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PROBABILITY THEORY AND MATHEMATICAL STATISTICS

КОСОЛАПОВ Ю.Ф.

ТЕОРІЯ ЙМОВІРНОСТЕЙ І МАТЕМАТИЧНА СТАТИСТИКА

Навчальний посібник по вивченню курсу
"Теорія ймовірностей і математична статистика"
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Викладаються основні поняття теорії випадкових подій та випадкових величин: правила знаходження ймовірностей; повторення випробувань, формула Бернуллі, локальна та інтеґральна теореми Лапласа, формула Пуассона; функція та щільність розподілу випадкової величини; основні розподіли випадкових величин та їх числові характеристики; двовимірні випадкові величини, функції однієї і двох випадкових величин. Вивчаються основи математичної статистики і теорії кореляції. Подаються численні приклади розв’язання типових задач. Дано завдання для самостійного розв’язання. До всіх основних тем курсу пропонуються англо-французько-німецько-російсько-українські термінологічні словники.

Велику допомогу в створенні посібника надали автору студенти факультету економіки і менеджменту ДонНТУ Константинова А., Хорунжая О., Маринова К., Мамічева В., Місіньова О., Фролофф Г., Карандей К., Фоменко О. (впорядкування лекційних конспектів та створення їх електронних версій, редагування англомовного тексту, робота над термінологічними словниками). Галя Фролофф перевірила розв"язки всіх задач, уважно вивчила підготовлений текст з позицій користувача-студента і внесла багато дуже корисних пропозицій щодо покращення посібника. Значний внесок в написання посібника внесла старший викладач Слов”янського педагогічного університету Косолапова Н. В. (підготовка великого ілюстративного матеріалу, робота над термінологічними словниками).

Автор висловлює щиру подяку всім своїм помічникам.

Автор щиро дякує кафедрі англійської мови ДонНТУ, викладачі якої ретельно переглянули рукопис і запобігли появі низки серьозних мовних помилок.

Для студентів і викладачів технічних вузів.

УКЛАДАЧ: Косолапов Ю.Ф.

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PROBABILITY THEORY AND MATHEMATICAL STATISTICS

THEORIE DES PROBABILITES ET STATISTIQUE MATHEMATIQUE

WAHRSCHEINLICHKEITSTHEORIE UND MATHEMATISCHE STATISTIK

LITERATURE

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LECTURE NO. 1. EVENT AND PROBABILITY

LE PREMIER COURS. EVENEMENT ET PROBABILITE

ERSTE VORLESUNG. EREIGNIS UND WAHRSCHEINLICHKEIT

POINT 1. TRIAL [EXPERIMENT] AND EVENT. Essai *m* [épreuve *f*, expérience *f*] et événement *m*. Versuch *m* [Experiment *n*; Probe *f*] und Ereignis *n*.

POINT 2. ELEMENTS OF COMBINATORICS. Eléments d'analyse combinatoire. Elemente der Kombinatórik.

POINT 3. CLASSIC DEFINITION OF PROBABILITY. Définition classique de probabilité. Klassische Definition von Wahrschéinlichkeit.

POINT 4. STATISTIC(AL) DEFINITION OF PROBABILITY. Définition statistique de probabilité. Statistische Definition von Wahrschéinlichkeit.

POINT 1. TRIAL AND EVENT

The probability theory is the science of rules of mass random phenomena. One can say that it's the learning dealing with regularities of mass random phenomena.

Known sources of the probability theory are: a) investigations of demographyc processes [of population laws]; b) games of chance [games of luck, hazards].

A trial and an event are the **main notions** of the probability theory.

Def. 1. A **trial** is called realization of some complex of conditions.

It's supposed that a trial can arbitrary many times be realized.

Def. 2. An **event** is called every fact, which can occur [appear, happen] or not occur in a trial.

Ex. 1 (see the table).

Trial	*Events*
1. Coin flip [coin tossing]	"head" (occurrence of a head), "tail"
2. Dice toss(ing), fair dice rolling	"1", "2", "3", "4", "5", "6"
3. Drawing a ball from an urn containing *a* white and *b* black balls	"white ball", "black ball"

Events are usually denoted by capitals (A, B, C, …).

There are impossible, certain and random events.

Def. 3. An event is called **impossible** one if it can't occur in any trial.

Ex. 2. Occurrence of a head **and** a tail in one coin tossing.

Def. 4. An event is called **certain** one if it necessary occurs in any trial.

Ex. 3. Occurrence of a head **or** a tail in one coin tossing. Occurrence of at least one of digits 1, 2, 3, 4, 5, 6 in one dice rolling.

Def. 5. An event is called **random** one if it can occur or not occur in a trial.

Ex. 4. All events fixed in Ex. 1.

There are joint or disjoint events.

Def. 6. Events A, B are called those **joint** [compatible] if they can occur together [or simultaneously] in a trial.

Ex. 5. "head", "head"; "tail", "tail"; "head", "tail"; "tail", "head" if a trial means the double coin tossing.

Def. 7. Events A, B are called those **disjoint** [incompatible, non-compatible] if they can't occur together [or simultaneously] in a trial.

Ex. 6. "head", "tail" in one coin toss.

Ex. 7. The events "1", "2", "3", "4", "5", "6" are pairwise disjoint in one dice rolling.

There are **dependent** and **independent** events. Corresponding definitions see in the following.

Def. 8. One says that events A, B,…, C form a total [complete] group (of events) [A, B,…, C are only possible events, A, B,…, C are exhaustive events] if at least one of them occurs in any trial.

Ex. 8. Events "head" and "tail" in one coin toss. All the events "1", "2", "3", "4", "5", "6" in one dice rolling.

Def. 9. Two events A and $\bar{A}$ (non A) are called those **opposite** if they are disjoint and form a total group.

Ex. 9. If A is "head", then $\overline{A}$ (non A) is "tail" (in one coin toss). If A is "1", then $\overline{A}$ (non A) is the occurrence of at least one of events "2", "3", "4", "5", "6", $\overline{A}$ = = {"2" or "3", or "4", or "5", or "6"} (in one fair dice rolling).

POINT 2. ELEMENTS OF COMBINATORICS

Theorem 1 (*fundamental principle of combinatorics*). Let an action A_1 can be done by n_1 ways, an action A_2 by n_2 ways,…, an action A_k by n_k ways, then all these actions can be done together [or simultaneously] by $n_1 \cdot n_2 \cdot ... \cdot n_k$ ways.

We'll illustrate the validity of this statement with the help of the next example.

Ex. 10. Let's suppose that one has a coins and b dies. Then he can take a coin and a die by $a \cdot b$ ways.

■Indeed, each coin generates $1 \cdot b = b$ pairs "coin-die". Therefore, a coins generate $a \cdot b$ pairs.■

Ex. 11. One has 2 coins, 3 ties and 5 books. He can take one coin, one tie and one book by $2 \cdot 3 \cdot 5 = 30$ ways.

Main notions of combinatorics

Let there be given some set M containing n elements.

Def. 10. Arrangement of n elements (taken) k at a time [k-fold arrangement of n elements] is called any ordered k-fold subset of the n-fold set M.

Various arrangements differ by at least one element or by the order of their elements.

Def. 11. Permutation of n elements is called any arrangement of all n elements of the n-fold set M.

Distinct permutations differ by the order of (the same) elements.

One can say that permutation of n elements is the ordered set of all elements of the set M.

Def. 12. Combination of n elements (taken) k at a time [k-fold combination of n elements] is called any k-fold subset of the n-fold set M.

Every combination differs from another one by at least one element.

Theorem 2. Numbers of all k-fold arrangements, of all permutations, of all k-fold combinations of n elements are respectively equal

$$A_n^k = n(n-1)(n-2)\cdot\ldots\cdot(n-(k-1)) \text{ (} k \text{ factors)}, \qquad (1)$$

$$P_n = n! = 1\cdot 2\cdot 3\cdot\ldots\cdot n, \qquad (2)$$

$$C_n^k = \frac{A_n^k}{P_k} = \frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-(k-1))}{k!} = \frac{n!}{k!(n-k)!}. \qquad (3)$$

■The first element of the k-fold arrangement can be taken by n ways, the second by $n-1$ ways, the third by $n-2$ ways, …, the kth by $n-(k-1)$ ways. On the base of the fundamental principle of combinatorics an arrangement in question can be generated by $A_n^k = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-(k-1))$ ways, and the formula (1) is proved. In particular we get the formula (2)

$$P_n = A_n^n = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-(n-1)) = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1 = n!$$

Validity of the formula (3) follows from the next evident fact $A_n^k = C_n^k \cdot P_k$.

In addition we get

$$C_n^k = \frac{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-(k-1))\cdot(n-k)\cdot(n-(k+1))\cdot\ldots\cdot 2\cdot 1}{k!(n-k)\cdot(n-(k+1))\cdot\ldots\cdot 2\cdot 1} =$$

$$= \frac{n\cdot(n-1)\cdot(n-2)\ldots 2\cdot 1}{k!(n-k)\cdot(n-(k+1))\cdot\ldots\cdot 2\cdot 1} = \frac{n!}{k!(n-k)!}. ■$$

Ex. 12. $A_{10}^6 = 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5 = 151200$, $P_6 = 6! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 = 720$,

$$C_{10}^6 = \frac{A_{10}^6}{P_6} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} = \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4} = \frac{A_{10}^4}{P_4} = C_{10}^4 = 210.$$

Ex. 13. A group containing 25 students can elect the leader and its assistant by

$$A_{25}^2 = 25\cdot 24 = 600$$

ways because these two students form 2-fold arrangement of 25 elements .

Ex. 14. One can invite any 4 students of the same group to do some work by

$$C_{25}^4 = \frac{25\cdot 24\cdot 23\cdot 22}{1\cdot 2\cdot 3\cdot 4} = 12650$$

ways because these 4 students form 4-fold combinations of 25 elements.

Ex. 15. 15 compétitors of a chess tóurnament must play

$$C_{15}^{2} = \frac{15 \cdot 14}{1 \cdot 2} = 105$$

games in one lap (every two chess players form 2-fold combination of 15 elements).

Ex. 16. 8 books can be placed in a bookshelf by

$$P_8 = 8! = 1 \cdot 2 \cdot \ldots \cdot 8 = 40320$$

ways because they form a permutation of 8 elements.

POINT 3. CLASSIC DEFINITION OF PROBABILITY

There are events for which we can subtract a set of **elementary events** (**chances, possibilities**) that is a total group of pairwise disjoint and equally possible events. A chance is called that **favourable** for an event A if A occurs when this chance occurs.

Let n be the number of all chances [of all elementare events, of all possibilities] and m be the number of those favourable for some event A. In this case the probability of this event is called the next ratio:

$$P(A) = \frac{m}{n}. \qquad (4)$$

Ex. 17. Find the probability of occurrence of the head in one coin-tossing.

Solution. Let A be an event which means that a head occurs. We can subtract the next n = 2 chances [elementary events, possibilities]: “head”, “tail”. There is m = 1 favourable chance namely “head”. By the formula (4)

$$P(A) = \frac{m}{n} = \frac{1}{2} = 0.5$$

Ex. 18. Find the probability of occurrence of an even number in one dice-rolling.

Solution. Let A be an event which consists in occurrence of even number in one dice-rolling. The chances [elementary events, possibilities] connected with the event A are “1”, “2”, “3”, “4”, “5”, “6”, $n = 6$. The favourable chances are “2”, “4”, “6”, $m = 3$. By the formula (4)

$$P(A) = \frac{m}{n} = \frac{3}{6} = 0.5.$$

Ex. 19. There are 6 white and 14 black balls in some urn. One at random takes 10 balls. Find the probability of drawing of 4 white and 6 black balls.

Solution. Let A be an event consisting in drawing of 4 white and 6 black balls. The chances [elementary events, possibilities] for the event A are various sets of 10 balls that is 10-fold combinations of 20 elements. Therefore the number of all chances is equal to

$$n = C_{20}^{10},$$

that is to number of all 10-fold combinations of 20 elements.

To determine the number m of favourable chances we must take into account that one can take 4 white balls (4-fold combination of 6 elements) by C_6^4 ways and 6 black balls (6-fold combination of 14 elements) by C_{14}^6 ways. Therefore to take 4 white and 6 black balls together he can, by virtue of the fundamental principle of combinatorics, by $C_6^4 \cdot C_{14}^6$ ways. It means that

$$m = C_6^4 \cdot C_{14}^6,$$

hence

$$P(A) = \frac{m}{n} = \frac{C_6^4 \cdot C_{14}^6}{C_{20}^{10}} = \frac{\frac{6\cdot5\cdot4\cdot3}{1\cdot2\cdot3\cdot4}\cdot\frac{14\cdot13\cdot12\cdot11\cdot10\cdot9}{1\cdot2\cdot3\cdot4\cdot5\cdot6}}{\frac{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14\cdot13\cdot12\cdot11}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}} =$$

$$= \frac{6\cdot5\cdot4\cdot3\cdot14\cdot13\cdot12\cdot11\cdot10\cdot9\cdot1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot14\cdot13\cdot12\cdot11\cdot1\cdot2\cdot3\cdot4\cdot1\cdot2\cdot3\cdot4\cdot5\cdot6} =$$

$$= \frac{10\cdot9\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}{20\cdot19\cdot18\cdot17\cdot16\cdot15\cdot1\cdot2} = \frac{6\cdot7\cdot9\cdot5}{2\cdot19\cdot2\cdot17\cdot2\cdot3} = \frac{7\cdot9\cdot5}{4\cdot19\cdot17} \approx 0.24.$$

Remark. Counting of the number of favourable chances it’s well to represent by the next table:

Actions to draw 4 white and 6 black balls	Number of ways to do these actions
1. Drawing of 4 white balls (4-fold combinations of 6 elements)	C_6^4
2. Drawing of 6 blach balls (6-fold combinations of 14 elements)	C_{14}^6
Drawing of 4 white and 6 black balls together	$C_6^4 \cdot C_{14}^6$ (by virtue of the main principle of combinatorics)

Ex. 20. It's necessary to place 8 books on a bookshelf. Find the probability for two certain books A, B to stand side by side.

Solution. Let an event C be the required position of our books. The chances [elementary events, possibilities] are various their locations which are permutations of 8 elements. Therefore the number of all chances is that of all possible permutations of 8 elements,

$$n = P_8 = 8!$$

To find the number of favourable chances (that is that of required situations of books) we'll introduce the next table

Actions to place A, B side by side	Number of ways to do these actions
1. Finding places for A, B	7
2. Location of A, B on these places (permutation of 2 elements)	$P_2 = 2!$
3. Disposition of the other 6 books (permutation of 6 elements)	$P_6 = 6!$
Getting of required disposition of all 8 books	$7 \cdot P_2 \cdot P_6$ (by virtue of the main combinatorial principle)

So the number of all favourable chances equals

$$m = 7 \cdot P_2 \cdot P_6 .$$

On the base of classical definition of probability

$$P(C) = \frac{m}{n} = \frac{7 \cdot P_2 \cdot P_6}{P_8} = \frac{7 \cdot 2 \cdot 1 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 2}{8 \cdot 7} = \frac{1}{4} = 0.25 .$$

POINT 4. STATISTIC DEFINITION OF PROBABILITY

Let some event A be studied and there be fulfilled very large number N of independent trials on A. Let's denote $N(A)$ the number of occurrences of A in these trials. The ratio

$$p^* = P_N^*(A) = N(A)/N \qquad (5)$$

is called a relative frequency (or sometimes frequency) of the event *A*.

Let's fulfil series of very large numbers $N_1, N_2, \ldots$ of independent trials on *A* and denote by

$$p_1^* = P_{N_1}^*(A),\ p_2^* = P_{N_2}^*(A), \ldots$$

corresponding relative frequencies of *A*.

There are many events for which relative frequencies possess a property of **statistic stability** that is they are approximately equal to some number *p*

$$p_1^* \approx p,\ p_2^* \approx p, \ldots$$

If our event *A* possesses such the stability property, we say that it has a probability (so-called **statistic probability**), and this probability equals

$$P(A) = p. \qquad (6)$$

Ex. 21. Many scientists have performed series of very large numbers of coin-tossings (see the table).

Scientist	*N*	*N*("head")	$p^* = P_N^*("head")$
Buffon G.L.L.[1] (1777)	4040	2048	0.507
de Morgan A.[2] (at the beginning of the 19th century)	4092	2048	0.5005
Pearson K.[3] (at the beginning of the 20th century)	12000	6019	0.5016
Pearson K.	24000	12012	0.50005

On the base of these results we conclude that the statistical probability of the event "head" (in one coin-tossing) equals *p* = *P* ("head") = 0.5 that is coincides with its "classic" probability.

[1] Buffon, G.L.L. (1707 - 1788), a French naturalist
[2] Morgan de (de Morgan, A.) (1806 – 1871), a Scottish mathematician and logician
[3] Pearson K. (Ch.) (1857 - 1936) - an English mathematician-stàtistícian, bìólogist and philósopher-pósitivist

LECTURE NO. 2. MAIN RULES OF EVALUATING PROBABILITIES

LE DEUXIEME COURS. REGLES PRINCIPALES DE CALCUL DE PROBABILITES

ZWEITE VORLESUNG. GRUNDRECHNUNGSREGELN VON WAHRSCHEINLICHKEITEN

POINT 1. SUM AND PRODUCT OF EVENTS. Somme et produit d'événements. Súmme und Prodúkt von Eréignissen

POINT 2. PROPERTIES OF RELATIVE FREQUENCIES. Propriétés de fréquences relatives. Eigenschaften von relatíven Frequénzen

POINT 3. AXIOMS OF PROBABILITY THEORY. COROLLARIES. Axiomes de théorie des probabilités. Corollaires. Axiome von Wahrschéinlichkeitstheorie. Fólgerungen.

POINT 4. FORMULAS OF TOTAL PROBABILITY AND BEYES. Formules de la probabilité complète et de Beyes. Fórmeln von totáler Wahrschéinlichkeit und von Bayes.

POINT 1. SUM AND PRODUCT OF EVENTS

Def. 1. Sum $A+B$ (*A or B*) of two events A and B is called an event which consists in occurrence of at least one of them [which means that at least one of these events occurs] (A but not B or B but not A or A and B together).

Ex. 1. Sum of an event A and its opposite one $\overline{A}$ is the certain event.

Ex. 2. If an event A is "1" in one dice rolling, then the opposite event $\overline{A}$ is the sum $\overline{A} = "2" + "3" + "4" + "5" + "6"$.

Def. 2. Product AB (*A and B*) of two events A and B is called an event consisting in occurrence of both these events [an event which means that both these events occur] together.

Ex. 3. Product of an event A and its opposite one $\overline{A}$ is the impossible event.

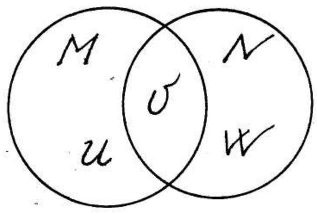

Fig. 1

Ex. 4 (*Euler*[1] *circles*). Let M and N be two circles having non-empty intersection $V = M \cap N$, also let $U = M \setminus N, W = N \setminus M$, and so $M = U \cup V, N = V \cup W$ (see fig. 1). If an event A means that a point P belongs to M, and B means that P belongs to N, then

$$A + B = \{P \in (U \cup V \cup W)\},\ AB = \{P \in V\}.$$

Solving probabilistic problems it is sometimes useful to represent an event in question as a sum of product of other events (with pairwise disjoint summands).

Ex. 5. $A + B = A \cdot \overline{B} + \overline{A} \cdot B + A \cdot B$.

Ex. 6. Let events A, B, C mean that the first, second, third device (correspond-dingly) works. In this case the events

$$D = ABC,\ E = \overline{A}\overline{B}\overline{C},\ F = A\overline{B}\overline{C} + \overline{A}B\overline{C} + \overline{A}\overline{B}C,\ G = AB\overline{C} + A\overline{B}C + \overline{A}BC,$$

$$F + G + D = A\overline{B}\overline{C} + \overline{A}B\overline{C} + \overline{A}\overline{B}C + AB\overline{C} + A\overline{B}C + \overline{A}BC + ABC = \overline{E}$$

mean respectively that all three devices work, no one device work, works only one device (and two other don't work), work two devices (and one don't work), at least one device works.

POINT 2. PROPERTIES OF RELATIVE FREQUENCIES

Let N be the number of independent trials and $N(A)$ is the number of occurrences of an event A. The ratio

$$P_N^*(A) = \frac{N(A)}{N} \tag{1}$$

we've called the relative frequency of the event A (see Lect. 1, the formula (5)).

We introduce also a notion of a conditional relative frequency which is connected with two events. Namely, the ratio

[1] Euler, L. (1707 - 1783), a great scientist (a Swiss by birth). He spent most of his life in Russia and died in St. Petersburg. L. Euler contributed many outstanding results to mathematical analysis, celestial mechanics, shipbuilding and other divisions of science

$$P_N^*(A/B)=\frac{N(AB)}{N(B)} \qquad (2)$$

is the relative frequency of the event *A* under condition that the event *B* occurs. Analogously

$$P_N^*(B/A)=\frac{N(AB)}{N(A)} \qquad (3)$$

is the relative frequency of the event B under condition that the event A occurs.

Properties of relative frequencies

1. Relative frequency of an impossible event *A* equals zero.

■For an impossible event $N(A)=0$. Dividing by *N* we get $P_N^*(A)=0$. ■

2. Relative frequency of a certain event *A* is equal to unity.

■For a certain event $N(A)=N \Rightarrow P_N^*(A)=1$.■

3. Relative frequency of a random event *A* is contained between zero and unity that is

$$0 \le P_N^*(A) \le 1.$$

■For a random event one has $0 \le N(A) \le N \Rightarrow 0 \le P_N^*(A) \le 1$.■

4. Relative frequency of a sum of two disjoint events *A* and *B* equals the sum of their relative frequencies that is

$$P_N^*(A+B)=P_N^*(A)+P_N^*(B).$$

■Because of disjointness of *A* and *B* we have $N(A+B)=N(A)+N(B)$. After division by *N* we prove the validity of the property.■

5. Relative frequency of a product of two events *A* and *B* equals the product of relative frequency of one event and corresponding conditional relative frequency of the other, namely

$$P_N^*(AB)=P_N^*(A)\cdot P_N^*(B/A)=P_N^*(B)\cdot P_N^*(A/B).$$

■By virtue of the formulas (2), (3)

$$P_N^*(AB)=\frac{N(AB)}{N}=\frac{N(AB)\cdot N(A)}{N\cdot N(A)}=\frac{N(A)}{N}\cdot\frac{N(AB)}{N(A)}=P_N^*(A)P_N^*(B/A),$$

$$P_N^*(AB)=\frac{N(AB)}{N}=\frac{N(AB)\cdot N(B)}{N\cdot N(B)}=\frac{N(B)}{N}\cdot\frac{N(AB)}{N(B)}=P_N^*(B)P_N^*(A/B).\blacksquare$$

POINT 3. AXIOMS OF PROBABILITY THEORY. COROLLARIES

We state axioms of the probability theory on the base of statistic definition of probability ($P(A)\approx P_N^*(A)$ for large number N of trials).

1. If A is an **impossible** event, then its probability equals zero,

$$P(A)=0 \ (A \text{ is impossible}).$$

2. If A is a **certain** event, then its probability equals unity,

$$P(A)=1 \ (A \text{ is certain}).$$

3. If A is a **random** event, then its probability is contained between zero and unity,

$$0\le P(A)\le 1 \ (A \text{ is random}).$$

4. If A and B are two disjoint events, then the probability of their sum is equal to the sum of probabilities of these events,

$$P(A+B)=P(A)+P(B) \ (A, B \text{ are disjoint}).$$

To formulate the last axiom let's introduce the notion of a **conditional probability** of an event. Namely, $P(B/A)$ is the probability of an event B by condition that an event A occurs. Analogously $P(A/B)$ is the probability of A if B occurs.

5. Probability of a product of two events equals the product of the probability of one event and the condition probability of the other,

$$P(AB)=P(A)\cdot P(B/A)=P(B)\cdot P(A/B).$$

Ex. 7. An urn contains 3 white and 2 black balls. One successively and at random takes two balls. Find the probability that they are white.

Solution. Let's denote by A an event which means that two drawn balls are white. Also let's denote by B and C events which mean that the first respectively the second drawn ball are white. It's evident that

$$A = BC$$

hence by virtue of the fifth axiom (and classic definition of probability)

$$P(A) = P(BC) = P(B)P(C/B) = \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{10} = 0.3$$

Some corollaries

1. If some events A, B, …, C are **pairwise disjoint**, then

$$P(A + B + \ldots + C) = P(A) + P(B) + ... + P(C).$$

If besides they **form a total group**, then

$$P(A) + P(B) + ... + P(C) = 1.$$

2. The sum of probabilities of two opposite events equal 1,

$$P(A) + P(\overline{A}) = 1$$

because the events $A, \overline{A}$ are disjoint and form a total group.

3. Probabilities of a product of three, four etc events are equal to

$$P(ABC) = P(A) \cdot P(B/A) \cdot P(C/AB),$$

$$P(ABCD) = P(A) \cdot P(B/A) \cdot P(C/AB) \cdot P(D/ABC), \ldots .$$

4. For two arbitrary events A and B the probability of their sum equals

$$P(A + B) = P(A) + P(B) - P(AB).$$

■By Ex. 5 one has

$$A + B = A \cdot \overline{B} + \overline{A} \cdot B + A \cdot B .$$

With pairwise disjoint summans in the right side. Is is evident that

$$A = A\overline{B} + AB; B = \overline{A} \cdot B + A \cdot B .$$

The pairs of events $A\overline{B}$, AB and $\overline{A} \cdot B$, $A \cdot B$ are disloint, and so

$$P(A) = P(A\overline{B}) + P(AB); P(B) = P(\overline{A} \cdot B) + P(A \cdot B),$$

hens

$$P(A\overline{B}) = P(A) - P(AB); P(\overline{A} \cdot B) = P(B) - P(A \cdot B).$$

Therefore

$$P(A+B)=P(A\cdot\overline{B})+P(\overline{A}\cdot B)+P(A\cdot B)=$$

$$=(P(A)-P(AB))+(P(B)-P(A\cdot B))+P(A\cdot B)=P(A)+P(B)-P(AB).\blacksquare$$

It's well to illustrate this fact on the fig.1 in P.1. Indeed, the sum

$$P(A)+P(B)$$

twice takes into account the probability of the event

$$(P\in V)=(P\in M\cap N),$$

that is the probability $P(AB)$.

Def 3. Two events *A, B* are called independent if probability of one of them doesn't depend on occurrence (or non-occurrence) of the other.

For 3, 4 … events one introduces a notion of mutual independence (independence in the aggregate, collectionwise independence).

Def 4. *n* events (for $n>2$) are called mutually independent if the probability of one of them doesn't depend on occurrence or non-occurrence of any group of the other.

5. If *A, B* are independent events, then

$$P(A/B)=P(A),\ P(B/A)=P(B),$$

and so

$$P(AB)=P(A)\cdot P(B),$$

that is the probability of a product of two **independent** events is equal to the product of their probabilities.

