $y=f(x)$ increases on (*a, b*).

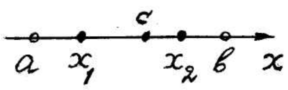


Fig. 2

■Let $f'(x)>0$ on the interval (*a, b*) and x_1, x_2 be two arbitrary points of (*a, b*) such that $x_1<x_2$ (fig. 2). By Lagrange theorem there exists a point $c\in(x_1,x_2)$ such that

$$f(x_2)-f(x_1)=f'(c)(x_2-x_1)>0 \text{ because of } f'(c)>0,\ x_2-x_1>0.$$

Therefore $f(x_1)<f(x_2)$ that is the function $y=f(x)$ increases on (*a, b*)■

Note 2. By analogy the inequality $f'(x)<0$ on the interval (*a, b*) is sufficient condition for décrease of a function $y=f(x)$ on (*a, b*).

Ex. 1. Prove that a function represented implicitly by an equation of an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

decreases in the first quadrant.

Solution. By the rule of differentiation of implicit function

$$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0,\ \frac{yy'}{b^2}=-\frac{x}{a^2},\ y'=-\frac{b^2x}{a^2y}<0 \text{ for } x>0,\ y>0\,.$$

Ex. 2. Prove that implicit functions represented by canonical equations of a hyperbola and a parabola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ y^2=2px$$

increase in the first quadrant.

POINT 2. LOCAL EXTREMA

Def.1. A point x_0 is called a **point of a local maximum** of a function $y=f(x)$

if there exists some neighbourhood U_{x_0} of x_0 (on the fig. 3 $U_{x_0} = (m, n)$) such that for any $x \in U'_{x_0} = U_{x_0} \setminus \{x_0\}$ the inequality

$$f(x) < f(x_0) \text{ or } \Delta f(x_0) = f(x) - f(x_0) < 0$$

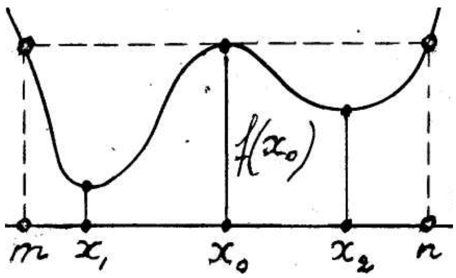


Fig. 3

holds. The value of the function at the point x_0, that is $f(x_0)$, is called **a local maximum** of the function.

By analogous way a point of a local minimum and a local minimum of a function are defined (points x_1, x_2 on the fig. 3 and corresponding values $f(x_1)$, $f(x_2)$ of the function).

The terms a local maximum and a local minimum are united by the common term a **local extremum**.

Theorem 3 (necessary condition for existence of a local extremum). If a function $y = f(x)$ has a local extremum at a point x_0 then $f'(x_0) = 0$ or $f'(x_0)$ doesn't exist.

Correctness of the theorem follows from Fermat theorem.

Def. 2. A point $x_0 \in D(f)$ of the domain of definition $D(f)$ of a function $f(x)$ is called its critical point if $f'(x_0) = 0$ or $f'(x_0)$ doesn't exist.

In particular

Def. 3. A point x_0 is called a stationary point of a function $y = f(x)$ if its derivative at this point equals zero: $f'(x_0) = 0$.

Note 3. It follows from the theorem 3 that a function can take on a local extremum only at its critical point. But a critical point is not necessary a point of a local extremum, that is the necessary condition for existing of a local extremum isn't that sufficient.

Ex. 3. The point $x = 0$ is a critical one (namely a stationary one) for a function $f(x) = x^3$ ($f'(x) = 3x^2, f'(0) = 0$) but it isn't a point of a local extremum because of $f(x) < f(0) = 0$ if $x < 0$ and $f(x) > f(0) = 0$ if $x > 0$.

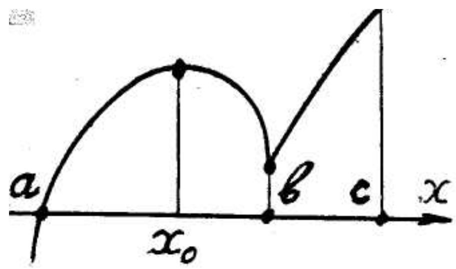


Fig. 4

Theorem 4 (the first sufficient condition for existence of a local maximum). If a function $f(x)$ is continuous at its critical point x_0, $f'(x)>0$ in an interval (a,x_0), $f'(x)<0$ in an interval (x_0,b) (fig. 4), then the function has a local maximum at the point x_0.

■Proving follows from the theorem 2 and the Note 2: the function $y=f(x)$ increases in the interval (a,x_0), decreases in the interval (x_0,b) and it's continuous at the point x_0■

Note 4. One can get the sufficient condition for existence of a local minimum substituting the inequalities of the theorem 4 by the next: $f'(x)<0$ in an interval (a,x_0), $f'(x)>0$ in an interval (x_0,b). A function represented on the figure 5 has a local minimum at the point *b*.

Theorem 5 (the second sufficient condition for existence of a local extremum at the stationary point). Let a function $y=f(x)$ be continuous at a stationary point x_0 (definition 3) and $f''(x_0)\neq 0$. The point x_0 is that of a local maximum if $f''(x_0)<0$ and a local minimum if $f''(x_0)>0$.

■Let for example

$$f''(x_0)=\lim_{\Delta x\to 0}\frac{f'(x_0+\Delta x)-f'(x_0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f'(x_0+\Delta x)}{\Delta x}<0.$$

It follows from the theory of limits that

$$\frac{f'(x_0+\Delta x)}{\Delta x}<0$$

for sufficiently small Δx. It means that $f'(x_0+\Delta x)>0$ for $\Delta x<0$ and $f'(x_0+\Delta x)<0$ for $\Delta x>0$. So the function increases from the left of the point x_0 and decreases from its right. Being continuous at the point x_0 the function takes on a local maximum at this point■

Note 5. It follows from the theory of limits that if some function $\varphi(x)$ is continuous at a point *a* ($\lim\limits_{\Delta x\to a}\varphi(x)=\varphi(a)$) and takes on positive value at this point, then the function is positive in certain neighbourhood of the point *a*.

■On the base of the Note 5 we can prove the theorem 5 in additional suppositions of existence of the second derivative of the function $y = f(x)$ in some neighbourhood U_{1,x_0} of the point x_0 and of its continuity at this point.

Let $f''(x_0) < 0$. By virtue of the Note 5 we have $f''(x) < 0$ in some other neighbourhood U_{2,x_0} of the point x_0. By Taylor formula the increment of the function at the point x_0 can be written, in the common part $U_{x_0} = U_{1,x_0} \bigcup U_{2,x_0}$ of U_{1,x_0} and U_{2,x_0}, as follows

$$\Delta f(x_0) = df(x_0) + \frac{1}{2!}d^2 f(c) = f'(x_0)\Delta x + \frac{1}{1\cdot 2}f''(c)\Delta x^2 = \frac{1}{2}f''(c)\Delta x^2$$

$(f'(x_0) = 0,\ f''(c) < 0)$. It means that $\Delta f(x_0)$ has the sign of $f''(c)$ namely is negative in U_{x_0}. Therefore

$$\Delta f(x_0) = f(x_0 + \Delta x) - f(x_0) < 0,\ f(x_0 + \Delta x) < f(x_0),$$

and the function has local maximum at the point x_0■

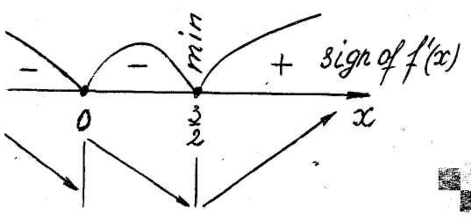


Fig.5

Ex. 4. Find intervals of increase, decrease and local extrema of the function

$$y = f(x) = \sqrt[3]{x}(x-6).$$

Solution. The domain of definition of the function is $\Re = (-\infty, +\infty)$. Its derivative equals

$$f'(x) = (\sqrt[3]{x})' \cdot (x-6) + \sqrt[3]{x} \cdot (x-6)' =$$

$$= \frac{1}{3\sqrt[3]{x^2}}(x-6) + \sqrt[3]{x} = \frac{4x-6}{3\sqrt[3]{x^2}};$$

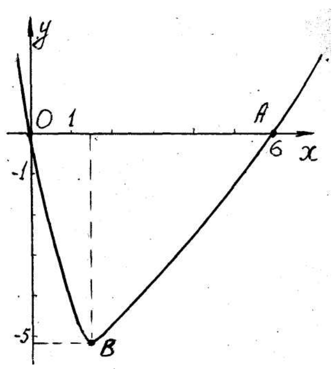


Fig 6

$f'(x) = 0$ for $x = 3/2$, $f'(x)$ doesn't exist at the point $x = 0$ and so the points $x = 0$, $x = 3/2$ are those critical of the function. We find the intervals of constant sign of the derivative by the interval method (fig. 5). The distribution of signs shows that the function increases on the interval $(3/2, \infty)$ and decreases on the interval $(-\infty, 3/2)$. It has a local minimum at the point $x = 3/2$ which equals

$$y_{\min} = f(3/2) = -9/2\sqrt{3/2} \approx -5.15.$$

Remark. The function $y = f(x) = \sqrt[3]{x}(x-6)$ is positive on $(-\infty, 0) \cup (6, \infty)$, negative on $(0, 6)$, its limit on $\pm\infty$ equals $+\infty$. Approximate graph of the function is represented on the fig. 6. It passes through the points $O(0; 0)$, $A(6; 0)$, $B\left(3/2; \frac{9}{2}\sqrt{\frac{3}{2}}\right)$.

POINT 3. ABSOLUTE EXTREMA

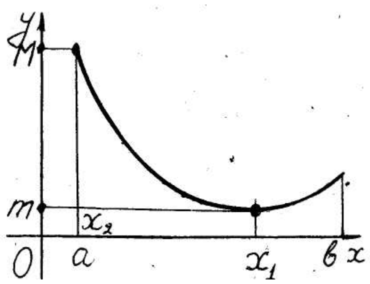


Fig. 7

Let a function $y = f(x)$ is continuous on a segment $[a, b]$. By virtue of the theorem 4 of the lecture No. 11 it takes on the least *m* and the greatest *M* values on $[a, b]$ that is there are points $x_1 \in [a, b]$, $x_2 \in [a, b]$ such that

$$f(x_1) = m = \min_{[a,b]} f(x), \; f(x_2) = M = \max_{[a,b]} f(x).$$

Numbers *m*, *M* are called **absolute extrema** of the function $y = f(x)$ on the segment $[a, b]$. It's necessary to find m, M.

Solving the problem of finding *m*, *M* we take into account that at least one of the points x_1, x_2 can lie inside the segment or can be an end point of the segment. In the first case by Fermat theorem the derivative at such the point equals zero or doesn't exist. For example a function represented by the fig. 7 takes on the least value *m* at the inner point x_1 (and $f'(x_1) = 0$) and the greatest value *M* at the end point *a* (that is $x_2 = a$).

On the base of these remarks we can state the next

Rule. To find the greatest and the least values (absolute extrema) of a function which is continuous on a segment it's sufficient to do as follows:

1. To find all inner critical points of the function (that is critical points which lie inside the segment).

2. To calculate the values of the function at all these points and at the end points of the segment.

3. To choose the greatest and the least of these values.

Ex. 5. Find absolute extrema of the function $f(x)=\sqrt[3]{x}(x-6)$ on the segment $[-1, 4]$.

Solution. The function has two critical points $x=0, x=3/2$ (see example 2) which are those interior. The values of the function at these points and at the points $x=-1,\ x=4$ are equal to

$$f(0)=0, f(3/2)=-9\sqrt[3]{3/2}\,/\,2\approx-5.15, f(-1)=7, f(4)=-2\sqrt[3]{4}\approx 3.17.$$

Therefore $m=\min\limits_{[-1,\ 4]} f(x)=f(3/2)=-9/2\cdot\sqrt[3]{3/2}\approx-5.15,\ M=\max\limits_{[-1,\ 4]} f(x)=f(-1)=7$.

POINT 4. CONVEXITY, CONCAVITY, INFLEXION POINTS

Def. 4. A curve L is called **convex** one if it lies below a tangent to L at any its point $M(x;y)$ (fig. 8 a).

Def. 5. A curve L is called **concave** one if it lies above a tangent to L at any its point $M(x;y)$ (fig. 8 b).

Def. 6. A point $M_0(x_0;y_0)$ is called **inflexion point** of a curve L if it separates the parts of convexity and concavity of the curve (fig. 8 c).

Theorem 6 (sufficient condition of convexity of a graph of a function)**.** If the second derivative $f''(x)<0$ on an interval (a,b) then the graph of the function $f(x)$ is convex one over this interval.

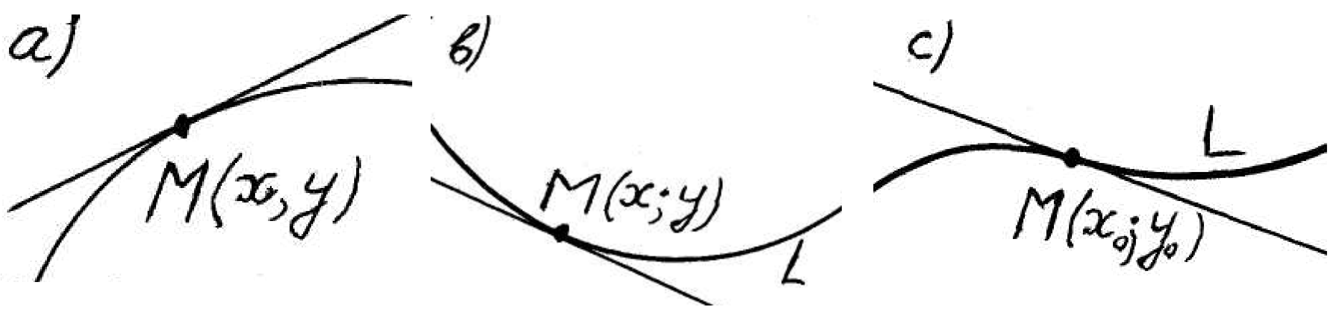


Fig. 8

■ Let $M_0(x_0;y_0)$, $y_0=f(x_0)$, $x_0\in(a,b)$, be some point of the graph of the function $y=f(x)$, M_0N be the tangent to the graph which equa-tion is $y=y_{\text{tang}}=f(x_0)+f'(x_0)(x-x_0)$. To prove convexity of the graph in the case $f''(x)<0$ we must prove that for any $x\in(a,b)$ $BM-BN=f(x)-y_{\text{tang}}<0$ (fig. 9). We'll do it in supposition $x_0<x$. Applying two times Lagrange theorem we get

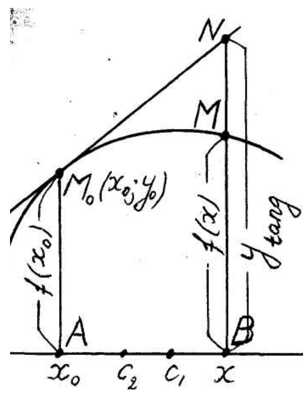


Fig. 9

$$f(x)-y_{\text{tang}}=f(x)-f(x_0)-f'(x_0)(x-x_0)=f'(c_1)(x-x_0)-f'(x_0)(x-x_0)=$$

$$=(f'(c_1)-f'(x_0))(x-x_0)=f''(c_2)(c_1-x_0)(x-x_0),\text{ where } x_0<c_2<c_1.$$

By virtue of $f''(c_2)<0$, $(c_1-x_0)(x-x_0)>0$ we have $f(x)-y_{\text{tang}}<0$ ■

Note 6. Sufficient condition of concavity of the graph of a function $y=f(x)$ is $f''(x)>0$.

Note 7. Convexity of the graph of a function $y=f(x)$ in some neighbourhood of a point x_0 in condition $f''(x)<0$ can be proved with the help of Taylor formula for $n=1$. Indeed,

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(c)(x-x_0)^2,\ c\in(x_0,x)$$

$$f(x)=y_{\text{tang}}+\frac{1}{2!}f''(c)(x-x_0)^2,\ f(x)-y_{\text{tang}}=\frac{1}{2!}f''(c)(x-x_0)^2<0.$$

Theorem 7 (necessary condition of existing of an inflexion point). If some point $M_0(x_0;y_0)$ is an inflexion point of a graph of a function $y=f(x)$ and the first derivative $f'(x)$ of the function is continuous in some neighbourhood of the point x_0, then $f''(x_0)=0$ or $f''(x_0)$ doesn't exist.

Theorem 8 (sufficient condition for existing of inflexion point). Let: a) a function $y=f(x)$ is continuous at a point x_0; b) $f''(x_0)=0$ or $f''(x_0)$ doesn't exist;

c) $f''(x) < 0$ (or $f''(x) > 0$) for $x < x_0$; d) $f''(x) > 0$ (corresp. $f''(x) < 0$) for $x > x_0$. In these conditions the point $M_0(x_0; y_0)$ is inflexion one of the graph of the function.

■Correctness of the theorem is simple corollary of the theorem 6■

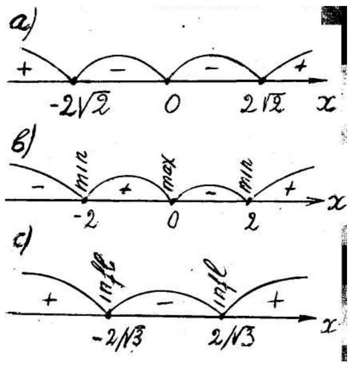


Fig. 10

Ex. 6. Investigate the function $y = \frac{1}{4}x^4 - 2x^2$ and plot its graph.

Solution. 1) Domain of definition of the function is the set of all reals [of all real numbers] $D(y) = \Re$.

2) The function is positive on $(-\infty, -2\sqrt{2}) \cup (2\sqrt{2}, \infty)$ and negative on $(-2\sqrt{2}, 0) \cup (0, 2\sqrt{2})$ (see fig. 10 a).

3) The graph of the function passes through the points $A(2\sqrt{2}; 0)$, $B(-2\sqrt{2}; 0)$, $O(0; 0)$.

4) $\lim\limits_{x \to \pm\infty} y = \lim\limits_{x \to \pm\infty}\left(\frac{1}{4}x^4 - 2x^2\right) = \lim\limits_{x \to \pm\infty} \frac{1}{4}x^4 = +\infty$.

5) $y' = x^3 - 4x = x(x+2)(x-2)$; $y' = 0$ if $x = 0, x = -2, x = 2$. The derivative is positive on $(-2, 0) \cup (2, +\infty)$ and negative on $(-\infty, -2) \cup (0, 2)$ (fig. 10 b). Therefore the function increases on $(-2, 0) \cup (2, +\infty)$, decreases on $(-\infty, -2) \cup (0, 2)$, has a local minimum -4 at the points $x = \pm 2$, a local maximum 0 at the point $x = 0$. Its graph passes through the points $C(2; -4)$, $D(-2; -4)$, $O(0; 0)$.

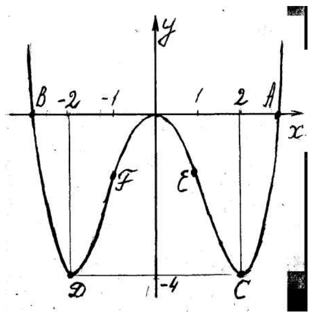


Fig. 11

6) $y'' = 3x^2 - 4$, $y'' = 0$ if $x = \pm 2/\sqrt{3}$. The second derivative is positive on the set $(-\infty, -2/\sqrt{3}) \cup (2/\sqrt{3}, +\infty)$ and negative on the interval $(-2/\sqrt{3}, 2/\sqrt{3})$. The graph of the function is concave over the union of intervals $(-\infty, -2/\sqrt{3}) \cup (2/\sqrt{3}, +\infty)$, convex over the interval $(-2/\sqrt{3}, 2/\sqrt{3})$ (fig. 10 c), it has two inflexion points, namely $E(2/\sqrt{3}; -20/9)$ and $F(-2/\sqrt{3}; -20/9)$.

The graph of the function is represented on the fig. 11.

Ex. 7. Investigate for convexity and concavity the graph of the function which we've studied in Ex. 4, that is of the function $y = f(x) = \sqrt[3]{x}(x-6)$.

Solution. $y'' = \dfrac{4(x+3)}{9x\sqrt[3]{x^2}}$, $y'' = 0$ if $x = -3$, y'' doesn't exist if $x = 0$. $y'' > 0$ on $(-\infty,-3)\cup(0,+\infty)$, $y'' < 0$ on the interval $(-3,0)$. So the graph of the function is convex over $(-3,0)$, concave over $(-\infty,-3)\cup(0,+\infty)$ and has two inflection points namely $O(0;\,0)$, $C(-3;\,9\sqrt[3]{3})$.

Ex. 8. Prove convexity of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ and the parabola $y^2 = 2px$ in upper half-plane (for $y > 0$).

Solution. In the case of the ellipse we have $y' = -\dfrac{b^2x}{a^2y}$ (see Ex. 1). The second derivative

$$y'' = -\frac{b^2}{a^2}\cdot\frac{y-xy'}{y^2} = -\frac{b^2}{a^2}\cdot\frac{y-x\cdot\left(-\dfrac{b^2x}{a^2y}\right)}{y^2} = -\frac{b^4}{a^2}\cdot\frac{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}}{y^3} = -\frac{b^4}{a^2}\cdot\frac{1}{y^3} < 0 \text{ for } y > 0.$$

Consider two other cases yourselves.

POINT 5. ASYMPTOTES

Def. 7. Let a current point $M(x;y)$ of a curve *L* retire in the infinity and simultaneously approache some straight line *l*. This straight line *l* is called an asymptote of the curve *L* (fig. 12).

We'll deal with asymptotes of the graphs of functions. One distinguishes three types of asymptotes namely those vertical, horizontal and oblique.

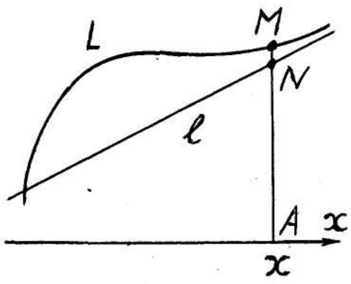


Fig. 12

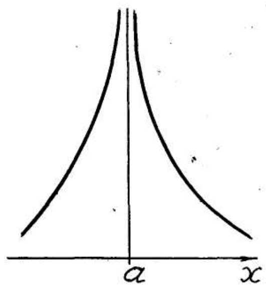


Fig. 13

1) If a function $y = f(x)$ is infinitely large for x tending to a point a ($x \to a$ or $x \to a-0$, or $x \to a+0$) then the straight line

$$x = a \qquad (1)$$

is the **vertical asymptote** of its graph (fig. 13).