6. If *A, B, ... C* are mutually independent events, then

$$P(A\cdot B\cdot\ldots\cdot C)=P(A)P(B)\cdot\ldots\cdot P(C)$$

Ex. 8. To pass an exam successfully a student has to know the proofs of 50 theorems but he knows only 40 of them. What is the probability for him to pass an exam if exam tasks contain 3 theorems?

Let *A* be an event which means that a student will pass an exam. Let's introduce the next three auxiliary events: B_1 that is a student knows the proof of the first theorem, B_2 of the second, B_3 of the third. Then by the fourth corollary (and classic definition of probability)

$$A = B_1 B_2 B_3 \Rightarrow P(A) = P(B_1 B_2 B_3) = P(B_1) \cdot P(B_2 / B_1) \cdot P(B_3 / B_1 B_2) = \frac{40}{50} \cdot \frac{39}{49} \cdot \frac{38}{48} \approx 0.5$$

Ex. 9. A device consists of 2 independent modules. Probability for these modules to work are 0.95 and 0.9 respectively. Find the probability that the device doesn't work because of: a) only one module; b) at least one module.

Let an event A mean that a device doesn't work because of only one module, and an event B because of at least one module. Let events C_1, C_2 mean that the first, the se-cond module works. By condition

$$P(C_1) = 0.95,\ P(C_2) = 0.9,$$

and so by the corollary 3

$$P(\overline{C_1}) = 1 - P(C_1) = 0.05,\ P(\overline{C_2}) = 1 - P(C_2) = 0.1.$$

We represent the events A, B and $\overline{B}$ as follows

$$A = C_1\overline{C_2} + \overline{C_1}C_2,\ B = C_1\overline{C_2} + \overline{C_1}C_2 + \overline{C_1}\,\overline{C_2} \Rightarrow \overline{B} = C_1 C_2.$$

All summands are pairwise disjoint, and all factors are independent in summands. Therefore

$$P(A) = P\left(C_1\overline{C_2} + \overline{C_1}C_2\right) = P\left(C_1\overline{C_2}\right) + P\left(\overline{C_1}C_2\right) = P(C_1)P\left(\overline{C_2}\right) + P\left(\overline{C_1}\right)P(C_2) =$$
$$= 0.05 \cdot 0.9 + 0.95 \cdot 0.1 = 0.14,$$

$$P\left(\overline{B}\right) = P(C_1 C_2) = P(C_1)P(C_2) = 0.95 \cdot 0.9 = 0.86 \Rightarrow P(B) = 1 - P\left(\overline{B}\right) = 1 - 0.86 = 0.14.$$

Ex. 10. Three independently working éngines are instálled in a workshop. Probabilities to work at a given time equal for them 0.6, 0.9, 0.7 respectively. Find probabilities of the next events: a) only one engine works; b) at least one engine works.

Solution. Let an event A mean that only one engine works and an event B mean that at least one engine works. Our problem is to find the probabilities of these events.

Let's introduce three auxiliary events, namely C_1 which means that the first engine works, C_2 the second engine works and C_3 the third engine works. By conditions of the problem

$$P(C_1)=0.6;\ P(\overline{C_1})=1-P(C_1)=1-0.6=0.4,$$
$$P(C_2)=0.9;\ P(\overline{C_2})=1-P(C_2)=1-0.9=0.1,$$
$$P(C_3)=0.7;\ P(\overline{C_3})=1-P(C_3)=1-0.7=0.3\ .$$

a) The event A can be represented as the sum of products

$$A=C_1\overline{C_2}\,\overline{C_3}+\overline{C_1}C_2\overline{C_3}+\overline{C_1}\,\overline{C_2}C_3$$

with pairwise disjoint summands and independent factors in every summand. Hence the probability of the event A equals

$$P(A)=P\left(C_1\overline{C_2}\,\overline{C_3}+\overline{C_1}C_2\overline{C_3}+\overline{C_1}\,\overline{C_2}C_3\right)=P\left(C_1\overline{C_2}\,\overline{C_3}\right)+P\left(\overline{C_1}C_2\overline{C_3}\right)+P\left(\overline{C_1}\,\overline{C_2}C_3\right)=$$
$$=P(C_1)P(\overline{C_2})P(\overline{C_3})+P(\overline{C_1})P(C_2)P(\overline{C_3})+P(\overline{C_1})P(\overline{C_2})P(C_3)=$$
$$=0.6\cdot0.1\cdot0.3+0.4\cdot0.9\cdot0.3+0.4\cdot0.1\cdot0.7=0.154\,.$$

b) To find the probability of the event B[1] we'll evaluate at first the probability of its opposite one $\overline{B}$ (which means that all three engines don't work, $\overline{B}=\overline{C_1}\,\overline{C_2}\,\overline{C_3}$). We'll obtain

$$P(\overline{B})=P(\overline{C_1}\,\overline{C_2}\,\overline{C_3})=P(\overline{C_1})P(\overline{C_2})P(\overline{C_3})=0.4\cdot0.1\cdot0.3=0.012$$

whence it follows that $P(B)=1-P(\overline{B})=1-0.012=0.988$.

Ex. 11a. An urn contains 3 white and 4 black balls. One succesfully [consecutively, in sequence, sequentially, successively] takes 3 balls. Find the probability that the first and third balls will be those white.

Clue. Let's introduce an event A which means that the first and third balls will be those white. Then (notations are clear) $A=w\cdot w\cdot w+w\cdot b\cdot w$, and so

$$P(A)=P(w\cdot w\cdot w)+P(w\cdot b\cdot w)=P(w)P(w/w)P(w/ww)+P(w)P(b/w)P(w/wb)=$$
$$=3/7\cdot2/6\cdot1/5+3/7\cdot4/6\cdot2/5=1/7\,.$$

---

[1] That's bad to write

$$B=C_1\overline{C_2}\,\overline{C_3}+\overline{C_1}C_2\overline{C_3}+\overline{C_1}\,\overline{C_2}C_3+C_1C_2\overline{C_3}+C_1\overline{C_2}C_3+\overline{C_1}C_2C_3+C_1C_2C_3.$$

POINT 4. FORMULAE OF TOTAL PROBABILITY AND BEYES

The formula of total probability

In practice we often deal with the next situation. An event A can occur only together with one of pairwise disjoint events $H_1, H_2, ..., H_n$, which form a total group. Let's call these events **hypotheses.** Their probabilities and corresponding conditional probabilities of the event A are known. In this case the probability of the event A can be found with the help of the next formula (the **formula of total probability**):

$$P(A) = P(H_1) \cdot P(A/H_1) + P(H_2) \cdot P(A/H_2) + ... + P(H_n) \cdot P(A/H_n). \quad (4)$$

■On the base of the condition we must represent the event A as the sum of its products with the hypotheses,

$$A = AH_1 + AH_2 + ... + AH_n.$$

By virtue of the fifth axiom of the probability theory it follows that

$$P(A) = P(AH_1) + P(AH_2) + ... + P(AH_n) =$$

$$= P(H_1)P(A/H_1) + P(H_2)P(A/H_2) + ... + P(H_n)P(A/H_n).■$$

Beyes[1] formulae

Let an event A, which can occur only together with one of given hypotheses $H_1, H_2, ..., H_n$, be occurred. In this case the next probabilities $P(H_i/A)$ of its occurrence together with each of these hypotheses can be evaluated with the help of known Beyes formulas

$$P(H_i/A) = \frac{P(H_i)P(A/H_i)}{P(A)}, \quad i = 1, 2, ..., n. \quad (5)$$

■Let's take an arbitrary hypothesis H_i and express the probability of a product AH_i by two ways,

[1] Beyes, T. (1702 - 1761), an English mathematician

$$P(AH_i) = P(A)P(H_i / A) = P(H_i)P(A / H_i).$$

Two last parts of this tripled equality give Beyes formulae.■

Beyes formulae (5) state the probability that namely the i-th hypothesis has occurred together with the event A in question.

Ex. 11. There are 6 white and 2 black balls in the first urn and 8 white and 3 black balls in the second urn. One at random moves a ball from the first urn to the second one, and then he (also at random) takes a ball from the second urn.

1. Find the probability for him to take a white ball from the second urn.
2. Let a white ball be taken from the second urn. A ball of which colour was the most probably moved from the first urn?

1. Solution of the first problem. Let an event A mean that one will take a white ball from the second urn. We can introduce the next two hypotheses: H_1 means that one has moved a white ball from the first urn; H_2 that he has moved a black ball from there. Their probabilities equal

$$P(H_1) = 6/8 = 3/4, \quad P(H_2) = 2/8 = 1/4$$

by condition, and corresponding conditional probabilities of the event A equal

$$P(A / H_1) = 9/12 = 3/4, \qquad P(A / H_2) = 8/12 = 2/3.$$

On the base of the formula (4) of total probability the probability of the event A equals

$$P(A) = P(H_1) \cdot P(A / H_1) + P(H_2) \cdot P(A / H_2) = 3/4 \cdot 3/4 + 1/4 \cdot 2/3 = 0.73.$$

2. To solve the second problem we must find and compare the next conditional probabilities $P(H_1 / A), P(H_2 / A)$. On the base of Beyes formulae (5)

$$P(H_1 / A) = \frac{P(H_1)P(A / H_1)}{P(A)} = \frac{3/4 \cdot 3/4}{0.73} \approx 0.77,$$
$$P(H_2 / A) = \frac{P(H_2)P(A / H_2)}{P(A)} = \frac{1/4 \cdot 2/3}{0.73} \approx 0.23.$$

We see that 0.77 > 0.23, therefore one has the most probably moved a white ball from the first urn to the second one.

Ex. 12. There are 10 and 15 products of the first and second factories respectively in the storage. The first factory makes 5% and the second 7% of defective products. One takes at random a product.

1) Find the probability of its defectiveness.

2) Suppose that this product is defective. Which factory has the most probably made it?

Let an event A mean that a product taken at random is defective. Let's introduce the next two hypotheses:

H_1 this product is done by the first factory;

H_2 it was done by the second factory.

On the base of the classical definition of probability

$$P(H_1)=\frac{m_1}{n}=\frac{10}{25}=\frac{2}{5}=0.4;\quad P(H_2)=\frac{m_2}{n}=\frac{15}{25}=\frac{3}{5}=0.6.$$

Corresponding conditional probabilities of the event A equal

$$P(A/H_1)=\frac{5}{100}=0.05;\quad P(A/H_2)=\frac{7}{100}=0.07.$$

1) Using the formula of total probability we'll get

$$P(A)=P(H_1)P(A/H_1)+P(H_2)P(A/H_2)=0.4\cdot 0.05+0.6\cdot 0.07=0.062.$$

2) Now with the help of Beyes formulas we find

$$P(H_1/A)=\frac{P(H_1)P(A/H_1)}{P(A)}=\frac{0.4\cdot 0.05}{0.062}=0.32,$$

$$P(H_2/A)=\frac{P(H_2)P(A/H_2)}{P(A)}=\frac{0.6\cdot 0.07}{0.062}=0.68.$$

Thus

$$P(H_2/A)>P(H_1/A).$$

Therefore the taken product was the most probably made by the second factory.

Ex. 13. Three shots have simultaneously shot at a target [-g-] with probabilities of hitting 0.6, 0.8, 0.7 correspondingly, and a hole has appeared in the target. Find the probability that the first of the shots has hit.

At first we'll introduce the next four events A, B, C, D:

A consists in appearance of a hole on the target;

B means that the hitting is done by the first shot;

C means that this hitting is done by the second shot;

D means that this hitting is done by the third shot. By conditions

$P(B) = 0.6,\ P(\overline{B}) = 0.4;\ P(C) = 0.8,\ P(\overline{C}) = 0.2;\ \ P(D) = 0.7,\ P(\overline{D}) = 0.3.$

All the events $B, \overline{B}, C, \overline{C}, D, \overline{D}$ are (mutually) independent. It is obviously that

$$A = B\overline{C}\,\overline{D} + \overline{B}C\overline{D} + \overline{B}\,\overline{C}D$$

with pairwise disjoint summands, and so

$$P(A) = P\left(B\overline{C}\,\overline{D} + \overline{B}C\overline{D} + \overline{B}\,\overline{C}D\right) = P\left(B\overline{C}\,\overline{D}\right) + P\left(\overline{B}C\overline{D}\right) + P\left(\overline{B}\,\overline{C}D\right) =$$
$$= P(B)P(\overline{C})P(\overline{D}) + P(\overline{B})P(C)P(\overline{D}) + P(\overline{B})P(\overline{C})P(D) =$$
$$= 0.6 \cdot 0.2 \cdot 0.3 + 0.4 \cdot 0.8 \cdot 0.3 + 0.4 \cdot 0.2 \cdot 0.7 \approx 0.188\,.$$

Now we can introduce two hypotheses

$H_1 = B\overline{C}\,\overline{D}$ (hitting of the first shot and misses of the other), $H_2 = \overline{H_1}$.

The probability of the first hypothesis and corresponding conditional probability of the event A equal respectively

$$P(H_1) = P\left(B\overline{C}\,\overline{D}\right) = P(B)P(\overline{C})P(\overline{D}) = 0.6 \cdot 0.2 \cdot 0.3 = 0.036, \quad P(A/H_1) = 1\,.$$

Hence by Beyes formulas (5) we get

$$P(H_1/A) = \frac{P(H_1)P(A/H_1)}{P(A)} = \frac{P(H_1)}{P(A)} = \frac{0.036}{0.188} \approx 0.19\,.$$

Thus with the probability 0.19 namely the first shot has hit in the target. By the same way we can find the other probabilities $P(H_2/A), P(H_3/A)$.

Ex. 14. There are 5 balls in the urn. One has taken at random 3 balls, and he has seen that they are white. Find the probability that the urn contains only white balls provided that all suppositions about number of white balls are equiprobable.

Let A be an event "to take 3 white balls from the urn". Let's introduce the next hypotheses $H_i, i = \overline{0, 5}$, which mean that the urn contains i white balls.

Our aim is to find the probability $P(H_5/A)$.

By conditions of the problem the probabilities of the hypotheses equal

$$P(H_0) = P(H_1) = P(H_2) = P(H_3) = P(H_4) = P(H_4) = P(H_5) = 1/6.$$

Corresponding conditional probabilities of the event A are equal to

$$P(A/H_0) = P(A/H_1) = P(A/H_2) = 0,\ P(A/H_3) = 1/C_5^3 = \frac{1\cdot 2\cdot 3}{5\cdot 4\cdot 3} = 0.1,$$

$$P(A/H_4) = C_4^3/C_5^3 = 0.4,\ P(A/H_5) = 1.$$

By virtue of the formula of total probability (4) and Beyes formulas (5) we obtain the probability to take three white balls from the urn,

$$P(A) = \sum_{i=3}^{5} P(H_i)P(A/H_i) = 1/6\cdot 0.1 + 1/6\cdot 0.4 + 1/6\cdot 1 = 0.25,$$

and the probability of the hypothesis that the urn contains only white balls

$$P(H_5/A) = \frac{P(H_5)P(A/H_5)}{P(A)} = \frac{1/6\cdot 1}{0.25} = 0.(6) \approx 0.67.$$

Thus, with the probability $0.(6) \approx 0.67$ the urn contains only white balls (in condition that three white balls were at random taken from there).

EVENT AND PROBABILITY: basic terminology RUEFD

1. аксиома	аксіома	áxiom	axiome *m*	Axióm (*n*–s,-e)
2. апостериорная вероятность	апостеріорна ймовірність	a postèrióri [postérior, after-the-evént, final] pròbabílity	probabilité *f* a posteriori	Wahrschéinlichkeit (*f* =) a posterióri
3. априорная вероятность	апріорна ймовірність	a prióry [príor, prémature, exístence] pròbabílity	probabilité *f* a priori	Wahrschéinlichkeit (*f* =) a prióri
4. безусловная [абсолютная] вероятность	безумовна [абсолютна] ймовірність	úncondítional [ábsolute] pròbabílity	probabilité *f* inconditionnelle [absolue]	únbedingte [absolúte] Wahrschéinlichkeit (*f* =)
5. благоприятствующее (следствие), благоприятствующий (результат, шанс)	сприятливий [наслідок, результат, шанс]	fávourable [occúrrence, óutcome, cáse, chance]; succéss	cas *m* [résultat *m*, issue *f* , chance *f*] favorable; succès *m*	günstiger Versúchsausgang (*m*-(e)s,.gänge) [Áusgang, Chance [´ʃaŋsə *u* ´ʃã:s(ə)] (*f* =,-n)]
6. бросание [подбрасывание] монеты	кидання [підкидання] монети	cointossing, toss [tossing, flípping] of a coin	lancer *m* [lancement *m*] d´une pièce	Wérfen (*n* –s) [Hóchwerfen (*n* –s)] einer Münze *f*
7. бросание [подбрасывание] игральной кости	кидання [підкидання] гральної кістки	throw [throwing] of a die [rólling of a fáirdie]	lancer *m* [lancement *m*] d'un dé	Wérfen (*n* –s) eines Würfels
8. бросать [подбрасывать] игральную кость	кидати [підкидати] гральну кістку	throw a die	lancer [jeter] un dé	würfeln
9. бросать [подбрасывать] монету	кидати [підкидати] монету	toss [throw, flip] a coin	lancer [jeter] une pièce	Eine Münze (*f*) wérfen* (wérfe, wirfst, wirft)
10.вероятно-	імовірність	pròbabílity	probabilité *f*	Wahrschéin-

сть	[ймовірність]			lichkeit (*f* =)
11.вероятность гипотезы	імовірність гіпотези	pròbabílity of the hypóthesis	probabilité *f* de l'hypothèse	Wahrschéinlichkeit einer Hypothése
12.вероятность гипотезы (если известен результат испытания)	імовірність гіпотези (якщо відомий результат випробування)	pròbabílity of a hypóthesis (if the resúlt of a tríal [of an expériment] is knówn)	probabilité *f* d'une hypothèse (si le résultat d'une expérience/épreuve [d'un essai] est connu(e))	Wahrschéinlichkeit der Hypothése (wenn den Versúchsausgang [das Versúchsergebnis] bestimmt ist)
13.вероятность наступления события	імовірність [ймовірність] настання події	pròbabílity of occúrrence of an evént	probabilité *f* d'apparition [d'arrivée *f*] d'un événement	Wahrschéinlichkeit von Éintreten(e) [von Éintritt(e)] eines Eréignisses
14.вероятность наступления хотя бы одного события	імовірність [ймовірність] настання хоча б однієї події	pròbabílity of occúrrence at least of one event	probabilité *f* d'apparition *f* [d'arrivée *f*] d'au [du] moins seul/un événement	Wahrschéinlichkeit (*f* =) von Éintreten(e)[von Éintritt(e)] míndestens des eines Eréignisses
15.вероятность события	імовірність [ймовірність] події	pròbabílity of an evént	probabilité *f* d´un événement	Eréigniswahrscheinlichkeit (*f* =); Wahrschéinlichkeit von Eréignisse
16.герб [орел]	герб [орел]	head	tête *f*	Kopf (*m* –(e)s, Köpfe)
17.гипотеза	гіпотеза	hypóthesis (*pl* -ses)	hypothèse *f*	Hypothése (*f* =, -n)
18.действие	дія	áction	action *f*	Aktión (*f*)
19.достоверное событие	вірогідна [достовірна] подія	cértain [persístent,sure] evént	evénement *m* certain	sícheres Eréignis
20.единственно возможные	єдино можливі парами не-	only póssible páirwise dis-	evénements uniquement	éinzig mögliche páarweise

попарно несовместные и равновозможные события	сумісні й рівноможливі події	jóint [ìncompátible] and équally [líkely] póssible evénts	possibles incompatibles deux à deux et également possibles	unveréinbare [únvereinbare, únverträgliche, unverträgliche] und gléichmöglichen Eréignisse
21.единственно возможные события	єдино можливі події	only póssible evénts	evénements *m* uniquement possibles	éinzig mögliche Eréignisse
22.зависимые события	залежні події	depéndent evénts	evénements *m* dépendants	abhängige Eréignisse
23.закон [формула] произведения вероятностей	закон [формула] добутку імовірностей	próduct law of probabílity	loi *f* de composition *f* des probabilités	Multiplikationsgesetz (*n*) der Wahrscheinlichkeiten
24.закон [формула] сложения вероятностей	закон [формула] додавання ймовірностей	law of addítion of probabílity	loi *f* d'addition *f* des probabilités	Additióngesetz (*n*) der Wahrscheinlichkeiten
25.игральная кость	гральна кістка	die *pl* dice [fáirdie]	dé *m* (à jouers)	Würfel (*m*–s,=)
26.испытание [эксперимент, опыт]	випробування	tríal [expériment]	essai *m* [épreuve *f*, expérience *f*]	Versúch (*m* –(e)s,-e) [Experimént (*n* –(e)s, -e); Próbe (*f* =,-n)]
27.классическая вероятность	класична ймовірність	clássic(al) pròbabílity	probabilité *f* classique	clássische Wahrschéinlichkeit
28.классическое определение вероятности	класичне означення ймовірності	clássic dèfinítion of pròbabílity	définition *f* classique de probabilité	klássische Definitión (*f*) von Wahrschéinlichkeit
29.количество всех перестановок из *n* элементов	кількість всіх перестановок з *n* елементів	númber of all permutations of *n* élements	nombre *m* de toutes les permutations de *n* éléments	Die Ánzahl (*f*) [die Gesámt‖zahl *f*] aller Permutatiónen von *n* Elemén-

				ten
30.количество всех размещений из *n* элементов по *k*	кількість всіх розміщень з *n* елементів по *k*	númber of all arrangements of *n* élements (taken) *k* at a time [of all arrángements of *k* (élements) out of *n* , of all *k*-fold arrángements of *n* élements]	nombre *m* de tous les arrangements de *n* éléments *k* à *k*	Die Ánzahl (*f*) [die Gesámt-‖zahl *f*] aller Variatiónen von *n* Elementen zur *k*-ten Klasse [von *n* Eleménten zu je *k*]
31.количество всех сочетаний из *n* элементов по *k*	кількість всіх комбінацій з *n* елементів по *k*	númber of all còmbinátions of *n* élements (taken) *k* at a time [of all còmbinátions of *k* (élements) out of *n*, of all *k*-fold còmbinátions of *n* élements]	nombre *m* de toutes les combinaisons de *n* éléments *k* à *k*	Die Ánzahl (*f*) [die Gesámt-‖zahl *f*] aller Kombinatiónen von *n* Eleménten zur *k*-ten Klásse [von *n* Eleménten zu je *k*]
32.количество наступлений [количество появлений] события в *n* независимых испытаниях	кількість настання [кількість появ] події в *n* незалежних випробуваннях	númber of occúrrences of an evént in *n* índepéndent tríals [expériments]	nombre *m* d'arrivées/ d'apparitions d'un événement à *n* expériences/épreuves indépendantes [à *n* d'essais indépendants]	Ánzahl (*f*) von Éintritten) [von Éintretenen] eines Eréignisses in *n* únabhängigen Versúchen [Experiménten (*n*), Próben (*f*)]
33.количество независимых испытаний	кількість незалежних випробувань	númber of índepéndent tríals [expériments]	nombre *m* d'expériences/ d'épreuves indépendantes [d'essais indépendants]	Ánzahl (*f* =) únabhängiger Versúche [Experiménte, Próben]
34.количество способов (сделать *что-*	кількість способів (зробити *щось*)	númber of ways [modes, méthods] (to do	nombre *m* de façons [modes, méthodes] de	Ánzahl (*f* =) von Árten [von Wéisen] (*etwas*

л.)		*smth*)	(faire *qch*)	máchen)
35.комбинаторика	комбінаторика	còmbinatórial [combìnatórial] análysis, còmbinatórics, combìnatórics	théorie *f* des combinaisons, analyse *f* combinatiore	Kombinatórik (*f* =) [Kombinatiónslehre (*f* =,-n)]
36.монета	монета	coin	pièce *f*	Münze (*f*)
37.наступление [появление] хотя бы одного события	настання [поява] хоча б однієї події	occúrrence of (at least of one) event	arrivée *f* [apparition *f*] d'au [du] moins seul /un evénement *m*	Éintritt (*m* – (e)s,-e) [Éintreten (*n*)] míndestens des eines Eréignisses
38.наступление события	настання події	occúrrence of an evént	arrivée *f* [apparition *f*] d'un événement	Éintritt (*m* – (e)s,-e) [Éintreten (*n*)] eines Eréignisses
39.наудачу [наугад]	навмання	at rándom	au *hasard [à tout *hasard, à l'aventure]	belíebig, zúfällig, zúfälligerweise
40.невозможное событие	неможлива подія	impóssible evént	evénement [évé-] *m* impossible	únmögliches [unmögliches] Eréignis
41.независимые в совокупности события	незалежні в сукупності події	indepéndent set of evénts	evénements indépendants dans leur ensemble	insgesámt [vollständig] únabhängige Eréignisse
42.независимые испытания	незалежні випробування	índepéndent tríals [expériments]	expériences *f* [épreuves *f*] indépendantes; essais *m* indépendants	únabhängige Versúche [Experiménte, Próben]
43.независимые события	незалежні події	indepéndent evénts	evénements *m* indépendants	únabhängige Eréignisse
44.несовместные события	несумісні події	disjóint [ìncompátible, mútually exclúsive] evénts	evénements *m* incompatibles	unveréinbare [únvereinbare, únverträgliche, unverträgliche] Eréignisse
45.образовы-	утворювати	form the com-	constituer/en-	Gesámtheit (*f*)

вать полную группу	повну групу подій	pléte group/set of evénts [form the full evént group/set, be the exháustive evénts]	gendrer un groupe complet	der Elementár-ereignisse erzéugen [(sich) bílden]
46.основной принцип комбинаторики	основний принцип комбінаторики	básic [fùndaméntal, main, básis] prínciple of combìnatórics [of còmbinatórial/combìnatórial análysis]	principe *m* fondamental de théorie *f* des combinaisons [d'analyse *f* combinatoire]	fundamentáles [grúndlegendes] Kombinatiónsprinzip (*n* -s,-e *u* -pi¦en) [Prinzíp von Kombinatórik]
47.относительная частота события	відносна частота події	rélative fréquency of an event	fréquence *f* relative d'un événement	relatíve Frequénz (*f* =,-en) [Häufigkeit (*f* =)] eines Eréignisses
48.перестановка из *n* элементов	перестановка з *n* елементів	pèrmutátion of *n* élements	permutation *f* de *n* éléments	Permutatión (*f*) von *n* Elemén-ten
49.полная вероятность	повна ймовірність	tótal [cómposite, áverage] pròbabílity	probabilité *f* complète [totale]	totále [vóllständige] Wahrschéinlichkeit
50.полная группа попарно несовместных равновозможных событий (об элементарных событиях, о шансах)	повна група парами несумісних рівноможливих подій (про елементарні події, про шанси)	compléte/tótal group of pairwise disjóint [ìncompátible] and equally [líkely] póssible evénts (of èleméntary events, of chances)	groupe *m* complet d'événements incompatibles deux à deux et également possibles (sur des evénements élémentaires, sur les chances)	Gesámtheit (*f* =) von páarweise unveréinbaren [únvereinbaren, únverträglichen, unverträglichen] und gléichmöglichen Eréignissen (über Elementárereignisse, über Chancen [ˊʃaŋsən, ˊʃã:s(ə)n])

51.полная группа событий	повна група подій	exháustive evénts [total [compléte] group of evénts]	groupe *m* complet d'événements	Gesámtheit (*f* =) der Elementárereignisse [von Elementárereignissen]
52.попарно независимые события	парами незалежні події	páirwise [in pairs, mútually] indepéndent evénts	evénements *m* indépendants deux à deux	páarweise únabhängige Eréignisse
53.попарно несовместные события	парами несумісні події	mútually [páirwise] disjóint [exclúsive, ìncompátible] evénts	evénements incompatibles deux à deux	páarweise unveréinbare [únvereinbare, únverträgliche, unverträgliche] Eréignisse
54.произведение (пересечение) событий	добуток (переріз) подій	próduct [còmposítion (ìnterséction)] of events	produit *m* (intersection *f*) d'événements	Prodúkt (*n*–e)s, -e) [Dúrchschnitt(*m*–(e)s,-e)] von Eréignissen; Dúrchschnittseréignis (*n* –ses,-se)
55.происходить (о событии)	відбуватися [наставати, з"являтися] (про подію)	occúr [appéar, happen, take place] (about an evént)	arriver [se présenter] (sur un événement)	Éintreten*(-trete,-trittst, -tritt) [éintreffen*(-treffe,-triffst,-trifft)] (über ein Eréignis)
56.противоположные события	протилежні події	ópposite [còmpleméntary, contrary] evénts	evénements *m* contraires	entgégengesetzte [komplementäre] Eréignisse
57.равновероятные [одинаково вероятные] события	рівноймовірні [однаково ймовірні] події	èquipróbable evénts	evénements *m*, équiprobables [ekŋi]	gléichwahrscheinliche Eréignisse
58.равновозможные [одинаково возможные]события	рівноможливі [однаково можливі] події	équally [líkely] póssible evénts	evénements également possibles	gléichmögliche Eréignisse
59.размеще-	розміщення з	arrángement of	arrangement *m*	Variatión [va-]

ние из *n* элементов] по *k*	*n* елементів по *k*	*n* élements (taken) *k* at a time, arrángement of *k* (éleménts) out of *n* [*k*-fold arrángement of *n* élements]	de *n* éléments *k* à *k*	(*f* =,-en) von *n* Eleménten zur *k*-ten Klásse [von *n* Eleménten zu je *k*]
60.результат [следствие] испытания	результат [наслідок] випробування	resúlt [occúrrence, óutcome] of a tríal [of an expériment]	résultat *m* [cas *m*, issue *f*] d'expérience [d'essai, d'épreuve]	Prüfergebnis (*n* -ses,-se) [Versúchsergebnis, Versúchsausgang (*m* –(e)s, ..gänge)]; Áusgang [Ergébnis, Resultát *n* (–(e)s,-e)] von Versúch(e) [von Experimént(e), von Próbe]
61.сделать *что-то n* способами	зробити *щось n* способами	do *smth* in/by *n* ways [méthods, modes]	faire *qch* de *n* façons [modes, méthodes]	*Etwas* auf (A) [mit (D)] *n* Árten [Wéisen] máchen
62.серия из *n* независимых испытаний	серія (з *n* незалежних) випробувань	séries of *n* índepéndent tríals [expériments]	série *f* de *n* expériences indépendantes	Réihe *f* [Séri¦e] von *n* únabhängigen Versúchen [Experiménten, Próben]
63.следствие [результат] испытания	наслідок [результат] (випробування)	resúlt [occúrrence, óutcome] (of a tríal)	résultat *m* [cas *m*, issue *f*] (d'essai/d'expérience)	Versúchsausgang (*m* –(e)s, ..gänge) [Ausgang (von Versúch(e),von Experimént(e), von Próbe]
64.случайное событие	випадкова подія	rándom [accidental, áleatory, casual, chance, stochas-	evénement *m* aléatoire	Zúfallsereignis (*n*–ses,-se), Zúfälliges Eréignis

		tic] evént; accident		
65.событие	подія	evént [act, occúrrence]	evénement [évé-] *m*	Eréignis (*n* – ses,-se)
66.совместные события	сумісні події	jóint [compátible] evénts	evénements compatibles	veréinbare [verträgliche, kompatíbele] Eréignisse
67.совпадение классической и статистической вероятностей	збіг класичної і статистичної ймовірностей	coíncidence of clássic and statístic(al) pròbabílities	coïncidence *f* de probabilités classique et statistique	Überéinstimmung klássischer und statístischer Wahrschéinlichkeiten
68.сочетание из *n* элементов по *k*	комбінація з *n* елементів по *k*	còmbinátion of *n* élements (taken) *k* at a time, còmbinátion of *k* (éléments) out of *n* [*k*-fold còmbinátion of *n* éléments]	combinaison *f* de *n* éléments *k* à *k*	Kombinatión (*f*) von *n* Eleménten zur *k*-ten Klásse [von *n* Eleménten zu je *k*]
69.статистическая вероятность	статистична ймовірність	statístic(al) pròbabílity	probabilité *f* statistique	statístische Wahrschéinlichkeit
70.статистическая устойчивость относительных частот	статистична стійкість відносних частот	statístic(al) stabílity of rélative fréquencies	stabilité statistique des fréquences relatives	statístische Stabilität relativer Frequenzen [Häufigkeiten]
71.статистическое определение вероятности	статистичне означення ймовірності	statístic(al) dèfinítion of pròbabílity	définition *f* statistique de probabilité	statístische Definitión von Wahrschéinlichkeit
72.сумма (объединение) событий	сума (об"єднання) подій	sum (únion) of evénts	somme *f* ((ré)-union *f*) d'événements	Súmme (*f*=,-en) [Veréinigung (*f* =,-en), Zusámmenfassung(*f* =,-en)] von Eréignis-

				sen; Súmmen-eréignis
73.сумма произведений [объединение пересечений] событий	сума добутків (об"єднання перерізів) подій	sum of próducts (únion of ìntersections) of evénts	somme *f* de produits ((ré)-union d'intersections) d'événements	Súmme (*f*) von Prodúkten (Veréinigung (*f* =,-en), [Zusámmenfassung (*f*)] von Dúrchschnitten) von Eréignissen
74.теория вероятностей	теорія ймовірностей	pròbabílity théory	théorie *f* [calcul *m*] des probabilités	Wahrschéinlichkeitslehre *f*, Wahrschéinlichkeitsrechnung *f*, Wahrschéinlichkeitstheorie *f*
75.условная вероятность	умовна ймовірність	condítional pròbabílity	probabilité *f* conditionnelle [conditionnée, liée]	bedíngte Wahrschéinlichkeit
76.условная относительная частота события	умовна відносна частота події	condítional rélative fréquency of an evént	condítional rélative fréquency of an event	bedíngte relatíve Frequénz (*f* =,-en) [~~ Häufigkeit *f*] eines Eréignisses
77.формула Бейеса	формула Бейеса	Beyes fórmula (*pl* fórmulae/ fórmulas)	formule *f* de Beyes	Fórmel (*f* =,-n) von Bayes [Bayessche Fórmel]
78.формула вероятностей гипотез	формула ймовірностей гіпотез	fórmula of pròbabílities of hypótheses	formule *f* de probabilités des hypothèses	Fórmel (*f* =,-n) über die Wahrschéinlichkeit von Hypothésen
79.формула полной вероятности	формула повної ймовірності	fórmula of the tótal [cómposite, áverage] pròbabílity	formule *f* de la probabilité complète/totale	Fórmel (*f* =,-n) von totáler [vóllständiger] Wahrschéinlichkeit

80.цифра [решка]	цифра [решка]	tail	pile *f*	Zahl (*f* =,-en)
81.частота события	частота події	fréquency of an evént	fréquence *f* d'un événement	Frequénz (*f* =,-en) [Häufigkeit (*f* =)] eines Eréignisses
82.шанс	шанс	chance	chance *f*	Chance [´ʃaŋsə и ´ʃã:s(ə)] (*f* =, -*n*)
83.элементарное событие	елементарна подія	èleméntary [simple] evént	evénement (é-) *m* élémentaire [simple]	Elementár-ereignis (*n* – ses,-se)
84.элементарный исход [результат] (испытания)	елементарний наслідок [результат] (випробування)	èleméntary [simple] resúlt [occúrrence, óutcome] (of a tríal)	résultat *m* [cas *m*, issue *f*] élémentaire/simple (d'essai/ d'expérience)	elementárer Áusgang (*m* – (e)s,..gänge) [Versúchsausgang (*m*)]

EVENT AND PROBABILITY: basic terminology EFDRU

1. a prióry [príor, prémature, exístence] pròbabílity	probabilité *f* a priori	Wahrschéinlichkeit (*f* =) a prióri	априорная вероятность	апріорна ймовірність
2. a postèrióri [postérior, after-the-evént, final] pròbabílity	probabilité *f* a posteriori	Wahrschéinlichkeit (*f* =) a posterióri	апостериорная вероятность	апостеріорна ймовірність
3. áction	action *f*	Aktión (*f*)	действие	дія
4. arrángment of *n* élements (taken) *k* at a time, arrángement of *k* (éléments) out of *n* [*k*-fold arrángement of *n* élements]	arrangement *m* de *n* éléments *k* à *k*	Variatión [va-] (*f* =,-en) von *n* Eleménten zur *k*-ten Klásse [von *n* Elementen zu je *k*]	размещение из *n* элементов] по *k*	розміщення з *n* елементів по *k*
5. at rándom	au *hasard [à tout *hasard, à l'aventure]	belíebig, zúfällig, zúfälligerweise	наудачу [наугад]	навмання
6. áxiom	axiome *m*	Axióm (*n*–s,-e)	аксиома	аксіома
7. básic [fùndaméntal,main, básis] príncіple of combìnatórics [of còmbinatórial/ combìnatórial análysis]	principe *m* fondamental de théorie *f* des combinaisons [d'analyse *f* combinatoire]	fundamentáles [grúndlegendes] Kombinatiónsprinzip (*n* -s,-e *u* -pi¦en) [Prinzíp von Kombinatórik]	основной принцип комбинаторики	основний принцип комбінаторики
8. Beyes fórmula (*pl* fórmulae/fórmulas)	formule *f* de Beyes	Fórmel (*f* =,-n) von Bayes [Bayessche Fórmel]	формула Бейеса	формула Бейеса
9. cértain [persístent, sure] evént	evénement *m* certain	sícheres Eréignis (*n*–ses, -se)	достоверное событие	вірогідна [достовірна] подія

10.chance	chance *f*	Chance [ˊʃaŋsə и ˊʃã:s(ə)] (*f* =,-*n*)	шанс	шанс
11.clássic dèfinítion of pròbabílity	définition *f* classique de probabilité	klássische Definitión von Wahrschéinlichkeit	классическое определение вероятности	класичне означення ймовірності
12.clássic(al) pròbabílity	probabilité *f* classique	clássische Wahrschéinlichkeit	классическая вероятность	класична ймовірність
13.coin	pièce *f*	Münze (*f*)	монета	монета
14.coíncidence of clássic and statístic(al) pròbabílities	coïncidence *f* de probabilités classique et statistique	Überéinstimmung klássischer und statístischer Wahrschéinlichkeiten	совпадение классической и статистической вероятностей	збіг класичної і статистичної ймовірностей
15.cointossing, toss [tossing, flípping] of a coin	lancer *m* [lancement *m*] dˊune pièce	Wérfen (*n* –s) [Hóchwerfen (*n* –s)] einer Münze	бросание [подбрасывание] монеты	кидання [підкидання] монети
16.còmbinátion of *n* élements (taken) *k* at a time, còmbinátion of *k* (élements) out of *n* [*k*-fold còmbinátion of *n* élements]	combinaison *f* de *n* éléments *k* à *k*	Kombinatión (*f*) von *n* Eleménten zur *k*-ten Klásse [von *n* Eleménten zu je *k*]	сочетание из *n* элементов по *k*	комбінація з *n* елементів по *k*
17.còmbinatórial [combìnatórial] análysis, còmbinatórics, combìnatórics	théorie *f* des combinaisons, analyse *f* combinatiore	Kombinatórik (*f* =) [Kombinatiónslehre (*f* =,-n)]	комбинаторика	комбінаторика
18.compléte/ tótal group of pairwise disjóint [ìncompátible] and equ-	groupe *m* complet d'événements incompatibles deux à deux et	Gesámtheit (*f* =) von páarweise unvereinbaren [únvereinbaren,	полная группа попарно несовместных равновозможных событий (об	повна група парами несумісних рівноможливих подій (про еле-

ally [líkely] póssible evénts (of èleméntary events, of chances)	également possibles (sur des evénements élémentaires, sur les chances)	únverträglichen, unverträglichen] und gléichmöglichen Eréignissen (über Elementárereignisse, über Chancen [ˊʃaŋsən, ˊʃã:s(ə)n])	элементарных событиях, о шансах)	ментарні події, про шанси)
19.condítional pròbabílity	probabilité *f* conditionnelle [conditionnée, liée]	bedíngte Wahrschéinlichkeit	условная вероятность	умовна ймовірність
20.condítional rélative fréquency of an evént	condítional rélative fréquency of an event	bedíngte relatíve Frequénz (*f* =,-en) [~~ Häufigkeit *f*] eines Eréignisses	условная относительная частота события	умовна відносна частота події
21.die *pl* dice [fáirdie]	dé *m* (à jouers)	Würfel (*m*–s,=)	игральная кость	гральна кістка
22.disjóint [ìncompátible, mútually exclúsive] evénts	evénements *m* incompatibles	unveréinbare [únvereinbare, únverträgliche, unverträgliche] Eréignisse	несовместные события	несумісні події
23.do *smth* in/by *n* ways [méthods, modes]	faire *qch* de *n* façons [modes, méthodes]	*Etwas* auf (A) [mit (D)] *n* Árten [Wéisen] máchen	сделать *что-то n* способами	зробити *щось n* способами
24.èleméntary [simple] evént	evénement *m* élémentaire [simple]	Elementárereignis (*n* – ses,-se)	элементарное событие	елементарна подія
25.èleméntary [simple] resúlt [occúrrence, óutcome] (of a tríal)	résultat *m* [cas *m*, issue *f*] élémentaire/simple (d'essai/ d'expérience)	elementárer Áusgang (*m* – (e)s,..gänge) [Versúchsausgang (*m*)]	элементарный исход [результат] (испытания)	елементарний наслідок [результат] (випробування)
26.équally [lí-	evénements	gléichmögli-	равновозмож-	рівноможливі

kely] póssible events	également possibles	che Eréignisse	ные [одинаково возможные]события	[однаково можливі] події
27.èquipróbable evénts	evénements *m*, équiprobables [ekŋi]	gléichwahrscheinliche Eréignisse	равновероятные [одинаково вероятные] события	рівноймовірні [однаково ймовірні] події
28.evént [act, occúrrence]	evénement [évé-] *m*	Eréignis (*n* – ses,-se)	событие	подія
29.exháustive evénts [total [compléte] set/ group of evénts]	groupe *m* complet d'événements	Gesámtheit (*f* =) der Elementárereignisse [von Elementárereignissen]	полная группа событий	повна група подій
30.fávourable [occúrrence, óutcome, cáse, chance]; succéss	cas *m* [résultat *m*, issue *f* , chance *f*] favorable; succès *m*	günstiger Versúchsausgang (*m*) [Áusgang (*m*-(e)s,.gänge, Chance [´ʃaŋsə *u* ´ʃã:s(ə)] (*f* =,-n)]	благоприятствующее (следствие), благоприятствующий (результат, шанс)	сприятливий [наслідок, результат, шанс]
31.form the compléte set /group of evénts [form the full evént group/set, be the exháustive evénts]	constituer/engendrer un groupe complet	Gesámtheit (*f*) der Elementárereignisse erzéugen [(sich) bílden]	образовывать полную группу	утворювати повну групу подій
32.fórmula of pròbabílities of hypótheses	formule *f* de probabilités des hypothèses	Fórmel (*f* =,-n) über die Wahrschéinlichkeit von Hypothésen	формула вероятностей гипотез	формула ймовірностей гіпотез
33.fórmula of the tótal [cómposite, áverage] pròbabílity	formule *f* de la probabilité complète/totale	Fórmel (*f* =,-n) von totáler [vóllständiger] Wahrschéinlichkeit	формула полной вероятности	формула повної ймовірності
34.fréquency	fréquence *f*	Frequénz (*f* =,-	частота собы-	частота події

of an evént	d’un événement	en) [Häufigkeit (*f* =)] eines Eréignisses	тия	
35.head	tête *f*	Kopf (*m* –(e)s, Köpfe)	герб [орел]	герб [орел]
36.hypóthesis (*pl* -ses)	hypothèse *f*	Hypothése (*f* =, -n)	гипотеза	гіпотеза
37.impóssible evént	evénement *m* impossible	únmögliches [unmögliches] Eréignis	невозможное событие	неможлива подія
38.depéndent evénts	evénements *m* dépendants	abhängige Eréignisse	зависимые события	залежні події
39.indepéndent evénts	evénements *m* indépendants	únabhängige Eréignisse	независимые события	незалежні події
40.indepéndent set of evénts	evénements indépendants dans leur ensemble	Insgesámt [vollständig] únabhängige Eréignisse	независимые в совокупности события	незалежні в сукупності події
41.índepéndent tríals [expériments]	expériences *f* [épreuves *f*] indépendantes; essais *m* indépendants	únabhängige Versúche [Experiménte, Próben]	независимые испытания	незалежні випробування
42.jóint [compátible] evénts	evénements compatibles	veréinbare [verträgliche, kompatíbele] Eréignisse	совместные события	сумісні події
43.law of addítion of probabílity	loi *f* d'addition des probabilités	Additióngesetz (*n*) der Wahrscheinlichkeiten	закон [формула] сложения вероятностей	закон [формула] додавання ймовірностей
44.mútually [páirwise] disjóint [exclúsive, ìncompátible] evénts	evénements incompatibles deux à deux	páarweise unveréinbare [únvereinbare, únverträgliche, unverträgliche] Eréignisse	попарно несовместные события	парами несумісні події
45.númber of all arrangements of *n* éle-	nombre *m* de tous les arrangements de *n*	Die Ánzahl (*f*) [die Gesámt‖zahl *f*] aller	количество всех размещений из *n*	кількість всіх розміщень з *n*

ments (taken) *k* at a time [of all arrángements of *k* (élements) out of *n* , of all *k*-fold arránge-ments of *n* éle-ments]	éléments *k* à *k*	Variatiónen von *n* Elemen-ten zur *k*-ten Klasse [von *n* Eleménten zu je *k*]	елементов по *k*	елементів по *k*
46.númber of all còmbiná-tions of *n* éle-ments (taken) *k* at a time [of all còmbinátions of *k*(élements) out of *n*, of all *k*-fold còmbi-nátions of *n* élements]	nombre *m* de toutes les com-binaisons de *n* éléments *k* à *k*	Die Ánzahl (*f*) [die Gesámt-‖zahl *f*] aller Kombinatiónen von *n* Elemén-ten zur *k*-ten Klásse [von *n* Eleménten zu je *k*]	количество всех сочета-ний из *n* эле-ментов по *k*	кількість всіх комбінацій з *n* елементів по *k*
47.númber of all permuta-tions of *n* éle-ments	nombre *m* de toutes les per-mutations de *n* éléments	Die Ánzahl (*f*) [die Gesámt-‖zahl *f*] aller Permutatiónen von *n* Elemén-ten	количество всех переста-новок из *n* элементов	кількість всіх перестановок з *n* елементів
48.númber of índepéndent tríals [expéri-ments]	nombre *m* d'expériences /d'épreuves in-dépendantes [d'essais indé-pendants]	Ánzahl (*f* =) únabhängiger Versúche [Ex-periménte, Pró-ben]	количество независимых испытаний	кількість не-залежних ви-пробувань
49.númber of occúrrences of an evént in *n* índepéndent tríals [expéri-ments]	nombre *m* d'ar-rivées/d'appa-ritions d'un événement à *n* ex-périences/ épreuves indé-pendantes [à *n* d'essais indé-pendants]	Ánzahl (*f*) von Éintritten [von Éintretenen] eines Eréignis-ses in *n* únab-hängigen Ver-súchen [Expe-riménten, Pró-ben]	количество наступлений [количество появлений] события в *n* независимых испытаниях	кількість на-стання [кіль-кість появ] події в *n* не-залежних ви-пробуваннях
50.númber of ways [modes,	nombre *m* de façons [modes,	Ánzahl (*f* =) von Árten [von	количество способов (сде-	кількість спо-собів (зробити

méthods] (to do *smth*)	méthodes] de (faire *qch*)	Wéisen] (*etwas* máchen)	лать *что-л.*)	*щось*)
51.occúr [ap-péar, happen, take place] (about an evént)	arriver [se pré-senter] (sur un événement)	éintreten*(-tre-te,-trittst, -tritt) [éintreffen*(-treffe,-triffst,-trifft)] (über ein Eréignis)	происходить (о событии)	відбуватися [наставати, з"являтися] (про подію)
52.occúrrence of (at least of one) event	arrivée *f* [ap-parition *f*] d'au [du] moins seul/un evéne-ment *m*	Éintritt (*m* – (e)s,-e) [Éin-treten (*n*)] mín-destens des ei-nes Eréignisses	наступление [появление] хотя бы одно-го события	настання [по-ява] хоча б однієї події
53.occúrrence of an evént	arrivée *f* [ap-parition *f*] d'un événe-ment	Éintritt (*m* – (e)s,-e) [Éin-treten (*n*)] ei-nes Eréignisses	наступление события	настання по-дії
54.only póssi-ble evénts	evénements *m* uniquement possibles	éinzig mögli-che Eréignisse	единственно возможные события	єдино можли-ві події
55.only póssi-ble páirwise disjóint [ìn-compátible] and équally [lí-kely] póssible evénts	evénements uniquement possibles in-compatibles deux à deux et également pos-sibles	éinzig mögli-che páarweise unveréinbare [únvereinbare, únverträgliche, unverträgliche] und gléichmö-glichen Eréig-nisse	единственно возможные попарно несо-вместные и равновозмож-ные события	єдино можли-ві парами не-сумісні й рів-griffith номожливі події
56.ópposite [còmplemén-tary, contrary] evénts	evénements *m* contraires	entgégenge-setzte [komple-mentäre] Eréi-gnisse	противопо-ложные собы-тия	протилежні події
57.páirwise [in pairs, mú-tually] inde-péndent evénts	evénements *m* indépendants deux à deux	páarweise ún-abhängige Eréignisse	попарно неза-висимые со-бытия	парами неза-лежні події
58.pèrmutátion of *n* élements	permutation *f* de *n* éléments	Permutatión (*f*) von *n* Elemén-ten	перестановка из *n* элемен-тов	перестановка з *n* елементів

59.pròbabílity	probabilité *f*	Wahrschéinlichkeit (*f* =)	вероятность	імовірність [ймовірність]
60.pròbabílity of a hypóthesis (if the resúlt of a tríal [of an expériment] is knówn)	probabilité *f* d'une hypothèse (si le résultat d'une expérience/épreuve [d'un essai] est connu(e))	Wahrschéinlichkeit der Hypothése (wenn den Versúchsausgang [das Versúchsergebnis] bestimmt ist)	вероятность гипотезы (если известен результат испытания)	імовірність гіпотези (якщо відомий результат випробування)
61.pròbabílity of an evént	probabilité d´un événement	Eréigniswahrscheinlichkeit (*f* =); Wahrschéinlichkeit von Eréignisse	вероятность события	імовірність [ймовірність] події
62.pròbabílity of occúrrence at least of one event	probabilité *f* d'apparition *f* [d'arrivée *f*] d'au [du] moins seul/un événement	Wahrschéinlichkeit (*f* =) von Éintreten(e) [von Éintritt(e)] míndestens des eines Eréignisses	вероятность наступления хотя бы одного события	імовірність [ймовірність] настання хоча б однієї події
63.pròbabílity of occúrrence of an evént	probabilité *f* d'apparition [d'arrivée] d'un événement	Wahrschéinlichkeit von Éintreten(e) [von Éintritt(e)] eines Eréignisses	вероятность наступления события	імовірність [ймовірність] настання події
64.pròbabílity of the hypóthesis	probabilité *f* de l'hypothèse	Wahrschéinlichkeit einer Hypothése	вероятность гипотезы	імовірність гіпотези
65.pròbabílity théory	théorie *f* [calcul *m*] des probabilités	Wahrschéinlichkeitslehre *f*, Wahrschéinlichkeitsrechnung *f*, Wahrschéinlichkeitstheorie *f*	теория вероятностей	теорія ймовірностей
66.próduct	produit *m* (in-	Prodúkt (*n*–e)s,	произведение	добуток (пе-

[còmposítion (ìnterséction)] of events	tersection *f*) d'événements	-e) [Dúrchschnitt(*m*–(e)s,-e)] von Eréignissen; Dúrchschnittseréignis (*n* –ses,-se)	(пересечение) событий	реріз) подій
67.próduct law of probabílity	loi *f* de composition des probabilités	Multiplikationsgesetz (*n*) der Wahrscheinlichkeiten	закон [формула] произведения вероятностей	закон [формула] добутку імовірностей
68.rándom [accidental, áleatory, casual, chance, stochastic] evént; accident	evénement [évé-] *m* aléatoire	Zúfallsereignis (*n* –ses,-se), Zúfälliges Eréignis	случайное событие	випадкова подія
69.rélative fréquency of an event	fréquence *f* relative d'un événement	relatíve Frequénz (*f* =,-en) [Häufigkeit (*f* =)] eines Eréignisses	относительная частота события	відносна частота події
70.resúlt [occúrrence, óutcome] (of a tríal)	résultat *m* [cas *m*, issue *f*] (d'essai/d'expérience)	Versúchsausgang (*m* –(e)s, ..gänge) [Ausgang (*m*) (von Versúch(e),von Experimént(e), von Próbe]	следствие [результат] испытания	наслідок [результат] (випробування)
71.resúlt [occúrrence, óutcome] of a tríal [of an expériment]	résultat *m* [cas *m*, issue *f*] d'expérience *f* [d'essai *m*, d'épreuve *f*]	Prüfergebnis (*n* -ses,-se) [Versúchsergebnis, Versúchsausgang (*m* –(e)s, ..gänge)]; Áusgang [Ergébnis *n*, Resultát *n* (–(e)s,-e)] von Versúch(e) [von Experimént(e , von Próbe]	результат [следствие] испытания	результат [наслідок] випробування

72.séries of *n* índepéndent tríals [expériments]	série *f* de *n* expériences indépendantes	Réihe *f* [Séri¦e] von *n* únabhängigen Versúchen [Experiménten, Próben]	серия из *n* независимых испытаний	серія (з *n* незалежних) випробувань
73.statístic(al) dèfinítion of pròbabílity	définition *f* statistique de probabilité	statístische Definitión von Wahrschéinlichkeit	статистическое определение вероятности	статистичне означення ймовірності
74.statístic(al) pròbabílity	probabilité *f* statistique	statístische Wahrschéinlichkeit	статистическая вероятность	статистична ймовірність
75.statístic(al) stabílity of rélative fréquencies	stabilité statistique des fréquences relatives	statístische Stabilität relativer Frequenzen [Häufigkeiten]	статистическая устойчивость относительных частот	статистична стійкість відносних частот
76.sum (únion) of evénts	somme *f* ((ré)-union *f*) d'événements	Súmme (*f*=,-en) [Veréinigung (*f* =,-en), Zusámmenfassung(*f* =,-en)] von Eréignissen; Súmmeneréignis *n*–ses,-se	сумма (объединение) событий	сума (об"єднання) подій
77.sum of próducts (únion of ìntersések tions) of evénts	somme *f* de produits ((ré)-union d'intersections) d'événements	Súmme (*f*) von Produkten (Veréinigung (*f* =,-en), [Zusámmenfassung (*f*)] von Dúrchschnitten) von Ereignissen	сумма произведений [объединение пересечений] событий	сума добутків (об"єднання перерізів) подій
78.tail	pile *f*	Zahl (*f* =,-en)	цифра [решка]	цифра [решка]
79.throw [throwing] of a die [rólling of a fáirdie]	lancer *m* [lancement *m*] d'un dé	Wérfen (*n* –s) eines Würfels	бросание [подбрасывание] игральной кости	кидання [підкидання] гральної кістки

80.throw a die	lancer [jeter] un dé	Würfeln	бросать [подбрасывать] игральную кость	кидати [підкидати] гральну кістку
81.toss [throw, flip] a coin	lancer [jeter] une pièce	eine Münze (*f*) wérfen* (wérfe, wirfst, wirft)	бросать [подбрасывать] монету	кидати [підкидати] монету
82.tótal [cómposite, áverage] pròbabílity	probabilité *f* complète [totale]	totále [vóllständige] Wahrschéinlichkeit	полная вероятность	повна ймовірність
83.tríal [expériment]	essai *m* [épreuve *f*, expérience *f*]	Versúch (*m* – (e)s,-e) [Experimént (*n* – (e)s,-e); Próbe (*f* =,-n)]	испытание [эксперимент, опыт]	випробування
84.úncondítional [ábsolute] pròbabílity	probabilité *f* inconditionnelle [absolue]	únbedingte [absolúte] Wahrschéinlichkeit	безусловная [абсолютная] вероятность	безумовна [абсолютна] ймовірність

LECTURE NO. 3. RANDOM VARIABLES
LE TROUSIEME COURS. VARIABLES ALEATOIRES
DRITTE VORLESUNG. ZUFALLSGRÖßEN

POINT 1. A RANDOM VARIABLE. Variable aléatoire. Zufallsgröße.