Ex. 9. Graphs of functions $\ln x, \tan x$ have correspondingly the vertical asymptotes $x = 0, x = \pi/2 + \pi n \ (n \in \mathbb{Z})$ because of

$$\ln x \to -\infty \text{ if } x \to 0+0,\ |\tan x| \to +\infty \text{ if } x \to \frac{\pi}{2} + \pi n.$$

2) If there exists a finite limit $\lim\limits_{x\to+\infty} f(x) = b \left(\lim\limits_{x\to-\infty} f(x) = b\right)$ then a straight line

$$y = b \qquad (2)$$

is the **horizontal asymptote** for the right part (corresp. for the left part) of the graph of the function.

Ex. 10. Left parts of the graphs of the functions $y = e^x$ and $y = a^x$ for $a > 1$ have the horizontal asymptote $y = 0 \ (Ox - \text{axis})$ because of $\lim\limits_{x\to-\infty} e^x = 0,\ \lim\limits_{x\to-\infty} a^x = 0$. On the contrary for $0 < a < 1$ the horizontal asymptote $y = 0$ possesses the right part of the graph of the function $y = a^x$ because of in this case $\lim\limits_{x\to+\infty} a^x = 0$.

3) Equation of an **oblique asymptote** of the graph of a function $y = f(x)$ we find in the next form

$$y = kx + b \qquad (3)$$

with unknown k, b.

For the **right part** of the graph we must have (see fig. 12)

$$NM \to 0 \text{ or } f(x) - kx - b \to 0 \text{ if } x \to +\infty.$$

Dividing by x we have in addition

$$\frac{f(x)}{x} - k - \frac{b}{x} \to 0 \text{ if } x \to +\infty$$

and therefore

$$k = \lim_{x\to+\infty} \frac{f(x)}{x}, \quad b = \lim_{x\to+\infty} \left(f(x) - kx\right). \qquad (4)$$

Oblique asymptote of the left part of the graph one seeks in the form (3) but finds parameters k, b by the formulas

$$k = \lim_{x\to-\infty} \frac{f(x)}{x}, \quad b = \lim_{x\to-\infty} \left(f(x) - kx\right) \qquad (5)$$

If at least one of limits (4), (5) is infinite or doesn't exist then corresponding asymptote doesn't exist.

Ex. 11. Find asymptotes of the graph of the function $y = \frac{x^3}{x^2 - 9}$.

Solution. Straight lines $x = -3, x = 3$ are vertical asymptotes because of $y \to \infty$ for $x \to \pm 3$. To find an oblique asymptote $y = kx + b$ we get

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^2}{x^2 - 9} = \lim_{x\to\pm\infty} \frac{x^2}{x^2} = 1;\ k = 1;$$

$$b = \lim_{x\to\pm\infty} (f(x) - kx) = \lim_{x\to\pm\infty} \left(\frac{x^3}{x^2 - 9} - x\right) = \lim_{x\to\pm\infty} \frac{9x}{x^2 - 9} = 9 \lim_{x\to\pm\infty} \frac{x}{x^2} = 0;\ b = 0$$

Answer: both parts of the graph have the same oblique asymptote $y = x$.

Ex. 12. Graph of the function $f(x) = 3\arctan x - x$ hasn't vertical asymptote but its left and right sides have different oblique asymptotes. Indeed

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \left(3\frac{\arctan x}{x} - 1\right) = 0 - 1 = -1;$$

$$b = b_{\text{left}} = \lim_{x\to-\infty} \left(f(x) - kx\right) = \lim_{x\to-\infty} \left(3\arctan x - x - (-1)x\right) = \lim_{x\to-\infty} 3\arctan x = -\frac{3\pi}{2};$$

$$b = b_{\text{right}} = \lim_{x\to+\infty} \left(f(x) - kx\right) = \lim_{x\to+\infty} \left(3\arctan x - x - (-1)x\right) = \lim_{x\to+\infty} 3\arctan x = \frac{3\pi}{2}.$$

Answer. $y = -x - \frac{3\pi}{2}$, $y = -x + \frac{3\pi}{2}$ for the left and the right sides correspondingly.

POINT 6. GENERAL SCHEME FOR INVESTIGATION OF FUNCTIONS

Investigation of a function and plotting its graph can be often fulfill by the next general scheme.

I. The first part. Preliminary sketch of the graph of a function.

1. Determination the domain of definition and continuity of the function, fixing the points of infinite discontinuity and corresponding vertical asymptotes.

2. Determination intervals of constant sign of the function that is intervals where it is positive or negative.

3. Evaluation the left-hand and right-hand limits of the function at the points of infinite discontinuity.

4. Finding intersection points of the graph with coordinate axes.

5. Finding limits of the function as $x \to \pm\infty$, fixing eventual horizontal asymptotes and their intersection points with the graph.

6. Determination oblique asymptotes of the graph in the case of infinite limit of the function on $-\infty$ or $+\infty$ and their intersection points with the graph.

It's useful (as the rule from the beginning) to bring to light the next two questions.

7. Whether the function is even or odd one. Evenness or oddness of the function means symmetry of its graph with respect to the *Oy*-axis or the origin of coordinates respectively and permits to regard the function only in the interval $[0, \infty)$.

8. Whether the function is periodic or non-periodic one. Periodicity of the function permits to graph it only on some one period.

9. Tracing a preliminary sketch of the graph with the help of results of preceding study.

II. The second part. Investigation of the function for monotonicity and local extrema (with the help of the first derivative $y' = f'(x)$) and as result the first correction of the preliminary draft of the graph.

III. The third part. Investigation of functions for convexity, concavity, inflexion points (with the help of the second derivative $y'' = f''(x)$). Second correction of the graph.

IV. The fourth part. Final plotting the graph of the function.

Ex. 13. Investigate and graph a function

$$y = \frac{x^3}{x^2 - 9}.$$

I. The first part.

1. The function is defined and continuous for all values $x \neq \pm 3$. The domain of its definition and continuity is the union of the intervals $(-\infty,-3) \cup (-3,3) \cup (3,\infty)$. Points $x = \pm 3$ are those of infinite discontinuity of the function and the straight lines $x = \pm 3$ are the vertical asymptotes of its graph.

2. Obviously

$$f(-x) = \frac{(-x)^3}{(-x)^2 - 9} = -\frac{x^3}{x^2 - 9} = -f(x).$$

Therefore the function is odd one and its graph is symmetric with respect to the origin of coordinates. It's sufficient to investigate the function only in the interval $[0,\infty)$.

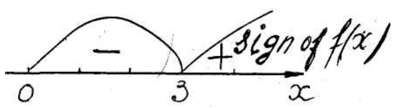


Fig 14

3. Determination intervals of constant sign of the function.

The function equals zero for $x = 0$, it doesn't exist for $x = 3$. By the interval method we ascertain that the function is positive in the interval $(3,\infty)$ and negative in the interval $(0,3)$ (fig. 14).

4. Evaluation the left-hand and right-hand limits of the function at the point $x = 3$ corresponding to vertical asymptote. We have

$$f(3-0) = \lim_{x \to 3-0} f(x) = -\infty, \; f(3+0) = \lim_{x \to 3+0} f(x) = +\infty$$

because the function is negative from the left of the point $x = 3$ and positive on its right.

5. There exists unique cross point of the graph with coordinate axes. It's the origin of coordinates $O(0;0)$.

6. Limit of the function on $+\infty$,

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^3}{x^2-9} = \lim_{x\to+\infty} \frac{x^3}{x^2} = \lim_{x\to+\infty} x = +\infty .$$

It follows that it's necessary to find out oblique asymptote of the graph.

7. We seek an oblique asymptote of (the right part of) the graph in the form

$$y = kx + b$$

getting

$$k = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x^2}{x^2-9} = 1,$$

$$b = \lim_{x\to+\infty} (f(x) - kx) = \lim_{x\to+\infty} \left(\frac{x^3}{x^2-9} - 1\cdot x \right) = \lim_{x\to+\infty} \frac{9x}{x^2-9} = 0 .$$

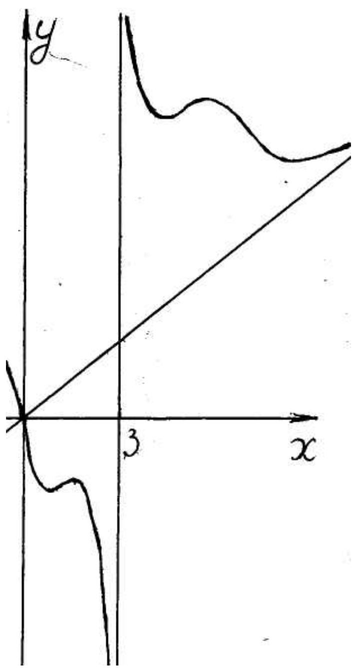


Fig. 15

Thus (the right part of) the graph of the function possesses the oblique asymptote of the equation

$$y = x .$$

8. To find out whether there are intersection points of the graph with the oblique asymptote we must solve the system of equations

$$\begin{cases} y = \dfrac{x^3}{x^2-9}, \\ y = x, \end{cases}$$

It has unique solution $(0, 0)$ and so the asymptote $y = x$ crosses the graph of the function only at the origin $O(0;0)$.

9. Now we can plot preliminary (very approximate) sketch of the graph (see for example fig. 15).

II. The second part. Investigation the function for íncrease, décrease, local extrema by means of the first-order derivaive.

10. Making use of the rule of diffentiation of a ratio we get

$$f'(x)=\frac{3x^2\left(x^2-9\right)-2x\cdot x^3}{\left(x^2-9\right)^2}=\frac{x^4-27x^2}{\left(x^2-9\right)^2}=\frac{x^2\left(x^2-27\right)}{\left(x^2-9\right)^2}.$$

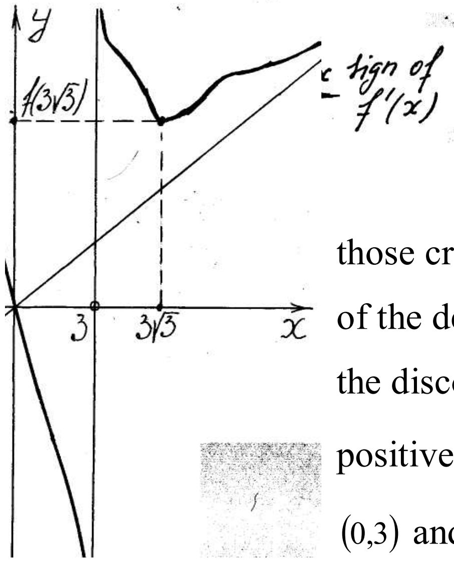


Fig. 16

The derivative turns into zero for $x=0, x=3\sqrt{3}\approx 5.2$ ($3\sqrt{3}$ approximately equals 5.2). It doesn't exist for $x=3$. The points 0, $3\sqrt{3}$ are those critical of the function. We seek intervals of constant sign of the derivative using interval method and taking into account the discontinuity point $x=3$ of the function. The derivative is positive in the interval $\left(3\sqrt{3},\infty\right)$ and negative in the intervals $(0,3)$ and $\left(3,3\sqrt{3}\right)$ (see fig. 16). It follows that the function increases in the interval $\left(3\sqrt{3},\infty\right)$ and decreases in the intervals $(0,3)$ and $\left(3,3\sqrt{3}\right)$. At the point $x=3\sqrt{3}$ it has a local minimum

$$y_{\min}=f\left(3\sqrt{3}\right)=\frac{\left(3\sqrt{3}\right)^3}{\left(3\sqrt{3}\right)^2-9}\approx 7.8$$

Corresponding point of the graph is $A\left(3\sqrt{3};f\left(3\sqrt{3}\right)\right)$.

11. We can do the first correction of preliminary sketch of the graph (see fig. 17).

Fig. 17

III. **The third part.** Investigation the graph of the function for convexity, concavity, finding inflexion points making use of the second-order derivative.

12. The second-order derivative of the function equals

$$f''(x)=\frac{18x\left(x^2+27\right)}{\left(x^2-9\right)^3}.$$

It vanishes at the point $x=0$ and doesn't exist for $x=3$. It's negative in the interval $(0,3)$ and positive in the interval $(3,\infty)$. Therefore its graph is convex one over the interval $(0,3)$ and concave over the interval $(3,\infty)$. For $x\in(0,\infty)$ it doesn't possess the inflexion

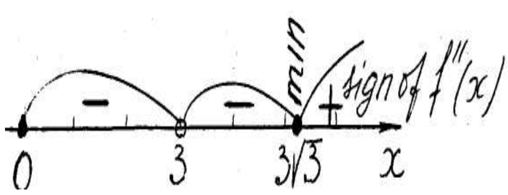


Fig. 18

points. But in view of its symmetry with respect to the origin the graph has single inflexion point namely the origin $O(0,0)$.

13. The slope of the tangent to the graph at inflection point $O(0,0)$ equals zero because of $f'(0)=0$. Therefore the graph touches the Ox-axis at the point $O(0,0)$.

14. At will one may form a table of variation of the function using all results (that is tabulate the results of done investigations).

Now we can fulfill the second correction of the graph and pass to final part.

IV. The fourth part. Final plotting the graph of the function (see fig. 19).

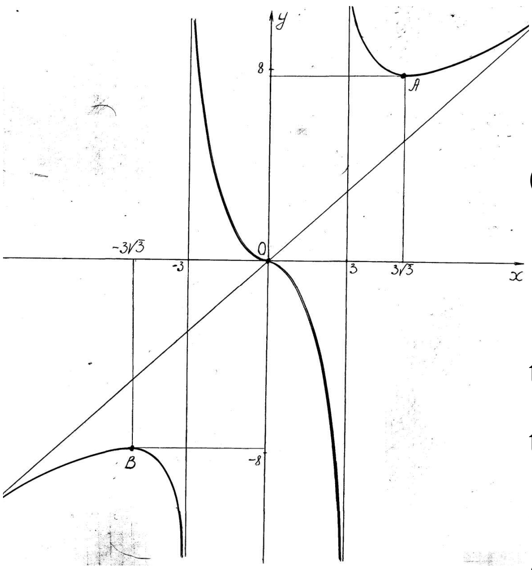


Fig. 19

Ex. 14. Investigate and graph the function

$$y = e^{-x^2}$$

(this graph is called **Gaussian curve**).

I. The first part.

1. Domain of definition and continuity of the function is $\Re = (-\infty, +\infty)$. Its graph hasn't vertical asymptotes.

2. The function is even one and therefore its graph is symmetric with respect to the Oy-axis. We can investigate the function only over the interval $[0,+\infty)$.

3. The function is positive for all $x \in [0,+\infty)$.

4. The point $A(0;1) \in Oy$ is unique common point of the graph with coordinate axes.

5. $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} e^{-x^2} = 0$ and therefore (the right part of) the graph has the horizontal asymptote $y=0$ (Ox-axis). It doesn't intersect this asymptote.

II. The second part.

6. The first derivative of the function $y' = -2xe^{-x^2} = -2xy < 0$ for $x \in [0,+\infty)$. Hence the function decreases on the interval $[0,+\infty)$ and hasn't local extrema.

III. The third part.

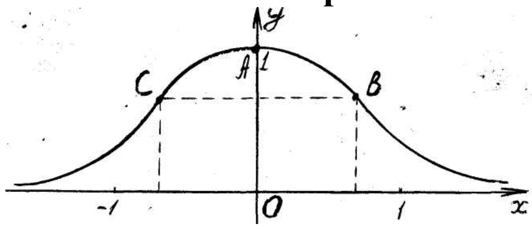


Fig. 20

7. The second derivative of the function

$$y'' = -2(y + xy') = -2(y + x(-2xy)) = 2y(2x^2 - 1).$$

It equals zero at the point $x = 1/\sqrt{2}$, is negative over the interval $(0, 1/\sqrt{2})$ and is positive over the interval $(1/\sqrt{2}, +\infty)$. The graph of the function is convex over the interval $(0, 1/\sqrt{2})$, concave over the interval $(1/\sqrt{2}, +\infty)$ and has an inflexion point for $x = 1/\sqrt{2}$ that is the point

$$B(1/\sqrt{2}\,; f(1/\sqrt{2})) = B(1/\sqrt{2}\,; 1/\sqrt{e}).$$

IV. The fourth part. Graph of the function is represented on the fig. 20.

Ex. 15. Investigate and graph the function

$$y = 3\arctan x - x\,.$$

I. The first part.

1. Domain of definition and continuity of the function is $\Re = (-\infty, +\infty)$. Its graph hasn't vertical asymptotes.

2. $f(-x) = 3\arctan(-x) - (-x) = -3\arctan x + x = -(3\arctan x - x) = -f(x)$.

The function is odd one and therefore its graph is symmetric with respect to the origin of coordinates. We'll investigate the function only on the interval $[0, +\infty)$.

3. It's known one zero of the function on the interval $[0, +\infty)$, that is $x = 0$, and the graph of the function passes through the origin $O(0; 0)$. We don't know whether there are other zeros and hence we can't find intervals of fixed signs of the function and intersection points of the graph with Ox-axis on $(0, +\infty)$.

4. We must seek oblique asymptote of the graph because of

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}(3\arctan x - x) = 3\lim_{x\to+\infty}\arctan x - \lim_{x\to+\infty} x = 3\pi/2 - \lim_{x\to+\infty} x = -\infty$$

5. Finding the equation of oblique asymptote in the form $y = kx + b$ we get (see Ex. 12)

$$y = -x + \frac{3\pi}{2}.$$

6. The graph of the function doesn't intersect oblique asymptote because of corresponding system of equations

$$\begin{cases} y = 3\arctan x - x, \\ y = -x + \dfrac{3\pi}{2} \end{cases}$$

has no solutions.

7. Let's plot a preliminary sketch of the graph in supposition that there aren't intersection points with Ox-axis distinct from the origin $O(0;0)$ (fig. 21).

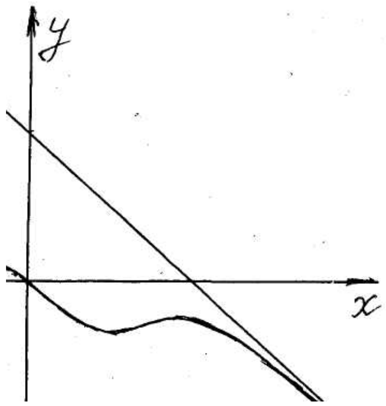


Fig. 21

II. The second part. The first derivative of the function equals

$$y' = \frac{3}{1+x^2} - 1 = \frac{2-x^2}{1+x^2}.$$

Critical (stationary) point is $x = \sqrt{2}$. On the interval $(0, \sqrt{2})$ $y' > 0$, and the function increases. On the interval $(\sqrt{2}, +\infty)$ $y' < 0$ and the function decreases. It means that the function has a lo-cal maximum at the point $x = \sqrt{2}$ which equals $y_{\max} = f(\sqrt{2}) = 3\arctan\sqrt{2} - \sqrt{2} \approx 3 \cdot 0.96 - 1.41 \approx 1.45$ Corresponding point of the graph is $A(\sqrt{2}; f(\sqrt{2}))$ and therefore the graph crosses the Ox-axis in some point with abscissa lying in the interval $\left(\sqrt{2}, \frac{3\pi}{2}\right)$.

III. The third part. The second deriva-tive of the function

$$y'' = \frac{-2x(1+x^2) - 2x(2-x^2)}{(1+x^2)^2} = \frac{-6x}{(1+x^2)^2} < 0$$

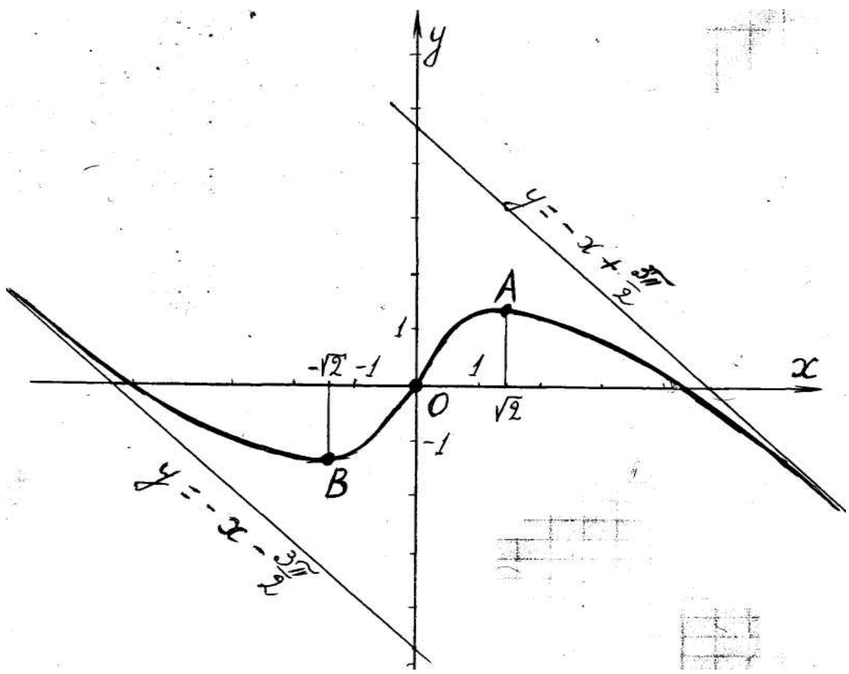


Fig. 22

for any $x > 0$, hence the graph of the function is convex on the interval $(0,+\infty)$. It has an inflexion point $O(0;0)$ with the slope of the tangent $f'(0) = 2$ at this point.

The fourth part. Final graph of the function is represented on the fig. 22.

POINT 7. EXTREMAL PROBLEMS

There are many word problems that ask for the maximum or minimum value of a certain quantity. Solving such problems consists of the next tree parts.

A. Translation a problem to a purely mathematical one.

Typically, we can follow a three-step procedure:

(1) *Drawing a picture* with the quantities given in the problem and with as many unknowns as we need.

(2) *Finding an expression for the quantity to be maximized (or minimized).* This expression as usually involves two or more variables. Using the picture, we find equations relating these variables to each other to eliminate all but one variable in the expression in question.

(3) *Notation any restrictions on this variable that are imposed by the problem.*

Now the problem is entirely translated to a mathematical extremum problem.

• Usually the translation process is the most difficult task.

B. Solving a mathematical problem on extremum.

Suppose we find the maximum (or minimum) value of a differentiable function $f(x)$ on a certain interval. We find its critical points on this interval. If there is only one such a point $x = a$ and if $f(x)$ has no vertical asymptotes, then it's well to take into account the following:

If the function has a local maximum (minimum) at this point, it is its absolute maximum (minimum).

We study the function for a local extremum at the point $x = a$ by examining the sign of the first derivative $f'(x)$ on both sides of $x = a$ or the sign of the second derivative $f''(a)$ at this point.

Instead investigation a function on a local extremum we often can seek the absolute maximum (or minimum) of the function in question if we'll define it as continuous one on some segment (bounded closed interval).

C. Answering the question asked in the problem.

Ex. 16. A cone with a slant height [a generator, a genera-trix, a ruling] l is to be constructed. What is the largest possible volume of such a cone?