POINT 2. BERNOULLI [BINOMIAL] DISTRIBUTION. Distribution de Bernoulli [distribution binomiale]. Bernoullische Verteilung [Binomialverteilung].

POINT 3. LAPLACE LOCAL AND INTEGRAL THEOREMS. Théorèmes local et intégral de Laplace. Laplaceschere lokaler und integraler Theoreme.

POINT 4. POISSON FORMULA AND DISTRIBUTION. Formule et distribution de Poisson. Poissonsche Formel und Verteilung.

POINT 1. A RANDOM VARIABLE

Def. 1. A random variable is called a variable which takes on some value in any trial and this value isn't known beforehand [in advance].

We'll denote random variables by letters $\xi, \eta, \zeta, \ldots$ and their possible values by $x, y, z, \ldots$.

We'll study discrete and continuous random variables.

Def. 2. A random variable is called **discrete** one if it can take on only separate isolated possible values (with some probabilities).

Ex.1. A number ξ of occurrences of an event A in one trial: $\xi = 1$ if A occurs, and $\xi = 0$, if A doesn't occur (that is an opposite event $\overline{A}$ occurs);

$P(\xi = 1) = P(A),\ P(\xi = 0) = P(\overline{A}) = 1 - P(A).$

Ex. 2. The number of students on the lecture.

Ex. 3. The daily production of some factory (in items).

Definition of a **continuous** random variable will be given below. Now we'll only say that its possible values completely fill some interval.

Ex. 4. The human height and weight.

Ex. 5. The size of an item.

Ex. 6. The error of measurement.

Def. 3. The distribution [the distribution law, the law of distribution, the law] of a random variable is called a rule which sets a correspondence between its possible values and corresponding probabilities.

The distribution law of a random variable can be expressed:

1) analytically by a formula (for example $P(\xi = k) = C_n^k p^k q^{n-k}$);

2) tabularly (by the **distribution table** for discrete random variables);

3) geometrically (by **distribution polygon** for discrete random variables, by graph of the distribution function or density (see below)).

The distribution table of a discrete random variable ξ (with finite number n of possible values) has the next form:

ξ	x_1	x_2	$\dots$	x_n
p_i	p_1	p_2	$\dots$	p_n

Its first row contains all possible values of the random variable, and the second row contains corresponding probabilities of these values. The notation $\xi = x_i$ means that the random variable ξ takes on a value x_i. Events

$$(\xi = x_1), (\xi = x_2), \dots, (\xi = x_n)$$

are pairwise disjoint and form a total group. Therefore, the sum of their probabilities $p_1 = P(\xi = x_1), p_2 = P(\xi = x_2), \dots, p_n = P(\xi = x_n)$ equals 1,

$$p_1 + p_2 + \dots + p_n = 1.$$

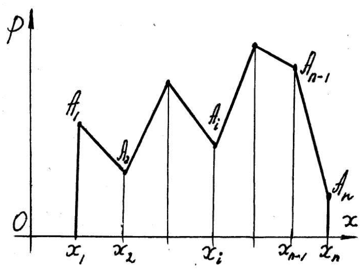

Fig. 1

The distribution polygon of a discrete random variable is a broken line [a polygonal line, an open polygon] which is generated by successive joining of the points $A_1(x_1; p_1), A_2(x_2; p_2), \dots, A_n(x_n; p_n)$ (fig. 1).

Ex. 7. An urn contains 7 balls (namely 3 white and 4 black balls). One at random draws 3 balls. Find the distribution law of the number ξ of white balls which can be taken from the urn.

Possible values of the random variable ξ are 0, 1, 2, 3. We determine corresponding probabilities

$$p_1 = P(\xi = 0),\ p_2 = P(\xi = 1),\ p_3 = P(\xi = 2),\ p_4 = P(\xi = 3)$$

with the help of the classical definition of probability. Elementary events (chances) for every of these four cases are sets of 3 balls that is 3-fold combinations of 7 elements. Hence the general number of chances equals

$$n = C_7^3 = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35.$$

Numbers of favourable chances are represented in the table

Event	Number of favorable chances	Explication
$\xi = 0$	$m_1 = C_4^3 = C_4^1 = 4$	One can take 0 white (and so 3 black) balls by the number of 3-fold combinations of 4 elements
$\xi = 1$	$m_2 = 3 \cdot C_4^2 = 18$	One can take 1 white ball by 3 ways and 2 black balls by C_4^2 ways
$\xi = 2$	$m_3 = C_3^2 \cdot 4 = 12$	One can take 2 white balls by C_3^2 ways and 1 black ball by 4 ways
$\xi = 3$	$m_4 = 1$	One can take 3 white balls in one unique way

Numbers m_2, m_3 are calculated with the help of the main principle of combinatorics.

The distribution law of the random variable ξ is represented by the next distribution table:

ξ	0	1	2	3
p_i	$p_1 = m_1/n = 4/35$	$p_2 = m_2/n = 18/35$	$p_3 = m_3/n = 12/35$	$p_4 = m_4/n = 1/35$

The sum of obtained probabilities equal 1:

$$p_1 + p_2 + p_3 + p_4 = 4/35 + 18/35 + 12/35 + 1/35 = 1.$$

The most probable value of the random variable is $\xi = 1$.

The distribution polygon of the random variable ξ is shown on the fig. 2 a.

Ex. 8 (**trials to the first success**). Let's fulfil trials until the first success *A* (that is until the first occurrence of some event *A*). Probability of the success is constant one ($P(A) = \text{const} = p$), and the number of trials is limited by some number *n*.

Determine the distribution law of a random variable ξ namely the number of really realized trials.

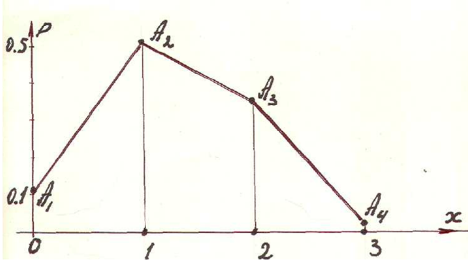

Fig. 2 a

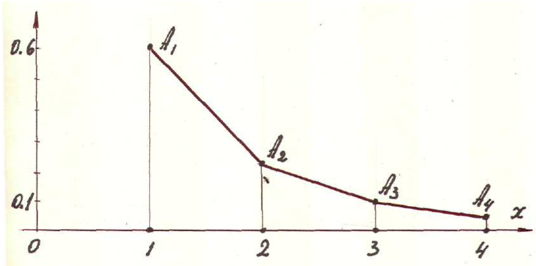

Fig. 2 b

We suppose for example that one has 4 cartridges and shoots at a target to the first hitting with the hitting probability 0.6. Find the distribution law of a random variable ξ which means the number of shots which can be done in reality.

Let the success A be a hitting in a target, so $\bar{A}$ be a miss. Possible values of the random variable ξ are 1, 2, 3, 4. It can be easily seen the validity of equalities

$$(\xi = 1) = A, (\xi = 2) = \bar{A}A, (\xi = 3) = \bar{A}\bar{A}A, (\xi = 4) = \bar{A}\bar{A}\bar{A} = \bar{A}\bar{A}\bar{A}A + \bar{A}\bar{A}\bar{A}\bar{A}.$$

Probabilities of these events are equal

$$p_1 = P(\xi = 1) = P(A) = 0.6;\ p_2 = P(\xi = 2) = P(\bar{A}A) = P(\bar{A})P(A/\bar{A}) = 0.4 \cdot 0.6 = 0.24;$$
$$p_3 = P(\xi = 3) = P(\bar{A}\bar{A}A) = P(\bar{A})P(\bar{A}/\bar{A})P(A/\bar{A}\bar{A}) = 0.4 \cdot 0.4 \cdot 0.6 = 0.096;$$
$$p_4 = P(\xi = 4) = P(\bar{A}\bar{A}\bar{A}) = P(\bar{A})P(\bar{A}/\bar{A})P(\bar{A}/\bar{A}\bar{A}) = 0.4 \cdot 0.4 \cdot 0.4 = 0.064.$$

Now we can compile the distribution table of the random variable ξ.

ξ	1	2	3	4
p_i	$p_1 = 0.6$	$p_2 = 0.24$	$p_3 = 0.096$	$p_4 = 0.064$

The most probable value of ξ is 1.

Let's test the distribution table by adding the probabilities of all values of the random variable ξ:

$$p_1 + p_2 + p_3 + p_4 = 0.6 + 0.24 + 0.096 + 0.064 = 1.$$

The distribution polygon of the random variable ξ is shown on the fig. 2 b.

POINT 2. BERNOULLI [BINOMIAL] DISTRIBUTION

Def.4. Let a random variable ξ be the number of successes (the number of occurrences of some event *A*) in *n* independent trials with constant probability of the success *A* in any trial

$$P(A) = p = \text{const} \Rightarrow P(\overline{A}) = q = 1 - P(A) = 1 - p \ .$$

One says that ξ is distributed binomially (by Bernoulli[1] [binomial] law) or simply: ξ is Bernoulli (binomial) distribution (briefly: ξ **distr. *B***).

Let's find a probability $P(\xi = k) = P_n(k)$, that is the probability of *k* successes. We'll get so-called Bernoulli formula:

$$P(\xi = k) = P_n(k) = C_n^k p^k q^{n-k} . \qquad (1)$$

■For simplicity let $n = 3, k = 2$, and it is a matter of the probability $P(\xi = 2)$. The event $(\xi = 2)$ can be represented as follows

$$(\xi = 2) = AA\overline{A} + A\overline{A}A + \overline{A}AA .$$

It is a sum of $3 = C_3^2$ pairwise disjoint summands[2] with the same probability. For example, $P(AA\overline{A}) = P(A)P(A)P(\overline{A}) = p \cdot p \cdot q = p^2 \cdot q = p^2 \cdot q^{3-2}$. Hence,

$$P(\xi = 2) = 3p^2 q^{3-2} = C_3^2 p^2 q^{3-2} .■$$

Note 1. For $k = 0, 1, 2, \ldots$ the value of the probability (1) first increases and then decreeses, and therefore there exists so-called the most probable number k_0 of successses. It is defined by the next double inequality

$$np - q \le k_0 < np + p . \qquad (2)$$

Note 2 (to the name of binomial distribution). Numbers C_n^k in Bernoulli formula are coefficients of Newton binomial (so-called binomial coefficients)

$$(a + b)^n = a^n + C_n^1 a^{n-1} b + C_n^2 a_n^{n-2} b^2 + \ldots + b^n , a^n = 1 \cdot a^n = C_n^0 a^n , b^n = C_n^n b^n ,$$

[1] Bernoulli, Jacob (1654 - 1705), the famous Swiss mathematician
[2] The number of 2-fold combinations of 3 places.

$$a^n = 1 \cdot a^n \equiv C_n^0 \cdot a^n, b^n = 1 \cdot b^n \equiv C_n^n \cdot b^n.$$

Note 3. Probability of no less than k_1 and no greater than k_2 successes equals

$$P(k_1 \le \xi \le k_2) = P_n(k_1, k_2) = P(\xi = k_1) + P(\xi = k_1 + 1) + ... + P(\xi = k_2) \qquad (3)$$

because of $(k_1 \le \xi \le k_2) = (\xi = k_1) + (\xi = k_1 + 1) + ... + (\xi = k_2)$, and the addends of the sum are pairwise disjoint events. Their probabilities we can find by Bernoulli formula (1).

Ex. 9. 6 independently working engines are installed in a shop. Probability for any engine to work at a given moment is 0.8. Find the distribution of a random variable ξ which is a number of working engines at this moment. Find the probabilities of at least one engine to work, of no less than 2 and no greater than 5 engines to work and the most probable value of ξ.

We can consider setting of an engine as a trial. So we have $n = 6$ independent trials. Let a success A mean that an engine works.

$$P(A) = 0.8 = const = p, q = 1 - p = 0.2.$$

The random variable ξ has Bernoulli distribution (briefly "ξ distributed B"), it can take on the values $0, 1, 2, 3, 4, 5, 6$, which one calculates by Bernoulli formula (1). For example

$$P(\xi = 0) = P_6(0) = C_6^0 p^0 q^{6-0} = q^6 = 0.2^6 = 0.00006,$$

$$P(\xi = 3) = P_6(3) = C_6^3 p^3 q^{6-3} = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} \cdot 0.8^3 \cdot 0.2^3 = 0.08192.$$

By the same way the other probabilities are calculated, and the distribution table of the random variable is the next one

ξ	0	1	2	3	4	5	6
p_i	0.00006	0.00154	0.01536	0.08192	0.24576	0.39321	0.26214

The probability of at least one engine to work equals

$$P(\xi > 0) = P(\xi \ge 1) = 1 - P(\overline{\xi > 0}) = 1 - P(\xi = 0) = 1 - q^6 = 1 - 0.00006 = 0.99994.$$

The most probable value of ξ, $k_0 = 5$, we see in the table. Evaluation of k_0 by the formula (2) gives $np - q \le k_0 < np + p, 6 \cdot 0.8 - 0.2 \le k_0 < 6 \cdot 0.8 + 0.8$, whence it follows that $4.6 \le k_0 < 5.6 \Rightarrow k_0 = 5$.

Probability of no less than 2 and no greater than 5 engines to work on the base of the formula (3) equals

$$P(2 \le \xi \le 5) = P_6(2, 5) = P(\xi = 2) + P(\xi = 3) + P(\xi = 4) + P(\xi = 5) =$$

$$= 0.01536 + 0.08192 + 0.24576 + 0.39321 = 0.73625 \approx 0.74$$

Ex. 10. Probability of a boy birth is 0.51. Find the probability to have two boys for a family with 5 children and the most probable number of boys.

It is useful to solve the problem by the next scheme.

1. A trial is a child birth.
2. We nave $n = 5$ independent trials.
3. A success A is a boy birth.
4. The probability of the success

$$P(A) = 0.51 = const = p \Rightarrow q = 1 - p = 0.49.$$

5. Let’s introduce a random variable ξ, namely the number of boys.
6. ξ distributed B, and we must find the probability $P(\xi = 2)$.

On the base of Bernoulli formula (1)

$$P(\xi = 2) = P_5(2) = C_5^2 p^2 q^{5-2} = C_5^2 p^2 q^3 = \frac{5 \cdot 4}{1 \cdot 2} \cdot 0.51^2 \cdot 0.49^3 = 0.5.$$

The most probable number of boys

$$5 \cdot 0.51 - 0.49 \le k_0 < 5 \cdot 0.51 + 0.51, 2.06 \le k_0 < 3.6 \Rightarrow k_0 = 3.$$

Ex. 11 (inverse problem). Probability of a boy birth equals 0.51. How many children must have a family to have at least one boy with probability no less than 0.99?

A random variable ξ is a number of boys among n children. We have to find n from the condition

$$P(\xi > 0) \ge 0.99.$$

But

$$P(\xi > 0) = 1 - P(\overline{\xi > 0}) = 1 - P(\xi = 0) = 1 - C_n^0 p^0 q^{n-0} = 1 - q^n = 1 - 0.49^n,$$

and it's a matter of solving the inequality $1-0.49^n \geq 0.99,\, 0.49^n \leq 0.01$. Taking the logarithm we get

$$lg\,0.49^n \leq lg\,0.01,\, n\,lg\,0.49 \leq -2 \Rightarrow n > \frac{-2}{lg\,0.49} = 3.22.$$

Answer*:* the minimal value of n is $n_{\min} = 4$, therefore $n \geq 4$.

Bernoulli formula (1) isn't convenient for large number n of trials. There are some approximate formulas.

POINT 3. LAPLACE[1] LOCAL AND INTEGRAL THEOREMS

Laplace local theorem gives an approximate value of the probability

$$P(\xi = k) = P_n(k)$$

of k successes in n independent trials (with constant probability p of the success in any trial). Namely, for large n we can substitute Bernoulli formula (1) by the next approximate one:

$$P(\xi = k) = P_n(k) \approx \frac{1}{\sqrt{npq}} \varphi(x_0), \tag{3}$$

where

$$x_0 = \frac{k - np}{\sqrt{npq}} \tag{4}$$

and

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}} \tag{5}$$

is so-called **small Laplace function**. Let's investigate and plot it.

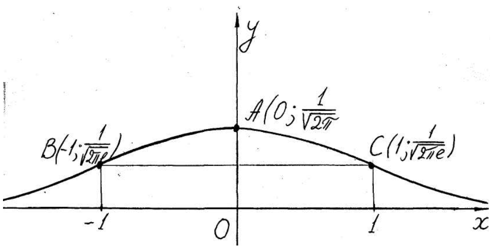

Fig. 3

1. Domain of definition and continuity of the function is $\Re = (-\infty, +\infty)$. Its graph doesn't have vertical asymptotes.

[1] Laplace, P.S. (1749 - 1827), a French mathematician, physicist, and astronomer

2. The function is even one, that is

$$\varphi(-x) = \varphi(x),$$

and therefore its graph is symmetric with respect to the Oy-axis. We can investigate the function only over the interval $[0,+\infty)$.

3. The function is positive for all $x \in [0,+\infty)$.

4. The point $A(0; 1/\sqrt{2\pi}) \in Oy$ is unique common point of the graph with coordinate axes.

5. $\lim\limits_{x\to+\infty} \varphi(x) = \frac{1}{\sqrt{2\pi}} \lim\limits_{x\to+\infty} e^{-\frac{x^2}{2}} = 0$ and therefore (the right part of) the graph has the horizontal asymptote $y = 0$ (the Ox-axis).

6. The first derivative of the function

$$\varphi'(x) = -\frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}} = -x\varphi(x) < 0$$

for $x \in [0, +\infty)$ and so the function decreases on the interval $[0, +\infty)$ and doesn't have local extrema.

7. The second derivative of the function

$$\varphi''(x) = -(\varphi(x) + x\varphi'(x)) = -(\varphi(x) + x(-x\varphi(x))) = \varphi(x)(x^2 - 1).$$

It equals zero at the point $x = 1$, is negative over the interval $(0, 1)$ and positive over the interval $(1, +\infty)$. The graph of the function is convex over the interval $(0, 1)$, concave over the interval $(1, +\infty)$ and has an inflexion point for $x = 1$ that is the point $C(1; \varphi(1)) = C(1; 1/\sqrt{2\pi e}), 1/\sqrt{2\pi e} \approx 0.24$.

The graph of the function $y = \varphi(x)$ is represented on the fig. 3. The function is tabulated one. For example

$$\varphi(2) = 0.0540, \varphi(3) = 0.0044, \varphi(3.9) = 0.0002, \varphi(3.99) = 0.0001.$$

We can suppose $\varphi(x) = 0$ for $|x| \geq 4$.

Integral Laplace theorem gives an approximate value of the probability of hitting of values of Bernoulli distribution in a segment $[k_1, k_2]$. Namely for large n we can substitute the exact formula (3) by the next approximate formula:

$$P(k_1 \le \xi \le k_2) = P(k_1, k_2) \approx \Phi(x_2) - \Phi(x_1), \qquad (6)$$

where

$$x_1 = \frac{k_1 - np}{\sqrt{npq}}, \quad x_2 = \frac{k_2 - np}{\sqrt{npq}}, \qquad (7)$$

and a function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt \qquad (8)$$

is so-called **Laplace function** (or **normed Laplace function**).

Let's investigate Laplace function and plot its graph.

1. Domain of definition and continuity of the function is the set of all real numbers. Its graph doesn't have vertical asymptotes.

2. The function is odd one, that is $\Phi(-x) = -\Phi(x)$.

$$\blacksquare\ \Phi(-x) = \frac{1}{\sqrt{2\pi}} \int_0^{-x} e^{-\frac{t^2}{2}} dt = \left| \begin{array}{l|c|c} t = -z, \ t & 0 & -x \\ \hline dt = -dz \ z & 0 & x \end{array} \right| = -\frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{z^2}{2}} dz = -\Phi(x). \blacksquare$$

Oddeness of the function means that its graph is symmetric in respect to the origin, and we can investigate the function only over the interval $[0, +\infty)$.

3. The function is positive for all positive values of x because its integrand is continuous and positive one.

4. The origin of coordinates is unique common point of the graph with coordinate axes.

5. Limit of the function for $x \to +\infty$ equals 0.5, and therefore (the right part of) the graph has a horizontal asymptote $y = 0.5$.

■To prove the property we'll take into account Poisson integral

$$\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \qquad (9)$$

Now we'll do as follows:

$$\lim_{x\to+\infty}\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}e^{-\frac{t^2}{2}}dt=\left|\begin{matrix} t/\sqrt{2}=y, & t & 0 & +\infty \\ dt=\sqrt{2}dy & y & 0 & +\infty\end{matrix}\right|=\frac{\sqrt{2}}{\sqrt{2\pi}}\int_0^{+\infty}e^{-y^2}dy=\frac{1}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2}=\frac{1}{2}.\blacksquare$$

6. The first derivative of Laplace function is equal to that of small Laplace function,

$$\Phi'(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}=\varphi(x)>0$$

For $x\geq 0$, and so the function increases on the interval $[0,+\infty)$ and hasn't local extrema.

7. The second derivative of Laplace function equals

$$\Phi''(x)=\varphi'(x)=-x\varphi(x)<0 \text{ for } x>0,$$

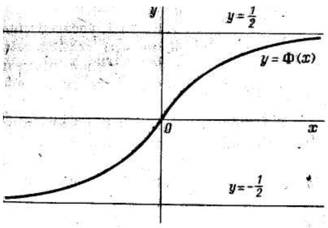

Fig. 4

and therefore its graph is a convex line over the interval $(0,+\infty)$.

The graph of the function $y=\Phi(x)$ is represented on the fig. 4.

Laplace function is that tabulated. For example,

$$\Phi(2)=0.4772, \Phi(3)=0.49865, \Phi(4)=0.499968.$$

It means that the function increases to 0.5 very quickly, and we can suppose

$$\Phi(x)=0.5 \text{ for } x\geq 5.$$

Ex. 12. Probability of boy birth 0.51. Find probabilities for a) 50, b) not less than 50 boys to born if 100 children were born.

Solving the problem we use the same scheme as in Ex. 9, 10.

1. A trial is a child birth.
2. We nave n = 100 independent trials.
3. Success A is a boy birth.
4. Probability of the success $P(A)=0.51=const=p\Rightarrow q=1-p=0.49$.
5. Let's introduce a random variable ξ, namely the number of boys.
6. ξ distributed B, and we must find the probabilities $P(\xi=50)$, $P(\xi\geq 50)=$

$=P(50\leq\xi\leq 100)$.

Using Laplace local and integral theorems we have

$$np = 100 \cdot 0.51 = 51,$$

$$npq = 100 \cdot 0.51 \cdot 0.49 = 24.99, \sqrt{npq} = 5,$$

$$P(\xi = 50) = \left| \begin{array}{l} k = 50, \\ x_0 = \dfrac{k - np}{\sqrt{npq}} = -0.2 \end{array} \right| \approx \frac{1}{\sqrt{npq}} \varphi(x_0) = \frac{1}{5} \varphi(-0.2) = \frac{\varphi(0.2)}{5} = \frac{0.3910}{5} \approx 0.08,$$

$$P(50 \le \xi \le 100) = \Phi(x_2) - \Phi(x_1) = \left| \begin{array}{l} k_1 = 50, x_1 = \dfrac{k_1 - np}{\sqrt{npq}} = \dfrac{50 - 51}{5} = -0.2, \\ k_2 = 100, x_2 = \dfrac{k_2 - np}{\sqrt{npq}} = \dfrac{100 - 51}{5} = 9.8 \end{array} \right| =$$
$$= \Phi(9.8) - \Phi(-0.2) = 0.5 + 0.0793 \approx 0.58.$$

Ex.13 (inverse problem). Probability of a success A in independent trials equals $p = 0.05$. How many trials it's necessary to fulfil to have no less than 5 successes with probability not less than 0.8.

Let ξ be a number of successes. We have to find n (a number of trials) to satisfy the next inequality

$$P(5 \le \xi \le n) \ge 0.8.$$

Here

$$x_1 = \frac{5 - np}{\sqrt{npq}} = \frac{5 - 0.05n}{0.22\sqrt{n}} = -\frac{0.05n - 5}{0.22\sqrt{n}}, \; x_2 = \frac{n - np}{\sqrt{npq}} = \frac{0.95n}{0.22\sqrt{n}} \approx 4.3\sqrt{n},$$

and by Laplace integral theorem

$$P(5 \le \xi \le n) = \Phi(4.3\sqrt{n}) + \Phi\left(\frac{0.05n - 5}{0.22\sqrt{n}}\right).$$

We can suppose n so large that $4.3\sqrt{n} \ge 5$ (it's sufficient to suppose $n \ge 2$; by the condition $n \ge 5$ and so $4.3\sqrt{n} \ge 9.6$). Hence we must solve the inequality

$$P(5 \le \xi \le n) = 0.5 + \Phi\left(\frac{0.05n - 5}{0.22\sqrt{n}}\right) \ge 0.8 \text{ or } \Phi\left(\frac{0.05n - 5}{0.22\sqrt{n}}\right) \ge 0.3.$$

Using the table of Laplace function and the property of its increasing we get

$$\frac{0.05n - 5}{0.22\sqrt{n}} \ge 0.84 \Rightarrow \sqrt{n} \ge 12, n \ge 144.$$

Answer. Minimal value of n is 144.