Solving. Let's label the slant height $AB = l$, the height of the cone $OB = H$, the radius of its base $OA = R$ (fig. 23). The volume of the cone equals

$$V = 1/3\pi R^2 H$$

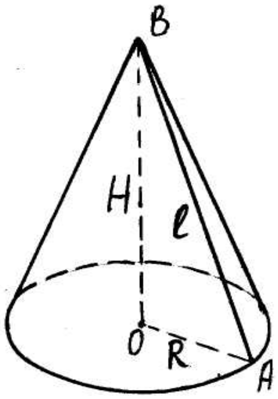


Fig. 23

and depends on two variables R, H. But by Pythagorean theorem we express R in terms of H from the triangle OAB, $R^2 = l^2 - H^2$. So we get V as a function only of one variable H,

$$V = f(H) = 1/3\pi\left(l^2 - H^2\right)H = 1/3\pi\left(l^2 H - H^3\right),$$

where $0 < H < l$. Putting $f(0) = f(l) = 0$ we define $f(H)$ as continuous function on the segment $[0, l]$. The problem in question is translated to mathematical problem of finding the greatest value of this function on this segment.

But $f'(H) = 1/3\pi\left(l^2 - 3H^2\right)$, $f'(H) = 0$ if $l^2 - 3H^2 = 0$, whence $H = l/\sqrt{3}$;

$$f(l/\sqrt{3}) = 1/3\pi\left(l^2 - l^2/3\right)l/\sqrt{3} = 2\pi l^3\sqrt{3}/27 > 0,\ f(0) = f(l) = 0$$

and therefore the maximal volume of the cone equals

$$V_{\max} = \max_{[0,\,l]} f(H) = f\left(l/\sqrt{3}\right) = 2\pi l^3 \sqrt{3}/27 .$$

Note. We can not put $f(0) = f(l) = 0$ but prove that the point $H = l/\sqrt{3}$ is that of local maximum of the function $V = f(H) = 1/3\pi\left(l^2 - H^2\right)H = 1/3\pi\left(l^2 H - H^3\right)$. Indeed, its first derivative is positive of the interval $\left(0, l/\sqrt{3}\right)$ and negative on the interval $\left(l/\sqrt{3}, l\right)$. On the other hand the second derivative of the function at the point $l/\sqrt{3}$ is negative: $f''(H) = -2\pi H,\ f''\left(l/\sqrt{3}\right) = -2\pi\, l/\sqrt{3} < 0$. Because of uniqueness of the critical point a local maximum of the function is that absolute.

Ex. 17. Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius R.

Solution. Let (fig. 24) $AB = x, BC = y, AC = 2R$. The area of the inscribed rectangle $ABCD$ equals

$$S = xy$$

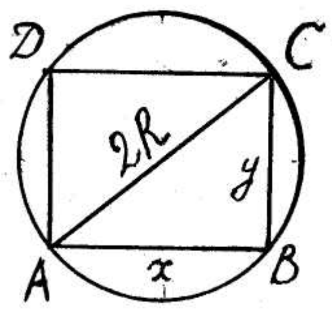


Fig. 24

and depends on two variables *x* and *y*. From the triangle *ABC* by Pythagorean theorem

$$y = BC = \sqrt{AC^2 - AB^2} = \sqrt{4R^2 - x^2} ,$$

and

$$S = f(x) = x\sqrt{4R^2 - x^2}, 0 < x < 2R .$$

We'll define the function $f(x) = S$ as continuous one on the segment $[0, 2R]$ if we put $f(0) = f(2R) = 0$ and we must find its greatest value on $[0, 2R]$.

$$f'(x) = \sqrt{4R^2 - x^2} + x \cdot \frac{-2x}{2\sqrt{4R^2 - x^2}} = \frac{4R^2 - 2x^2}{\sqrt{4R^2 - x^2}};\ f'(x) = 0 \text{ if } x = R\sqrt{2};$$

$$f(0) = f(2R) = 0 \text{ and } f\left(R\sqrt{2}\right) = R\sqrt{2} \cdot \sqrt{4R^2 - 2R^2} = 2R^2 > 0 .$$

Thus the area in question takes on the largest value if

$$x = R\sqrt{2}, y = \sqrt{4R^2 - x^2} = R\sqrt{2}$$

that is if the rectangle $ABCD$ is a square with the length of its sides $R\sqrt{2}$.

Note. Point $x = R\sqrt{2}$ is that of local maximum of the function

$$S = f(x) = x\sqrt{4R^2 - x^2},\, 0 < x < 2R$$

(why?) which is unique one. Therefore a local maximum at this point is that absolute.

Ex. 18. Solve yourselves the next problem. A cylindrical can with a top and bottom is constructed using S in^2 of tin. What is the largest volume such a can might contain?

Ex. 19. One needs to transport a cargo along the path ABC (see fig. 25 where $AO \perp OC,\, AO = 9\ km,\, OC = 15\ km$). The expenses of transportation of the unit of a cargo per unit of distance are in the ratio 5 : 4 along AB and BC correspondingly. Where B must be situated the expenses to be least?

Solving. Let $OB = x$, then

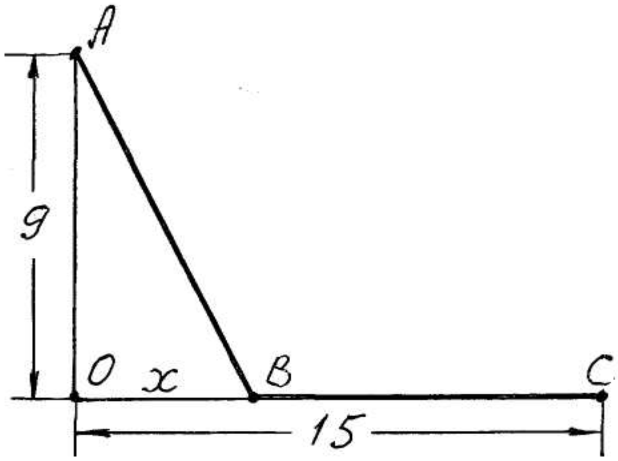


Fig. 25

$$AB = \sqrt{81 + x^2},\, BC = 15 - x,$$

and the expenses of transportation of T units of the cargo along AB and BC are equal respectively to

$$S_{AB} = 5kT \cdot \sqrt{81 + x^2},\, S_{BC} = 4kT \cdot (15 - x),$$

where k is some proportionality coefficient. So the function

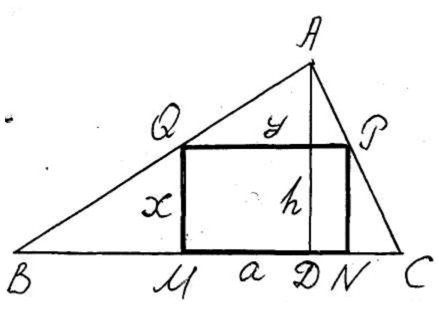
$$f(x) = S_{AB} + S_{BC} =$$

$$= 5kT \cdot \sqrt{81 + x^2} + 4kT \cdot (15 - x),\, 0 < x < 15,$$

gives the expenses along ABC and it's required to find its minimum. The derivative

$$f'(x) = \frac{5kTx}{\sqrt{81 + x^2}} - 4kT = kT \cdot \frac{\left(5x - 4\sqrt{81 + x^2}\right)}{\sqrt{81 + x^2}};\ f'(x) = 0 \text{ if } 5x - 4\sqrt{81 + x^2} = 0,\, x = 12.$$

For $x = 12$ the function $f(x) = S_{AB} + S_{BC}$ reaches the minimum because of $f'(x) < 0$ for $0 < x < 12$, $f'(x) > 0$ for $12 < x < 15$ and the critical point is unique one.



Ex. 20. Inscribe the rectangle of the greatest area in a triangle with a base a and an altitude h if one side of the rectangle lies on the base of the triangle.

Fig. 26 Solution. Let $BC = a, AD = h$, $x = MQ, y = PQ$ be the sides of the inscribed rectangle $MNPQ$ (fig. 26). It follows from the similitude of the triangles ABC, APQ that

$$PQ : BC = (h - x) : h,\ y : a = (h - x) : h,\ PQ = y = a/h \cdot (h - x)$$

whence the area of the rectangle $MNPQ$ is represented by the next function

$$S = f(x) = a/h \cdot x(h - x), 0 < x < h.$$

Its derivative $f'(x) = a/h \cdot (h - 2x)$ turns into zero for $x = h/2$ which is a point of the maximum of the function $f(x)$ (why?).

Ex. 21. A ray of light travels from point A to point B, where A and B are in different media (fig. 27). Suppose that the common boundary of the two media is a plane. Fermat's principle in optics states that the light will travel along the path for which the time of travel is a minimum. Show that if v_1 and v_2 are the velocities of light in media 1 and 2, respectively, then the light will travel a path that crosses the boundary in accordance with Snell's law:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

where θ_1 and θ_2 are the angles noted in fig. 27.

In short words we have to prove that for a ray of light, to pass from a point A of the medium 1 to a point B of the medium 2 in shortest time (see fig. 27), Snell's law must be satisfied where v_1 and v_2 are the velocities of light in media 1 and 2, respectively.

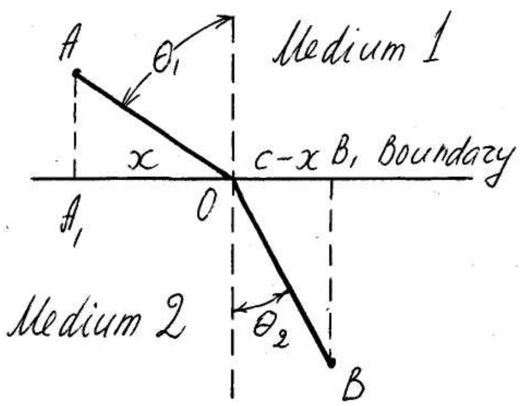


Fig. 27

Solution. Let (fig. 27) $AA_1 \perp A_1B_1, BB_1 \perp A_1B_1, AA_1 = a, BB_1 = b, A_1B_1 = c$ and $x = A_1O$. Then $OB_1 = c - x, AO = \sqrt{a^2 + x^2}, OB = \sqrt{b^2 + (c - x)^2}$. If T is the time of travel of a ray from A to B then

$$T=\frac{AO}{v_1}+\frac{OB}{v_2}=\frac{\sqrt{a^2+x^2}}{v_1}+\frac{\sqrt{b^2+(c-x)^2}}{v_2},\ T'=\frac{x}{v_1\sqrt{a^2+x^2}}-\frac{c-x}{v_2\sqrt{b^2+(c-x)^2}}=$$
$$=\frac{1}{v_1}\cdot\frac{A_1O}{AO}-\frac{1}{v_2}\cdot\frac{OB_1}{OB}=\frac{1}{v_1}\cdot\sin\theta_1-\frac{1}{v_2}\cdot\sin\theta_2=\frac{\sin\theta_1}{v_1}-\frac{\sin\theta_2}{v_2}=0\text{ if }\frac{\sin\theta_1}{\sin\theta_2}=\frac{v_1}{v_2}.$$

Obviously, this gives the minimum for *T*.

Ex. 22. Prove that for $x>0$

$$\ln(1+x)<x$$

■Let's introduce a function

$$f(x)=\ln(1+x)-x$$

with the domain of definition $D(f)=(-1,+\infty)$ and investigate it for monotonicity and local extrema.

$$f'(x)=\frac{1}{1+x}-1=\frac{-x}{1+x};$$

$f'(x)=0$ if $x=0$; $f'(x)>0$ on $(-1,0)$, $f'(x)<0$ on $(0,+\infty)$.

It follows that the function

$$f(x)=\ln(1+x)-x$$

has a local maximum at the point $x=0$ which equals $f_{\max}=f(0)=\ln 1-0=0$. Therefore $f(x)<0$ for $-1<x\neq 0$ and so

$$\ln(1+x)-x<0,\ \ln(1+x)<x,$$

in particularly for $x>0$.■

Ex. 23. Prove that

$$2x\arctan x\geq\ln(1+x^2).$$

■For a function

$$f(x)=2x\arctan x-\ln(1+x^2)$$

we have

$$f'(x)=2\arctan x+\frac{2x}{1+x^2}-\frac{2x}{1+x^2}=2\arctan x;\ f'(x)=0\ \text{if}\ 2\arctan x=0,\ x=0;$$

$$f''(x)=\frac{2}{1+x^2},\ f''(0)=2>0.$$

Thus the function

$$f(x)=2x\arctan x-\ln\left(1+x^2\right)$$

has the minimum at the point $x=0$, which equals $f(0)=0$, and so $f(x)\geq 0$ for any x that is

$$2x\arctan x-\ln\left(1+x^2\right)\geq 0$$

and

$$2x\arctan x\geq \ln\left(1+x^2\right).\blacksquare$$

Ex. 24. Prove yourselves that for $x\neq 0$

$$e^x>1+x.$$

Ex. 25. Using the result of preceding example prove that for $x>0$

$$e^x>1+x+\frac{x^2}{2}.$$

Solution. A function

$$f(x)=e^x-1-x-\frac{x^2}{2}$$

possesses the derivative

$$f'(x)=e^x-1-x$$

which equals zero at the point $x=0$ and is positive for $x\neq 0$ because of Ex. 24. It means that the function $f(x)$ increases on its domain of definition. But $f(0)=0$ and therefore $f(x)>0$ for $x>0$ and so the inequality in question is fulfilled.

Ex. 26. Solve the equation $x^4+4x^3+6x^2+4x+\sqrt{x^2+2x+37}=5$.

Instructions. Represent the derivative of a function

$$f(x)=x^4+4x^3+6x^2+4x+\sqrt{x^2+2x+37}$$

in the form

$$f'(x)=(x+1)\left(4(x+1)^2+\frac{1}{\sqrt{x^2+2x+37}}\right)$$

and prove that the function possesses a local (and an absolute) minimum 5 at the point $x=-1$. As result you'll get $x=-1$.

APPLICATIONS OF DIFFERENTIAL CALCULUS: basic terminology

1. Ábsolute (extrémum, mínimum, máximum)	Абсолютний (екстремум, мінімум, максимум)	Абсолютный (экстремум, минимум, максимум)
2. Àngular póint of a domáin [région]	Кутова точка області	Угловая точка области
3. Appróach [tend to] *smth* (about a point of a graph/curve)	Наближатися до *чогось* (про точку кривої, графіка)	Приближаться к *чему-то* (о точке кривой, графика)
4. Appróximate válue	Наближене значення	Приближённое значение
5. Ascend/rise (from left to right) (about a graph, about a curve)	Сходити/підійматися (зліва направо) (про графік, про криву)	Восходить/подниматься (слева направо) (о графике, о кривой)
6. Ascénding/rísing (from left to right) (about a graph/curve)	Висхідний (зліва направо)	Восходящий, поднимающийся (слева направо)
7. Assúmed [propósal, presuppósed] extrémum (*pl* extréma)	Передбачуваний/можливий екстремум	Предполагаемый [возможный] экстремум
8. Ásymptote (horizóntal, vértical, oblíque/inclíned)	Асимптота (горизонтальна, вертикальна, похила)	Асимптота (горизонтальная, вертикальная, наклонная)
9. Be [lie, be found, situate, be situated]	Знаходитись, бути розташованим	Находиться/располагаться, быть расположенным
10. Be [lie, be found, situate, be situated] from/on the right *of smth*	Лежати справа/праворуч від *чогось*	Лежать справа *от чего-либо*
11. Be [lie, be found, situate, be situated] lówer/belów/únder *of smth*	Лежати нижче *чогось*	Лежать ниже *чего-то*
12. Be [lie, be found, situate, be situated] from/on the left *of smth*	Лежати зліва/ліворуч від *чогось*	Лежать слева *от чего-либо*
13. Be [lie, be found, situate, be situated] over/above *smth*	Лежати вище *чогось*	Лежать/находиться выше *чего-то*
14. Be síituated [locáted, dispósed, arránged], be	Розміщуватися, бути розташованим	Располагаться, быть расположенным
15. Behávior (of a fúnction, curve)	Поведінка (функції, кривої)	Поведение (функции, кривой)
16. Concáve	Угнутий	Вогнутый
17. Concáve (graph, part/	Угнутий [угнута] (гра-	Вогнутый [вогнутая]

piece of a graph, curve)	фік, частина/ділянка графіка, крива)	(график, часть/участок графика, кривая)
18.Concávity	Угнутість	Вогнутость
19.Condítional (extrémum, mínimum, máximum	Умовний (екстремум, мінімум, максимум)	Условный (экстремум, минимум, максимум)
20.Constrúct [plot, trace, sketch] a cúrve, a graph póint by póint	Будувати, побудувати криву, графік по точках	Строить, построить кривую, график по точкам
21.Constrúct [plot, trace, sketch] a graph of a fúnction, graph a fúnction	Будувати, побудувати графік функції	Строить, построить график функции
22.Constrúction [constrúcting, trácing] graph of a fúnction [gráphing a fúnction]	Побудова графіка функції	Построение графика функции
23.Constrúction a graph póint by póint	Побудова графіка по точках	Построение графика по точкам
24.Cònvéx [cónvex]	Опуклий	Выпуклый
25.Cònvéx [cónvex] (graph, part/piece of a graph, of a curve)	Опуклий [опукла] (графік, частина/ділянка графіка, крива)	Выпуклый [выпуклая] (график, часть/участок графика, кривая)
26.Convéxity	Опуклість	Выпуклость
27.Còrrespónd to the extrémum (abóut a point of a cúrve, of a graph)	Відповідати екстремуму (про точку кривої, графіка	Соответствовать экстремуму (о точке кривой, графика)
28.Crítical póint	Критична точка	Критическая точка
29.Cúspidal póint	Точка звороту	Точка возврата
30.Decréase	Спадати	Убывать
31.Décrease	Спадання	Убывание
32.Decréasing/decay	Спадаючий	Убывающий
33.Dependence (línear, nònlínear/cùrvilínear, quadrátic, pàrabólic(al) *etc*) between váriables …	Залежність (лінійна, нелінійна, квадратична, параболічна *і т.ін.*) між змінними…	Зависимость (линейная, нелинейная, квадратическая, параболическая *и т.д.*) между переменными…
34.Descénd/drop (from left to right) (about a graph, about a curve)	Спадати/опускатися/ спускатися (зліва направо) (про графік, криву)	Нисходить/опускаться (слева направо) (о графике, о кривой)
35.Descénding/dropping (from left to right) (about a graph, about a curve)	Низхідний, той, що опускається (зліва направо)	Нисходящий, опускающийся (слева направо)
36.Desígn [draft, drawing, fréehànd/rough draw-	Ескіз графіка функції	Эскиз, набросок графика функции

ing, sketch, vérsion] of a graph/plot of a fúnction		
37.Desígn, dráwing, fígure	Рисунок	Рисунок
38.Dìsposítion [sìtuátion, locátion] (*for exámple* of a line)	Положення, розташування (*напр.* лінії)	Положение, расположение (*напр*. линии)
39.Draft [ɑ:] , do a draft	Робити рисунок	Делать чертёж, рисунок
40.Dráwing, fígure, draft	Креслення	Чертёж
41.Drop/descénd (from left to right) (about a graph/curve)	Спадати/опускатися/ спускатися (зліва направо) (про графік, про криву)	Опускаться/нисходить (слева направо) (о графике, о кривой)
42.Drópping/descénding (from left to right) (about a graph/curve)	Низхідний [той, що опускається] (зліва направо) (про графік, про криву)	Опускающийся, нисходящий (слева направо) (о графике, о кривой)
43.Empíric(al) relátion [de-péndence,connéction, còrre-látion] (betwéen váriables …)	Емпіричне співвідношення [емпірична залежність, емпіричний зв"язок] (між змінними)	Эмпирическое соотношение [эмпирическая зависимость, эмпирическая связь] (между переменными)
44.Estáblish (a relátion [depéndence,connéction, còrrelátion] between váriables …)	Установити (співвідношення, зв"язок між змінними)	Установить (соотношение, связь между переменными)
45.Estáblish a condítion	Встановити умову	Установить условие
46.Exact desígn/dráwing/ fígure/draft	Точний рисунок	Точный чертёж/рисунок
47.Exístence	Існування	Существование
48.Exístence condítion, condítion of exístence	Умова існування	Условие существования
49.Extrémum (*pl* extréma) of a fúnction of one [two, three, *n*, séveral] váriables (lócal, rélative, ábsolute, condítional)	Екстремум функції однієї [двох, трьох, *n*, декількох] змінних (локальний, відносний, абсолютний, умовний)	Экстремум функции одной [двух, трёх, *n*, нескольких] переменных (локальный, относительный, абсолютный, условный)
50.Extrémum próblem	Екстремальна задача	Экстремальная задача
51.Extrémum, *pl* extréma (lócal, rélative, absolute/ global, condítional)	Екстремум (локальний, відносний, абсолютний /глобальний, умовний)	Экстремум (локальный, относительный, абсолютный/глобальный, условный)
52.Find *smth* in the best way	Знайти *щось* якнайкраще	Найти *что-л.* наилучшим образом

53.Find the (lócal, rélative, ábsolute, condítional) extréma [mínima, maxima)] of a given fúnction	Знайти (локальні, відносні, абсолютні, умовні) екстремуми [мінімуми, максимуми] даної функції	Найти (локальные, относительные, абсолютные, условные) экстремумы [минимумы, максимумы] данной функции
54.Géneral schéme/plan for invèstigátion/invèstigáting fúnctions and constrúcting graphs	Загальна схема [загальний план] дослідження функцій і побудови графіків	Общая схема [общий план] исследования функций и построения графиков
55.Glóbal [ábsolute] (extrémum, mínimum, máximum)	Глобальний/абсолютний (екстремум, мінімум, максимум)	Глобальный/абсолютный (экстремум, минимум, максимум)
56.Graph [chart, curve, graphical chart, curve, plot] of a fúnction, plótted fúnction, fúnction graph	Графік функції	График функции
57.Gréatest and léast válues of a fúnction contínuous óver/in the bóunded clósed domáin/région	Найбільше й найменше значення функції, неперервної на відрізку [в замкненій обмеженій області]	Наибольшее и наименьшее значение функции, непрерывной на отрезке [в замкнутой ограниченной области]
58.Gréatest válue of a fúnction	Найбільше значення функції	Наибольшее значение функции
59.Gréatest válue of a fúnction which is contínuous one óver/in/on a ségment [bóunded clósed domáin/ région] (ábsolute maximum)	Найбільше значення функції, неперервної на відрізку [в замкненій обмеженій області] (абсолютний максимум)	Наибольшее значение функции, непрерывной на отрезке [в замкнутой ограниченной области] (абсолютный максимум)
60.Héssian	Гессіан, визначник (детермінант) Гессе	Гессиан, определитель (детерминант) Гессе
61.Héssian mátrix	Матриця Гессе	Матрица Гессе
62.Horizóntal ásymptote	Горизонтальна асимптота	Горизонтальная асимптота
63.Hypóthesis (*pl* hypótheses)	Гіпотеза	Гипотеза
64.Hypóthesize	Будувати [утворювати, висловлювати] гіпотезу	Строить [образовывать, высказывать] гипотезу
65.Incréase	Зростати	Возрастать
66.Íncrease	Зростання	Возрастание
67.Incréasing	Зростаючий	Возрастающий
68.Infléction/infléxion (of a graph of a fúnction)	Перегин (графіка функції)	Перегиб (графика функции)
69.Infléction/infléxion/	Точка перегину	Точка перегиба

flex póint, póint of inflect-tion/infléxion [flex, infléxion, póint of cóntrary fléxure]		
70.Ínterval of décrease of a fúnction	Інтервал спадання функції	Интервал убывания функции
71.Ínterval of íncrease of a fúnction	Інтервал зростання функції	Интервал возрастания функции
72.Ínterval of mònotonícity [monotoneness, monótony] of a fúnction	Інтервал монотонності функції	Интервал монотонности функции
73.Invéstigate [find out] (a fúnction, the behávior of a function, a crítical/ státionary póint *etc*)	Дослідити (функцію, поведінку функції, критичну/стаціонарну точку *і т.ін.*)	Исследовать (функцию, поведение функции, критическую/стационарную точку *и т.д*)
74.Invèstigátion [finding out] (of a fúnction, of the behávior of a function, of a crítical/státionary póint *etc*)	Дослідження (функції, поведінки функції, критичної/стаціонарної точки *і т.ін.*)	Исследование (функции, поведения функции, критической/стационарной точки *и т.д*)
75.Léast válue of a fúnction	Найменше значення функції	Наименьшее значение функции
76.Léast válue of a fúnction which is contínuous one óver/in/on a ségment [bóunded clósed domáin/ région] (ábsolute minimum)	Найменше значення функції, неперервної на відрізку [в замкненій обмеженій області] (абсолютний мінімум)	Наименьшее значение функции, непрерывной на отрезке [в замкнутой ограниченной области] (абсолютный минимум)
77.Léast-squares méthod [méthod of léast squáres]	Метод найменших квадратів	Метод наименьших квадратов
78.Line of regréssion of y on x	Лінія реґресії y на x	Линия регрессии y на x
79.Lócal (extrémum, mínimum, máximum)	Локальний (екстремум, мінімум, максимум)	Локальный (экстремум, минимум, максимум)
80.Màximizátion	Максимізація	Максимизация
81.Máximize *smth*	Максимізувати	Максимизировать
82.Maximum (*pl* maxima) (lócal, rélative, ábsolute/global, condítional) of a fúnction	Максимум функції (локальний, відносний, абсолютний/глобальний, умовний)	Максимум функции (локальный, относительный, абсолютный/глобальный, условный)
83.Máximum póint, póint of máximum	Точка максимуму	Точка максимума
84.Méthod of Lagrange's ìndetérminate/úndetermi-	Метод невизначених множників Лагранжа	Метод неопределённых множителей Лагранжа

ned múltipliers		
85.Mínimizátion	Мінімізація	Минимизация
86.Mínimize *smth*	Мінімізувати	Минимизировать
87.Minimum (*pl* mínima) (lócal, rélative, ábsolute/ global, condítional) of a fúnction	Мінімум функції (локальний, відносний, абсолютний/глобальний, умовний)	Минимум функции (локальный, относительный, абсолютный/глобальный, условный)
88.Mínimum póint, póint of mínimum	Точка мінімуму	Точка минимума
89.Mónotòne/mónotonic	Монотонний	Монотонный
90.Mònotónically (incréase, decréase)	Монотонно (зростати, спадати)	Монотонно (возрастать, убывать)
91.Mònotonícity [mónotoneness, monótony]	Монотонність	Монотонность
92.Nécessary condítion	Необхідна умова	Необходимое условие
93.Nécessary condítion of exístence	Необхідна умова існування	Необходимое условие существования
94.Négative définite quadrátic form	Від"ємно-визначена квадратична форма	Отрицательно определённая квадратичная форма
95.Nórmal sýstem of (the) léast-squares méthod	Нормальна система методу найменших квадратів	Нормальная система метода наименьших квадратов
96.Not to decréase	Не спадати	Не убывать
97.Not to incréase	Не зростати	Не возрастать
98.Oblíque [inclíned] ásymptote	Похила асимптота	Наклонная асимптота
99.Part/piece of concá-vity	Частина/ділянка угнутості	Участок/часть вогнутости
100. Part/piece of convéxity	Частина/ділянка опуклості	Участок/часть выпуклости
101. Pass through the point	Проходити через точку	Проходить через точку
102. Póint of (assúmed/ propósal/presuppósed) extrémum	Точка можливого екстремуму	Точка (возможного) экстремума
103. Póint of a cúrve, of a graph còrrespónding to the extrémum, bénding póint	Точка кривої, графіка, яка відповідає екстремуму	Точка кривой, графика, соответствующая экстремуму
104. Póint of extrémum, extréme póint	Екстремальна точка, точка екстремуму	Экстремальная точка, точка экстремума
105. Pósitive définite quadrátic form	Додатно-визначена квадратична форма	Положительно определённая квадратичная

		форма
106. Prelíminary/téntative desígn [draft, drawing, freehand/rough drawing, sketch, vérsion] of a graph /plot of a fúnction (graph/ plot *ad interim лат.*)	Попередній ескіз графіка функції	Предварительный эскиз, набросок графика функции
107. Príncipal mínor of the first [second, third, *n*-th] órder; príncipal mínor of órder one [two, three, *n*]; first-[second-, third- *n*-th] órder príncipal mínor	Головний мінор першого [другого, третього, *n*-го] порядку	Главный минор первого [второго, третьего, *n*-го] порядка
108. Quadrátic form	Квадратична форма	Квадратичная форма
109. Rélative (extrémum, mínimum, máximum)	Відносний (екстремум, мінімум, максимум)	Относительный (экстремум, минимум, максимум)
110. Rèpresént (*for exámple* a cúrve)	Зображати/зобразити (*напр.* криву)	Изображать/изобразить (*напр.* кривую)
111. Rèpresentátion (*for exámple* of a cúrve)	Зображення (*напр.* кривої)	Изображение (*напр.*, кривой)
112. Rise/ascénd (from left to right) (about a graph /curve)	Сходити/підійматися (зліва направо) (про криву, про графік)	Подниматься/ восходить (слева на-право) (о графике, о кривой)
113. Rísing/ascénding (from left to right) (about a graph/curve)	Висхідний, той, що підіймається (зліва направо) (про криву, про графік)	Поднимающий-ся, восходящий (слева направо) (о графике, о кривой)
114. Schemátic desígn [dráwing, fígure, draft]	Схематичний рисунок	Схематический чертёж/рисунок
115. Séparate a part/piece of convéxity of a curve and that of its concávity	Відокремлювати ділянку /частину опуклості кривої від ділянки/частини її угнутості	Отделять участок/часть выпуклости кривой от участка/части её вогнутости
116. Solve the próblem for a(n) (lócal, rélative, ábsolute, condítional) extrémum	Розв"язати задачу на (локальний, відносний, абсолютний, умовний) екстремум	Решить задачу на (локальный, относительный, абсолютный, условный) экстремум
117. Stage/step of investtigátion	Етап дослідження	Этап исследования
118. Státionary póint	Стаціонарна точка	Стационарная точка
119. Straight line of regréssion of *y* on *x*	Пряма реґресії *y* на *x*	Прямая регрессии *y* на *x*
120. Strict (mònotonícity	Строгий [строга] (моно-	Строгий [строгая] (мо-

[monotoneness, monóto-ny], íncrease, décrease, ex-trémum, mínimum, maxi-mum)	тонність, зростання, спадання, екстремум, мінімум, максимум)	нотонность, возрастание, убывание, экстремум, минимум, максимум)
121. Stríctly (incréase, decréase, mónotòne/mónotonic, incréasing, decréasing/decay)	Строго (зростати, спадати, монотон-ний, зростаючий, спадаючий)	Строго (возрастать, убывать, монотонный, возрастающий, убывающий)
122. Suffícient condítion	Достатня умова	Достаточное усло-вие
123. Suffícient condítion of exístence	Достатня умова існування	Достаточное условие существования
124. Suggést (a depéndence between variables … of the form…)	Наводити на думку, підказувати (залежність між змінними … вигляду…)	Наводить на мысль, подсказывать (зависимость между переменными … вида…)
125. Sum of squáres of (the) érrors	Сума квадратів помилок/похибок	Сумма квадратов оши бок/погрешностей
126. Tángent (líne) at the póint of infléction/infléxion	Дотична в точці перегину	Касательная в точке перегиба
127. Test/invéstigate a fúnction for a(n) (lócal, rélative, ábsolute, condítional) extrémum	Дослідити функцію на (локальний, відносний, абсолютний, умовний) екстремум	Исследовать функцию на (локальный, относительный, абсолютный, условный) экстремум
128. Vértical ásymptote	Вертикальна асимптота	Вертикальная асимптота

LECTURE NO.18. EXTREMA OF FUNCTIONS OF SEVERAL VARIABLES

POINT 1. LOCAL EXTREMA

POINT 2. LEAST SQUARES METHOD

POINT 3. CONDITIONAL EXTREMA

POINT 4. ABSOLUTE EXTREMA

POINT 1. LOCAL EXTREMA

Remark. In this lecture we consider only **twice continuously differentiable** functions of several variables.

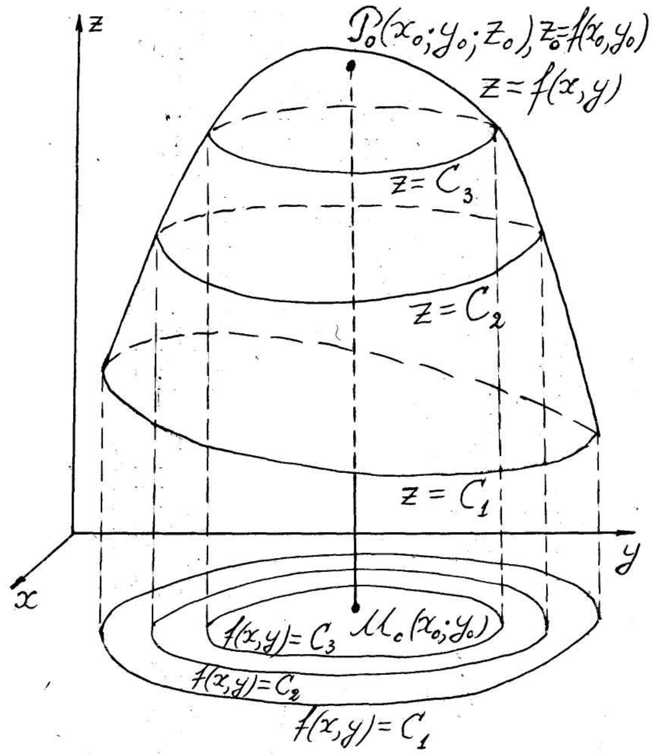


Fig. 1

Def.1. A point $x_0 = (x_{10}, x_{20}, ..., x_{n0}) \in \Re^n$ is called a **point of a local maximum** of a function of n variables $f(x) = f(x_1, x_2, ..., x_n)$ if there exists some neighbourhood U_{x_0} of x_0 such that for any $x \in U'_{x_0} = U_{x_0} \setminus \{x_0\}$ the inequality

$$f(x) < f(x_0) \text{ or } \Delta f(x_0) = f(x) - f(x_0) < 0 \tag{1}$$

holds. The value of the function at the point x_0, that is $f(x_0)$, is called a **local maxi-**

mum of the function.

By analogous way a point of a **local minimum** and a local minimum of a function of *n* variables are defined. The terms a local maximum and a local minimum we as usually unite by common term a **local extremum**.

The case of a local maximum of a function on two variables $z = f(M) = f(x, y)$ is represented on the fig. 1. A point $M_0(x_0; y_0)$ is a point of a local maximum. The latter equals $z_0 = f(M_0) = f(x_0, y_0) = M_0P_0$ where $P_0(x_0; y_0; z_0)$ is the point of a surface $z = f(x, y)$ which is the graph of the function. Some level lines of the func-tion namely $f(x, y) = C_1, f(x, y) = C_2, f(x, y) = C_3$ are shown of the same figure.

Def. 2. A point $x_0 = (x_{10}, x_{20}, ..., x_{n0}) \in \Re^n$ is called that **stationary** of a function of *n* variables $f(x) = f(x_1, x_2, ..., x_n)$ if all its first order partial derivatives equal zero at this point,

$$f'_{x_1}(x_0) = 0,\ f'_{x_2}(x_0) = 0, ..., f'_{x_n}(x_0) = 0. \qquad (2)$$

Note 1. The differential of the function $f(x) = f(x_1, x_2, ..., x_n)$ equals zero at the stationary point,

$$df(x_0) = f'_{x_1}(x_0)dx_1 + f'_{x_2}(x_0)dx_2 + ... + f'_{x_n}(x_0)dx_n = 0. \qquad (3)$$

Theorem 1 (necessary condition of existence of a local extremum). If a function of *n* variables $f(x), x \in \Re^n$ possesses a local extremum at a point $x_0 \in \Re^n$ then this latter is a stationary point for the function, that is the equalities (2), (3) hold.

■Let $x_2 = x_{20}, x_3 = x_{30}, ..., x_n = x_{n0}$ and $\varphi(x_1) = f(x_1, x_{20}, x_{30}, ..., x_{n0})$ be a function of one variable x_1. If a function $f(x) = f(x_1, x_2, ..., x_n)$ has a local extremum at the point $x_0 = (x_{01}, x_{02}, ..., x_{0n})$ then the function $\varphi(x_1)$ has a local extremum at the point x_{01} and so $\varphi'(x_{01}) = 0$. It means that $f'_{x_1}(x_{01}, x_{20}, x_{30}, ..., x_{n0}) = f'_{x_1}(x_0) = 0$. In the same way we can prove that $f'_{x_2}(x_0) = 0, ..., f'_{x_n}(x_0) = 0$.■

Note 2. It follows from the theorem 1 that a (twice continuously differentiable) function $f(x) = f(x_1, x_2, ..., x_n)$ can possess a local extremum only at a stationary point. But a stationary point is not necessary a point of a local extremum that is the necessary condition for existing of a local extremum isn't that sufficient.

Ex. 1. The point $O(0;0)$ is that stationary for a function of two variables $z = f(x,y) = xy$ $\left(f'_x = y, f'_y = x,\ f'_x = f'_y = 0 \text{ if } x = y = 0\right)$ but it isn't a point of a local extremum because of $f(x,y) < f(0;0) = 0$ for $xy < 0$ (in the second and forth quadrants) and $f(x,y) > f(0;0) = 0$ for $xy > 0$ (in the first and third quadrants).

To state a sufficient condition for existence of a local extremum we'll take into consideration some facts of theory of quadratic forms.

Def. 3. The quadratic form of *n* variables $x_1, x_2, \ldots, x_n$ is called an expression

$$F(x) = F(x_1, x_2, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j,\ a_{ij} = a_{ji}. \qquad (4)$$

It's easy to prove that it can be written in a matrix form

$$F(x_1, x_2, \ldots, x_n) = (x_1\ x_2 \ldots x_n)\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix},\ A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, \qquad (5)$$

and A is called the matrix of the quadratic form. It's symmetric one with respect its leading [main, principal] diagonal, that is $a_{ij} = a_{ji}$.

Ex. 2. The quadratic form of two variables x_1, x_2 is an expression

$$F(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = \sum_{i,j=1}^{2} a_{ij}x_ix_j = (x_1\ x_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ a_{12} = a_{21} \qquad (6)$$

with the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix},\ a_{21} = a_{12} \qquad (7)$$

Ex. 3. The quadratic form of tree variables x_1, x_2, x_3 is an expression

$$F(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 =$$

$$= \sum_{i,j=1}^{3} a_{ij}x_ix_j = (x_1\ x_2\ x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\ a_{ji} = a_{ij}.$$

Def 4. Quadratic form (4) if called **positive (negative) definite** if it takes on only positive (negative) values for any $x \neq 0,$ that is if $x_1^2 + x_2^2 + \ldots + x_n^2 \neq 0$, and **undetermined** if it can take on both positive and negative values.

Def. 5. Principal minors of the matrix (5) of the quadratic form (4) are called its diagonal minors,

$$\Delta_1 = a_{11}, \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \ldots, \Delta_n = |A| \equiv \det A. \qquad (8)$$

Theorem 2 (Sylvester[1]**).** Quadratic form (4) is **positive definite** if and only if all its principal minors are positive,

$$\Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0, \ldots, \Delta_n > 0. \qquad (9)$$

It is **negative definite** if and only if these minors have alternating signs in the next manner

$$\Delta_1 < 0, \Delta_2 > 0, \Delta_3 < 0, \Delta_4 > 0, \ldots \qquad (10)$$

If all principal minors are those non-zero and distribution of their signs differs from (9), (10), then the quadratic form (4) is undetermined one.

Def. 6. Hesse[2] matrix for a function $f(x) = f(x_1, x_2, \ldots, x_n)$ (at arbitrary point $x = (x_1, x_2, \ldots, x_n)$) is called the next one

$$H(f,x) = \begin{pmatrix} f''_{x_1x_1}(x) & f''_{x_1x_2}(x) & f''_{x_1x_3}(x) & \ldots & f''_{x_1x_n}(x) \\ f''_{x_2x_1}(x) & f''_{x_2x_2}(x) & f''_{x_2x_3}(x) & \ldots & f''_{x_2x_n}(x) \\ f''_{x_3x_1}(x) & f''_{x_3x_2}(x) & f''_{x_3x_3}(x) & \ldots & f''_{x_3x_n}(x) \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots & \ldots\ldots\ldots \\ f''_{x_nx_1}(x) & f''_{x_nx_2}(x) & f''_{x_nx_3}(x) & \ldots & f''_{x_nx_n}(x) \end{pmatrix}. \qquad (11)$$

We'll in the future suppose that at least one second order partial derivative of the function $f(x)$ doesn't equal zero at the stationary point x_0. It means that the matrix $H(f, x_0)$ is supposed to be non-zero.

Theorem 3 (sufficient condition for existence of a local extremum at a sta-

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[1] Sylvester J.J. (1814 - 1897), an English mathematician.
[2] Hesse, L.O. (1811 - 1874), a German mathematician

tionary point). Let a point $x_0 = (x_{10}, x_{20}, ..., x_{n0}) \in \Re^n$ be stationary one of a function $f(x) = f(x_1, x_2, ..., x_n)$, and $H(f, x_0)$ - is the value of Hesse matrix (11) at this point with non-zero principal minors.

a) If all principal minors of the matrix $H(f, x_0)$ are positive,

$$\Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0, ..., \Delta_n > 0, \qquad (12)$$

then the point $x_0 = (x_{10}, x_{20}, ..., x_{n0})$ is that of a local minimum;

b) If the signs of principal minors of the matrix $H(f, x_0)$ are alternating such that

$$\Delta_1 < 0, \Delta_2 > 0, \Delta_3 < 0, \Delta_4 > 0, ..., \qquad (13)$$

then the point $x_0 = (x_{10}, x_{20}, ..., x_{n0})$ is that of a local maximum.

c) No local extrema in the other cases.

■ By Taylor formula the increment of the function at the point x_0 equals

$$\Delta y = \Delta f(x_0) = f(x) - f(x_0) = df(x_0) + \frac{1}{2!} d^2 f(c),$$

where $c = (c_1, c_2, ..., c_n)$ is some point. By virtue of the condition (3) we have $df(x_0) =$ $= 0$, and so

$$\Delta f(x_0) = f(x) - f(x_0) = \frac{1}{2} d^2 f(c). \qquad (14)$$

A sign of the right side in (14) coincides, in some neighbourhood U_{x_0} of the point x_0, with a sign of $d^2 f(x_0)$ because of continuity of the second order partial derivatives of the function *f*. But the differential $d^2 f(x_0)$ equals (see (35) in Lecture No.16)

$$d^2 f(x_0) = d^2 f(x_{10}, x_{20}, ..., x_{n0}) = \sum_{i,j=1}^{n} f''_{x_i x_j}(x_0) dx_i dx_j \qquad (15)$$

therefore it's a quadratic form of variables dx_i with the matrix $H(f, x_0)$ (see (11)). It is positive (negative) definite in U_{x_0} because of the conditions (12) ((13)). In the first case we have $d^2 f(x_0) < 0$ and so $\Delta f(x_0) < 0$ in U_{x_0}, and the function has a local minimum at the point x_0. In the second case the inequality $d^2 f(x_0) < 0$ holds, so

$\Delta f(x_0)<0$ in U_{x_0}, and the function has a local maximum at the point x_0. If the conditions (12) (13) don't fulfill (but $\Delta_i \neq 0, i=\overline{1,n}$) the quadratic form (15) is undetermined one, therefore the differential $d^2 f(x_0)$ and the increment $\Delta f(x_0)$ of the function don't conserve their signs in any neighbourhood of the point x_0. It means that the function $f(x)$ doesn't have a local extremum at the point x_0.■

Note 3. The proof of the theorem is relieved for the case of a function of two variables $f(x)=f(x_1,x_2)$. It doesn't require the theory of quadratic forms because the sign of the differential $d^2 f(x_0)=d^2 f(x_{10},x_{20})$ at the stationary point x_0 is determined by the theory of the quadratic trinomial. Indeed, in this case

$$d^2 f(x_0)=f''_{x_1x_1}(x_0)\Delta x_1^2+2f''_{x_1x_2}(x_0)\Delta x_1\Delta x_2+f''_{x_2x_2}(x_0)\Delta x_2^2.$$

As usually $\Delta x_1=dx_1=x_1-x_{10}, \Delta x_2=dx_2=x_2-x_{20}$. For example if $\Delta x_2\neq 0$, then

$$d^2 f(x_0)=\Delta x_2^2\left(f''_{x_1x_1}(x_0)\left(\frac{\Delta x_1}{\Delta x_2}\right)^2+2f''_{x_1x_2}(x_0)\frac{\Delta x_1}{\Delta x_2}+f''_{x_2x_2}(x_0)\right).$$

Quadratic trinomial (with respect to $\Delta x_1/\Delta x_2$) is positive (negative) for every $\Delta x_1, \Delta x_2$ ($\Delta x_1^2+\Delta x_2^2\neq 0$) if $\Delta_1=f''_{x_1x_1}(x_0)>0$ (respectively $\Delta_1=f''_{x_1x_1}(x_0)<0$) and if its discriminant

$$D=4\left(f''_{x_1x_2}(x_0)\right)^2-4f''_{x_1x_1}(x_0)\cdot f''_{x_2x_2}(x_0)=4\left(\left(f''_{x_1x_2}(x_0)\right)^2-f''_{x_1x_1}(x_0)\cdot f''_{x_2x_2}(x_0)\right)=$$

$$=4\begin{vmatrix} f''_{x_1x_2}(x_0) & f''_{x_1x_1}(x_0) \\ f''_{x_2x_2}(x_0) & f''_{x_1x_2}(x_0)\end{vmatrix}=-4\begin{vmatrix} f''_{x_1x_1}(x_0) & f''_{x_1x_2}(x_0) \\ f''_{x_1x_2}(x_0) & f''_{x_2x_2}(x_0)\end{vmatrix}=-4\det H(f,x_0)=-4\Delta_2,$$

is negative one (and therefore the main minor Δ_2 is that positive). The function has a local minimum in the case $\Delta_1>0, \Delta_2>0$ and a local maximum in the case $\Delta_1<0$, $\Delta_2>0$. In the other cases ($\Delta_1\neq 0$ but $\Delta_2<0$) it doesn't have a local extremum at the stationary point $x_0=(x_{10},x_{20})$.