Ex. 14. **The probability of the deviation of the relative frequency of an event *A* from its probability** $p = P(A)$ (in n independent trails with constant probability $P(A)$ = const = p of the event) can be find by the next formula

$$P\left(\left|P_n^*(A) - P(A)\right| \le \varepsilon\right) = P\left(\left|P_n^*(A) - p\right| \le \varepsilon\right) = 2\Phi\left(\varepsilon\sqrt{\frac{n}{pq}}\right) \qquad (10)$$

■Let a random variable ξ be a number of occurrences of the event A. Then its relative frequency equals $P_n^*(A) = \dfrac{\xi}{n}$, and with the help of Laplace integral theorem we obtain

$$P\left(\left|P_n^*(A) - P(A)\right| \le \varepsilon\right) = P\left(\left|\frac{\xi}{n} - p\right| \le \varepsilon\right) = P\left(\left|\xi - np\right| \le n\varepsilon\right) =$$

$$= P\left(-n\varepsilon \le \xi - np \le n\varepsilon\right) = P(np - n\varepsilon \le \xi \le n\varepsilon + np) = \left|\begin{array}{l} x_2 = \dfrac{np + n\varepsilon - np}{\sqrt{npq}} = \varepsilon\sqrt{\dfrac{n}{pq}} \\ x_1 = \dfrac{np - n\varepsilon - np}{\sqrt{npq}} = -\varepsilon\sqrt{\dfrac{n}{pq}} \end{array}\right| =$$

$$= \Phi\left(\varepsilon\sqrt{\frac{n}{pq}}\right) - \Phi\left(-\varepsilon\sqrt{\frac{n}{pq}}\right) = 2\Phi\left(\varepsilon\sqrt{\frac{n}{pq}}\right). \blacksquare$$

If for example

$$n = 100,\ P(A) = p = 0.03,\ \varepsilon = 0.02,$$

then the formula (10) gives

$$P(\left| P_{100}^*(A) - 0.03 \right| \le 0.02) = 2\Phi\left(0.02\sqrt{\frac{100}{0.03 \cdot 0.97}}\right) = 2\Phi(1.17) \approx 2 \cdot 0.3790 \approx 0.76.$$

Ex. 15. Probability for an item to be imperfect is 0.03. How many imperfect items are contained in a batch of 100 items with probability 0.9?

Let a random variable ξ be a number of imperfect items in a batch, and a success A means that an item, taken at random, is imperfect one. The problem can be reduced to finding of some number ε and possible values of ξ from the next equality

$$P\left(\left|\frac{\xi}{n}-p\right|\le\varepsilon\right)=0.9$$

or, by virtue of the formuls (10),

$$P\left(\left|\frac{\xi}{n}-p\right|\le\varepsilon\right)=2\Phi\left(\varepsilon\sqrt{\frac{n}{pq}}\right)=0.9\,.$$

By conditions of the problem we have $p = P(A) = 0.03$, $q = 1 - p = 0.97$, $n = 100$, and therefore

$$P\left(\left|\frac{\xi}{100}-0.03\right|\le\varepsilon\right)=2\Phi\left(\varepsilon\sqrt{\frac{100}{0.03\cdot 0.97}}\right)=0.9\,.$$

a) Using the equality

$$2\Phi\left(\varepsilon\sqrt{\frac{100}{0.03\cdot 0.97}}\right)=0.9$$

we at first find the value of ε, namely

$$\Phi(58.62\varepsilon)=0.45,\quad 58.62\varepsilon=1.65\Rightarrow\varepsilon=0.028\,.$$

b) Now we find possible values of ξ with the help of the relation

$$P\left(\left|\frac{\xi}{100}-0.03\right|\le 0.028\right)=0.9\,,$$

from which with the probability 0.9

$$\left|\frac{\xi}{100}-0.03\right|\le 0,028,\ |\xi-3|\le 2.8, -2.8\le\xi-3\le 2.8, 0.2\le\xi\le 5.8\,.$$

Answer. With the probability 0.9 there are no less than 1 and no greater than 5 imperfect items in a batch.

Ex. 16. Probability for an item to be imperfect is 0.03. How many items is it necessary to take in order that the deviation of the relative frequency of imperfect items from probability of item imperfection were no greater than ε = 0.02 with probability no less than 0.9.

Let an event A mean that an item is imperfect, $p = P(A) = 0.03$, $q = 0.97$.

$$P\left(\left|P_n^*(A)-P(A)\right|\le 0.02\right)=2\Phi\left(0.02\sqrt{\frac{n}{0.03\cdot 0.97}}\right)\ge 0.9,\ \Phi\left(0.117\sqrt{n}\right)\ge 0.45,$$

$$0.117\sqrt{n} \ge 1.65,\ \sqrt{n} \ge 14.10,\ n \ge 198.81 \Rightarrow n_{min} = 199.$$

POINT 4. POISSON FORMULA AND DISTRIBUTION

Let a random variable ξ be distributed *B*. Let's suppose that the number *n* of trials tends to infinity, the probability *p* of a success *A* goes to zero, but a product *np* retains constant,

$$n \to \infty,\ p = P(A) \to 0,\ np = const = a.$$

In this case the limit of the probability $P(\xi = k) = P_n(k)$, which is defined by Bernoulli formula (1), equals

$$\lim_{\substack{n\to\infty, p\to 0\\ np=a}} P(\xi = k) = \lim_{\substack{n\to\infty, p\to 0\\ np=a}} C_n^k p^k q^{n-k} = \frac{a^k}{k!} e^{-a}. \tag{11}$$

■ It follows from the equality $np = a$ that $p = a/n$, $q = 1 - p = 1 - a/n$. Hence

$$\lim_{\substack{n\to\infty, p\to 0\\ np=a}} C_n^k p^k q^{n-k} = \lim_{n\to\infty} \frac{n(n-1)(n-2)\cdot\ldots\cdot(n-(k-1))}{k!} \cdot \left(\frac{a}{n}\right)^k \cdot \left(1 - \frac{a}{n}\right)^{n-k} =$$

$$= \frac{a^k}{k!} \lim_{n\to\infty} \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \ldots \cdot \frac{n-(k-1)}{n} \cdot \frac{\left(1 - \frac{a}{n}\right)^n}{\left(1 - \frac{a}{n}\right)^k} =$$

$$= \frac{a^k}{k!} \lim_{n\to\infty} \frac{\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdot\ldots\cdot\left(1 - \frac{k-1}{n}\right)}{\left(1 - \frac{a}{n}\right)^k} \cdot \lim_{n\to\infty}\left(1 - \frac{a}{n}\right)^n = \frac{a^k}{k!} \cdot 1 \cdot \lim_{n\to\infty}\left(1 - \frac{a}{n}\right)^n =$$

$$= \frac{a^k}{k!} \lim_{n\to\infty}\left(\left(1 - \frac{a}{n}\right)^{-\frac{n}{a}}\right)^{-a} = \frac{a^k}{k!}\left(\lim_{n\to\infty}\left(1 - \frac{a}{n}\right)^{-\frac{n}{a}}\right)^{-a} \left|\begin{array}{c}\text{by the second}\\ \text{standard}\\ \text{limit}\end{array}\right| = \frac{a^k}{k!} e^{-a}$$ ■

Def. 5. One says that a discrete random variable ξ (with non-negative integer possible values) has Poisson[1] distribution with a parameter *a* if its distribution law is

[1] Poisson, Simeon Denis (1781 - 1840), a French mechanic, physicist, and mathematician

given by the next formula (Poisson formula):

$$P(\xi = k) = \frac{a^k}{k!} e^{-a} . \qquad (12)$$

Probabilities in the formula (12) are tabulated (see table 1).

Table 1. Probabilities $P(\xi = m) = \dfrac{a^m}{m!} e^{-a}$ (Poisson distribution)

m	a=0,1	a=0,2	a=0,3	a=0,4	a=0,5	a=0,6	a=0,7	a=0,8	a=0,9	
0	0,9048	0,8187	0,7408	0,6703	0,6065	0,5488	0,4966	0,4493	0,4066	
1	0,0905	0,1638	0,2222	0,2681	0,3033	0,3293	0,3476	0,3595	0,3659	
2	0,0045	0,0164	0,0333	0,0536	0,0758	0,0988	0,1217	0,1438	0,1647	
3	0,0002	0,0019	0,0033	0,0072	0,0126	0,0198	0,0284	0,0383	0,0494	
4		0,0001	0,0002	0,0007	0,0016	0,0030	0,0050	0,0077	0,0111	
5				0,0001	0,0002	0,0004	0,0007	0,0012	0,0020	
6							0,0001	0,0002	0,0003	

m	a=1	a=2	a=3	a=4	a=5	a=6	a=7	a=8	a=9	a=10
0	0,3679	0,1353	0,0498	0,0183	0,0067	0,0025	0,0009	0,0003	0,0001	0,0000
1	0,3679	0,2707	0,1494	0,0733	0,0337	0,0149	0,0064	0,0027	0,0011	0,0005
2	0,1839	0,2707	0,2240	0,1465	0,0842	0,0446	0,0223	0,0107	0,0050	0,0023
3	0,0613	0,1804	0,2240	0,1954	0,1404	0,0892	0,0521	0,0286	0,0150	0,0076
4	0,0153	0,0902	0,1680	0,1954	0,1755	0,1339	0,0912	0,0572	0,0337	0,0189
5	0,0031	0,0361	0,1008	0,1563	0,1755	0,1606	0,1277	0,0916	0,0607	0,0378
6	0,0005	0,0120	0,0504	0,1042	0,1462	0,1606	0,1490	0,1221	0,0911	0,0631
7	0,0001	0,0037	0,0216	0,0595	0,1044	0,1377	0,1490	0,1396	0,1171	0,0901
8		0,0009	0,0081	0,0298	0,0653	0,1033	0,1304	0,1396	0,1318	0,1126
9		0,0002	0,0027	0,0132	0,0363	0,0688	0,1014	0,1241	0,1318	0,1251
10			0,0008	0,0053	0,0181	0,0413	0,0710	0,0993	0,1186	0,1251
11			0,0002	0,0019	0,0082	0,0225	0,0452	0,0722	0,0970	0,1137
12			0,0001	0,0006	0,0034	0,0126	0,0263	0,0481	0,0728	0,0948
13				0,0002	0,0013	0,0052	0,0142	0,0296	0,0504	0,0729
14				0,0001	0,0005	0,0022	0,0071	0,0169	0,0324	0,0521
15					0,0002	0,0009	0,0033	0,0090	0,0194	0,0347
16						0,0003	0,0014	0,0045	0,0109	0,0217
17						0,0001	0,0006	0,0021	0,0058	0,0128
18							0,0002	0,0009	0,0029	0,0071
19							0,0001	0,0004	0,0014	0,0037
20								0,0002	0,0006	0,0019
21								0,0001	0,0003	0,0009
22									0,0001	0,0004
23										0,0002
24										0,0001

The next conclusion follows from the formulas (11), (12).

Let a random variable ξ have Bernoulli distribution (short: ξ is distributed B), the number of trials n is large, the probability p of a success is small and $np \le 10$. In this case we can consider ξ as having Poisson distribution with the parameter $a = np$ and use the formula (12) instead (1). By this reason Poisson distribution is sometimes called the **law of rare events**.

Ex. 18. 500 items are sent. Probability of damage of an item in the trade is 0.002. Find the probabilities: a) 3, b) less than 3, c) more than 2 items are damaged; d) at least one item is damaged.

We have $n = 500$ independent trials, a success A is a damage of an item, $p = P(A) = 0.002 =$ const, a random variable ξ is a number of damaged items. ξ is distributed B, but n is large, p is small, $np = 1$. Therefore we can consider ξ as Poisson distribution with $a = 1$ and make use of Poisson formula (12).

a) $P(\xi = 3) = 0.0613$; b) $P(\xi < 3) = P(\xi \le 2) = P(\xi = 0) + P(\xi = 1) + P(\xi = 2) = 0.3679 + 0.3679 + 0.1839 = 0.9197$; c) $P(\xi > 2) = 1 - P(\xi \le 2) = 1 - 0.9197 = 0.0803$

$$\text{d) } P(\xi \ge 1) = P(\xi > 0) = 1 - P(\xi = 0) = 1 - 0.3679 = 0.6321.$$

LECTURE NO. 4. THE DISTRIBUTION FUNCTION AND DENSITY
LE QUATRIEME COURS. FONCTION ET DENCITE DE DISTRIBUTION D'UNE VARIABLE ALEATOIRE.
VIERTE VORLESUNG. VERTEILUNGSFUNKTION UND VERTEILUNGSDICHTE VON ZUFALLSGRÖßE

POINT 1. THE DISTRIBUTION FUNCTION OF A RANDOM VARIABLE. La fonction de distribution d'une variable aléatoire. Die Verteilungsfunktion von Zufallsgrösse.

POINT 2. THE DISTRIBUTION DENSITY OF A RANDOM VARIABLE. La dencité de distribution d'une variable aléatoire. Die Verteilungsdichte von Zufallsgrösse.

POINT 3. A FUNCTION OF A RANDOM VARIABLE. Une fonction d'une variable aléatoire. Eine Funktion einer Zufallsgrösse.

POINT 1. THE DISTRIBUTION FUNCTION OF A RANDOM VARIABLE

The distribution function is the most general form of the distribution law of a random variable ξ.

Def. 1. The distribution function[1] of a random variable ξ is called a function:

$$F(x) = P(\xi < x) = P(\xi \in (-\infty, x)). \quad (1)$$

The distribution function $F(x)$ is the probability for the random variable ξ to take on values which are less than x or the probability of hitting of the random variable in the infinite interval (the half-axis) $(-\infty, x)$ (fig. 1)

Fig. 1

Ex.1. Distribution function of a discrete random variable with finite number of possible values. Its distribution law is given by a distribution table.

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[1] Or: accumulated distribution, cumulative distribution, cumulative distribution function, cumulative frequency function, íntegral dìstribútion [partítion] fúnction, partition function.

ξ	x_1	x_2	…	x_i	x_{i+1}	…	x_n
p_i	p_1	p_2	…	p_i	p_{i+1}	…	p_n

It's known that

$$\sum_{i=1}^{n} p_i = p_1 + p_2 + ... + p_n = 1.$$

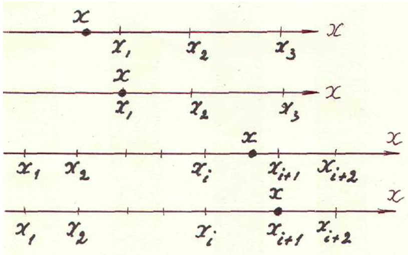


Fig. 2

We'll as the rule suppose that all values of the random variable are arrainged in order of magnitude, that is

$$x_1 < x_2 < ... < x_n.$$

At first let $x \le x_1$ (fig. 2). In this case the event $\xi < x$ is impossible one, and therefore

$$F(x) = P(\xi < x) = 0.$$

Now let $x_i < x \le x_{i+1}$ (see fig. 2). In this general case

$$F(x) = P(\xi < x) = P((\xi = x_1) + (\xi = x_2) + ... + (\xi = x_i)) =$$
$$= P(\xi = x_1) + P(\xi = x_2) + ... + P(\xi = x_i) = p_1 + p_2 + ... + p_i.$$

If finally $x > x_n$, then

$$F(x) = P(\xi < x) = P(\xi = x_1) + P(\xi = x_2) + ... + P(\xi = x_n) = p_1 + p_2 + ... + p_n = 1.$$

We have established the next result:

$$F(x) = \begin{cases} 0 & \text{if} \quad x \le x_1, \\ p_1 & \text{if} \quad x_1 < x \le x_2, \\ p_1 + p_2 & \text{if} \quad x_2 < x \le x_3, \\ p_1 + p_2 + p_3 & \text{if} \quad x_3 < x \le x_4, \\ \dots\dots\dots & \dots \quad \dots\dots \\ p_1 + p_2 + ... + p_{n-1} & \text{if} \quad x_{n-1} < x \le x_n, \\ p_1 + p_2 + ... + p_{n-1} + p_n = 1 & \text{if} \quad x > x_n. \end{cases} \qquad (2)$$

or in the compact form

$$F(x) = \sum_{x_i < x} p_i . \qquad (3)$$

We see that the distribution function $F(x)$ of a discrete random variable ξ is equal to the sum of probabilities p_i of such possible values of ξ which are less than *x*.

The graph of the distribution function (fig. 3) is a system of segments which are parallel to the Ox-axis and have deleted left end-points, namely:

$$y = 0\ (x \le x_1),\ y = p_1\ (x_1 < x \le x_2), \ldots,\ y = p_1 + p_2\ (x_2 < x \le x_3), \ldots,\ y = 1\ (x > x_n).$$

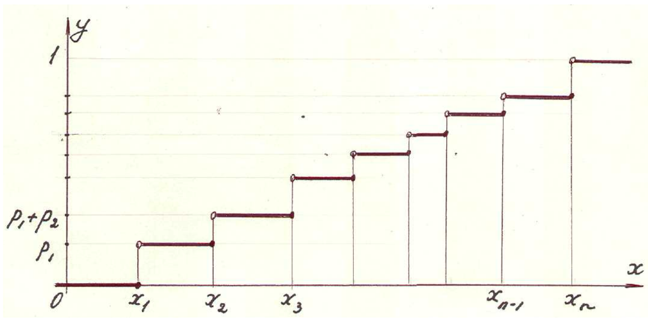


Fig. 3

Ex.2. Firing at a target to the first hitting with hitting probability 0.6 and with four cartridges (see Ex. 8 of the preceding Lecture).

ξ	1	2	3	4
p_i	0.6	0.24	0.096	0.064

The distribution function of the random variable ξ, namely of the number of shots which can be done in reality, and its graph are given as follows:

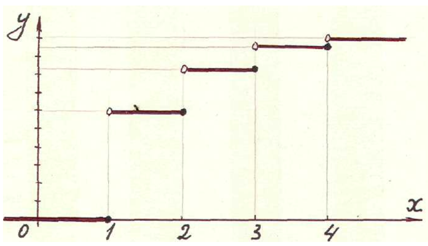


Fig. 4

$$F(x) = \begin{cases} 0 & \text{if} \quad x \le 1, \\ 0.6 & \text{if} \quad 1 < x \le 2, \\ 0.6 + 0.24 = 0.84 & \text{if} \quad 2 < x \le 3, \\ 0.84 + 0.096 = 0.936 & \text{if} \quad 3 < x \le 4, \\ 1 & \text{if} \quad x > 4. \end{cases}$$

Properties of the distribution function

1. The distribution function, being a probability, lies between 0 and 1 [ranges from 0 to 1]:

$$0 \le F(x) \le 1.$$

2. The event $\xi < -\infty$ is impossible one, and so we can accept by definition that

$$F(-\infty)=0.$$

3. The event $\xi < +\infty$ is certain one, and we accept by definition that

$$F(+\infty)=1.$$

4. Probability of hitting of a random variable ξ in a half-interval $[a,b)$:

$$P(a \le \xi < b) = P(\xi \in [a,b)) = F(b) - F(a). \qquad (4)$$

■ The event

$$(\xi < b) \equiv (\xi \in (-\infty, b))$$

equals the sum of two disjoint [exclusive, incompatible] events, namely $(\xi < a) \equiv (\xi \in (-\infty, a))$ and $(a \le \xi < b) \equiv (\xi \in [a,b)) \Rightarrow (\xi < b) = (\xi < a) + (a \le \xi < b)$ and therefore

$$P(\xi < b) = P((\xi < a) + (a \le \xi < b)) = P(\xi < a) + P(a \le \xi < b).$$

On the base of the formula (1) we can rewrite this latter equality in the form

$$F(b) = F(a) + P(a \le \xi < b)$$

whence it follows the required formula (4).■

Corollaries:

$$P(a < \xi < b) = P(\xi \in (a,b)) = P(a \le \xi < b) - P(\xi = a), \qquad (5)$$

$$P(a \le \xi \le b) = P(\xi \in [a,b]) = P(a \le \xi < b) + P(\xi = b), \qquad (6)$$

$$P(a < \xi \le b) = P(\xi \in (a,b]) = P(a \le \xi < b) - P(\xi = a) + P(\xi = b). \qquad (7)$$

5. Probability of a separate value of a random variable ξ.

Let's represent the probability of a separate value a of a random variable ξ as the next limit (the right limit) at the point a,

$$P(\xi = a) = \lim_{b \to a} P(a \le \xi < b) = \lim_{b \to a+0} P(a \le \xi < b).$$

By virtue of the formula (4) we'll get either zero if the distribution function $F(x)$ is continuous one at the point a, or some non-zero number A, namely

$$P(\xi = a) = \lim_{b \to a}(F(b) - F(a)) = \lim_{b \to a} F(b) - F(a) = \begin{cases} 0 & \text{if} \quad \lim_{b \to a} F(b) = F(a), \\ A \ne 0 & \text{if} \quad \lim_{b \to a} F(b) \ne F(a). \end{cases} \qquad (8)$$

Def. 2. A random variable is called that continuous if its distribution function *F(x)* is continuous at all points of the set of real numbers.

For continuous random variable

$$P(a \le \xi < b) = P(a < \xi < b) = P(a \le \xi \le b) = P(a < \xi \le b) = F(b) - F(a)$$

or

$$P(\xi \in [a, b)) = P(\xi \in (a, b)) = P(\xi \in [a, b]) = P(\xi \in (a, b]) = F(b) - F(a).$$

In this case we can substitute all the formulas (4) – (7) by the next one

$$P(\xi \in < a, b >) = F(b) - F(a) \qquad (9)$$

if we'll denote by $< a, b >$ an interval with the end points *a, b* (no matter the points *a, b* belong or not to the interval).

6. Distribution function never decreases.

■Let x_1, x_2 be two values of *x* and $x_1 < x_2$. Making use of the formula (4) we obtain

$$F(x_2) - F(x_1) = P(x_1 \le \xi < x_2) \ge 0, F(x_2) \ge F(x_1),$$

and therefore the distribution function doesn't decrease.■

The graph of distribution function lies in the strip between the straight lines $y = 0, y = 1$ and doesn't steep [doesn't go down]. In the case of continuous random variable it's a continuous non-steeping curve.

Ex. 3. In Ex. 9 of the preceding Lecture we've found the distribution law of the random number ξ of working engines

ξ	0	1	2	3	4	5	6
p_i	0.00006	0.00154	0.01536	0.08192	0.24576	0.39321	0.26214

Let's compile the distribution function of the random variable ξ and with its help find the next probabilities: a) no less than 2 and less than 5 engines work; b) greater than 2 and less than 5 engines work; c) no less than 2 and no greater than 5 engines work; d) less than 2 and no greater than 5 engines work.

On the base of the formula (2) the distribution function of ξ is

$$F(x)=\begin{cases} 0 & if \quad x\le 0,\\ 0.00006 & if \quad 0<x\le 1,\\ 0.00006+0.00154=0.00160 & if \quad 1<x\le 2,\\ 0.00160+0.01536=0.01696 & if \quad 2<x\le 3,\\ 0.01696+0.08192=0.09888 & if \quad 3<x\le 4,\\ 0.09888+0.24576=0.34464 & if \quad 4<x\le 5,\\ 0.34464+0.39321=0.73785 & if \quad 5<x\le 6,\\ 0.73785+0.26214=0.99999=1 & if \quad x>6. \end{cases}$$

Probabilities a) – d) we calculate by virtue of the formulas (4) - (7) namely:

a) $P(2\le\xi<5)=F(5)-F(2)=0.34464-0.00160=0.34304$;

b) $P(2<\xi<5)=P(2\le\xi<5)-P(\xi=2)=0.34304-0.01536=0.32768$;

c) $P(2\le\xi\le 5)=P(2\le\xi<5)+P(\xi=5)=0.34304+0.39321=0.73625$;

d) $P(2<\xi\le 5)=P(2<\xi<5)+P(\xi=5)=0.32768+0.39321=0.72089$.

We notice that the result of c) coincides with that in Ex. 9 of the Lecture No. 3.

Ex. 4. Let's calculate the known probability $P(\xi=x_i)=p_i$ of a discrete random variable of Ex. 1 with the help of the formula (8). We'll have

$$P(\xi=x_i)=\lim_{x\to x_i+0}F(x)-F(x_i)=(p_1+p_2+...+p_{i-1}+p_i)-(p_1+p_2+...+p_{i-1})=p_i.$$

Ex. 5. Given the distribution function of a continuous random variable ξ

$$F(x)=\begin{cases} 0 & \text{if} \quad x<-1,\\ a\arcsin x+b & \text{if} \quad -1\le x\le 1,\\ 1 & \text{if} \quad x>1. \end{cases}$$

It's required to find: a) the values of the parameters a and b; b) probability of hitting of the random variable ξ in the interval $<-1/2,\sqrt{3}/2>$ (that is in the interval with the end points $-1/2,\sqrt{3}/2$).

a) By virtue of continuity of the random variable its distribution function is continuous for all x and therefore must have the same unilateral limits at the points $x=-1, x=1$,

$$\lim_{x\to -1-0}F(x)=\lim_{x\to -1+0}F(x),\ \lim_{x\to 1-0}F(x)=\lim_{x\to 1+0}F(x).$$

But

$$\lim_{x\to -1-0} F(x) = 0,\ \lim_{x\to -1+0} F(x) = -\frac{\pi}{2}a + b,\ \lim_{x\to 1-0} F(x) = \frac{\pi}{2}a + b,\ \lim_{x\to 1+0} F(x) = 1,$$

and we get the next system of equations in a, b:

$$\begin{cases} -\frac{\pi}{2}a + b = 0, \\ \frac{\pi}{2}a + b = 1. \end{cases}$$

It is obviously that $a = \frac{1}{\pi}$, $b = \frac{1}{2}$, and so

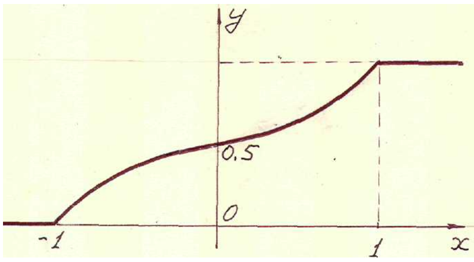


Fig. 5

$$F(x) = \begin{cases} 0 & \text{if} \quad x < -1, \\ \frac{1}{\pi}\arcsin x + \frac{1}{2} & \text{if} \quad -1 \le x \le 1, \\ 1 & \text{if} \quad x > 1. \end{cases}$$

The graph of the distribution function is represented on the fig. 5. The function increases on the segment $[-1, 1]$, its graph is convex over the interval $(-1, 0)$ and concave over the interval $(0, 1)$. We can see these facts if we study signs of the first and second derivatives of the function,

$$F'(x) = \begin{cases} 0 \quad \text{if} \quad |x| > 1, \\ \frac{1}{\pi}\cdot\frac{1}{\sqrt{1-x^2}} > 0 \text{ if } -1 < x < 1; \end{cases}$$

$$F''(x) = \begin{cases} 0 & \text{if } |x| > 1, \\ \frac{1}{\pi}\cdot\frac{x}{(1-x^2)\sqrt{1-x^2}} & \begin{cases} < 0 & \text{if } -1 < x < 0, \\ > 0 & \text{if } 0 < x < 1. \end{cases} \end{cases}$$

b) Now we find the required probability using the formula (9)

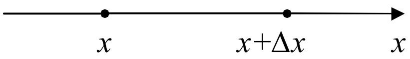
$$P\left(\xi \in < -\frac{1}{2}, \frac{\sqrt{3}}{2} >\right) = F\left(\frac{\sqrt{3}}{2}\right) - F\left(-\frac{1}{2}\right) = \left(\frac{1}{\pi}\arcsin\frac{\sqrt{3}}{2} + \frac{1}{2}\right) - \left(\frac{1}{\pi}\arcsin\left(-\frac{1}{2}\right) + \frac{1}{2}\right) =$$

$$= \left(\frac{1}{\pi}\cdot\frac{\pi}{3} + \frac{1}{2}\right) - \left(\frac{1}{\pi}\cdot\left(-\frac{\pi}{6}\right) + \frac{1}{2}\right) = \frac{1}{2}.$$

POINT 2. THE DISTRIBUTION DENSITY OF A RANDOM VARIABLE

Let ξ be continuous random variable and $F(x)$ its distribution function. The probability of hitting of the random variable in an infinitely small interval

$$< x, x+\Delta x > \text{ (fig. 6)}$$



by the formula (9) equals

Fig. 6

$$P(\xi \in < x, x+\Delta x >) \underset{(9)}{=} F(x+\Delta x) - F(x) = \Delta F(x).$$

The average density of this probability on the interval $< x, x+\Delta x >$ equals

$$f_{av}(x) = \frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{\Delta F(x)}{\Delta x};$$

its limit as Δx goes to zero is the density of probability at the point x,

$$f(x) = \lim_{\Delta x \to 0} f_{av}(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta F(x)}{\Delta x} = F'(x).$$

Def. 3. The distribution density[1] (the density of probability) of a continuous random variable ξ is called the derivative of its distribution function,

$$f(x) = F'(x). \qquad (10)$$

It follows from the definition 3 that:

1. The distribution function of a continious random variable is a primitive (an antiderivative) of its distribution density.