Note 4. Theorem 3 is valid if $d^2 f(x_0)$ doesn't equal zero identically (with respect to $dx_i, i=\overline{1,n}$). Otherwise we must resort to more general theory which involves higher order differentials.

Note 5. In practice we often deal with cases when at least one main minor of the matrix (11) equals zero. We consider such the cases as those doubtful. But it's possible to close the question completely for functions of two variables $f(x)=$ $=f(x_1,x_2)$.

It's sufficient to study two possibilities for the stationary point $x_0=(x_{10},x_{20})$ namely: a) $\Delta_1=0$, but $\Delta_2\neq 0$; b) $\Delta_2=0$.

a) If $\Delta_1=0$, but $\Delta_2\neq 0$, then $f''_{x_1x_2}(x_0)\neq 0$, $\Delta_2<0$, and the formula (15) takes on the form

$$d^2f(x_0)=2f''_{x_1x_2}(x_0)dx_1dx_2+f''_{x_2x_2}(x_0)dx_2^2.$$

It's evident that $d^2f(x_0)$ doesn't conserve a constant sigh in any neighbourhood of the stationary point, and so the function doesn't have a local extremum at this point. We've seen that it also doesn't exist if $\Delta_1\neq 0$ and $\Delta_2<0$.

b) The case $\Delta_2=0$, when the trinomial $d^2f(x_0)$ has two real equal roots, is that doubtful for each value of the minor Δ_1 ($\Delta_1\neq 0$ or $\Delta_1=0$).

Now we can state sufficient condition of existing of a local extremum of the **function of two variables** $f(x)=f(x_1,x_2)$ at the stationary point $x_0=(x_{10},x_{20})$ in the form of the next theorem.

Theorem 4. Let $x_0=(x_{10},x_{20})$ be a stationary point of a function of two variables $f(x)=f(x_1,x_2)$.

a) If

$$\Delta_1=f''_{x_1x_1}(x_0)>0 \quad \left(\Delta_1=f''_{x_1x_1}(x_0)<0\right) \text{ and } \Delta_2=\det H(f,x_0)=\begin{vmatrix} f''_{x_1x_1}(x_0) & f''_{x_1x_2}(x_0) \\ f''_{x_1x_2}(x_0) & f''_{x_2x_2}(x_0) \end{vmatrix}>0,$$

then a function has a local minimum (respectively maximum) at this point.

b) In the case

$$\Delta_2=\det H(f,x_0)=\begin{vmatrix} f''_{x_1x_1}(x_0) & f''_{x_1x_2}(x_0) \\ f''_{x_1x_2}(x_0) & f''_{x_2x_2}(x_0) \end{vmatrix}<0$$

a local extremum doesn't exist.

c) The case

$$\Delta_2 = \det H(f, x_0) = 0$$

is that doubtful. One must resort to more general theory involving higher order differentials.

Ex. 4. Find local extrema of the function $z = x^3 + y^3 - 9xy + 27$.

The first step: finding stationary points of the function.

$$\begin{matrix} z'_x = 3x^2 - 9y, \\ z'_y = 3y^2 - 9x; \end{matrix} \quad \begin{cases} z'_x = 0, \\ z'_y = 0; \end{cases} \quad \begin{cases} 3x^2 - 9y = 0, \\ 3y^2 - 9x = 0; \end{cases} \quad \begin{cases} x^2 - 3y = 0, \\ y^2 - 3x = 0; \end{cases} \quad \begin{matrix} x_1 = 0, y_1 = 0; & O(0;\,0), \\ x_2 = 3, y_2 = 3; & M(3;\,3). \end{matrix}$$

The second step: studying the stationary points $O(0;\,0), M(3;\,3)$. For this purpose we can use both the conditions (12), (13) of the general theory and those (12 a), (13 a) for the case of a function of two variables. We'll begin from the general theory.

Let's form at first Hesse matrix for the given function:

$$\begin{matrix} z''_{xx} = 6x,\, z''_{xy} = -9, \\ z''_{yx} = z''_{xy} = -9,\, z''_{yy} = 6y; \end{matrix} \quad H(z,(x,y)) = \begin{pmatrix} z''_{xx} & z''_{xy} \\ z''_{yx} & z''_{yy} \end{pmatrix} = \begin{pmatrix} 6x & -9 \\ -9 & 6y \end{pmatrix}.$$

a) For the point $M(3;\,3)$ the corresponding value of Hesse matrix is

$$H(z, M(3,3)) = \begin{pmatrix} 18 & -9 \\ -9 & 18 \end{pmatrix};$$

all its principal minors are positive

$$\Delta_1 = 18 > 0, \quad \Delta_2 = \begin{vmatrix} 18 & -9 \\ -9 & 18 \end{vmatrix} > 0;$$

by virtue of the theorem 3 the function has a local minimum at the point $M(3;\,3)$.

b) For the point $O(0;\,0)$ Hesse matrix and its principal minors are

$$H(z, O(0;\,0)) = \begin{pmatrix} 0 & -9 \\ -9 & 0 \end{pmatrix}; \; \Delta_1 = 0, \quad \Delta_2 = \begin{vmatrix} 0 & -9 \\ -9 & 0 \end{vmatrix} = -81$$

and by the theorem 4 a local extremum doesn't exist at the point $O(0;\,0)$.

Ex. 5. Find local extrema of a function of three variables

$$u = -x^2 - y^2 - 10z^2 + 4xz + 3yz - 2x - y + 13z + 5.$$

1. Finding stationary points of the function. There is one stationary point because of

$$u'_x = -2x + 4z - 2,\ u'_y = -2y + 3z - 1,\ u'_z = -20z + 4x + 3y + 13,$$

$$\begin{cases} -2x + 4z - 2 = 0, \\ -2y + 3z - 1 = 0, \\ -20z + 4x + 3y + 13; \end{cases} \quad \begin{cases} -2x & & +4z & = & 2, \\ & -2y & +3z & = & 1, \\ 4x & +3y & -20z & = & -13; \end{cases} \quad \begin{matrix} x = y = z = 1, \\ M_0(1;1;1). \end{matrix}$$

2. Investigation the stationary point $M_0(1;1;1)$. The second order partial derivatives of the given function

$$u''_{xx} = -2,\ u''_{xy} = u''_{yx} = 0,\ u''_{xz} = u''_{zx} = 4,\ u''_{yy} = -2,\ u''_{yz} = u''_{zy} = 3,\ u''_{zz} = -20$$

generate Hesse matrix with constant elements, so

$$H(u, M(x;y;z)) = H(u, M_0(x_0;y_0;z_0)) = H(u, M_0(1;1;1)) =$$

$$= \begin{pmatrix} u''_{xx}(M_0) & u''_{xy}(M_0) & u''_{xz}(M_0) \\ u''_{yx}(M_0) & u''_{yy}(M_0) & u''_{yz}(M_0) \\ u''_{zx}(M_0) & u''_{zy}(M_0) & u''_{zz}(M_0) \end{pmatrix} = \begin{pmatrix} -2 & 0 & 4 \\ 0 & -2 & 3 \\ 4 & 3 & -20 \end{pmatrix};$$

the principal minors of the value of Hesse matrix at the stationary point $M_0(1;1;1)$ are equal to

$$\Delta_1 = -2 < 0,\ \Delta_2 = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0,\ \Delta_3 = \begin{vmatrix} -2 & 0 & 4 \\ 0 & -2 & 3 \\ 4 & 3 & -20 \end{vmatrix} = -30 < 0,$$

and therefore the given function possesses a local maximum $u_{\max} = u(M_0) = 10$ at the point $M_0(1;1;1)$.

Ex. 6. Find local extrema of the function $u = x + y/x + z/y + 2/z$.

1. $u'_x = 1 - y/x^2$, $u'_y = 1/x - z/y^2$, $u'_z = 1/y - 2/z^2$;

$$\begin{cases} 1 - y/x^2 = 0, \\ 1/x - z/y^2 = 0, \\ 1/y - 2/z^2 = 0; \end{cases} \begin{cases} x^2 = y, \\ y^2 = xz, \\ z^2 = 2y; \end{cases} \begin{cases} x^2 = y, \\ x^4 = xz, \\ z^2 = 2x^2; \end{cases} \begin{cases} x^2 = y, \\ x^3 = z, \\ z^2 = 2x^2; \end{cases} \begin{cases} x^2 = y, \\ x^3 = z, \\ x^6 = 2x^2; \end{cases} \begin{cases} x = \pm 2^{1/4}, \\ y = 2^{1/2}, \\ z = \pm 2^{3/4}. \end{cases}$$

There are two stationary points $M_1(2^{1/4}; 2^{1/2}; 2^{3/4})$, $M_2(-2^{1/4}; 2^{1/2}; -2^{3/4})$.

2. Now we must study the stationary points for existing of a local extrema. The second order partial derivatives of the function at arbitrary point $(x;y;z)$ are equal to

$$\begin{aligned} u_{xx} &= 2y/x^3, \quad u_{xy} = -1/x^2, \quad u_{xz} = 0, \\ u_{yx} &= -1/x^2, \quad u_{yy} = 2z/y^3, \quad u_{yz} = -1/y^2, \\ u_{zx} &= 0, \quad u_{zy} = -1/y^2, \quad u_{zz} = 4/z^3. \end{aligned}$$

a) Hesse matrix and principal minors for the point $M_1(2^{1/4};\, 2^{1/2};\, 2^{3/4})$ are

$$H\left(u, M_1\left(2^{1/4};2^{1/2};2^{3/4}\right)\right) = \begin{pmatrix} 2^{3/4} & -2^{-1/2} & 0 \\ -2^{-1/2} & 2^{1/4} & -2^{-1} \\ 0 & -2^{-1} & 2^{-1/4} \end{pmatrix}, \quad \Delta_1 = 2^{3/4} > 0, \quad \Delta_2 = \begin{vmatrix} 2^{3/4} & -2^{-1/2} \\ -2^{-1/2} & 2^{1/4} \end{vmatrix} = 2 - 2^{-1} > 0,$$

$$\Delta_3 = \begin{vmatrix} 2^{3/4} & -2^{-1/2} & 0 \\ -2^{-1/2} & 2^{1/4} & -2^{-1} \\ 0 & -2^{-1} & 2^{-1/4} \end{vmatrix} = 2^{3/4} - 2\cdot 2^{-5/4} = 2^{-5/4}\left(2^2 - 2\right) = 2^{-1/4} > 0.$$

Hence we have a local minimum at the point $M_1(2^{1/4};\, 2^{1/2};\, 2^{3/4})$.

b) Hesse matrix and principal minors for the point $M_2(-2^{1/4};\, 2^{1/2};\, -2^{3/4})$ are

$$H\left(u, M_2\left(-2^{1/4};2^{1/2};-2^{3/4}\right)\right) = \begin{pmatrix} -2^{3/4} & -2^{-1/2} & 0 \\ -2^{-1/2} & -2^{1/4} & -2^{-1} \\ 0 & -2^{-1} & -2^{-1/4} \end{pmatrix}, \quad \Delta_1 = -2^{3/4} < 0, \quad \Delta_2 = \begin{vmatrix} -2^{3/4} & -2^{-1/2} \\ -2^{-1/2} & -2^{1/4} \end{vmatrix} > 0,$$

$$\Delta_3 = \begin{vmatrix} -2^{3/4} & -2^{-1/2} & 0 \\ -2^{-1/2} & -2^{1/4} & -2^{-1} \\ 0 & -2^{-1} & -2^{-1/4} \end{vmatrix} = -2^{3/4} + 2\cdot 2^{-5/4} = 2^{-5/4}\left(2 - 2^2\right) = -2^{-1/4} < 0.$$

We have a local maximum at the point $M_2(-2^{1/4};\, 2^{1/2};\, -2^{3/4})$.

Ex. 7. Functions

$$z = f_1(x,y) = x^4 + y^4, z = f_2(x,y) = -x^4 - y^4, z = f_3(x,y) = x^4 - y^4$$

have the same stationary point $O(0;\,0)$. Their second order differentials

$d^2 f_1 = 12x^2dx^2 + 12y^2dy^2$, $d^2 f_2 = -12x^2dx^2 - 12y^2dy^2$, $d^2 f_3 = 12x^2dx^2 - 12y^2dy^2$,

identically equal zero at the stationary point and the theorem 3 is inapplicable one for these functions. One can easily see that $f_1(x,y)$ has a maximum, $f_2(x,y)$ has a minimum, $f_3(x,y)$ hasn't a local extremum at the point $O(0;\,0)$. Indeed, $f_1(x,y)>0$, $f_2(x,y)<0$ at any point $M(x;y)\neq O(0;\,0)$ while $f_3(x,y)>0$ as $|x|>|y|$, $f_3(x,y)<0$ as $|x|<|y|$ and $f_3(x,y)=0$ as $|x|=|y|$.

POINT 2. LEAST SQUARES METHOD

Let we study two variables x, y and we seek the form of a functional dependence between them. For this purpose we fulfil n experiments on x, y and represent obtained results by a table of pairs $(x_i; y_i)$ and by corresponding points $A_i(x_i; y_i)$ of the $x0y$–plane (see the table 1 and fig. 2).

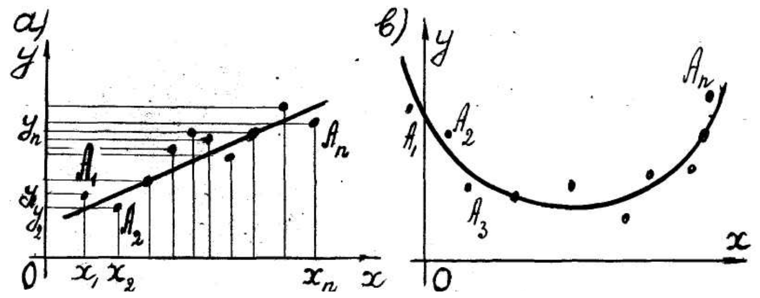

Fig. 2

Table 1

x	x_1	x_2	x_3	...	x_n
y	y_1	y_2	y_3	...	y_n
Point	A_1	A_2	A_3	...	A_n

The disposition of points $A_1, A_2, ..., A_n$ sometimes helps us to hypothesize concerning a form $y = f(x,a,b,...)$ of dependence in question. For example a fig. 2a leads to the hypothesis about linear dependence between x, y, namely $y = ax + b$. On the other hand a fig. 2b generates the hypothesis about parabolic (of the second degree) dependence $y = ax^2 + bx + c$.

Our aim is to find parameters $a, b,...$ by the best (in a certain sense) way. This way is the least squares method (LSM).

Let in general we hypothesize

$$y = f(x,a,b,...). \qquad (16)$$

We introduce the next quantities (so-called errors)

$$\varepsilon_i = f(x_i,a,b...) - y_i \qquad (17)$$

which are the differences between theoretic and empiric results of experiments on the variables x and y. Least squares method which was devises by Legendre[1] and Gauss[2] and justified by Gauss consists in follows: we find $a,b,...$ in such a way to make minimal (or to minimize) the sum of squares of the errors. It means that we have to find a minimum of the next function of the variables $a,b,...$

$$\Phi(a,b,...)=\sum_{i=1}^{n}\varepsilon_i^2=\sum_{i=1}^{n}\left(f(x_i,a,b...)-y_i\right)^2 . \qquad (18)$$

To find a, b, ... we must solve the next system of equations

$$\begin{cases}\Phi'_a(a,b,...)=0,\\ \Phi'_b(a,b,...)=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\end{cases} \qquad (19)$$

which is called a **normal system** of least squares method.

We'll limit ourselves to two hypotheses generated by dispositions of points $A_i(x_i;y_i)$ on the fig. 1 a, b, namely $y=ax+b$ and $y=ax^2+bx+c$.

If we suppose

$$y=ax+b \qquad (20)$$

then we have to minimize the next function

$$\Phi(a,b)=\sum_{i=1}^{n}\varepsilon_i^2=\sum_{i=1}^{n}(ax_i+b-y_i)^2 . \qquad (21)$$

Its partial derivatives with respect to a and b equal

$$\Phi'_a=\sum_{i=1}^{n}2(ax_i+b-y_i)x_i=2\left(a\sum_{i=1}^{n}x_i^{\,2}+b\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}x_iy_i\right),$$

$$\Phi'_b=\sum_{i=1}^{n}2(ax_i+b-y_i)=2\left(a\sum_{i=1}^{n}x_i+bn-\sum_{i=1}^{n}y_i\right)$$

and we have to solve the next normal system of linear equations in a, b

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[1] Legendre, A.M. (1752 - 1833), a French mathematician
[2] Gauss, K.F. (1777 - 1855), a great German mathematician, astronomer, physicist, and land-surveyor

$$\begin{cases}\Phi'_a = 0, \\ \Phi'_b = 0;\end{cases} \quad \begin{cases} a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i; \\ a\sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i. \end{cases} \qquad (22)$$

In the case of a hypothesis

$$y = ax^2 + bx + c \qquad (23)$$

we must minimize a function of three variables *a*, *b*, *c*

$$\Phi(a,b,c) = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} (ax_i^2 + bx_i + c - y_i)^2 \qquad (24)$$

with the next partial derivatives with respect to *a*, *b*, *c*

$$\Phi'_a = \sum_{i=1}^{n} 2(ax_i^2 + bx_i + c - y_i)x_i^2 = 2\left(a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i^2 y_i \right),$$

$$\Phi'_b = \sum_{i=1}^{n} 2(ax_i^2 + bx_i + c - y_i)x_i = 2\left(a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i y_i \right),$$

$$\Phi'_c = \sum_{i=1}^{n} 2(ax_i^2 + bx_i + c - y_i) = 2\left(a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i + cn - \sum_{i=1}^{n} y_i \right).$$

Therefore a system of linear equations in *a*, *b*, *c* to be solved

$$\begin{cases}\Phi'_a = 0, \\ \Phi'_b = 0, \\ \Phi'_c = 0;\end{cases}$$

$$\begin{cases} a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 y_i, \\ a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i, \\ a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i + cn = \sum_{i=1}^{n} y_i. \end{cases} \qquad (25)$$

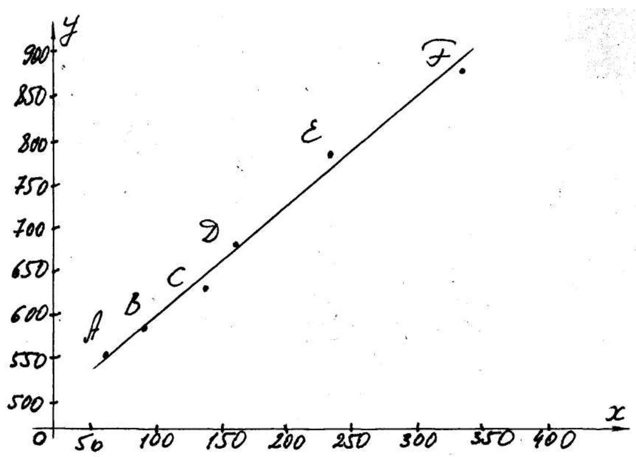

Fig. 3

Ex. 8. Amount of goods x (in thousands of i.c.u.) and costs of circulation y (in i.c.u) are given by the table 2.

Disposition of points *A*, *B*, *C*, *D*, *E*, *F* (fig. 3) permits us to hypothesize that

$$y = ax + b,$$

that is costs of circulation y and amount of goods x are connected by linear dependence. By virtue of (22) we must solve the next system of equations

Table 2

№	x_i	y_i	Points	$x_i y_i$	x_i^2
1	60	551	*A*	33060	3600
2	80	576	*B*	46080	6400
3	140	628.5	*C*	87990	19600
4	160	673	*D*	107680	25600
5	240	768.5	*E*	184440	57600
6	320	863	*F*	276160	102400
$\sum$	1000	4080		735410	215200

$$\begin{cases} 215200a + 1000b = 735410, \\ 1000a + 6b = 4080. \end{cases}$$

The solution of the system is $a \approx 1.13$, $b \approx 489.71$ and so the depen-dence in question is given by the next equation

$$y = 1.13x + 489.71.$$

POINT 3. CONDITIONAL EXTREMA

Simplest problem on a conditional extremum:

Find extrema of a function of two variables

$$z = f(M) = f(x, y) \tag{26}$$

provided that x and y are connected by the equation [condition, constraint, relation]

$$\varphi(x, y) = 0 \tag{27}$$

Geometric sense of this problem consists in finding an extremum of the function $z = f(x, y)$ at the points of a curve of the equation (27).

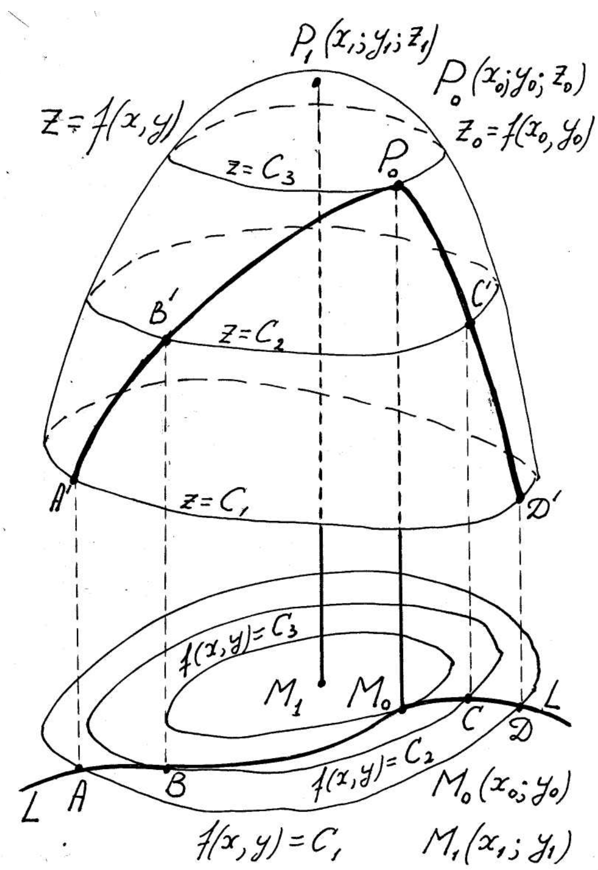

Fig. 4

A condition maximum of a function $f(x,y)$ along a curve $L: ABM_0CD$ is represented on fig. 4. It equals $z_0 = f(M_0) = f(x_0, y_0) = M_0P_0$, and the function achieves it at the point $M_0(x_0; y_0) \in L$. For comparison fig. 4 gives the local maximum of the same function $z_1 = f(M_1) = f(x_1, y_1) = M_1P_1$
which differs from the condition maximum.

General problem on a conditional extremum:

Find extrema of a function of n variables

$$u = f(x) = f(x_1, x_2, ..., x_n) \qquad (28)$$

provided that the variables $x_1, x_2, ..., x_n$ are connected by the next k $(k < n)$ equations [conditions, constraints, relations]

$$\begin{aligned}&\varphi_1(x_1, x_2, ..., x_n) = 0,\\ &\varphi_2(x_1, x_2, ..., x_n) = 0,\\ &\dots\dots\dots\dots\dots\dots\\ &\varphi_k(x_1, x_2, ..., x_n) = 0.\end{aligned} \qquad (29)$$

A. Necessary condition for existing of a conditional extremum.

Case 1. The simplest problem (26), (27) on a conditional extremum.

Let a conditional extremum (26), (27) is attained at a point $M_0(x_0; y_0)$ and at least one of the first order partial derivatives of the function $\varphi(x, y)$ doesn't equal zero at this point, for example

$$\varphi'_y(M_0) = \varphi'_y(x_0; y_0) \neq 0. \qquad (30)$$

In this case the equation (27) determines y as an implicit function of x in some neighbourhood of the point $M_0(x_0; y_0)$,

$$y = y(x) \ \left(\varphi(x, y(x)) \equiv 0, \varphi(x_0, y_0) = 0, \, y_0 = y(x_0)\right). \qquad (31)$$

If we can directly find y from the equation (27) we get a problem on usual local extremum for a function $z = z(x) = f(x, y(x))$ of one variable x. The necessary condition for existing of such the extremum is $z'(x_0) == 0,$ or in the full form

$$f'_x(x_0, y_0) + f'_y(x_0, y_0) \cdot y'(x_0) = 0. \qquad (32)$$

In reality it isn't necessary to express y through x from the equation (27). It's sufficiently only to take into account that y is a function of x implicitly defined by this equation, and therefore to consider the equality (27) as identity with respect to x. By its differentiation we get at the point $M_0(x_0; y_0)$

$$\varphi'_x(x_0, y_0) + \varphi'_y(x_0, y_0) \cdot y'(x_0) = 0. \qquad (33)$$

Now from (32) and (33) we find

$$y'(x_0) = -\frac{\varphi'_x(x_0, y_0)}{\varphi'_y(x_0, y_0)}, \, y'(x_0) = -\frac{f'_x(x_0, y_0)}{f'_y(x_0, y_0)} \Rightarrow \frac{\varphi'_x(x_0, y_0)}{\varphi'_y(x_0, y_0)} = \frac{f'_x(x_0, y_0)}{f'_y(x_0, y_0)},$$

$$\frac{f'_x(x_0, y_0)}{\varphi'_x(x_0, y_0)} = \frac{f'_y(x_0, y_0)}{\varphi'_y(x_0, y_0)} \qquad (34)$$

If we denote equal ratios (34) by $-\lambda$, where λ be some number which is called **Lagrange's multiplier**, we'll get

$$\frac{f'_x(x_0, y_0)}{\varphi'_x(x_0, y_0)} = \frac{f'_y(x_0, y_0)}{\varphi'_y(x_0, y_0)} = -\lambda \Rightarrow f'_x(x_0, y_0) = -\lambda\varphi'_x(x_0, y_0),\ f'_y(x_0, y_0) = -\lambda\varphi'_y(x_0, y_0),$$

$$f'_x(x_0, y_0) + \lambda\varphi'_x(x_0, y_0) = 0,\ f'_y(x_0, y_0) + \lambda\varphi'_y(x_0, y_0) = 0.$$

We have proved the next theorem.

Theorem 4 (necessary condition of existing of the conditional extremum (26), (27))**.** If a function $z = f(M) = f(x, y)$ of two variables attains the conditional extremum (26), (27) at a point $M_0(x_0; y_0)$, then its coordinates satisfy the next system of equations in x, y, λ:

$$\begin{cases} f'_x(x, y) + \lambda\varphi'_x(x, y) = 0, \\ f'_y(x, y) + \lambda\varphi'_y(x, y) = 0, \\ \varphi(x, y) = 0 \end{cases} \qquad (35)$$

One can easy remember the system (35) by introducing the next auxiliary function (**Lagrange function**)

$$L = L(\lambda, x, y) = f(x, y) + \lambda\varphi(x, y). \qquad (36)$$

The necessary condition of existing of a conditional extremum (26), (27) goes over

$$\begin{cases} L'_x(\lambda, x, y) = 0, \\ L'_y(\lambda, x, y) = 0, \\ \varphi(x, y) = 0; \end{cases} \quad \text{or} \quad \begin{cases} L'_x(\lambda, x, y) = 0, \\ L'_y(\lambda, x, y) = 0, \\ L'_\lambda(\lambda, x, y) = 0. \end{cases} \qquad (37)$$

Def. 6. Every solution $P = (\lambda_0, x_0, y_0)$ of the system (37) is called a **stationary point** of the Lagrange function (36). Corresponding geometric point $M_0(x_0, y_0)$ can be named a stationary point of the function $z = f(x, y)$ (for the simplest problem on a condition extremum (26), (27)).

It follows from the definition 6 and the theorem 4 that a function $z = f(x, y)$ can reach a condition extremum only at a stationary point of Lagrange function.

Case 2. The general problem (28), (29) on a conditional extremum.

In general problem (28), (29) on a conditional extremum one introduces **La-**

grange function

$$L = L(\lambda, x) = L(\lambda_1, \lambda_2, ..., \lambda_k, x_1, x_2, ..., x_n) = f + \lambda_1\varphi_1 + \lambda_2\varphi_2 + ... + \lambda_k\varphi_k \qquad (38)$$

Theorem 5 (necessary condition of existing of a conditional extremum (28), (29))**.** If a function of *n* variables $u = f(x) = f(x_1, x_2, ..., x_n)$ attains the conditional extremum (28), (29) at a point $x_0 = (x_{10}, x_{20}, ..., x_{n0}) \in \Re^n$ then its coordinates satisfy the next system of equations in $\lambda_1, \lambda_2, ..., \lambda_k, x_1, x_2, ..., x_n$

$$\begin{cases} L'_{x_i} = 0 & (i = \overline{1, n}), \\ \varphi_j = 0 & (j = \overline{1, k}); \end{cases} \quad \text{or} \quad \begin{cases} L'_{x_i} = 0 & (i = \overline{1, n}), \\ L'_{\lambda_j} = 0 & (j = \overline{1, k}). \end{cases} \qquad (39)$$

Def. 7. Every solution $(\lambda_0, x_0) = (\lambda_{10}, \lambda_{20}, ..., \lambda_{k0}, x_{10}, x_{20}, ..., x_{n0})$ of the system (39) is called a **stationary point** of Lagrange function (38). Corresponding geometric point $x_0 = (x_{10}, x_{20}, ..., x_{n0})$ is often named a stationary point of the function $u = f(x) = f(x_1, x_2, ..., x_n)$ (for the general problem on a condition extremum (28), (29)).

The function $u = f(x) = f(x_1, x_2, ..., x_n)$ can reach a condition extremum only at a stationary point of Lagrange function.

B. Sufficient condition for existing of a conditional extremum

Case 1. The simplest problem (26), (27) on a conditional extremum.

Let $P = (\lambda_0, x_0, y_0)$ be some stationary point of Lagrange function (36) for a function $z = f(x, y)$, that is one of solutions of the system (37). Let's introduce Hesse matrix for Lagrange function at arbitrare point $P(\lambda; x; y)$ for two cases:

a) in the first case, when $L''_{\lambda x}(\lambda_0, x_0, y_0) = \varphi'_x(x_0, y_0) \neq 0$, one has

$$H(f, P(\lambda, x, y)) = H(f, \lambda, x, y) = \begin{pmatrix} L''_{\lambda\lambda}(x, y) & L''_{\lambda x}(x, y) & L''_{\lambda y}(x, y) \\ L''_{x\lambda}(x, y) & L''_{xx}(\lambda, x, y) & L''_{xy}(\lambda, x, y) \\ L''_{y\lambda}(x, y) & L''_{yx}(\lambda, x, y) & L''_{yy}(\lambda, x, y) \end{pmatrix}$$

or

$$H(f, P(\lambda, x, y)) = H(f, \lambda, x, y) = \begin{pmatrix} 0 & \varphi'_x(x, y) & \varphi'_y(x, y) \\ \varphi'_x(x, y) & L''_{xx}(\lambda, x, y) & L''_{xy}(\lambda, x, y) \\ \varphi'_y(x, y) & L''_{yx}(\lambda, x, y) & L''_{yy}(\lambda, x, y) \end{pmatrix}; \quad (40\text{ a})$$

b) in the second case, when

$$L''_{\lambda x}(\lambda_0, x_0, y_0) = \varphi'_x(x_0, y_0) = 0 \text{ but } L''_{\lambda y}(\lambda_0, x_0, y_0) = \varphi'_y(x_0, y_0) \neq 0,$$

$$H(f, P(\lambda, x, y)) = H(f, \lambda, y, x) = \begin{pmatrix} L''_{\lambda\lambda}(\lambda, x, y) & L''_{\lambda y}(\lambda, x, y) & L''_{\lambda x}(\lambda, x, y) \\ L''_{y\lambda}(\lambda, x, y) & L''_{yy}(\lambda, x, y) & L''_{yx}(\lambda, x, y) \\ L''_{y\lambda}(\lambda, x, y) & L''_{yx}(\lambda, x, y) & L''_{yy}(\lambda, x, y) \end{pmatrix},$$

$$H(f, P(\lambda, x, y)) = H(f, \lambda, y, x) = \begin{pmatrix} 0 & \varphi'_y(x, y) & \varphi'_x(x, y) \\ \varphi'_y(x, y) & L''_{yy}(\lambda, x, y) & L''_{yx}(\lambda, x, y) \\ \varphi'_x(x, y) & L''_{xy}(\lambda, x, y) & L''_{xx}(\lambda, x, y) \end{pmatrix}. \quad (40\text{ b})$$

The first main minor of Hesse matrix equals zero, $\Delta_1 = 0$, and the second one is negative, $\Delta_2 < 0$, at any point. Let's consider the value of the third main minor at the stationary point $P = (\lambda_0, x_0, y_0)$, namely

$$\Delta_3(\lambda_0, x_0, y_0) = \det H(f, \lambda_0, x_0, y_0) = \begin{vmatrix} 0 & \varphi'_x(x_0, y_0) & \varphi'_y(x_0, y_0) \\ \varphi'_x(x_0, y_0) & L''_{xx}(\lambda_0, x_0, y_0) & L''_{xy}(\lambda_0, x_0, y_0) \\ \varphi'_y(x_0, y_0) & L''_{yx}(\lambda_0, x_0, y_0) & L''_{yy}(\lambda_0, x_0, y_0) \end{vmatrix}$$

for the matrix (40 a) and

$$\Delta_3(\lambda_0, x_0, y_0) = \det H(f, \lambda_0, y_0, x_0) = \begin{vmatrix} 0 & \varphi'_y(x_0, y_0) & \varphi'_x(x_0, y_0) \\ \varphi'_y(x_0, y_0) & L''_{yy}(\lambda_0, x_0, y_0) & L''_{yx}(\lambda_0, x_0, y_0) \\ \varphi'_x(x_0, y_0) & L''_{xy}(\lambda_0, x_0, y_0) & L''_{xx}(\lambda_0, x_0, y_0) \end{vmatrix}$$

for the matrix (40 b).

Theorem 6. If

$$\Delta_3(\lambda_0, x_0, y_0) < 0$$

that is sign of $\Delta_3(\lambda_0, x_0, y_0)$ coincides with that of Δ_2, then the function $z = f(x, y)$ possesses a **condition minimum** at the (geometrical stationary) point $M_0(x_0, y_0)$.

If

$$\Delta_3(\lambda_0, x_0, y_0) > 0,$$

then the function reaches a **condition maximum** at the point $M_0(x_0, y_0)$.

Ex. 9. Find conditional extrema of the function $z = x^2 - y^2$ under the next condition $x^2 + y^2 = 4$ that is on the circle with the radius 2 centered at the origin.

The first step: introduction of Lagrange function and finding its stationary points.

$$f(x,y) = x^2 - y^2, \varphi(x,y) = x^2 + y^2 - 4;$$
$$L(\lambda,x) = f(x,y) + \lambda\varphi(x,y) = x^2 - y^2 + \lambda(x^2 + y^2 - 4);$$
$$L'_x(\lambda,x) = 2x + 2\lambda x = 2x(1+\lambda), L'_y(\lambda,x) = -2y + 2\lambda y = 2y(-1+\lambda), L'_\lambda(\lambda,x) = \varphi(x,y);$$
$$\begin{cases} L'_x(\lambda,x) = 0, \\ L'_y(\lambda,x) = 0, \\ L'_\lambda(\lambda,x) = \varphi(x,y) = 0; \end{cases} \quad \begin{cases} x(1+\lambda) = 0, & (a) \\ y(-1+\lambda) = 0, & (b) \\ x^2 + y^2 - 4 = 0. & (c) \end{cases}$$

On the base of the equation (a) we can study two cases.

1 case: $x = 0$ in the equation (a); $(c) \Rightarrow y = \pm 2, (b) \Rightarrow \lambda = 1$.

2 case: $\lambda = -1$ in the equation (a); $(b) \Rightarrow y = 0, (c) \Rightarrow x = \pm 2$.

We've got four stationary points of Lagrange function and of the given function, namely:

$$P_1(\lambda_1; x_1; y_1) = P_1(-1; 2; 0), M_1(2; 0); P_2(\lambda_2; x_2; y_2) = P_2(-1; -2; 0), M_2(-2; 0);$$
$$P_3(\lambda_3; x_3; y_3) = P_3(1; 0; 2), M_3(0; 2); P_4(\lambda_4; x_4; y_4) = P_4(1; 0; -2), M_4(0; -2).$$

The second step: investigation the stationary points for existence of a conditional extremum.

Second order partial derivatives of Lagrange function are

$$L''_{\lambda\lambda}(\lambda,x,y) = \varphi'_\lambda(x,y) = 0, L''_{\lambda x}(\lambda,x,y) = \varphi'_x(x,y) = 2x, L''_{\lambda y}(\lambda,x,y) = \varphi'_y(x,y) = 2y,$$

$$L''_{xx}(\lambda,x,y) = 2 + 2\lambda, L''_{xy}(\lambda,x,y) = L''_{yx}(\lambda,x,y) = 0, L''_{yy}(\lambda,x,y) = -2 + 2\lambda.$$

A. For points $P_1(-1; 2; 0), M_1(2; 0)$ и $P_2(-1; -2; 0), M_2(-2; 0)$ we must take Hesse matrix for Lagrange function in the form (40 a) because of the partial derivative $L''_{\lambda x}(\lambda,x,y) = \varphi'_x(x,y) = 2x$ doesn't equal zero at $M_1(2; 0), M_2(-2; 0)$. We have

$$H(f, P(\lambda, x, y)) = \begin{pmatrix} 0 & \varphi'_x(x,y) & \varphi'_y(x,y) \\ \varphi'_x(x,y) & L''_{xx}(\lambda,x,y) & L''_{xy}(\lambda,x,y) \\ \varphi'_y(x,y) & L''_{yx}(\lambda,x,y) & L''_{yy}(\lambda,x,y) \end{pmatrix} = \begin{pmatrix} 0 & 2x & 2y \\ 2x & 2+2\lambda & 0 \\ 2y & 0 & -2+2\lambda \end{pmatrix},$$

or simply

$$H(f,P(\lambda,x,y))=H(f,\lambda,x,y)=\begin{pmatrix} L''_{\lambda\lambda}(x,y) & L''_{\lambda x}(x,y) & L''_{\lambda y}(x,y) \\ L''_{x\lambda}(x,y) & L''_{xx}(\lambda,x,y) & L''_{xy}(\lambda,x,y) \\ L''_{y\lambda}(x,y) & L''_{yx}(\lambda,x,y) & L''_{yy}(\lambda,x,y) \end{pmatrix}=$$

$$=\begin{pmatrix} 0 & 2x & 2y \\ 2x & 2+2\lambda & 0 \\ 2y & 0 & -2+2\lambda \end{pmatrix},$$

a) For the point $P_1(-1;2;0)$ (respectively for $M_1(2;0)$)

$$\Delta_3(-1,2,0)=\det H(f,P_1(-1,2,0))=\det H(f,-1,2,0)=\begin{vmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & -4 \end{vmatrix}=64>0.$$

б) For the point $P_2(-1;-2;0)$ (respectively for $M_2(-2;0)$)

$$\Delta_3(-1,-2,0)=\det H(f,P_2(-1,-2,0))=\det H(f,-1,-2,0)=\begin{vmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & -4 \end{vmatrix}=64>0$$

On the base of the theorem 6 the given function has a conditional maximum at the points $M_1(2;0)$ and $M_2(-2;0)$ which equals $(\pm 2)^2-0^2=4$.

Б. For the other pair of stationary points

$$P_3(\lambda_3;x_3;y_3)=P_3(1;0;2),\ M_3(0;2) \text{ и } P_4(\lambda_4;x_4;y_4)=P_4(1;0;-2),\ M_4(0;-2),$$

we take Hesse matrix in the form (40 b), because $L''_{\lambda x}(\lambda,x,y)=\varphi'_x(x,y)=2x$ equals zero but $L''_{\lambda y}(\lambda,x,y)=\varphi'_y(x,y)=2y$ at the points $M_3(0;2)$ and $M_4(0;-2)$. We have

$$H(f,P(\lambda,x,y))=\begin{pmatrix} 0 & \varphi'_y(x,y) & \varphi'_x(x,y) \\ \varphi'_y(x,y) & L''_{yy}(\lambda,x,y) & L''_{yx}(\lambda,x,y) \\ \varphi'_x(x,y) & L''_{xy}(\lambda,x,y) & L''_{xx}(\lambda,x,y) \end{pmatrix}=\begin{pmatrix} 0 & 2y & 2x \\ 2y & -2+2\lambda & 0 \\ 2x & 0 & 2+2\lambda \end{pmatrix},$$

or simply

$$H(f,P(\lambda,x,y))=H(f,\lambda,y,x)=\begin{pmatrix} L''_{\lambda\lambda}(x,y) & L''_{\lambda y}(x,y) & L''_{\lambda x}(x,y) \\ L''_{y\lambda}(x,y) & L''_{yy}(\lambda,x,y) & L''_{yx}(\lambda,x,y) \\ L''_{y\lambda}(x,y) & L''_{yx}(\lambda,x,y) & L''_{yy}(\lambda,x,y) \end{pmatrix}=$$

$$= \begin{pmatrix} 0 & 2y & 2x \\ 2y & -2+2\lambda & 0 \\ 2x & 0 & 2+2\lambda \end{pmatrix}.$$

a) For the point $P_3(1;0;2)$ (and respectively for $M_3(0;2)$)

$$\Delta_3(1;0;2) = \det H(f, P_3(1;0;2)) = \det H(f,1,2,0) = \begin{vmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 4 \end{vmatrix} = -64 < 0.$$

On the base of the same theorem the function possesses a conditional minimum at the point $M_3(0;2)$, namely $0^2 - 2^2 = -4$.

б) For the point $P_4(1;0;-2)$ (and respectively for $M_4(0;-2)$)

$$\Delta_3(1;0;-2) = \det H(f, P_4(1;0;-2)) = \det H(f,1,-2,0) = \begin{vmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 4 \end{vmatrix} = -64 < 0.$$

So the function possesses a conditional minimum $0^2 - (-2)^2 = -4$ at the point M_4.

Thus the given function $z = x^2 - y^2$ attains a conditional maximum 4 at the points $M_1(2;0)$ and $M_2(-2;0)$ of the circle $x^2 + y^2 = 4$ and a conditional minimum -4 at its points $M_3(0;2)$ and $M_4(0;-2)$.

Case 2. The general problem (28), (29) on a conditional extremum.

Let $(\lambda_0, x_0) = (\lambda_{10}, \lambda_{20}, ..., \lambda_{k0}, x_{10}, x_{20}, ..., x_{n0})$ be some stationary point of Lagrange function (38), that is one of solutions of the system (39). To formulate the sufficient condition for existing of a conditional extremum at corresponding geometrical point $x_0 = (x_{10}, x_{20}, ..., x_{n0})$ we'll introduce two matrices.

a) The first matrix is that

$$\Phi(x) = \Phi(x_1, x_2, ..., x_n)$$

of partial derivatives of the functions (29) (see the formula (40) on the next page).

$$\Phi(x)=\Phi(x_1,x_2,...,x_n)=\begin{pmatrix} \frac{\partial\varphi_1(x)}{\partial x_1} & \frac{\partial\varphi_1(x)}{\partial x_2} & \cdots & \frac{\partial\varphi_1(x)}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\varphi_k(x)}{\partial x_1} & \frac{\partial\varphi_k(x)}{\partial x_2} & \cdots & \frac{\partial\varphi_k(x)}{\partial x_n} \end{pmatrix}. \qquad (40)$$

Here k is the number of conditions (29). It's supposed that the value of the matrix at the point $x_0=(x_{10},x_{20},...,x_{n0})$, that is

$$\Phi(x_0)=\Phi(x_{10},x_{20},...,x_{n0}),$$

has the rank k and so contains at least one non-zero k-th order minor. We'll dwell on the case when the next minor (so-called **jacobian**[1])

$$\frac{D(\varphi_1,\varphi_2,...,\varphi_k)}{D(x_1,x_2,...,x_k)}=\begin{vmatrix} \frac{\partial\varphi_1(x_0)}{\partial x_1} & \frac{\partial\varphi_1(x_0)}{\partial x_2} & \cdots & \frac{\partial\varphi_1(x_0)}{\partial x_k} \\ \frac{\partial\varphi_2(x_0)}{\partial x_1} & \frac{\partial\varphi_2(x_0)}{\partial x_2} & \cdots & \frac{\partial\varphi_2(x_0)}{\partial x_k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\varphi_k(x_0)}{\partial x_1} & \frac{\partial\varphi_k(x_0)}{\partial x_2} & \cdots & \frac{\partial\varphi_k(x_0)}{\partial x_k} \end{vmatrix} \qquad (41)$$

doesn't equal zero.

c) The second matrix to be introduced is Hesse one for Lagrange function (38) that is

$$H(L,\lambda,x)=H(L,\lambda_1,\lambda_2,...,\lambda_k,x_1,x_2,...,x_n). \qquad (42)$$

$$H(L,\lambda,x)=H(L,\lambda_1,\lambda_2,...,\lambda_k,x_1,x_2,...,x_n)=$$

$$=\begin{pmatrix} L''_{\lambda_1\lambda_1} & L''_{\lambda_1\lambda_2} & \cdots & L''_{\lambda_1\lambda_k} & L''_{\lambda_1x_1} & \cdots & L''_{\lambda_1x_n} \\ L''_{\lambda_2\lambda_1} & L''_{\lambda_2\lambda_2} & \cdots & L''_{\lambda_2\lambda_k} & L''_{\lambda_2x_1} & \cdots & L''_{\lambda_2x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L''_{\lambda_k\lambda_1} & L''_{\lambda_k\lambda_2} & \cdots & L''_{\lambda_k\lambda_k} & L''_{\lambda_kx_1} & \cdots & L''_{\lambda_kx_n} \\ L''_{x_1\lambda_1} & L''_{x_1\lambda_2} & \cdots & L''_{x_1\lambda_k} & L''_{x_1x_1} & \cdots & L''_{x_1x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_{x_n\lambda_1} & L''_{x_n\lambda_2} & \cdots & L''_{x_n\lambda_k} & L''_{x_nx_1} & \cdots & L''_{\lambda_1x_n} \end{pmatrix}=\begin{pmatrix} 0 & \cdots & 0 & L''_{\lambda_1x_1} & \cdots & L''_{\lambda_1x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & L''_{\lambda_kx_1} & \cdots & L''_{\lambda_kx_n} \\ L''_{x_1\lambda_1} & \cdots & L''_{x_1\lambda_k} & L''_{x_1x_1} & \cdots & L''_{x_1x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L''_{x_n\lambda_1} & \cdots & L''_{x_n\lambda_k} & L''_{x_nx_1} & \cdots & L''_{x_nx_n} \end{pmatrix}.$$

We have zeros on intersection of k first rows and columns because all first order partial derivatives of Lagrange function with restect to λ_1, λ_2, …, λ_n are the functions

[1] Jacobi, K.G.J. (1804 - 1851), a German mathematician. We use a known notation of a jacobian from the left in (41)

(29) which don't depend on $\lambda_1,\ \lambda_2, \ldots,\ \lambda_n$. First k main minors of Hesse matrix are equal to zero

$$\Delta_1 = \Delta_2 = \ldots = \Delta_k = 0.$$

Theorem 6 (sufficient condition for existence of a conditional extremum)**.** Let for a stationary point $(\lambda_0, x_0) = (\lambda_{10}, \lambda_{20}, \ldots, \lambda_{k0}, x_{10}, x_{20}, \ldots, x_{n0})$ of Lagrange function:

1. The Jacobian (41) doesn't equal zero;

2. $\Delta_i, i > k,$ is the first nonzero main minor of the value $H(L, \lambda_0, x_0)$ of Hesse matrix (42) at the point $(\lambda_0, x_0) = (\lambda_{10}, \lambda_{20}, \ldots, \lambda_{k0}, x_{10}, x_{20}, \ldots, x_{n0})$;

3. $sign\Delta_i = sign(-1)^k$, where k is the number of conditions (29).