2. Accurate [with an accuracy] to infinitely small of higher order

$$P(\xi \in < x, x+dx >) = f(x)dx; \qquad (11)$$

the express in the right side of the formula (11) , namely

$$f(x)dx, \qquad (12)$$

is called a **probability element**. It's the differential of the distribution function of the random variable which we consider,

$$f(x)dx = dF(x) = F'(x)dx.$$

[1] Or: density, density function, density of distribution, dìstribútion density of probabilities, dìfferéntial distríbutive fúnction, elementary probability law, frequency distribution function, partítion dénsity, partítion dénsity of probabilities, probability density (function), relative distribution function, relative frequency function.

Properties of the distribution density

1. The distribution density is non-negative function,

$$f(x) \geq 0. \qquad (13)$$

■This fact immediately follows from non-decreasing of the distribution function and the definition 3.■

2. The probability of hitting of a continuous random variable ξ in an interval $< a, b >$ with end-points a and b can be represented in terms of its distribution density

$$P(\xi \in < a, b >) = \int_a^b f(x)dx \qquad (14)$$

■Validity of the property follows from the formula (9) and Newton-Leibniz formula because the distribution function is a primitive of the distribution density. It can be proved in other way, that is with the help of summation [accumulation, adding, addition] of probability elements (12).■

The definite integral in the formula (14)

$$\int_a^b f(x)dx$$

must be considered as improper one if the distribution density $f(x)$ has discontinuity points.

The formula (14) can be extended on the cases $a = -\infty, b = +\infty$.

3. Expressing the distribution function through the distribution density. Extending the formula (14) on the case $a = -\infty$ we get

$$F(x) = P(\xi \in (-\infty, x)) = \int_{-\infty}^{x} f(x)dx. \qquad (15)$$

4. If we extend the formula (14) on the case $a = -\infty, b = +\infty$ and take into account the formula (15) and the property 3 of the distribution function, we'll obtain very important formula

$$\int_{-\infty}^{\infty} f(x)dx = 1 \ . \qquad (16)$$

If a non-negative function $f(x)$ satisfies the condition (16), it can be considered as the distribution density of some continuous random variable.

Ex. 6. Let there be given a function $f(x) = Ae^{-x^2}$. Find the value of the parameter A so that the function can be the distribution density of some continuous random variable. Find and graph its distribution function.

The function in question is non-negative one. To be the distribution density it must satisfy the condition (16). Using Poisson integral we find the value of A

$$1 = \int_{-\infty}^{\infty} f(x)dx = A\int_{-\infty}^{\infty} e^{-x^2}dx = A\sqrt{\pi} \Rightarrow A = \frac{1}{\sqrt{\pi}},$$

$$f(x) = \frac{1}{\sqrt{\pi}}e^{-x^2},$$

and by virtue of the formula (15) the distribution function of the random variable is

$$F(x) = \int_{-\infty}^{x} f(x)dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-x^2}dx .$$

The graph of the distribution density $f(x)$, up to the scale coefficient $1/\sqrt{\pi}$, coincides with Gaussian curve (see Lect. No. 17, Ex. 14, fig. 19). To plot the distribution function $F(x)$ let's observe:

a) $\lim\limits_{x\to-\infty} F(x) = 0, \lim\limits_{x\to\infty} F(x) = 1, F(0) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{0} e^{-x^2}dx = \frac{1}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = 0.5$, that is the graph of the function has two horizontal asymptotes $y = 0$ (for the left part), $y = 1$ (for the right part) and passes through the point $(0; 0.5)$ of the Oy-axis;

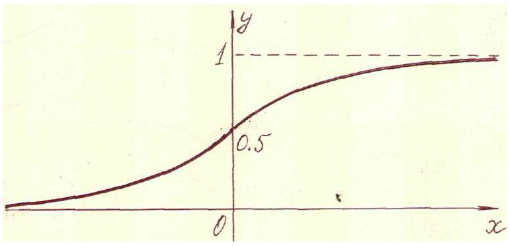


Fig. 7

b) $F'(x) = f(x) = \frac{1}{\sqrt{\pi}}e^{-x^2} > 0$, so the function $F(x)$ increases and has a rising [an ascending] graph;

c) $F''(x) = f'(x) = \left(\frac{1}{\sqrt{\pi}} e^{-x^2}\right)' = (-2x) \cdot \frac{1}{\sqrt{\pi}} e^{-x^2} = (-2x) \cdot f(x) \begin{cases} > 0 \; for \; x < 0, \\ < 0 \; for \; x < 0, \end{cases}$

hence the graph of the function is concave over the interval $(-\infty, 0)$ and convex over the interval $(0, \infty)$.

The distribution function $F(x)$ is plotted on the fig. 7.

Ex. 7. The same problem for the function

$$f(x) = \begin{cases} A\cos x \text{ if } |x| < \pi/2, \\ 0 \quad \text{if } |x| > \pi/2. \end{cases}$$

On the base of the condition (16)

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{-\frac{\pi}{2}} 0dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} A\cos x dx + \int_{\frac{\pi}{2}}^{\infty} 0dx = A\sin x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = A(1-(-1)) = 2A \Rightarrow A = \frac{1}{2}$$

,

$$f(x) = \begin{cases} \frac{1}{2}\cos x & \text{if} \quad |x| < \pi/2, \\ 0 & \text{if} \quad |x| > \pi/2. \end{cases}$$

Futher we make use of the formula (15) for three cases:

a) $x < -\pi/2$; b) $-\pi/2 \le x \le \pi/2$; c) $x > \pi/2$ (fig. 8.1 a, b, c).

a) In the first case ($x < -\pi/2$)

$$F(x) = \int_{-\infty}^{x} 0dx = 0;$$

b) in the second case ($-\pi/2 \le x \le \pi/2$)

$$F(x) = \int_{-\infty}^{-\frac{\pi}{2}} 0dx + \frac{1}{2}\int_{-\frac{\pi}{2}}^{x} \cos x dx = \frac{1}{2} \cdot \sin x \Big|_{-\frac{\pi}{2}}^{x} = \frac{1}{2}(\sin x - (-1)) = \frac{1}{2}(1 + \sin x);$$

c) in the last case ($x > \pi/2$) by the sense of finding of the parameter A

$$F(x) = \int_{-\infty}^{-\frac{\pi}{2}} 0dx + \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\infty} 0dx = 1.$$

Thus the distribution function of the random variable is given by the formula

$$F(x)=\begin{cases} 0 & \text{if} \quad x<-\pi/2, \\ \frac{1}{2}(1+\sin x) & \text{if} \quad -\pi/2 \le x \le \pi/2, \\ 1 & \text{if} \quad x>\pi/2. \end{cases}$$

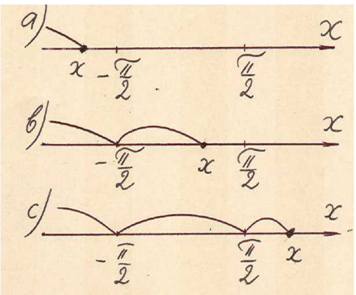


Fig. 8.1

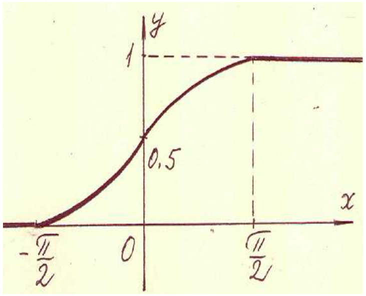


Fig. 8.2

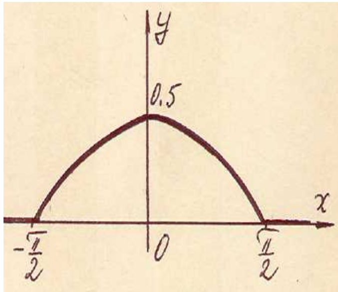


Fig. 8.3

To plot the distribution function and density we find their corresponding derivatives.

a) $F'(x)=f(x)=\begin{cases} \frac{1}{2}\cos x>0 & \text{if} \quad |x|<\frac{\pi}{2}, \\ 0 & \text{if} \quad |x|>\frac{\pi}{2}; \end{cases}$ $F(x)$ increases on $(-\pi/2, \pi/2)$.

b) $F''(x)=f'(x)=\begin{cases} -\frac{1}{2}\sin x \begin{cases} >0 \text{ if } -\pi/2<x<0, \\ <0 \text{ if } \ 0<x<\pi/2, \end{cases} \\ 0 \quad \text{if} \quad |x|>\pi/2; \end{cases}$

the graph of the function $F(x)$ is concave over $(-\pi/2, 0)$ and convex over $(0, \pi/2)$; the function $f(x)$ increases on $(-\pi/2, 0)$, decreases on $(0, \pi/2)$ and has a local maximum $1/2$ at the point $x=0$.

c) $f''(x)=\begin{cases} -\frac{1}{2}\cos x<0 & \text{if} \quad |x|<\frac{\pi}{2}, \\ 0 & \text{if} \quad |x|>\frac{\pi}{2}; \end{cases}$

the graph of the function $f(x)$ is convex over the interval $(-\pi/2, \pi/2)$.

The graphs of the functions $F(x)$, $f(x)$ are represented on the fig. 8.2, 8.3.

Ex. 8. Solve the same problem for the function

$$f(x)=\begin{cases} Ax^2 \text{ if } x\in(0,2); \\ 0 \quad \text{if } x\notin(0,2). \end{cases}$$

Answer. $A=3/8$,

$$F(x)=\begin{cases} \int\limits_{-\infty}^{x} 0dx = 0 \text{ for } x<0, \\ \int\limits_{-\infty}^{0} 0dx+\int\limits_{0}^{x}\frac{3}{8}x^2dx=\frac{3}{8}\cdot\frac{x^3}{3}\Big|_0^x=\frac{1}{8}x^3 \text{ for } 0\le x\le 2, \\ \int\limits_{-\infty}^{0} 0dx+\int\limits_{0}^{2}\frac{3}{8}x^2dx+\int\limits_{2}^{x}0dx=1 \text{ for } x>2. \end{cases}$$

Ex. 9. Find and graph the distribution density of the random variable which we've studied in Ex. 5.

The distribution function of the random variable is

$$F(x)=\begin{cases} 0 & \text{if} \quad x<-1, \\ \frac{1}{\pi}\arcsin x+\frac{1}{2} & \text{if} \quad -1\le x\le 1, \\ 1 & \text{if} \quad x>1. \end{cases}$$

Its distribution density by the definition 3 (see the formula (10)) equals

$$f(x)=F'(x)=\begin{cases} 0 \quad \text{if} \quad |x|>1, \\ \frac{1}{\pi}\cdot\frac{1}{\sqrt{1-x^2}} \text{ if } -1<x<1. \end{cases}$$

To graph the distribution density $f(x)$ we'll notice:

a) $\lim\limits_{x\to-1+0} f(x)=\lim\limits_{x\to 1-0} f(x)=+\infty$; the part of the graph over the interval $(-1,1)$ has two vertical asymptotes namely $x=-1, x=1$;

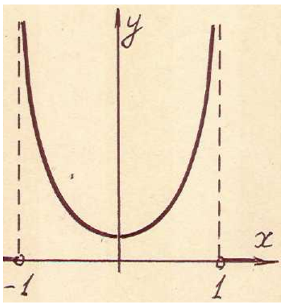


Fig. 9

b)

$$f'(x)=F''(x)=\begin{cases} 0 & \text{if } |x|>1, \\ \frac{1}{\pi}\cdot\frac{x}{(1-x^2)\sqrt{1-x^2}} & \begin{cases} <0 \text{ if } -1<x<0, \\ >0 \text{ if } 0<x<1. \end{cases} \end{cases}$$

the function $f(x)$ decreases on the interval $(-1,0)$, increases on $(0,1)$ and has a local extremum $1/\pi$ at the point $x=0$; its graph descends over $(-1,0)$ and ascends over $(0,1)$ and passes through the point $(0; 1/\pi)$ of the Oy-axis.

$$\text{c) } f''(x) = \begin{cases} 0 \quad \text{if} \quad |x| > 1, \\ \dfrac{1}{\pi} \cdot \dfrac{1 + 2x^2}{(1 - x^2)^2 \sqrt{1 - x^2}} > 0 \text{ if } -1 < x < 1; \end{cases}$$

the curve of the function $f(x)$ is concave over the interval $(-1, 1)$. Its graph is represented on fig. 9.

POINT 3. A FUNCTION OF A RANDOM VARIABLE

Let a random variable η be a function of a random variable ξ,

$$\eta = \varphi(\xi).$$

It's necessary to find the distribution law of η knowing that of ξ.

At first let ξ be a discrete random variable with the distribution table

ξ	x_1	x_2	$\dots$	x_n
$p_i = P(\xi = x_i)$	$p_1 = P(\xi = x_1)$	$p_2 = P(\xi = x_2)$	$\dots$	$p_n = P(\xi = x_n)$

a) If a function $y = \varphi(x)$ is monotone one, then to any value $\xi = x_i$ of ξ there corresponds unique value $y_i = \varphi(x_i)$ of η with the same probability as that of $\xi = x_i$,

$$q_i = P(\eta = y_i) = P(\eta = \varphi(x_i)) = P(\xi = x_i) = p_i .$$

Hence the distribution table of the function $\eta = \varphi(\xi)$ in this case is

$\eta = \varphi(\xi)$	$y_1 = \varphi(x_1)$	$y_2 = \varphi(x_2)$	$\dots$	$y_n = \varphi(x_n)$
$q_i = P(\eta = y_i) = P(\eta = \varphi(x_i))$	$q_1 = p_1$	$q_2 = p_2$	$\dots$	$q_n = p_n$

b) If a function $y = \varphi(x)$ isn't monotone one, then to some value $\eta = y_i$ of the random variable $\eta = \varphi(\xi)$ there can correspond several values $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ of the random variable ξ. In this case to find the probability of the value $\eta = y_i$ we must take the sum of all the probabilities of those values of ξ,

$$P(\eta = y_i) = P((\xi = x_{i_1}) + (\xi = x_{i_2}) + \dots + (\xi = x_{i_k})) = p_{i_1} + p_{i_2} + \dots + p_{i_k} .$$

Ex. 10. Compiling the distribution law of the function $\eta = \varphi(\xi) = \xi^2$ of a discrete random variable ξ (see the next two tables).

ξ	-2	-1	0	1	2	3	$\eta = \xi^2$	0	1	4	9
p_i	0.1	0.2	0.1	0.3	0.2	0.1	q_j	0.1	0.5	0.3	0.1

Here $P(\eta = 0) = P(\xi = 0) = 0.1$, $P(\eta = 9) = P(\xi = 3) = 0.1$, and

$$P(\eta = 1) = P((\xi = -1) + (\xi = 1)) = P(\xi = -1) + P(\xi = 1) = 0.2 + 0.3 = 0.5,$$

$$P(\eta = 4) = P((\xi = -2) + (\xi = 2)) = P(\xi = -2) + P(\xi = 2) = 0.1 + 0.2 = 0.3.$$

Now let ξ be a continuous random variable with the distribution density $f(x)$, and let

$$\eta = \varphi(\xi)$$

be its function. It's required to find the distribution function and, if it's possible, the distribution density of this function.

Let's denote by $M_{x,y}$ the next set of reals:

$$M_{x,y} = \{x : \varphi(x) < y\}.$$

Then the distribution function of the random variable $\eta = \varphi(\xi)$ can be found as follows

$$G(y) = P(\eta < y) = P(\varphi(\xi) < y) = P(\xi \in M_{x,y}) = \int_{M_{x,y}} f(x)dx = \int_{\varphi(x)<y} f(x)dx. \quad (17)$$

If the latter integral possesses the derivative, then the function $\eta = \varphi(\xi)$ has the distribution density, namely

$$g(y) = G'(y). \quad (18)$$

The formula (17) introduces a number function $y = \varphi(x)$. Its domain of definition depends on the values which can take on a random variable ξ.

There are formulas which express $g(y)$ in terms of inverse function $x = \phi(y)$ of the function $y = \varphi(x)$ if the latter is increasing or decreasing one. Namely,

$g(y) = f(\phi(y))\phi'(y)$ if the function $y = \varphi(x)$ increases,

$g(y) = -f(\phi(y))\phi'(y)$ if it decreases.

There are also general notices for treating with non-monotone (non-increasing or non-decreasing) function $y=\varphi(x)$.

We'll put aside these general considerations and study some interesting examples.

Ex. 11. Let η be a linear function of ξ,

$$\eta=\varphi(\xi)=a\xi+b, a\neq 0. \qquad (19)$$

Corresponding number function is $y=\varphi(x)=ax+b$. It's defined on the whole set of reals and can take on all real values. By the formula (17) the distribution function of the random variable η equals

$$G(y)=\int_{ax+b<y} f(x)dx.$$

a) If $a>0$, then the inequality $ax+b<y$ implies $x<(y-b)/a$, whence it follows that

$$G(y)=\int_{-\infty}^{\frac{y-b}{a}} f(x)dx$$

and, in condition of continuity of the function $f(x)$,

$$g(y)=G'(y)=\left(\int_{-\infty}^{\frac{y-b}{a}} f(x)dx\right)'_y=f\left(\frac{y-b}{a}\right)\left(\frac{y-b}{a}\right)'_y=\frac{1}{a}f\left(\frac{y-b}{a}\right).$$

b) If $a<0$, then the inequality $ax+b<y$ implies $x>(y-b)/a$, hence

$$G(y)=\int_{\frac{y-b}{a}}^{\infty} f(x)dx=-\int_{\infty}^{\frac{y-b}{a}} f(x)dx.$$

and

$$g(y)=G'(y)=\left(-\int_{\infty}^{\frac{y-b}{a}} f(x)dx\right)'_y=-f\left(\frac{y-b}{a}\right)\left(\frac{y-b}{a}\right)'_y=-\frac{1}{a}f\left(\frac{y-b}{a}\right).$$

As general result we can write

$$g(y)=\frac{1}{|a|}f\left(\frac{y-b}{a}\right). \tag{20}$$

Ex. 12. Let $\eta=\varphi(\xi)=\xi^2$ and a random variable can take on all real values.

Corresponding number function $y=\varphi(x)=x^2$ has the domain of definition $D(y)=(-\infty,\infty)$ and takes on non-negative values. With the help of the formula (17) (after solving the inequality $x^2<y$, fig. 10 a) we obtain the distribution function of the random variable $\eta=\xi^2$

$$G(y)=\int_{x^2<y}f(x)dx=\int_{-\sqrt{y}}^{\sqrt{y}}f(x)dx$$

provided that $y\geq 0$. It's evident that $G(0)=0$, and we must put $G(y)=0$ for $y<0$. Therefore

$$G(y)=\begin{cases}\int_{-\sqrt{y}}^{\sqrt{y}}f(x)dx \text{ for } y>0,\\ 0 \quad \text{for} \quad y\leq 0,\end{cases}$$

and so the distribution density of the function in question equals

$$g(y)=G'(y)=\begin{cases}\left(\int_{-\sqrt{y}}^{\sqrt{y}}f(x)dx\right)'_y \text{ for } y>0,\\ 0 \quad \text{for} \quad y<0.\end{cases}$$

Since (if $f(x)$ is a continuous function)

$$\left(\int_{-\sqrt{y}}^{\sqrt{y}}f(x)dx\right)'_y=\left(\int_0^{\sqrt{y}}f(x)dx-\int_0^{-\sqrt{y}}f(x)dx\right)'_y=f(\sqrt{y})(\sqrt{y})'-f(-\sqrt{y})(-\sqrt{y})'=$$

$$=f(\sqrt{y})\frac{1}{2\sqrt{y}}+f(-\sqrt{y})\frac{1}{2\sqrt{y}}=\frac{1}{2\sqrt{y}}\left(f(\sqrt{y})+f(-\sqrt{y})\right)$$

we finally get

$$g(y)=\begin{cases}\frac{1}{2\sqrt{y}}\left(f(\sqrt{y})+f(-\sqrt{y})\right) \text{for } y>0,\\ 0 \qquad\qquad\qquad \text{for} \quad y<0.\end{cases}$$

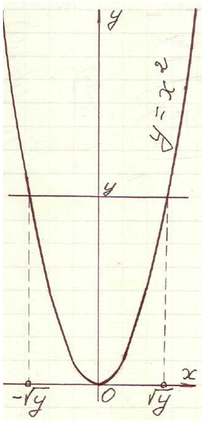


Fig. 10 a

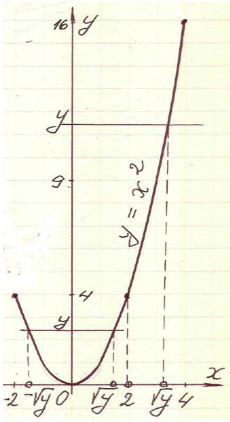


Fig. 10 b

Ex. 13. Let $\eta = \varphi(\xi) = \xi^2$ as in the preceding example, but let a random variable ξ take on its values only on the interval $(-2, 4)$ having the distribution density

$$f(x) = \begin{cases} f_1(x) \; if \; -2 < x < 4, \\ 0 \; if \; x < -2 \; or \; x > 4. \end{cases}$$

Corresponding function $y = \varphi(x) = x^2$ is defined on the interval $(-2, 4)$. For $x \in (-2, 2)$ the values of y are included in the interval $[0, 4)$, and for $x \in [2, 4)$ in the interval $[4, 16)$ (see fig. 10 b). On the interval $x \in (-2, 2)$ (for $y \in [0, 4)$) the inequality $\varphi(x) = x^2 < y$ implies $-\sqrt{y} < x < \sqrt{y}$ (as in Ex. 12). On the interval $[2, 4)$ (for $y \in [4, 16)$) the inequality $\varphi(x) = x^2 < y$ gives $2 \le x < \sqrt{y}$ (fig. 10b). By the formula (17) the unknown distribution function (for $0 < y < 16$) equals

$$G(y) = P(\eta < y) = P(\xi^2 < y) = \int\limits_{x^2 < y} f(x)dx = \begin{cases} \int\limits_{-\sqrt{y}}^{\sqrt{y}} f_1(x)dx \text{ if } 0 < y < 4, \\ \int\limits_{2}^{\sqrt{y}} f_1(x)dx \text{ if } 4 < y < 16. \end{cases}$$

We must suppose $G(y) = 0$ for $y \le 0$ and $G(y) = 1$ for $y > 16$. As to the distribution density of the function $\eta = \varphi(\xi) = \xi^2$ we obtain its value by analogy with Ex. 12,

$$g(y) = G'(y) = \begin{cases} \frac{1}{2\sqrt{y}}\left(f_1(\sqrt{y}) + f_1(-\sqrt{y})\right) & \text{if} \quad 0 < y < 4, \\ \frac{1}{2\sqrt{y}} f_1(\sqrt{y}) & \text{if} \quad 4 < y < 16, \\ 0 & \text{if} \quad y \le 0 \text{ or } y > 16. \end{cases}$$

RANDOM VARIABLES: basic terminology RUEFD

1. бином Ньютона	біном Н"ютона	Newton's binómial	binôme *m* de Newton	Newtonscher Binóm(*n*–s,-e)
2. биномиальная формула	біномна формула, формула бінома	binómial fórmula	formule *f* binomiale	Binomiálformel (*f* =,-n)
3. биномиально распределенная случайная величина	біномно розподілена випадкова величина	binómially distríbuted rándom váriable	variable *f* aléatoire distribuée [répartie] binomiallement	binomiálverteilte Zúfallsgröße
4. биномиальное распределение	біномний розподіл	binómial dìstribútion [partítion, law]	distribution *f* [répartition *f* , loi *f*] binomiale	Binomiálverteilung (*f* =)
5. биномиальный коэффициент	біномний коефіцієнт	binómial coefficient	coefficient *m* binomial	Binomiálkoeffizient (*m* –en, -en)
6. быть распределенной по данному закону	бути розподіленою за даним законом	be distríbuted by [accórding to, in accórdance, accordingly] the given law	etre distribuée [répartie] suivant [en, par, conformément] la loi donnée	nach gegébenem Gesétz vertéilt sein
7. вероятность отдельного значения случайной величины	імовірність окремого значення випадкової величини	pròbabílity of séparate/ísolated válue of a rándom váriable	probabilité *f* d’une valeur séparée [isolée, particulière] d’une variable aléatoire	Wahrschéinlichkeit (*f* =) von Éinzelwert(e) einer Zúfallsgröße
8. вероятность попадания случайной величины на полупрямую [в интервал, на отрезок, в полуинтервал]	імовірність попадання випадкової величини на півпряму [в інтервал, на відрізок, в півінтервал]	pròbabílity of hit/hitting of a rándom váriable on a halfline [in an ínterval, in a ségment, in a half-ínterval]	probabilité *f* du coup [d’impact, d’atteinte] d’une variable aléatoire sur une demi-droite [dans un intervalle, dans un segment, dans un	Wahrschéinlichkeit (*f*) von Tréffer einer Zúfallsgröße in/auf/an einer Hálbegerade [in ein Interváll, in/auf/an einem Ábschnitt(e) [einem Segmént(e),

			semi-intervalle]	einer Strécke], in ein Hálbintervall (*n*)]
9. вероятность успеха	імовірність успіху	succéss pròbabílity	probabilité *f* d'un succès	Erfólgswahrscheinlichkeit (*f* =)
10.возможное значение (случайной величины)	можливе значення (випадкової величини)	póssible válue (of a rándom váriable)	valeur *f* possible (d'une variable aléatoire)	möglicher Wert (*m* –(e)s, -e) einer Zúfallsgrößе
11.дискретная случайная величина	дискретна випадкова величина	discréte rándom váriable	variable *f* aléatoire discrète	diskréte Zúfallsgröße
12.дифференциальная функция распределения	диференціальна функція розподілу	dìfferéntial dìstríbutive fúnction	fonction *f* différentielle de répartition [de distribution]	differentiále Vertéilungsfunktion
13.закон [распределение] вероятностей	закон [розподіл] імовірностей	law of probabílities	loi *f* des probabilités	Wahrschéinlickeitsgesetz (*n* –es,-e)
14.закон распределения	закон розподілу	dìstribútion [partítion]law, law of dìstribútion [of partítion]	loi *f* de répartition [de distribution]	Vertéilungsgesetz (*n* –es, -e)
15.закон распределения (вероятностей) случайной величины	закон розподілу (ймовірностей) випадкової величини	law of (pròbabílity) dìstribútion, dìstríbutive law (of pròbabílities) (of a rándom váriable)	loi *f* de distribution [répartition] (des probabilités) d'une variable aléatoire	Vertéilungsgesetz (*n*) (von Wahrschéinlichkeiten) einer Zúfallsgröße
16.закон распределения Бернулли	закон розподілу Бернуллі	Bernoulli(´s) law of dìstribútion, Bernoulli distributive law	loi *f* de distribution [de répartition] de Bernoulli	Bernoullisches Vertéilungsgesetz (*n* –es,-e)
17.закон редких событий	закон рідких подій	law of rare evénts	loi *f* d'événements rares	Gesétz *n* der séltenen Eréignisse
18.значение случайной	значення випадкової ве-	válue of a rándom váriable	valeur *f* d'une variable aléa-	Zúfallsgröße-wert (*m*–(e)s,-

величины	личини		toire	e)
19.иметь данное распределение (о случайной величине)	мати даний розподіл (про випадкову величину)	have a given dìstribútion [partítion,law] (of a rándom váriable)	avoir une distribution [répartition, loi] donnée (de variable *f* aléatoire)	gegébene Vertéilung haben* (habe, hast, hat) (über eine Zúfallsgröße)
20.интегральная теорема Лапласа	інтеґральна теорема Лапласа	Laplace integral théorem	théorème *m* intégrale de Laplace	Laplacesches Integráltheorem (*n*), Laplacescher Grénzwertsatz (*m*), Laplacescher integráler Grénzsatz
21.интегральная функция распределения	інтеґральна функція розподілу	íntegral dìstribútion [partítion]fúnction	fonction *f* intégrale de répartition [de distribution]	Integrálverteilungsfunktion (*f*), Integrále Vertéilungsfunktion
22.испытания до первого успеха (до первого наступления события)	випробування до першого успіху (до першого настання[першої появи] події)	tríals/expériments till (=untíl) [up to, to] the first succéss [to the first occúrrence of an evént]	expériences [épreuves, essais] jusqu'au premier succès [jusqu'à la première arrivée [apparition] d'événement]	Versúche (*m*) [Experiménte (*n*), Próben (*f*)] bis den ersten Erfólg [bis den ersten Éintritt des Eréignisses]
23.количество независимых испытаний до первого успеха [до первого наступления события]	число/кількість незалежних випробувань до першого успіху [до першого настання, до першої появи події]	númber of índepéndent tríals (un)till/to the first succéss[(un)till/to the first occúrrence of an evént]	nombre *m* des expériences [épreuves] indépendantes [des essais indépendants] jusqu'au premier succès [jusqu'à la première arrivée/apparition d'événement]	die Ánzahl (*f* =) von únabhängigen Versúchen/Experiménten/Próben bis den ersten Erfólg [bis den ersten Éintritt/Éintreten des Eréignisses]
24. количество успехов	число/кількість успіхів	númber of succésses (of	nombre *m* des succès [des	die Ánzahl (*f* =) von Erfól-

(наступлений события) в *n* независимых испытаниях (при постоянной вероятности события/успеха)	(появ/настань події) в *n* незалежних випробуваннях (при сталій ймовірності події/успіху)	occúrrences of an evént) in *n* índependént tríals (as the pròbabílity of the succéss/ evént is cónstant)	arrivées/apparitions d’événement] à *n* expériences/ épreuves indépendantes [essais indépendants] (si la probabilité du succès [d’événement] est constante)	gen [die Ánzahl von Éintritten/Éintretenen eines Eréignisses] in *n* únabhängigen Versúchen/Experiménten/Próben (wenn die Wahrschéinlichkeit des Erfólges [des Eréignisses] konstánt ist)
25.кривая распределения	крива розподілу	dìstribútion [partítion] cúrve	courbe *f* de répartition [de distribution *f*]	Vertéilungskurve ([-və и -fə] *f* =,-n)
26.локальная теорема Лапласа	локальна теорема лапласа	laplace lócal théorem	théorème *m* local de laplace	Laplacescher lokáler Grénz(wert)satz (*m*)
27.многоугольник распределения	многокутник розподілу	pólygon of dìstribútion [partítion, law], distribútion pólygon	polygone *m* de distribution [de répartition , de loi]	Häufigkeitspolygon *n*, Vertéilungspolygon *n*
28.наивероятнейшее число	найімовірніше число	the most probable númber	nombre *m* le plus probable	wahrschéinlichste Ánzahl
29.независимые испытания	незалежні випробування	índependént tríals [expériments]	expériences *f* [épreuves *f*] indépendantes [essais *m* indépendants]	únabhängige Versúche (*m*) [Experiménten, Próben (*f*)]
30.непрерывная случайная величина	неперервна випадкова величина	contínuous rándom váriable	variable *f* aléatoire continue	stétige [kontinuíerliche]Zúfallsgröße (*f*)
31.ограниченное количество испытаний	обмежена кількість випробувань	límited númber of tríals	nombre *m* limité des expériences	Begrénzte/beschränkte/limitíerbare Ánzahl von Versúchen [Expe-

				riménten, Próben]
32.отклонение (относительной) частоты события от его вероятности	відхилення (відносної) частоти події від її ймовірності	dèviátion of a (rélative) fréquency of an evént from its pròbabílity	déviation *f* [écart *m*]d'une fréquence (relative) d'événement de sa probabilité	Ábweichung (*f* =,-en) (relatíver) Frequénz [einer Häufigkeit (*f*)] eines Eréignisses von ihr Wahrschéinlichkeit
33.плотность вероятности	щільність ймовірності	pròbabílity dénsity, dénsity of pròbabílity	densité *f* de probabilité	Wahrschéinlichkeitsdichte (*f*)
34.плотность распределения	щільність розподілу	dénsity of dìstribútion [of partítion], dìstribútion [partítion] dénsity	densité *f* de répartition *f* [de distribution *f*]	Vertéilungsdichte (*f*)
35.плотность распределения вероятностей	щільність розподілу ймовірностей	dénsity of dìstribútion [partítion] [dìstribútion [partítion] dénsity] of probabilities	densité *f* de répartition [de distribution] des probabilités	Vertéilungsdichte (*f*) von Wahrschéinlichkeiten (*f*)
36.принимать значение (о случайной величине)	набувати значення (про випадкову величину)	take on a válue (of a rándom váriable)	prendre une valeur (d'une variable aléatoire)	Wert (*m* –(e)s, -e) ánnehmen* (über eine Zúfallsgröße)
37.принимать отдельные изолированные возможные значения	набувати окремих ізольованих можливих значень	take on séparate ísolated póssible válues	prendre des valeurs possibles séparées isolées	éinzelne isolíerte mögliche Werte ánnehmen*
38.происходить *k* раз в *n* независимых испытаниях (о событии)	відбуватися *k* раз в *n* незалежних випробуваннях (про подію)	occúr [appear, happen, take place] *k* times in *n* índepéndent tríals (about an evént)	arriver [se présenter] *k* fois à *n* expériences/épreuves indépendantes [essais indépendants] (sur un événe-	Éintreten*(-trete,-trittst,-tritt) *k* Male in *n* únabhängigen Versúchen/Experiménten/Próben (über ein

			ment)	Eréignis)
39.распределение	розподіл	dìstribútion [partítion,law]	distribution *f* [répartition *f*, loi *f*]	Vertéilung (*f*)
40.распределение Бернулли	розподіл Бернуллі	Bernoulli(´s) dìstribútion [partítion,law]	distribution *f* [répartition *f*, loi *f*] de Bernoulli	Bernoullische Vertéilung
41.распределение вероятностей	розподіл ймовірностей	pròbabílity dìstribútion[partítion, law], dìstribútion [partítion,law] of pròbabílities	distribution *f* [répartition *f*, loi *f*] (des probabilités)	Wahrschéinlichkeitsverteilung (*f* =)
42.распределение Пуассона	розподіл Пуассона	Poisson(´s) dìstribútion [partítion,law]	distribution *f* [répartition *f*, loi *f*] de Poisson	Poissonverteilung, Poissonsche Vertéilung
43.распределение редких событий (распределение Пуассона)	розподіл рідких подій (розподіл Пуассона)	dìstribútion [partítion,law] of rare evénts	distribution *f* [répartition *f*, loi *f*] d'événements rares	Vertéilung (*f* =) der séltenen Eréignisse
44.связь [соотношение, соответствие] между возможными значениями случайной величины и соответствующими вероятностями	зв"язок [співвідношення, відповідність] між можливими значеннями випадкової величини та відповідними ймовірностями	connéction [relátion, correlátion, correspóndence] between póssible válues of a rándom váriable and correspónding probabílities	lien *m* [corrélation *f*, correspondance *f*] entre valeurs possibles d’une variable aléatoire et des probabilités correspondantes	Zusámmengang *m* [Verbíndung *f*, Verhältnis *n*] zwíschen möglichen Werten einer Zúfallsgröße und entsprèchenden Wahrschéinlichkeiten
45.случайная величина	випадкова величина	rándom/áleatory váriable	variable *f* aléatoire, aléa *m* numérique	Zúfallsgröße (*f* =,-n) [aleatórische Größe
46.таблица распределения (случайной величины)	таблиця розподілу (випадкової величини)	table of the dìstribútion [partítion,law] of a rándom váriable	table *f* [tableau *m*] de distribution *f* [de répartition *f*]	Vertéilungstabelle (*f*) (einer Zúfallsgröße

			d’une variable aléatoire	
47.успех	успіх	succéss	succès *m*	Erfőlg (*m* – (e)s,-e)
48.формула Пуассона	формула Пуассона	Poisson's fórmula	formule *f* de Poisson	Poissonsche Formel (*f*)
49.функция распределения	функція розподілу	distribútion [partítion] fúnction	fonction *f* de répartition *f* [de distribution *f*]	Vertéilungsfunktion (*f*)
50.функция распределения вероятностей	функція розподілу ймовірностей	dìstribútion [partítion] fúnction of probabilities	fonction *f* de répartition *f* [de distribution *f*] des probabilités	Vertéilungsfunktion von Wahrschéinlichkeiten (*f*)
51.фунция Лапласа	функція Лапласа	Laplace fúnction	fonction *f* de Laplace	Laplacesche Funktión
52.целиком заполнять некоторый интервал (о возможных значениях непрерывной случайной величины)	цілком заповнювати деякий інтервал (про можливі значення неперервної випадкової величини)	fill completely [entirely] some ínterval (of póssible válues of a contínuous rándom váriable)	remplir (de façon complète) un intervalle certain (de valeurs possibles d’une variable aléatoire continue)	ein gewíssere Intervàll (*n*) ganz [vóllständig, völlig] füllen (über mögliche Werte einer stétigen Zúfallsgröße (*f*))
53.элемент вероятности	елемент імовірності	pròbabílity élement	elément *m* de probabilité	Wahrschéinlichkeitselement (*n*–(e)s,-e)

RANDOM VARIABLES: basic terminology EFDRU

1. be distríbuted by [accórding to, in accórdance, accordingly] the given law	etre distribuée [répartie] suivant [en, par, conformément] la loi donnée	nach gegébenem Gesétz vertéilt sein	быть распределенной по данному закону	бути розподіленою за даним законом
2. Bernoulli(΄s) dìstribútion [partítion, law]	distribution *f* [répartition *f*, loi *f*] de Bernoulli	Bernoullische Vertéilung (*f*)	распределение Бернулли	розподіл Бернуллі
3. Bernoulli(΄s) law of dìstribútion, bernoulli distributive law	loi *f* de distribution *f* [de répartition *f*] de Bernoulli	Bernoullisches Vertéilungsgesetz (*n* –es,-e)	закон распределения Бернулли	закон розподілу Бернуллі
4. binómial coefficient	coefficient *m* binomial	Binomiálkoeffizient (*m* –en, -en)	биномиальный коэффициент	біномний коефіцієнт
5. binómial dìstribútion [partítion, law]	distribution *f* [répartition *f* , loi *f*] binomiale	Binomiálverteilung (*f* =)	биномиальное распределение	біномний розподіл
6. binómial fórmula	formule *f* binomiale	Binomiálformel (*f* =,-n)	биномиальная формула	біномна формула, формула бінома
7. binómially distríbuted rándom váriable	variable *f* aléatoire distribuée [répartie] binomiallement	binomiálverteilte Zúfallsgröße (*f* =,-n)	биномиально распределенная случайная величина	біномно розподілена випадкова величина
8. connéction [relátion, correlátion, correspóndence] between póssible válues of a rándom váriable and correspónding probabílities	lien *m* [corrélation *f*, correspondance *f*] entre valeurs possibles d'une variable aléatoire et des probabilités correspondantes	Zusámmengang *m* [Verbíndung *f*, Verhältnis *n*] zwíschen möglichen Werten einer Zúfallsgröße und entspréchenden Wahrschéinlichkeiten	связь [соотношение, соответствие] между возможными значениями случайной величины и соответствующими вероятностями	зв"язок [співвідношення, відповідність] між можливими значеннями випадкової величини та відповідними ймовірностями

9. contínuous rándom váriable	variable *f* aléatoire continue	stétige [kontinuíerliche] Zúfallsgröße	непрерывная случайная величина	неперервна випадкова величина
10.dénsity of dìstribútion [of partítion], dìstribútion [partítion] dénsity	densité *f* de répartition [de distribution]	Vertéilungsdichte (*f*)	плотность распределения	щільність розподілу
11.dénsity of dìstribútion [partítion], dìstribútion [partítion] dénsity of probabilities	densité *f* de répartition [de distribution] des probabilités	Vertéilungsdichte (*f*) von Wahrschéinlichkeiten	плотность распределения вероятностей	щільність розподілу ймовірностей
12.dèviátion of a (rélative) fréquency of an evént from its pròbabílity	déviation *f* [écart *m*]d'une fréquence (relative) d'événement de sa probabilité	Ábweichung (*f* =,-en) (relatíver) Frequénz [einer Häufigkeit] eines Eréignisses von ihr Wahrschéinlichkeit	отклонение (относительной) частоты события от его вероятности	відхилення (відносної) частоти події від її ймовірності
13.dìfferéntial dìstríbutive fúnction	fonction *f* différentielle de répartition [de distribution]	differentiále Vertéilungsfunktion	дифференциальная функция распределения	диференціальна функція розподілу
14.discréte rándom váriable	variable *f* aléatoire discrète	diskréte Zúfallsgröße	дискретная случайная величина	дискретна випадкова величина
15.dìstribútion [partítion, law]	distribution *f* [répartition *f*, loi *f*]	Vertéilung (*f*)	распределение	розподіл
16.dìstribútion [partítion, law] of rare evénts	distribution *f* [répartition *f*, loi *f*] d'événements rares	Vertéilung der séltenen Eréignisse	распределение редких событий (распределение пуассона)	розподіл рідких подій (розподіл пуассона)
17.dìstribútion [partítion] cúrve	courbe *f* de répartition [de distribution]	Vertéilungskurve ([-və и –fə] *f* =,-n)	кривая распределения	крива розподілу
18.dìstribú-	fonction *f* de	Vertéilungs-	функция рас-	функція роз-

tion [partítion] fúnction	répartition [de distribution]	funktion (*f*)	пределения	поділу
19.dìstribútion [partítion] fúnction of probabilities	fonction *f* de répartition [de distribution] des probabilités	Vertéilungsfunktion von Wahrschéinlichkeiten	функция распределения вероятностей	функція розподілу ймовірностей
20.dìstribútion [partítion] law, law of dìstribútion [of partítion]	loi *f* de répartition [de distribution]	Vertéilungsgesetz (*n* –es, -e)	закон распределения	закон розподілу
21.fill completely [entirely] some ínterval (of póssible válues of a contínuous rándom váriable)	remplir (de façon complète) un intervalle certain (de valeurs possibles d'une variable aléatoire continue)	ein gewíssere Intervállˌ (*n*) ganz [vóllständig, völlig] füllen (über mögliche Werte einer stétigen Zúfallsgröße)	целиком заполнять некоторый интервал (о возможных значениях непрерывной случайной величины)	цілком заповнювати деякий інтервал (про можливі значення неперервної випадкової величини)
22.have a given dìstribútion [partítion, law] (of a rándom váriable)	avoir une distribution [répartition, loi] donnée (de variable aléatoire)	gegébene Vertéilung(*f*=) haben* (habe, hast, hat)(über eine Zúfallsgröße)	иметь данное распределение (о случайной величине)	мати даний розподіл (про випадкову величину)
23.índepéndent tríals [expériments]	expériences *f* [épreuves *f*] indépendantes [essais *m* indépendants]	únabhängige Versúche (*m*) [Experiménte *n*, Próben (*f*)]	независимые испытания	незалежні випробування
24.íntegral dìstribútion [partítion] fúnction	fonction *f* intégrale de répartition [de distribution]	Integrálverteilungsfunktion (*f*), Integrále Vertéilungsfunktion (*f*)	интегральная функция распределения	інтеґральна функція розподілу
25.Laplace fúnction	fonction *f* de Laplace	Laplacesche Funktión	фунция Лапласа	функція Лапласа
26.Laplace integral théorem	théorème *m* intégrale de Laplace	Laplacesches Integráltheorem, Laplacescher Grénzwertsatz (*m*),	интегральная теорема Лапласа	інтеґральна теорема Лапласа

		Laplacescher integráler Grénzsatz		
27.Laplace lócal théorem	théorème *m* local de Laplace	Laplacescher lokáler Grénz(wert)satz (*m*)	локальная теорема Лапласа	локальна теорема Лапласа
28.law of (pròbabílity) dìstribútion, dìstríbutive law (of pròbabílities) (of a rándom váriable)	loi *f* de distribution [répartition] (des probabilités) d'une variable aléatoire	Vertéilungsgesetz (*n*) (von Wahrschéinlichkeiten) einer Zúfallsgröße	закон распределения (вероятностей) случайной величины	закон розподілу (ймовірностей) випадкової величини
29.law of probabílities	loi *f* des probabilités	Wahrschéinlickeitsgesetz (*n* –es,-e)	закон [распределение] вероятносей	закон [розподіл] імовірностей
30.law of rare evénts	loi *f* d'événements rares	Gesétz *n* der séltenen Eréignisse	закон редких событий	закон рідких подій
31.límited númber of tríals	nombre *m* limité des expériences	begrénzte/beschränkte/limitíerbare Ánzahl von Versúchen [Experiménten, Próben]	ограниченное количество испытаний	обмежена кількість випробувань
32.Newton's binómial	binôme *m* de Newton	Newtonscher Binóm(*n*–s,-e)	бином Ньютона	біном Н"ютона
33.númber of índepéndent tríals (un)till/ to the first succéss[(un)til /to the first occúrrence of an evént]	nombre *m* des expériences [épreuves] indépendantes [essais indépendants] jusqu'au premier succès [jusqu'à la première arrivée/apparition d'événement]	Die Ánzahl (*f* =) von únabhängigen Versúchen/Experiménten/Próben bis den ersten Erfólg [bis den ersten Éintritt/Éintreten des Eréignisses]	количество независимых испытаний до первого успеха [до первого наступления события]	число/кількість незалежних випробувань до першого успіху [до першого настання, до першої появи події]
34.númber of succésses (of	nombre *m* des succès [des	die Ánzahl (*f* =) von Erfól-	количество успехов (на-	число/кількість успіхів

occúrrences of an evént) in *n* índepéndent tríals (as the pròbabílity of the succéss/ evént is cónstant)	arrivées/apparitions d'événement] à *n* expériences/ épreuves indépendantes [essais indépendants] (si la probabilité du succès [d'événement] est constante)	gen [die Ánzahl von Éintritten/Éintretenen eines Eréignisses *n*] in *n* únabhängigen Versúchen/Experiménten/Próben (wenn die Wahrschéinlichkeit des Erfólges [des Eréignisses] konstánt ist)	ступлений события) в *n* независимых испытаниях (при постоянной вероятности события/успеха)	(появ/настань події) в *n* незалежних випробуваннях (при сталій імовірності події/успіху)
35.occúr [appear, happen, take place] *k* times in *n* índepéndent tríals (about an evént)	arriver [se présenter] *k* fois à *n* expériences/épreuves indépendantes [essais indépendants] (sur un événement)	éintreten*(-trete,-trittst,-tritt) *k* Male in *n* únabhängigen Versúchen/Experiménten/Próben (über ein Eréignis)	происходить *k* раз в *n* независимых испытаниях (о событии)	відбуватися *k* раз в *n* незалежних випробуваннях (про подію)
36.Poisson(´s) dìstribútion [partítion,law]	distribution *f* [répartition *f*, loi *f*] de Poisson	Poissonverteilung (*f* =), Poissonsche Vertéilung	распределение Пуассона	розподіл Пуассона
37.Poisson's fórmula	formule *f* de Poisson	Poissonsche Formel (*f*)	формула Пуассона	формула Пуассона
38.pólygon of dìstribútion [partítion, law], dìstribútion pólygon	polygone *m* de distribution *f* [de répartition, de loi]	Häufigkeitspolygon *n*, Vertéilungspolygon *n*	многоугольник распределения	многокутник розподілу
39.póssible válue (of a rándom váriable)	valeur *f* possible (d'une variable aléatoire)	möglicher Wert (*m* –(e)s, -e) einer Zúfallsgröße	возможное значение (случайной величины)	можливе значення (випадкової величини)
40.pròbabílity dénsity, dénsity of pròbabílity	densité *f* de probabilité	Wahrschéinlichkeitsdichte (*f*)	плотность вероятности	щільність ймовірності
41.pròbabílity	distribution *f*	Wahrschéin-	распределе-	розподіл імо-

dìstribútion [partítion, law], dìstribútion [partítion, law] of pròbabílities	[répartition *f*, loi *f*] (des probabilités)	lichkeitsverteilung (*f* =)	ние вероятностей	вірностей
42.pròbabílity élement	elément *m* de probabilité	Wahrschéinlichkeitselement (*n*–(e)s,-e)	элемент вероятности	елемент імовірності
43.pròbabílity of hit/hitting of a rándom váriable on a halfline [in an ínterval, in a ségment, in a half-ínterval]	probabilité *f* du coup [d'impact [-kt] *m*, d'atteinte *f*] d'une variable aléatoire sur une demi-droite [dans un intervalle, dans un segment, dans un semi-intervalle]	Wahrschéinlichkeit (*f*) von Tréffer einer Zúfallsgröße in/auf/an einer Hálbegerade [in ein Interváll , in/auf/an einem Ábschnitt(e) [einem Segmént(e), eine Strécke], in ein Hálbintervall]	вероятность попадания случайной величины на полупрямую [в интервал, на отрезок, в полуинтервал]	імовірність попадання випадкової величини на півпряму [в інтервал, на відрізок, в півінтервал]
44.pròbabílity of séparate/ ísolated válue of a rándom váriable	probabilité *f* d'une valeur séparée [isolée, particulière] d'une variable aléatoire	Wahrschéinlichkeit (*f* =) von Éinzelwert(e) einer Zúfallsgröße	вероятность отдельного значения случайной величины	імовірність окремого значення випадкової величини
45.rándom/ áleatory váriable	variable *f* aléatoire, aléa *m* numérique	Zúfallsgröße (*f* =,-n) [aleatórische Größe]	случайная величина	випадкова величина
46.succéss	succès *m*	Erfólg (*m* –(e)s,-e)	успех	успіх
47.succéss pròbabílity	probabilité *f* d'un succès	Erfólgswahrscheinlichkeit (*f* =)	вероятность успеха	імовірність успіху
48.table of the dìstribútion [partítion,law] of a rándom	table *f* [tableau *m*] de distribution *f* [de répartition *f*]	Vertéilungstabelle (*f*) (einer Zúfallsgröße)	таблица распределения (случайной величины)	таблиця розподілу (випадкової величини)

váriable	d'une variable aléatoire			
49.take on a válue (of a rándom váriable)	prendre une valeur (d'une variable aléatoire)	Wert (*m* –(e)s, -e) ánnehmen* (über eine Zúfallsgröße (*f*))	принимать значение (о случайной величине)	набувати значення (про випадкову величину)
50.take on séparate ísolated póssible válues	prendre des valeurs possibles séparées isolées	Éinzelne isolíerte mögliche Werte (*m pl*) ánnehmen*	принимать отдельные изолированные возможные значения	набувати окремих ізольованих можливих значень
51.the most probable númber	nombre *m* le plus probable	wahrschéinlichste Ánzahl	наивероятнейшее число	найімовірніше число
52.tríals/expériments till (=untíl) [up to, to] the first succéss [to the first occúrrence of an evént]	expériences [épreuves, essais] jusqu'au premier succès [jusqu'à la première arrivée [apparition] d'événement]	Versúche (*m*) [Experiménte (*n*), Próben (*f*)] bis den ersten Erfólg [bis den ersten Éintritt des Eréignisses]	испытания до первого успеха (до первого наступления события)	випробування до першого успіху (до першого настання[першої появи] події)
53.válue of a rándom váriable	valeur *f* d'une variable aléatoire	Zúfallsgrößewert (*m*–(e)s,-e)	значение случайной величины	значення випадкової величини

LECTURE NO. 5. NUMBER CHARACTERISTICS OF RANDOM VARIABLES

LE CINQUIEME COURS. CARACTERISTIQUES NUMERIQUES DE VARIABLES ALEATOIRES

FÜNFTE VORLESUNG. ZAHLENMÄßIGE CHARACTERISTIKEN VON ZUFALLSGRÖßEN

POINT 1. THE MATHEMATICAL EXPECTATION OF A RANDOM VARIABLE. Espérance mathématique d'une variable aléatoire. Erwartungswert einer Zufallsgröße.

POINT 2. THE DISPERSION AND ROOT-MEAN-SQUARE DEVIATION OF A RANDOM VARIABLE. Dispersion et écart quadratique moyen. Dispersion und durchschnittliche quadratische Abweichung einer Zufallsgröße.

POINT 3. MOMENTS OF A RANDOM VARIABLE. Moments d'une variable aléatoire. Momente einer Zufallsgröße.