Then:

a) if all successive main minors Δ_j of $H(L, \lambda_0, x_0)$ have the same sign,

$$sign\Delta_j = sign(-1)^k,\ j = i+1, i+2, \ldots, n,$$

then the geometrical point $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})$ is that of a conditional minimum;

b) if the principal minors $\Delta_i, \Delta_{i+1}, \Delta_{i+2}, \ldots, \Delta_n$ are alternating,

$$sign\Delta_i = (-1)^k, sign\Delta_{i+1} = (-1)^{k+1}, sign\Delta_{i+2} = (-1)^{k+2}, \ldots,$$

then the point $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})$ is that of a conditional maximum;

c) if at least one of principal minors $\Delta_j, i < j \le n,$ equals zero, we get so-called doubtful case which requires a more complicated theory;

d) no extrema in the other cases.

Ex. 10. Find conditional extrema of the function $u = xyz$ with two constraints

$$x + y + z = 5 \quad (\varphi_1(x, y, z) = x + y + z - 5),$$
$$xy + yz + zx = 8 \quad (\varphi_2(x, y, z) = xy + yz + zx - 8).$$

The first step: introduction of Lagrange function and finding its stationary points. Lagrange function is

$$L = L(\lambda_1, \lambda_2, x, y, z) = f + \lambda_1\varphi_1 + \lambda_2\varphi_2 = xyz + \lambda_1(x + y + z - 5) + \lambda_2(xy + yz + zx - 8).$$

Its first partial derivatives

$$L'_{\lambda_1} = \varphi_1 = x + y + z - 5,\ L'_{\lambda_2} = \varphi_2 = xy + yz + zx - 8,$$
$$L'_x = yz + \lambda_1 + \lambda_2(y+z),\ L'_y = xz + \lambda_1 + \lambda_2(x+z),\ L'_z = xy + \lambda_1 + \lambda_2(x+y),$$

Necessary condition for a conditional extremum is represented by the system

$$\begin{cases} L'_x = 0, \\ L'_y = 0, \\ L'_z = 0, \\ L'_{\lambda_1} = \varphi_1 = 0, \\ L'_{\lambda_2} = \varphi_2 = 0; \end{cases} \quad \begin{cases} yz + \lambda_1 + \lambda_2(y+z) = 0, & (a) \\ xz + \lambda_1 + \lambda_2(x+z) = 0, & (b) \\ xy + \lambda_1 + \lambda_2(x+y) = 0, & (c) \\ x + y + z - 5 = 0, & (d) \\ xy + yz + zx - 8 = 0. & (e) \end{cases}$$

Adding together the equations (a), (b), (c) and keeping in mind (d), (e) we get

$$3\lambda_1 + 10\lambda_2 + 8 = 0. \qquad (f)$$

Subtracting the equation (*b*) from (*a*) and then (*c*) from (*b*) we get

$$\begin{matrix} (y-x)(z+\lambda_2) = 0, & (g) \\ (z-y)(x+\lambda_2) = 0. & (h) \end{matrix}.$$

Remark. One can obtain the equations (g), (h) by the other way. Namely we have from (a), (b), (c)

$$\begin{matrix} yz + \lambda_2(y+z) = -\lambda_1, \\ xz + \lambda_2(x+z) = -\lambda_1, \\ xy + \lambda_2(x+y) = -\lambda_1, \end{matrix} \quad \begin{matrix} yz + \lambda_2(y+z) = xz + \lambda_2(x+z), & (y-x)z + \lambda_2(y-x) = 0, \\ xz + \lambda_2(x+z) = xy + \lambda_2(x+y), & (z-y)x + \lambda_2((z-y)) = 0, \end{matrix}$$

hence

$$\begin{matrix} (y-x)(z+\lambda_2) = 0, & (g) \\ (z-y)(x+\lambda_2) = 0. & (h) \end{matrix}$$

We must study the next cases:

$$1)\, y = x, z = y;\ 2)\, y = x,\ x = -\lambda_2;\ 3)\, z = -\lambda_2, z = y; 4)\, z = -\lambda_2, x = -\lambda_2.$$

1) This case $x = y = z$ is impossible by virtue of the equations (d), (e).

2) In the case $x = y = -\lambda_2$ the equation (*c*) gives $\lambda_1 = \lambda_2^2$, hence the equation (*f*) leads to the quadratic $3\lambda_2^2 + 10\lambda_2 + 8 = 0$ with roots $\lambda_{21} = -2,\ \lambda_{22} - \frac{4}{3}$. It follows that $\lambda_{11} = (\lambda_{21})^2 = 4,\ \lambda_{12} = (\lambda_{22})^2 = \frac{16}{9}$. Corresponding values of *x*, *y* and *z* (by the equation

(d)) are following: $2, 2, 1$ and $4/3, 4/3, 7/3$. Finally we get two stationary points of Lagrange function $P_1(4; -2; 2; 2; 1)$, $P_2\left(\frac{16}{9}; -\frac{4}{3}; \frac{4}{3}; \frac{4}{3}; \frac{7}{3}\right)$ and corresponding stationary points of the given function $M_1(2; 2; 1)$, $M_2\left(\frac{4}{3}; \frac{4}{3}; \frac{7}{3}\right)$.

In the cases 3) and 4) we analogously get another four stationary points

$$P_3(4, -2, 1, 2, 2),\ P_4(4, -2, 2, 1, 2),\ P_5\left(\frac{16}{9}, -\frac{4}{3}, \frac{7}{3}, \frac{4}{3}, \frac{4}{3}\right),\ P_6\left(\frac{16}{9}, -\frac{4}{3}, \frac{4}{3}, \frac{7}{3}, \frac{4}{3}\right)$$

(pairwise $P_3, P_5; P_4, P_6$) and corresponding geometrical points (stationary points of the given function)

$$M_3(1; 2; 2),\ M_4(2; 1; 2),\ M_5\left(\frac{7}{3}; \frac{4}{3}; \frac{4}{3}\right),\ M_6\left(\frac{4}{3}; \frac{7}{3}; \frac{4}{3}\right).$$

The second step: investigation the stationary points for existence of a conditional extremum. The number of conditions $k = 2$, so $(-1)^k = (-1)^2 = 1 > 0$.

A. The matrix of the first partial derivatives of the functions φ_1, φ_2 is

$$\Phi(x, y, z) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x} & \frac{\partial \varphi_1}{\partial y} & \frac{\partial \varphi_1}{\partial z} \\ \frac{\partial \varphi_2}{\partial x} & \frac{\partial \varphi_2}{\partial y} & \frac{\partial \varphi_2}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ y+z & x+z & x+y \end{pmatrix}.$$

Values of the matrix $\Phi(x, y, z)$ at the stationary points $M_1 - M_6$ of the function and corresponding Jacobians are represented below:

$$\Phi(M_1) = \Phi(x_1, y_1, z_1) = \begin{pmatrix} 1 & 1 & 1 \\ y_1+z_1 & x_1+z_1 & x_1+y_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \end{pmatrix},$$

$$\frac{D(\varphi_1, \varphi_2)}{D(x, y)} = \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 0 \text{ but } \frac{D(\varphi_1, \varphi_2)}{D(y, z)} = \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} \neq 0;$$

$$\Phi(M_2) = \Phi(x_2, y_2, z_2) = \begin{pmatrix} 1 & 1 & 1 \\ y_2+z_2 & x_2+z_2 & x_2+y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 11/3 & 11/3 & 8/3 \end{pmatrix},$$

$$\frac{D(\varphi_1, \varphi_2)}{D(x, y)} = \begin{vmatrix} 1 & 1 \\ 11/3 & 11/3 \end{vmatrix} = 0 \text{ but } \frac{D(\varphi_1, \varphi_2)}{D(y, z)} = \begin{vmatrix} 1 & 1 \\ 11/3 & 8/3 \end{vmatrix} \neq 0;$$

$$\Phi(M_3)=\begin{pmatrix}1&1&1\\4&3&3\end{pmatrix},\ \frac{D(\varphi_1,\varphi_2)}{D(x,y)}=\begin{vmatrix}1&1\\4&3\end{vmatrix}\neq 0;\quad \Phi(M_4)=\begin{pmatrix}1&1&1\\3&4&3\end{pmatrix},\ \frac{D(\varphi_1,\varphi_2)}{D(x,y)}\neq 0;$$

$$\Phi(M_5)=\begin{pmatrix}1&1&1\\\frac{8}{3}&\frac{11}{3}&\frac{11}{3}\end{pmatrix},\frac{D(\varphi_1,\varphi_2)}{D(x,y)}=\begin{vmatrix}1&1\\\frac{8}{3}&\frac{11}{3}\end{vmatrix}\neq 0;\Phi(M_6)=\begin{pmatrix}1&1&1\\\frac{11}{3}&\frac{8}{3}&\frac{11}{3}\end{pmatrix},\frac{D(\varphi_1,\varphi_2)}{D(x,y)}\neq 0$$

It follows that we must use Hesse matrix $H_1(L;\lambda_1,\lambda_2,x,y,z)$ for the stationary points P_3-P_6 of Lagrange function, but investigation of the points P_1, P_2 requires the other Hesse matrix, namely $H_2(L;\lambda_1,\lambda_2,y,z,x)$. For corresponding points M_1-M_6 the rank of the matrix $\Phi(x,y,z)$ equals 2.

B. Compiling Hesse matrices at arbitrary point $(\lambda_1,\lambda_2,x,y,z)$.

$$L''_{\lambda_1\lambda_1}=L''_{\lambda_1\lambda_2}=L''_{\lambda_2\lambda_1}=L''_{\lambda_2\lambda_2}=0;\ L''_{\lambda_1x}=L''_{x\lambda_1}=L''_{\lambda_1y}=L''_{y\lambda_1}=L''_{\lambda_1z}=L''_{z\lambda_1}=1;\ L''_{xx}=L''_{yy}=L''_{zz}=0;$$
$$L''_{\lambda_2x}=L''_{x\lambda_2}=y+z;\ L''_{\lambda_2y}=L''_{y\lambda_2}=x+z;\ L''_{\lambda_2z}=L''_{z\lambda_2}=x+y;$$
$$L''_{xy}=L''_{yx}=z+\lambda_2;\ L''_{xz}=L''_{zx}=y+\lambda_2;\ L''_{yz}=L''_{zy}=z+\lambda_2.$$

$$H_1(L,\lambda_1,\lambda_2,x,y,z)=$$
$$=\begin{pmatrix}L''_{\lambda_1\lambda_1}&L''_{\lambda_1\lambda_2}&L''_{\lambda_1x}&L''_{\lambda_1y}&L''_{\lambda_1z}\\L''_{\lambda_2\lambda_1}&L''_{\lambda_2\lambda_2}&L''_{\lambda_2x}&L''_{\lambda_2y}&L''_{\lambda_2z}\\L''_{x\lambda_1}&L''_{x\lambda_2}&L''_{xx}&L''_{xy}&L''_{xz}\\L''_{y\lambda_1}&L''_{y\lambda_2}&L''_{yx}&L''_{yy}&L''_{yz}\\L''_{z\lambda_1}&L''_{z\lambda_2}&L''_{zx}&L''_{zy}&L''_{zz}\end{pmatrix}=\begin{pmatrix}0&0&1&1&1\\0&0&y+z&x+z&x+y\\1&y+z&0&z+\lambda_2&y+\lambda_2\\1&x+z&z+\lambda_2&0&x+\lambda_2\\1&x+y&y+\lambda_2&x+\lambda_2&0\end{pmatrix},$$

$$H_2(L,\lambda_1,\lambda_2,y,z,x)=$$
$$=\begin{pmatrix}L''_{\lambda_1\lambda_1}&L''_{\lambda_1\lambda_2}&L''_{\lambda_1y}&L''_{\lambda_1z}&L''_{\lambda_1x}\\L''_{\lambda_2\lambda_1}&L''_{\lambda_2\lambda_2}&L''_{\lambda_2y}&L''_{\lambda_2z}&L''_{\lambda_2x}\\L''_{y\lambda_1}&L''_{y\lambda_2}&L''_{yy}&L''_{yz}&L''_{yx}\\L''_{z\lambda_1}&L''_{z\lambda_2}&L''_{zy}&L''_{zz}&L''_{zx}\\L''_{x\lambda_1}&L''_{x\lambda_2}&L''_{xy}&L''_{xz}&L''_{xx}\end{pmatrix}=\begin{pmatrix}0&0&1&1&1\\0&0&x+z&x+y&y+z\\1&x+z&0&x+\lambda_2&z+\lambda_2\\1&x+y&x+\lambda_2&0&y+\lambda_2\\1&y+z&z+\lambda_2&y+\lambda_2&0\end{pmatrix}$$

C. Testing stationary points of Lagrange function for existing of conditional extrema. There are $k=2$ conditions, so $(-1)^k=(-1)^2=1>0$.

a) For the point $P_1(4,-2,2,2,1)$ (and the point $M_1(2;2;1)$)

$$H_2(L;P_1)=H_2(L;4,-2,2,2,1)=\begin{pmatrix}0&0&1&1&1\\0&0&3&4&3\\1&3&0&0&-1\\1&4&0&0&0\\1&3&-1&0&0\end{pmatrix};\quad \Delta_1=0,\ \Delta_2=\begin{vmatrix}0&0\\0&0\end{vmatrix}=0,$$

$$\Delta_3=\begin{vmatrix}0&0&1\\0&0&3\\1&3&0\end{vmatrix}=0;\ \Delta_4=\begin{vmatrix}0&0&1&1\\0&0&3&4\\1&3&0&0\\1&4&0&0\end{vmatrix}=1>0,\ \Delta_5=\begin{vmatrix}0&0&1&1&1\\0&0&3&4&3\\1&3&0&0&-1\\1&4&0&0&0\\1&3&-1&0&0\end{vmatrix}=2>0.$$

b) For the point $P_2\left(\frac{16}{9},-\frac{4}{3},\frac{4}{3},\frac{4}{3},\frac{7}{3}\right)$ (and the point $M_2(4/3;4/3;7/3)$)

$$H_2(L;P_2)=H_2\left(L;\frac{16}{9},-\frac{4}{3},\frac{4}{3},\frac{4}{3},\frac{7}{3}\right)=\begin{pmatrix}0&0&1&1&1\\0&0&\frac{11}{3}&\frac{8}{3}&\frac{11}{3}\\1&\frac{11}{3}&0&0&1\\1&\frac{8}{3}&0&0&0\\1&\frac{11}{3}&1&0&0\end{pmatrix};\quad \begin{matrix}\Delta_1=\Delta_2=\Delta_3=0,\\ \Delta_4=1>0,\\ \Delta_5=-2<0.\end{matrix}$$

The function has a conditional minimum 4 at the point $M_1(2;2;1)$ and a conditional maximum $112/27$ at the point $M_2(4/3;4/3;7/3)$.

Note. If we tried to investigate the points

$$P_1(4;-2;2;2;1),\ P_2\left(\frac{16}{9};-\frac{4}{3};\ \frac{4}{3};\ \frac{4}{3};\ \frac{7}{3}\right)$$

with the help of Hesse matrix $H_1(L,\lambda_1,\lambda_2,x,y,z)$, we should get

$$\Delta_1=\Delta_2=\Delta_3=\Delta_4=0$$

and only

$$\Delta_5\neq 0,\ \Delta_5>0\ \left(sign\Delta_5=sign(-1)^k,\ k=2\right).$$

If we even asserted for the function to reach conditional extrema at these points, we could say nothing as to their character.

c) For the point $P_3(4,-2,1,2,2)$ (and the point $M_3(1;2;2)$)

$$H_1(L;P_3)=H_1(L;4,-2,1,2,2)=\begin{pmatrix}0&0&1&1&1\\0&0&4&3&3\\1&4&0&0&0\\1&3&0&0&-1\\1&3&0&-1&0\end{pmatrix},\quad \begin{matrix}\Delta_1=\Delta_2=\Delta_3=0,\\ \Delta_4=1>0,\\ \Delta_5=2>0.\end{matrix}$$

d) For the point $P_4(4,-2,2,1,2)$ (and the point $M_4(2;1;2)$)

$$H_1(L;P_4)=H_1(L;4,-2,2,1,2)=\begin{pmatrix}0&0&1&1&1\\0&0&3&4&3\\1&3&0&0&-1\\1&4&0&0&0\\1&3&-1&0&0\end{pmatrix},\quad \begin{matrix}\Delta_1=\Delta_2=\Delta_3=0,\\ \Delta_4=1>0,\\ \Delta_5=2>0.\end{matrix}$$

The function has conditional minima 4 at the points $M_3(1;2;2)$, $M_4(2;1;2)$.

e), f) By the same way we ascertain that for the points

$$P_5\left(\frac{16}{9},-\frac{4}{3},\frac{7}{3},\frac{4}{3},\frac{4}{3}\right),\ P_6\left(\frac{16}{9},-\frac{4}{3},\frac{4}{3},\frac{7}{3},\frac{4}{3}\right)$$

$$\Delta_1=\Delta_2=\Delta_3=0,\ \Delta_4=1>0,\ \Delta_5=-2<0$$

and therefore the function attains condition maxima $\frac{112}{27}$ at the points

$$M_5\left(\frac{7}{3},\frac{4}{3},\frac{4}{3}\right),\ M_6\left(\frac{7}{3},\frac{4}{3},\frac{4}{3}\right).$$

Answer. The given function achieves the conditional minimum 4 at the points M_1, M_3, M_4 and the condition maximum $\frac{112}{27}$ at the points M_2, M_5, M_6.

POINT 4. ABSOLUTE EXTREMA

Let a function $z=f(M)=f(x,y)$ of two variables is continuous one in a closed bounded domain *D*. By virtue of the theorem 5 of the lecture 11 it takes on the greatest M and the least m values in D. There are points $M_1(x_1, y_1) \in D, M_2(x_2, y_2) \in D$ such that

$$f(M_1) = f(x_1, y_1) = m = \min_D f(M) = \min_D f(x, y),$$
$$f(M_2) = f(x_2, y_2) = M = \max_D f(M) = \max_D f(x, y)$$

The numbers m, M are called **absolute extrema** of the function in the domain D. It's necessary to find them.

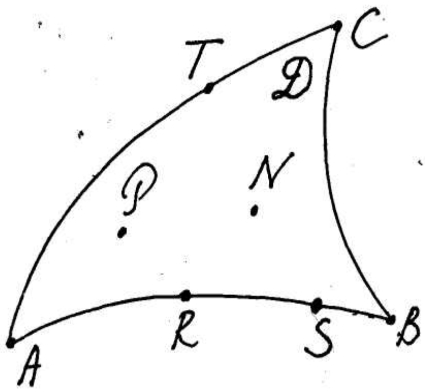

Fig. 5

Solving the problem of finding m, M we must take into account that each of the points $M_1(x_1, y_1), M_2(x_2, y_2)$ can lie as inside the domain D as on its boundary. In the first case it is that stationary of the function.

On the base of these remarks we can state the next

Rule. To find the greatest and the least values (absolute extrema) of a function $z = f(M) = f(x, y)$ of two variables, which is continuous in a closed bounded domain D, it's sufficient to do as follows:

1. To find all inner stationary points of the function (for ex. points N, P on the fig. 5).

2. To find stationary points of the function on the boundary of the domain (for ex. points R, S, T on the fig. 5).

3. To calculate the values of the function at all these points and at angular points of the boundary of the domain if they exist (for ex. points A, B, C on the fig. 5)

4. To choose the greatest and the least of these values.

Finding stationary points of the function on the boundary of the domain D is a part of the problem on a conditional extremum and can be done by using of Lagrange function.

If a boundary of the domain D consists of some separate parts (for ex. AB, BC, CA on the fig. 5), it's necessary to find stationary points of the function on every of

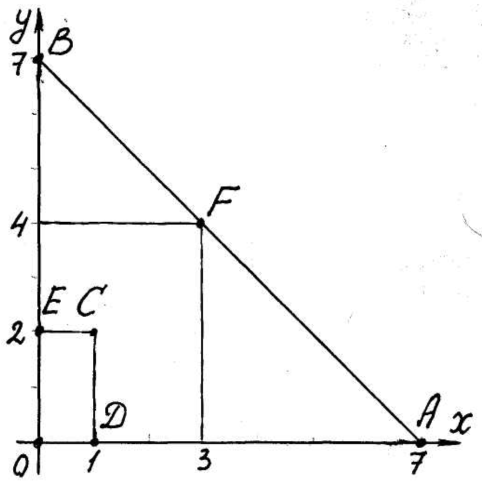

Fig. 6

these parts.

Ex. 11. Find the greatest and the least va-lues of the given function of two variables

$$z = f(x, y) = x^2 + y^2 - 2x - 4y$$

in the domain D which is determined by the ine-qualities $x \ge 0$, $y \ge 0$, $x + y \le 7$.

The function is continuous in a closed boun-ded domain D which is a triangle OAB generated by the coordinate axes and a straight line $x + y = 7$ (fig. 6).

1. $\begin{cases} z'_x = 2x - 2, \\ z'_y = 2y - 4; \end{cases} \begin{cases} 2x - 2 = 0, \\ 2y - 4 = 0; \end{cases} \begin{cases} x = 1, \\ y = 2. \end{cases}$

So the point $C(1; 2)$ is an inner stationary point of the function.

2. The boundary of the domain D contains three segments OA, OB, AB.

a) On the segment OA, $y = 0 \Rightarrow z = x^2 - 2x, z' = 2x - 2$, $z' = 0$ if $2x - 2 = 0$, $x = 1$ and so the point $D(1; 0)$ is that stationary on OA.

b) On OB, $x = 0 \Rightarrow z = y^2 - 4y$, $z' = 0$ if $2y - 4 = 0$, $y = 2$, and we get a sta-tionary point $E(0; 2)$.

c) On the segment AB

$y = 7 - x, z = x^2 + (7 - x)^2 - 2x - 4(7 - x) = 2x^2 - 12x + 21, z' = 0$ if $4x - 12 = 0, x = 3$, and there is a stationary point $F(3; 4) \in AB$.

3. Now we calculate the values of the function at the points C, D, E, F, O, A, B.

$$z(C) = z(1, 2) = -5; \ z(D) = z(1, 0) = -1, \ z(E) = z(0,2) = -4, \ z(F) = z(3,4) = 3,$$

$$z(O) = z(0,0) = 0, \ z(A) = z(7,0) = 35, \ z(B) = z(0,7) = 21.$$

4. Answer: $\min\limits_D z = z(C) = z(1; 2) = -5; \max\limits_D z = z(A) = z(7; 0) = 35.$

Ex. 12. Find the greatest and the least values of the function $z = x^2 - y^2$ in the domain D defined by the inequality $x^2 + y^2 \le 4$.

The function is continuous in a closed bounded domain D which is a circle of radius 2 centered at the origin $O(0;0)$ (fig. 7).

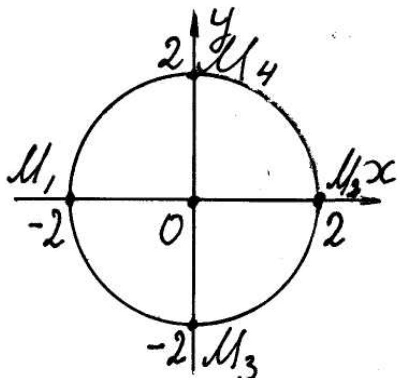

Fig. 7

1. The origin $O(0;0)$ is unique inner stationary point of the function ($z'_x = 2x,\ z'_y = -2y;\ z'_x = z'_y = 0$ if $x = y = 0$).

2. To find stationary points on the boundary of the domain we deal with a problem on conditional extremum for the given function with a boundary condition

$$x^2 + y^2 = 4.$$

Lagrange function of the problem is

$$L(\lambda, x) = f(x, y) + \lambda\varphi(x, y) = x^2 - y^2 + \lambda(x^2 + y^2 - 4),$$

and the corresponding system of equations, which represents the necessary existing condition for a conditional extremum, is

$$\begin{cases} L'_x(\lambda, x) = 0, \\ L'_y(\lambda, x) = 0, \\ L'_\lambda(\lambda, x) = \varphi(x, y) = 0; \end{cases} \qquad \begin{cases} x(1 + \lambda) = 0, \\ y(-1 + \lambda) = 0, \\ x^2 + y^2 - 4 = 0. \end{cases}$$

Solving the system (see Ex. 8) gives four stationary points, namely

$$M_1(-2;0), M_2(2;0), M_3(0;-2), M_4(0;2).$$

3. The values of the function at all found points

$$z(O) = z(0;0) = 0, z(M_1) = z(-2;0) = 0, z(M_2) = z(2;0) = 4,$$
$$z(M_3) = z(0;-2) = -4, z(M_4) = z(0;2) = -4.$$

4. Answer: $m = \min\limits_D z = z(M_3) = z(M_4) = -4;\ M = \max\limits_D z = z(M_1) = z(M_2) = 4$.

APPLICATIONS OF DIFFERENTIAL CALCULUS: basic terminology

129. Ábsolute (extré-mum, mínimum, máxi-mum)	Абсолютний (екстремум, мінімум, максимум)	Абсолютный (экстре-мум, минимум, макси-мум)
130. Àngular póint of a do-máin [région]	Кутова точка області	Угловая точка области
131. Appróach [tend to] *smth* (about a point of a graph/curve)	Наближатися до *чогось* (про точку кривої, графі-ка)	Приближаться к *чему-то* (о точке кривой, графи-ка)
132. Appróximate válue	Наближене значення	Приближённое значение
133. Ascend/rise (from left to right) (about a graph, about a curve)	Сходити/підійматися (зліва направо) (про гра-фік, про криву)	Восходить/подниматься (слева направо) (о графи-ке, о кривой)
134. Ascénding/rísing (from left to right) (about a graph/curve)	Висхідний (зліва напра-во)	Восходящий, поднимаю-щийся (слева направо)
135. Assúmed [propósal, presuppósed] extrémum (*pl* extréma)	Передбачуваний/можли-вий екстремум	Предполагаемый [воз-можный] экстремум
136. Ásymptote (horizóntal, vértical, oblíque/inclíned)	Асимптота (горизонта-льна, вертикальна, похи-ла)	Асимптота (горизонталь-ная, вертикальная, нак-лонная)
137. Be [lie, be found, situa-te, be situated]	Знаходитись, бути розта-шованим	Находиться/располагать-ся, быть расположенным
138. Be [lie, be found, situa-te, be situated] from/on the right *of smth*	Лежати справа/праворуч від *чогось*	Лежать справа *от чего-либо*
139. Be [lie, be found, situ-ate, be situated] lówer/be-lów/únder *of smth*	Лежати нижче *чогось*	Лежать ниже *чего-то*
140. Be [lie, be found, situa-te, be situated] from/on the left *of smth*	Лежати зліва/ліворуч від *чогось*	Лежать слева *от чего-либо*
141. Be [lie, be found, situ-ate, be situated] over/abo-ve *smth*	Лежати вище *чогось*	Лежать/находиться выше *чего-то*
142. Be síituated [locáted, dispósed, arránged], be	Розміщуватися, бути розташованим	Располагаться, быть рас-положенным
143. Behávior (of a fúnc-tion, curve)	Поведінка (функції, кри-вої)	Поведение (функции, кривой)
144. Concáve	Угнутий	Вогнутый

145. Concáve (graph, part/ piece of a graph, curve)	Угнутий [угнута] (графік, частина/ділянка графіка, крива)	Вогнутый [вогнутая] (график, часть/участок графика, кривая)
146. Concávity	Угнутість	Вогнутость
147. Condítional (extrémum, mínimum, máximum	Умовний (екстремум, мінімум, максимум)	Условный (экстремум, минимум, максимум)
148. Constrúct [plot, trace, sketch] a cúrve, a graph póint by póint	Будувати, побудувати криву, графік по точках	Строить, построить кривую, график по точкам
149. Constrúct [plot, trace, sketch] a graph of a fúnc-tion, graph a fúnction	Будувати, побудувати графік функції	Строить, построить график функции
150. Constrúction [constrúcting, trácing] graph of a fúnction [gráphing a fúnction]	Побудова графіка функції	Построение графика функции
151. Constrúction a graph póint by póint	Побудова графіка по точках	Построение графика по точкам
152. Cònvéx [cónvex]	Опуклий	Выпуклый
153. Cònvéx [cónvex] (graph, part/piece of a graph, of a curve)	Опуклий [опукла] (графік, частина/ділянка графіка, крива)	Выпуклый [выпуклая] (график, часть/участок графика, кривая)
154. Convéxity	Опуклість	Выпуклость
155. Còrrespónd to the ex-trémum (abóut a point of a cúrve, of a graph)	Відповідати екстремуму (про точку кривої, графіка	Соответствовать экстремуму (о точке кривой, графика)
156. Crítical póint	Критична точка	Критическая точка
157. Cúspidal póint	Точка звороту	Точка возврата
158. Decréase	Спадати	Убывать
159. Décrease	Спадання	Убывание
160. Decréasing/decay	Спадаючий	Убывающий
161. Dependence (línear, nònlínear/cùrvilínear, quadrátic, pàrabólic(al) *etc*) between váriables …	Залежність (лінійна, нелінійна, квадратична, параболічна *і т.ін.*) між змінними…	Зависимость (линейная, нелинейная, квадратическая, параболическая *и т.д.*) между переменными…
162. Descénd/drop (from left to right) (about a graph, about a curve)	Спадати/опускатися/ спускатися (зліва направо) (про графік, криву)	Нисходить/опускаться (слева направо) (о графике, о кривой)
163. Descénding/dropping (from left to right) (about a graph, about a curve)	Низхідний, той, що опускається (зліва направо)	Нисходящий, опускающийся (слева направо)

164. Desígn [draft, draw-ing, fréehànd/rough draw-ing, sketch, vérsion] of a graph/plot of a fúnc-tion	Ескіз графіка функції	Эскиз, набросок графика функции
165. Desígn, dráwing, fígure	Рисунок	Рисунок
166. Dìsposítion [sìtuátion, locátion] (*for exámple* of a line)	Положення, розташуван-ня (*напр.* лінії)	Положение, расположе-ние (*напр*. линии)
167. Draft [ɑ:] , do a draft	Робити рисунок	Делать чертёж, рисунок
168. Dráwing, fígure, draft	Креслення	Чертёж
169. Drop/descénd (from left to right) (about a graph/curve)	Спадати/опускатися/ спускатися (зліва напра-во) (про графік, про кри-ву)	Опускаться/нисходить (слева направо) (о графи-ке, о кривой)
170. Dróp-ping/descénding (from left to right) (about a graph/curve)	Низхідний [той, що опу-скається] (зліва направо) (про графік, про криву)	Опускающийся, нисхо-дящий (слева направо) (о графике, о кривой)
171. Empíric(al) relátion [de-péndence,connéction, còrre-látion] (betwéen váriables …)	Емпіричне співвідно-шення [емпірична за-лежність, емпіричний зв"язок] (між змінними)	Эмпирическое соотно-шение [эмпирическая за-висимость, эмпиричес-кая связь] (между пере-менными)
172. Estáblish (a relátion [depéndence,connéction, còrrelátion] between vária-bles …)	Установити (співвідно-шення, зв"язок між змін-ними)	Установить (соотноше-ние, связь между пере-менными)
173. Estáblish a condíti-on	Встановити умову	Установить условие
174. Exact desígn/dráwing/ fígure/draft	Точний рисунок	Точный чертёж/рисунок
175. Exístence	Існування	Существование
176. Exístence condítion, condítion of exístence	Умова існування	Условие существования
177. Extrémum (*pl* ex-tréma) of a fúnction of one [two, three, *n*, séveral] váriables (lócal, rélative,	Екстремум функції одні-єї [двох, трьох, *n*, декіль-кох] змінних (локальний, відносний, абсолютний,	Экстремум функции од-ной [двух, трёх, *n*, неско-льких] переменных (ло-кальный, относитель-

ábsolute, condítional)	умовний)	ный, абсолютный, условный)
178. Extrémum próblem	Екстремальна задача	Экстремальная задача
179. Extrémum, *pl* extréma (lócal, rélative, absolute/ global, condítional)	Екстремум (локальний, відносний, абсолютний /глобальний, умовний)	Экстремум (локальный, относительный, абсолютный/глобальный, условный)
180. Find *smth* in the best way	Знайти *щось* якнайкраще	Найти *что-л.* наилучшим образом
181. Find the (lócal, rélati-ve, ábsolute, condítional) extréma [mínima, maxi-ma)] of a given fúnction	Знайти (локальні, відносні, абсолютні, умовні) екстремуми [мінімуми, максимуми] даної функції	Найти (локальные, относительные, абсолютные, условные) экстремумы [минимумы, максимумы] данной функции
182. Géneral schéme/plan for in-vèstigátion/invèsti-gáting fúnctions and cons-trúcting graphs	Загальна схема [загальний план] дослідження функцій і побудови графіків	Общая схема [общий план] исследования функций и построения графиков
183. Glóbal [ábsolute] (ex-trémum, mínimum, máxi-mum)	Глобальний/абсолютний (екстремум, мінімум, максимум)	Глобальный/абсолютный (экстремум, минимум, максимум)
184. Graph [chart, curve, graphical chart, curve, plot] of a fúnction, plótted fúnction, fúnction graph	Графік функції	График функции
185. Gréatest and léast vá-lues of a fúnction contínuous óver/in the bóunded clósed domáin/région	Найбільше й найменше значення функції, неперервної на відрізку [в замкненій обмеженій області]	Наибольшее и наименьшее значение функции, непрерывной на отрезке [в замкнутой ограниченной области]
186. Gréatest válue of a fúnction	Найбільше значення функції	Наибольшее значение функции
187. Gréatest válue of a fún-ction which is contínuous one óver/in/on a ségment [bóunded clósed domáin/ région] (ábsolute maxim-um)	Найбільше значення функції, неперервної на відрізку [в замкненій обмеженій області] (абсолютний максимум)	Наибольшее значение функции, непрерывной на отрезке [в замкнутой ограниченной области] (абсолютный максимум)
188. Héssian	Гессіан, визначник (детермінант) Гессе	Гессиан, определитель (детерминант) Гессе
189. Héssian mátrix	Матриця Гессе	Матрица Гессе
190. Horizóntal ásymptote	Горизонтальна асимптота	Горизонтальная асимптота

191. Hypóthesis (*pl* hypó-theses)	Гіпотеза	Гипотеза
192. Hypóthesize	Будувати [утворювати, висловлювати] гіпотезу	Строить [образовывать, высказывать] гипотезу
193. Incréase	Зростати	Возрастать
194. Íncrease	Зростання	Возрастание
195. Incréasing	Зростаючий	Возрастающий
196. Infléction/infléxion (of a graph of a fúnction)	Перегин (графіка функції)	Перегиб (графика функции)
197. Infléction/infléxion/ flex póint, póint of inflecttion/infléxion [flex, infléxion, póint of cóntrary fléxure]	Точка перегину	Точка перегиба
198. Ínterval of décrease of a fúnction	Інтервал спадання функції	Интервал убывания функции
199. Ínterval of íncrease of a fúnction	Інтервал зростання функції	Интервал возрастания функции
200. Ínterval of mònotoníci-ty [monotoneness, monó-tony] of a fúnction	Інтервал монотонності функції	Интервал монотонности функции
201. Invéstigate [find out] (a fúnction, the behávior of a function, a crítical/ státionary póint *etc*)	Дослідити (функцію, поведінку функції, критичну/стаціонарну точку *і т.ін.*)	Исследовать (функцию, поведение функции, критическую/стационарную точку *и т.д*)
202. Invèstigátion [finding out] (of a fúnction, of the behávior of a function, of a crítical/státionary póint *etc*)	Дослідження (функції, поведінки функції, критичної/стаціонарної точки *і т.ін.*)	Исследование (функции, поведения функции, критической/стационарной точки *и т.д*)
203. Léast válue of a fúnc-tion	Найменше значення функції	Наименьшее значение функции
204. Léast válue of a fúnc-tion which is contínuous one óver/in/on a ségment [bóunded clósed domáin/ région] (ábsolute minim-um)	Найменше значення функції, неперервної на відрізку [в замкненій обмеженій області] (абсолютний мінімум)	Наименьшее значение функции, непрерывной на отрезке [в замкнутой ограниченной области] (абсолютный минимум)
205. Léast-squares méthod [méthod of léast squáres]	Метод найменших квадратів	Метод наименьших квадратов
206. Line of regréssion	Лінія реґресії *y* на *x*	Линия регрессии *y* на *x*

of y on x		
207. Lócal (extrémum, mí-nimum, máximum)	Локальний (екстремум, мінімум, максимум)	Локальный (экстремум, минимум, максимум)
208. Màximizátion	Максимізація	Максимизация
209. Máximize *smth*	Максимізувати	Максимизировать
210. Maximum (*pl* maxi-ma) (lócal, rélative, ábso-lute/global, condítional) of a fúnction	Максимум функції (локальний, відносний, абсолютний/глобальний, умовний)	Максимум функции (локальный, относительный, абсолютный/глобальный, условный)
211. Máximum póint, póint of máximum	Точка максимуму	Точка максимума
212. Méthod of Lagrange's ìndetérmi-nate/úndetermi-ned múltipliers	Метод невизначених множників Лаґранжа	Метод неопределённых множителей Лагранжа
213. Mínimizátion	Мінімізація	Минимизация
214. Mínimize *smth*	Мінімізувати	Минимизировать
215. Minimum (*pl* mínima) (lócal, rélative, ábsolute/ global, condítional) of a fúnction	Мінімум функції (локальний, відносний, абсолютний/глобальний, умовний)	Минимум функции (локальный, относительный, абсолютный/глобальный, условный)
216. Mínimum póint, póint of mínimum	Точка мінімуму	Точка минимума
217. Mónotò-ne/mónotonic	Монотонний	Монотонный
218. Mònotónically (incréase, decréase)	Монотонно (зростати, спадати)	Монотонно (возрастать, убывать)
219. Mònotonícity [mónoto-neness, monótony]	Монотонність	Монотонность
220. Nécessary condítion	Необхідна умова	Необходимое условие
221. Nécessary condítion of exístence	Необхідна умова існування	Необходимое условие существования
222. Négative défínite quad-rátic form	Від"ємно-визначена квадратична форма	Отрицательно определённая квадратичная форма
223. Nórmal sýstem of (the) léast-squares méthod	Нормальна система методу найменших квадратів	Нормальная система метода наименьших квадратов
224. Not to decréase	Не спадати	Не убывать
225. Not to incréase	Не зростати	Не возрастать
226. Oblíque [inclíned] ásymptote	Похила асимптота	Наклонная асимптота
227. Part/piece of concá-	Частина/ділянка угнуто-	Участок/часть вогнуто-

vity	сті	сти
228. Part/piece of convéxity	Частина/ділянка опуклості	Участок/часть выпуклости
229. Pass through the point	Проходити через точку	Проходить через точку
230. Póint of (assúmed/ propósal/presuppósed) extrémum	Точка можливого екстремуму	Точка (возможного) экстремума
231. Póint of a cúrve, of a graph còrrespónding to the extrémum, bénding póint	Точка кривої, графіка, яка відповідає екстремуму	Точка кривой, графика, соответствующая экстремуму
232. Póint of extrémum, extréme póint	Екстремальна точка, точка екстремуму	Экстремальная точка, точка экстремума
233. Pósitive définite quadrátic form	Додатно-визначена квадратична форма	Положительно определённая квадратичная форма
234. Prelíminary/téntative desígn [draft, drawing, freehand/rough drawing, sketch, vérsion] of a graph /plot of a fúnction (graph/ plot *ad interim лат.*)	Попередній ескіз графіка функції	Предварительный эскиз, набросок графика функции
235. Príncipal mínor of the first [second, third, *n*-th] órder; príncipal mínor of órder one [two, three, *n*]; first-[second-, third- *n*-th] órder príncipal mínor	Головний мінор першого [другого, третього, *n*-го] порядку	Главный минор первого [второго, третьего, *n*-го] порядка
236. Quadrátic form	Квадратична форма	Квадратичная форма
237. Rélative (extrémum, mínimum, máximum)	Відносний (екстремум, мінімум, максимум)	Относительный (экстремум, минимум, максимум)
238. Rèpresént (*for exámple* a cúrve)	Зображати/зобразити (*напр.* криву)	Изображать/изобразить (*напр.* кривую)
239. Rèpresentátion (*for exámple* of a cúrve)	Зображення (*напр.* кривої)	Изображение (*напр.*, кривой)
240. Rise/ascénd (from left to right) (about a graph /curve)	Сходити/підійматися (зліва направо) (про криву, про графік)	Подниматься/ восходить (слева на-право) (о графике, о кривой)
241. Rísing/ascénding (from left to right) (about a graph/curve)	Висхідний, той, що підіймається (зліва направо) (про криву, про графік)	Поднимающий-ся, восходящий (слева направо) (о графике, о кривой)

242. Schemátic desígn [dráwing, fígure, draft]	Схематичний рисунок	Схематический чертёж/рисунок
243. Séparate a part/piece of convéxity of a curve and that of its concávity	Відокремлювати ділянку /частину опуклості кривої від ділянки/частини її угнутості	Отделять участок/часть выпуклости кривой от участка/части её вогнутости
244. Solve the próblem for a(n) (lócal, rélative, ábsolute, condítional) extrémum	Розв"язати задачу на (локальний, відносний, абсолютний, умовний) екстремум	Решить задачу на (локальный, относительный, абсолютный, условный) экстремум
245. Stage/step of invest-tigátion	Етап дослідження	Этап исследования
246. Státionary póint	Стаціонарна точка	Стационарная точка
247. Straight line of regréssion of y on x	Пряма реґресії y на x	Прямая регрессии y на x
248. Strict (mònotonícity [monotoneness, monótony], íncrease, décrease, extrémum, mínimum, maximum)	Строгий [строга] (монотонність, зростання, спадання, екстремум, мінімум, максимум)	Строгий [строгая] (монотонность, возрастание, убывание, экстремум, минимум, максимум)
249. Stríctly (incréase, decréase, mónotòne/mónotonic, incréasing, decréasing/decay)	Строго (зростати, спадати, монотонний, зростаючий, спадаючий)	Строго (возрастать, убывать, монотонный, возрастающий, убывающий)
250. Suffícient condítion	Достатня умова	Достаточное условие
251. Suffícient condítion of exístence	Достатня умова існування	Достаточное условие существования
252. Suggést (a depéndence between variables … of the form…)	Наводити на думку, підказувати (залежність між змінними … вигляду…)	Наводить на мысль, подсказывать (зависимость между переменными … вида…)
253. Sum of squáres of (the) érrors	Сума квадратів помилок/похибок	Сумма квадратов ошибок/погрешностей
254. Tángent (líne) at the póint of infléction/infléxion	Дотична в точці перегину	Касательная в точке перегиба
255. Test/invéstigate a fúnction for a(n) (lócal, rélative, ábsolute, condítional) extrémum	Дослідити функцію на (локальний, відносний, абсолютний, умовний) екстремум	Исследовать функцию на (локальный, относительный, абсолютный, условный) экстремум
256. Vértical ásymptote	Вертикальна асимптота	Вертикальная асимптота

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