POINT 4. CHEBYSHEV INEQUALITY. LAW OF LARGE NUMBERS. Inégalité de Tchebichof. Loi des grands nombres. Tschebyschewsche Ungleichung. Gezetz der großen Zahlen.

Distribution function completely describes a random variable. Nevertheless there are some numerical quantities which characterize some its properties, for example: a) its mean value; b) concentration of its values about the mean value; c) symmetry or asymmetry of its distribution; d) its the most probable value (mode) etc.

In this lecture we'll study some of them.

POINT 1. THE MATHEMATICAL EXPECTATION OF A RANDOM VARIABLE

Definition of the mathematical expectation

Let's suppose that n independent trials on a random variable ξ are fulfilled and obtained results are represented by the next table:

ξ	x_1	x_2	x_3	$\dots$	x_k
n_i	n_1	n_2	n_3	$\dots$	n_k

Here x_i $\left(i=\overline{1,k}\right)$ are observed values of ξ, and n_i are numbers of their occurrences,

$$n_1+n_2+n_3+\dots+n_k=n.$$

The mean [average] value of these results equals

$$\overline{\xi}=\frac{x_1n_1+x_2n_2+\dots+x_nn_n}{n}=x_1\cdot\frac{n_1}{n}+x_2\cdot\frac{n_2}{n}+\dots+x_n\cdot\frac{n_n}{n}=x_1p_1^*+x_2p_2^*+\dots+x_np_n^*,$$

where

$$p_i^*=P_n^*(\xi=x_i)=n_i/n$$

is the relative frequency of occurrences of the value $\xi=x_i$.

Now let ξ be a discrete random variable with a distribution table

ξ	x_1	x_2	x_3	$\dots$	x_i	$\dots$	x_n
p_i	p_1	p_2	p_3	$\dots$	p_i	$\dots$	p_n

According to statistic definition of probability we introduce the next definition.

Def. 1. The mathematical expectation[1] of a discrete random variable ξ is called the next expression

$$m_\xi=M(\xi)=x_1p_1+x_2p_2+\dots+x_np_n=\sum_{i=1}^{n}x_ip_i \tag{1}$$

that is the sum of products of its possible values and corresponding probabilities of these values.

Let ξ be a continuous random variable with the distribution density $f(x)$. If we substitute a separable value x_i of a discrete random variable by the continuous variable x and the probability p_i of occurrence of the value $\xi=x_i$ by the probability element $f(x)dx$, that is by the probability of hitting of the random variable in an infinitely small interval $(x, x+dx)$, we'll get an element of the mathematical expectation

[1] Or: average of distribution, average value, distribution centre, expectation, expectation function, expectation value, mean of distribution, mean value, probabilistic average, statistical expectation

$$dM = xP(\xi \in (x, x+dx)) = xf(x)dx.$$

Adding all these elements we get the mathematical expectation of a continuous random variable in the form of the improper integral

$$m_\xi = M(\xi) = \int_{-\infty}^{\infty} xf(x)dx. \qquad (2)$$

If a continuous random variable ξ takes on all its values only on a finite interval $< a, b >$ that is

$$f(x) \equiv 0 \text{ for } x < a \text{ or } x > b,$$

then its mathematical expectation will be expressed by the next proper definite integral:

$$m_\xi = M(\xi) = \int_{a}^{b} xf(x)dx. \qquad (3)$$

Probability [probabilistic] sense of the mathematical expectation of a random variable: it is its mean [average] value.

Ex. 1. Let a discrete random variable ξ be the number of occurrences of an event A in one trial: $\xi = 1$ if A occurs, and $\xi = 0$, if A doesn't occur (that is an opposite event $\overline{A}$ occurs); such the random variable is often called the indicator of the event A.

ξ	1	0
p_i	$p_1 = P(A)$	$p_2 = P(\overline{A}) = 1 - P(A)$

Its mathematical expectation by the formula (1) equals

$$m_\xi = M(\xi) = 1 \cdot p_1 + 0 \cdot p_2 = p_1 = P(A).$$

Ex. 2. The mathematical expectation of the next discrete random variable

ξ	$x_1 = -2$	$x_2 = -1$	$x_3 = 0$	$x_4 = 1$	$x_5 = 3$
p_i	$p_1 = 0.1$	$p_2 = 0.3$	$p_3 = 0.2$	$p_4 = 0.1$	$p_5 = 0.3$

on the base of the same formula (1) equals

$$m_\xi = M(\xi) = \sum_{i=1}^{5} x_i p_i = (-2) \cdot 0.1 + (-1) \cdot 0.3 + 0 \cdot 0.2 + 1 \cdot 0.1 + 3 \cdot 0.3 = 0.5.$$

Ex. 3. A continuous random variable ξ is defined by the distribution density

$$f(x)=\frac{1}{\sqrt{\pi}}e^{-x^2}$$

(see Ex. 6 of the Lecture No. 4). By virtue of the formula (2) its mathematical expectation equals

$$m_\xi = M(\xi)=\int_{-\infty}^{\infty} xf(x)dx=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} xe^{-x^2}dx=\frac{1}{\sqrt{\pi}}\lim_{\substack{b\to\infty\\ c\to-\infty}}\int_c^b xe^{-x^2}dx=$$

$$=\frac{1}{\sqrt{\pi}}\lim_{\substack{b\to\infty\\ c\to-\infty}}\left(-\frac{1}{2}e^{-x^2}\right)\Bigg|_c^b=-\frac{1}{2\sqrt{\pi}}\lim_{\substack{b\to\infty\\ c\to-\infty}}\left(e^{-b^2}-e^{-c^2}\right)=0.$$

Remark. One can do simplier by introducing the notion of the principle value of an improper integral over the set $(-\infty,\infty)$ of all reals, namely:

$$m_\xi = M(\xi)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} xe^{-x^2}dx=\frac{1}{\sqrt{\pi}}p.v.\int_{-\infty}^{\infty} xe^{-x^2}dx=\frac{1}{\sqrt{\pi}}\lim_{b\to\infty}\int_{-b}^{b} xe^{-x^2}dx=$$

$$-\frac{1}{2\sqrt{\pi}}\lim_{b\to\infty}e^{-x^2}\Bigg|_{-b}^{b}=-\frac{1}{2\sqrt{\pi}}\lim_{b\to\infty}\left(e^{-b^2}-e^{-b^2}\right)=0.$$

One can also observe at once that

$$\int_{-b}^{b} xe^{-x^2}dx=0$$

because of oddness [oddity, unevenness] of the integrand.

Ex. 4. The distribution density of a continuous random variable ξ equals (see Ex. 7 of the Lecture 4)

$$f(x)=\begin{cases}\frac{1}{2}\cos x & \text{if} \quad -\pi/2<x<\pi/2,\\ 0 & \text{if} \quad x<-\pi/2 \text{ or } x>\pi/2.\end{cases}$$

In this case the mathematical expectation must be calculated by the formula (3),

$$m_\xi = M(\xi)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} xf(x)dx=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x dx==\begin{vmatrix} u=x & dv=\cos x dx\\ du=dx & v=\sin x\end{vmatrix}=$$

$$=\frac{1}{2}\left(\left(x\sin x\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin x dx\right)=\frac{1}{2}\left(\left(\frac{\pi}{2}\sin\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right)\right)+\cos x\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)=0.$$

The mathematical expectation of a function of a random variable

Let a random variable η be a function of a random variable ξ,

$$\eta=\varphi(\xi),$$

and it is required to find its mathematical expectation.

1. At first we'll consider the case of a discrete random variable ξ.

a) If a function $y=\varphi(x)$ is monotone one, then (see the beginning of P. 3 of the Lecture No. 4) the distribution table of the function $\eta=\varphi(\xi)$ is

$\eta=\varphi(\xi)$	$y_1=\varphi(x_1)$	$y_2=\varphi(x_2)$	$\dots$	$y_n=\varphi(x_n)$
$q_i=P(\eta=y_i)=P(\eta=\varphi(x_i))$	$q_1=p_1$	$q_2=p_2$	$\dots$	$q_n=p_n$

By the formula (1) the corresponding mathematical expectation will be equal

$$m_\eta=M(\eta)=M(\varphi(\xi))=\sum_{i=1}^{n}y_iq_i=\sum_{i=1}^{n}\varphi(x_i)p_i \qquad (4)$$

b) Now let's suppose that a function $y=\varphi(x)$ isn't monotone one and to some value $\eta=y_j$ of the random variable $\eta=\varphi(\xi)$ there correspond several values of the random variable ξ. For example let such values be $x_{j_1},x_{j_2},...,x_{j_k}$ for which

$$y_j=\varphi(x_{j_1})=\varphi(x_{j_2})=...=\varphi(x_{j_k}).$$

In this case by virtue of the same part of the Lecture 4

$$q_j=P(\eta=y_j)=P((\xi=x_{j_1})+(\xi=x_{j_2})+...+(\xi=x_{j_k}))=p_{j_1}+p_{j_2}+...+p_{j_k},$$

and the sum above will contain the summand

$$y_jq_j=y_jP(\eta=y_j).$$

We can transform it to the next form:

$$y_jq_j=y_j(p_{j_1}+p_{j_2}+...+p_{j_k})=\varphi(x_{j_1})p_{j_1}+\varphi(x_{j_2})p_{j_2}+...+\varphi(x_{j_k})p_{j_k},$$

and therefore the mathematical expectation of the function $\eta=\varphi(\xi)$ will be expressed by the same formula (4) in which the sum is extended on all possible values of the random variable ξ.

Ex. 5. Evaluate the mathematical expectation of the function $\eta = \varphi(\xi) = \xi^2$ of the discrete random variable ξ which we've studied in the Ex. 2. The distribution table of the function is

$\eta = \varphi(\xi) = \xi^2$	$y_1 = 0$	$y_2 = 1$	$y_3 = 4$	$y_5 = 9$
$q_i = P(\eta = y_i)$	$q_1 = 0.2$	$q_2 = 0.4$	$q_3 = 0.1$	$q_4 = 0.3$

Here

$$q_1 = P(\eta = y_1) = P(\eta = 0) = P(\xi = 0) = 0.2,$$

$$q_2 = P(\eta = y_2) = P(\eta = 1) = P(\xi = -1) + P(\xi = 1) = 0.3 + 0.1 = 0.4,$$

$$q_3 = P(\eta = y_3) = P(\eta = 4) = P(\xi = -2) = 0.1,$$

$$q_4 = P(\eta = y_4) = P(\eta = 9) = P(\xi = 3) = 0.3.$$

Henceforth we'll realize calculations by two ways.

a) Making use of the formula (1) and the distribution table of Ex. 5 we obtain

$$m_\eta = M(\eta) = \sum_{i=1}^{4} y_i q_i = 0 \cdot 0.2 + 1 \cdot 0.4 + 4 \cdot 0.1 + 9 \cdot 0.3 = 3.5.$$

b) The same result we get with the help of the formula (4) (where $\varphi(x_i) = x_i^2$) and the distribution table of Ex. 2,

$$M(\eta) = M(\xi^2) = \sum_{i=1}^{5} x_i^2 p_i = (-2)^2 \cdot 0.1 + (-1)^2 \cdot 0.3 + 0^2 \cdot 0.2 + 1^2 \cdot 0.1 + 3^2 \cdot 0.3 = 3.5$$

2. In the second case, when ξ is a continuous random variable, we apply the same device [stratagem, strategy, trick] as in passage from the formula (1) to the formula (2). Namely, we substitute a separable value x_i of a discrete random variable by a continuous variable x and the probability p_i of occurrence of the value $\xi = x_i$ by the probability element $f(x)dx$ in the formula (4). We'll get an element of the mathematical expectation

$$d(M(\eta)) = d(M(\varphi(\xi)) = \varphi(x) f(x) dx$$

and (after summation of elements) the mathematical expectation in question,

$$M(\eta) = M(\varphi(\xi)) = \int_{-\infty}^{\infty} \varphi(x) f(x) dx \qquad (5)$$

or

$$M(\eta) = M(\varphi(\xi)) = \int_{a}^{b} \varphi(x) f(x) dx \qquad (6)$$

if the random variable ξ takes on its values only on the interval $< a, b >$.

Ex. 6. Calculate the mathematical expectation of the function

$$\eta = \varphi(\xi) = \xi^3$$

of the random variable we've dealt in Ex. 3.

By the formula (5), where $\varphi(x) = x^3$, we get (with the help of the principal value of the integral)

$$M(\eta) = M(\varphi(\xi)) = M(\xi^3) = \int_{-\infty}^{\infty} x^3 f(x) dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^3 e^{-x^2} dx = \frac{1}{\sqrt{\pi}} p.v. \int_{-\infty}^{\infty} x^3 e^{-x^2} dx =$$

$$= \frac{1}{\sqrt{\pi}} \lim_{b \to \infty} \int_{-b}^{b} x^3 e^{-x^2} dx = 0$$

because of the integrand is an odd function.

Ex. 7. A continuous random variable is given by the distribution density

$$f(x) = \begin{cases} \frac{3}{8} x^2 & \text{if} \quad 0 < x < 2, \\ 0 & \text{if} \quad x < 0 \; or \; x > 2 \end{cases}$$

(see Ex. 8 in the Lecture 4). Find the mathematical expectation of the function

$$\eta = \varphi(\xi) = \sin \xi^3 .$$

Using the formula (6) for $\varphi(x) = \sin x^3, a = 0, b = 2$ we'll have

$$M(\eta) = M(\sin \xi^3) = \int_0^2 \sin x^3 \cdot \frac{3}{8} x^2 dx = \left| \begin{matrix} x^3 = t \\ x^2 dx = \frac{1}{3} dt \end{matrix} \quad \begin{array}{c|c|c} x & 0 & 2 \\ \hline t & 0 & 8 \end{array} \right| =$$

$$= \frac{3}{8} \cdot \frac{1}{3} \int_0^8 \sin t dt = -\frac{1}{8} \cos t \Big|_0^8 = \frac{1}{8} (\cos 0 - \cos 8) \approx \frac{1}{8} (1 - (-0.15)) \approx 0.14 .$$

Properties of the mathematical expectation

1. The mathematical expectation of a constant quantity C equals the same quantity,

$$M(C) = C.$$

■Let's interpret C as a random variable with unique possible value C the probability of which equals 1. In such the case

$$M(C) = C \cdot 1 = C \blacksquare$$

2. A constant quantity C can be taken outside the sign of the mathematical expectation,

$$M(C\xi) = CM(\xi).$$

■Let a random variable

$$\eta = \varphi(\xi) = C\xi$$

be a function of the random variable ξ.

a) If the random variable ξ is discrete one determined by the distribution table

ξ	x_1	x_2	x_3	...	x_i	$\cdots$	x_n
p_i	p_1	p_2	p_3	...	p_i	$\cdots$	p_n

then by virtue of the formulas (4) and (1)

$$M(\eta) = M(C\xi) = \sum_{i=1}^{n}(Cx_i)p_i = \sum_{i=1}^{n}Cx_i p_i = C\sum_{i=1}^{n}x_i p_i = CM(\xi).$$

b) If the random variable ξ is continuous one determined by the distribution density $f(x)$, then on the base of the formulas (5) and (2) one has

$$M(\eta) = M(C\xi) = \int_{-\infty}^{\infty}(Cx)f(x)dx = \int_{-\infty}^{\infty}Cxf(x)dx = C\int_{-\infty}^{\infty}xf(x)dx = CM(\xi). \blacksquare$$

Def. 2. A sum $\xi + \eta$ of two random variables ξ, η is called a random variable whose possible values are equal to sums of any possible value of ξ with any possible value of η.

3. The mathematical expectation of a sum of two random variables equals the sum of their mathematical expectations,

$$M(\xi+\eta)=M(\xi)+M(\eta).$$

■Now we can prove this property for discrete random variables with finite numbers of possible values. The case of those continuous will be studied in the Lecture No. 7. Let random variables ξ, η be represented by their distribution tables

ξ	x_1	x_2	$\cdots$	x_i	$\cdots$	x_n	η	y_1	y_2	$\cdots$	y_j	$\cdots$	y_m
p_i	p_1	p_2	$\cdots$	p_i	$\cdots$	p_n	q_j	q_1	q_2	$\cdots$	q_j	$\cdots$	q_m

For the sake of simplicity we'll suppose the sums x_i+y_j for all pairs $(\xi=x_i, \eta=y_j)$ to be different. By the definition of the mathematical expectation

$$M(\xi+\eta)=\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i+y_j)P((\xi=x_i)(\eta=y_j)).$$

After rearrangement of the summands we get

$$M(\xi+\eta)=\sum_{i=1}^{n}\sum_{j=1}^{m}x_iP((\xi=x_i)(\eta=y_j))+\sum_{j=1}^{m}\sum_{i=1}^{n}y_jP((\xi=x_i)(\eta=y_j))=$$

$$=\sum_{i=1}^{n}x_i\sum_{j=1}^{m}P((\xi=x_i)(\eta=y_j))+\sum_{j=1}^{m}y_j\sum_{i=1}^{n}P((\xi=x_i)(\eta=y_j))=$$

$$=\sum_{i=1}^{n}x_i\sum_{j=1}^{m}P(\eta=y_j)P((\xi=x_i)/(\eta=y_j))+\sum_{j=1}^{m}y_j\sum_{i=1}^{n}P(\xi=x_i)P((\eta=y_j)/(\xi=x_i)).$$

On the base of the formula of total probability [the probability invariance rule]

$$\sum_{j=1}^{m}P(\eta=y_j)P((\xi=x_i)/(\eta=y_j))=P(\xi=x_i)=p_i,$$

$$\sum_{i=1}^{n}P(\xi=x_i)P((\eta=y_j)/(\xi=x_i))=P(\eta=y_j)=q_j,$$

whence it follows that

$$M(\xi+\eta)=\sum_{i=1}^{n}x_ip_i+\sum_{j=1}^{m}y_jq_j=M(\xi)+M(\eta).■$$

Corollary 1. The property is extendable for a sum of several random variables, for example

$$M(\xi+\eta+\zeta)=M(\xi)+M(\eta)+M(\zeta).$$

Corollary 2. For any constant C

$$M(\xi + C) = M(\xi) + C.$$

Def. 3. A product $\xi\eta$ of two random variables ξ, η is called a random variable which possible values are equal to products of any possible value of ξ and any possible value of η.

Def. 4. Two random variables ξ, η are called those independent if the distribution law of one of them doesn't depend on values which the other random variable can take on.

4. The mathematical expectation of a product of two independent random variables equals the product of their mathematical expectations,

$$M(\xi\eta) = M(\xi)M(\eta).$$

One can prove this property only for discrete random variables (try to do it yourselves).

The key [the hint].

$$M(\xi\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j P((\xi = x_i)(\eta = y_j)) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j P(\xi = x_i)P(\eta = y_j) =$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j p_i q_j = \sum_{i=1}^{n} x_i p_i \sum_{j=1}^{m} y_j q_j = \sum_{i=1}^{n} x_i p_i \cdot \sum_{j=1}^{m} y_j q_j = M(\xi)M(\eta).$$

The proof of the property for the case of continuous random variables can be done in the Lecture No. 7.

Def. 5. Several random variables are called those mutually independent if the distribution laws of any number of them doesn't depend on values which the other random variables can take on.

The property 4 is extendable for a product of several mutually independent random variables, for example

$$M(\xi\eta\zeta) = M(\xi)M(\eta)M(\zeta).$$

Ex. 8. Two independent random variables ξ, η are given by their distribution tables

ξ	-1	0	2	3
p_{x_i}	0.2	0.3	0.1	0.4

η	-2	1	3	4
p_{y_j}	0.1	0.2	0.4	0.3

Find the distribution table and the mathematical expectation of the product $\zeta = \xi \cdot \eta$ of these random variables.

Solution. The product ζ of the random variables ξ, η can take on the next values: - 6, -4, -3, -1, 0, 2, 3, 6, 8, 9, 12 (products of all pairs of possible values of the variables ξ, η). We'll find their probabilities making use of independence of ξ, η.

a) $P(\zeta = 0) = P(\xi = 0) = 0.3$;

b) $P(\zeta = -6) = P((\xi = 3)(\eta = -2)) = P(\xi = 3)P(\eta = -2) = 0.4 \cdot 0.1 = 0.04$

and by analogy

$$P(\zeta = -3) = P(\xi = -1)P(\eta = 3) = 0.08,\ P(\zeta = -1) = P(\xi = -1)P(\eta = 1) = 0.04,$$
$$P(\zeta = 3) = P(\xi = 3)P(\eta = 1) = 0.08,\ P(\zeta = 6) = P(\xi = 2)P(\eta = 3) = 0.04,$$
$$P(\zeta = 8) = P(\xi = 2)P(\eta = 4) = 0.03,\ P(\zeta = 9) = 0.16,\ P(\zeta = 12) = 0.12.$$

c) $P(\zeta = -4) = P((\xi = -1)(\eta = 4) + (\xi = 2)(\eta = -2)) = P((\xi = -1)(\eta = 4)) +$
$+ P((\xi = 2)(\eta = -2)) = P(\xi = -1)P(\eta = 4) + P(\xi = 2)P(\eta = -2) = 0.07$

and by analogy

$$P(\zeta = 2) = P((\xi = -1)(\eta = -2) + (\xi = 2)(\eta = 1)) = P((\xi = -1)(\eta = -2)) +$$
$$+ P((\xi = 2)(\eta = 1)) = P(\xi = -1)P(\eta = -2) + P(\xi = 2)P(\eta = 1) = 0.04.$$

Now we can compile the distribution table of the random variable in question

$\zeta = \xi \cdot \eta$	-6	-4	-3	-1	0	2	3	6	8	9	12
p_{z_k}	0.04	0.07	0.08	0.04	0.30	0.04	0.08	0.04	0.03	0.16	0.12

The mathematical expectation of the variable $\zeta = \xi \cdot \eta$ we'll find by two ways namely directly and with the help of the corresponding property of the mathematical expectation of a product of two independent random variables.

a) $M(\zeta) = M(\xi\eta) = (-6)\cdot 0.04 + (-4)\cdot 0.07 + (-3)\cdot 0.08 + (-1)\cdot 0.04 +$
$+ 0 \cdot 0.30 + 2 \cdot 0.04 + 3 \cdot 0.08 + 6 \cdot 0.04 + 8 \cdot 0.03 + 9 \cdot 0.16 + 12 \cdot 0.12 = 2.88.$

b) $M(\xi) = (-1)\cdot 0.2 + 0 \cdot 0.3 + 2 \cdot 0.1 + 3 \cdot 0.4 = 1.2$,

$$M(\eta)=(-2)\cdot 0.1+1\cdot 0.2+3\cdot 0.4+4\cdot 0.3=2.4,$$

and therefore

$$M(\zeta)=M(\xi\eta)=M(\xi)M(\eta)=1.2\cdot 2.4=2.88.$$

Of course two results coincide.

Find yourselves the distribution function of the random variable $\zeta=\xi\cdot\eta$.

Ex. 9. Find the distribution table and the mathematical expectation of the sum $\chi=\xi+\eta$ of the same random variables as in Ex.8.

Answer. a) The distribution table of the random variable $\chi=\xi+\eta$ is

χ	-3	-2	0	1	2	3	4	5	6	7
p_{z_k}	0.02	0.03	0.05	0.10	0.08	0.20	0.17	0.04	0.19	0.12

b) $M(\chi)=M(\xi+\eta)=3.6$.

Find yourselves the distribution function of the random variable $\chi=\xi+\eta$.

POINT 2. THE DISPERSION AND ROOT-MEAN-SQUARE DEVIATION

Def. 6. The deviation of a random variable ξ from its mathematical expectation is called the next random variable

$$\overset{0}{\xi}=\xi-M(\xi). \qquad (7)$$

Its mathematical expectation equals zero,

$$M(\overset{0}{\xi})=M(\xi-M(\xi))=M(\xi)-M(M(\xi))=M(\xi)-M(\xi)=0.$$

Def. 7. A random variable with zero mathematical expectation is called centered one.

By virtue of the definition 7 the deviation (7) is a centered random variable.

Def. 8. The dispersion[1] of a random variable ξ is called the mathematical expectation of the square of its deviation (7) from its mathematical expectation

[1] Or: the variance

$$D_\xi = D(\xi) = M\left(\left(\overset{0}{\xi}\right)^2\right) = M(\xi - M(\xi))^2. \qquad (8)$$

Probability [probabilistic] sense of the dispersion: the dispersion characterises [discribes] a dissipation of a random variable about its mathematical expectation, that is about its mean [or average] value.

Let's denote by [*A*] the dimension of a quantity *A*. As can be easy seen

$$[D(\xi)] = [\xi]^2, \qquad (9)$$

that is the dimension of the dispersion of a random variable equals the square of its own dimension. By this reason it is well to introduce a more convenient characteristic of the dissipation of a random variable about its mathematical expectation which has the same dimension, namely the root-mean-square deviation.

Def. 9. The root-mean-square deviation[1] of a random variable ξ is called the square root of its dispersion, that is

$$\sigma(\xi) = \sigma_\xi = \sqrt{D_\xi} = \sqrt{D(\xi)}. \qquad (10)$$

It is obviously that

$$[\sigma(\xi)] = [\xi]. \qquad (11)$$

It follows from (11) that

$$D(\xi) = \sigma^2(\xi) \; or \; D_\xi = \sigma_\xi^2. \qquad (12)$$

Calculating formulas for the dispersion

Introducing the next function of a random variable ξ:

$$\eta = \varphi(\xi) = \left(\overset{0}{\xi}\right)^2 = (\xi - M(\xi))^2 \qquad (13)$$

we can avail ourselves of the formulas (5), (6).

a) For a discrete random variable ξ with the distribution table

ξ	x_1	x_2	x_3	$\cdots$	x_n
p_i	p_1	p_2	p_3	$\cdots$	p_n

[1] Or: quadratic mean deviation, mean-square deviation, mean-square distance, root-mean-square difference, sigma, squared error distance, standard deviate, standard deviation

we'll have by the formula (4)

$$D(\xi)=D_\xi=\sum_{i=1}^{n}(x_i-M(\xi))^2 p_i; \qquad (14)$$

b) For a continuous random variable ξ with the distribution density $f(x)$ the formula (5) (or (6) if ξ takes on its values only on the interval $<a,b>$) gives

$$D(\xi)=D_\xi=\int_{-\infty}^{\infty}(x-M(\xi))^2 f(x)dx \ \text{ (or } D(\xi)=D_\xi=\int_a^b (x-M(\xi))^2 f(x)dx\text{)}. \qquad (15)$$

But the best convenient formula directly follows from the definition (8) of the dispersion, namely

$$D(\xi)=M(\xi^2)-M^2(\xi). \qquad (16)$$

■By virtue of the formula (8)

$$D(\xi)=M(\xi-M(\xi))^2=$$
$$=M(\xi^2-2\xi\cdot M(\xi)+M^2(\xi))=M(\xi^2)-2M(\xi\cdot M(\xi))+M(M^2(\xi))=$$
$$=M(\xi^2)-2M(\xi)\cdot M(\xi)+M^2(\xi)=M(\xi^2)-2M^2(\xi)+M^2(\xi)=M(\xi^2)-M^2(\xi).$$

We've taken the constant quantities $M(\xi), M^2(\xi)$ outside the sign of the mathematical expectation.■

The formula (16) can be written in the form

$$D(\xi)=D_\xi=\sum_{i=1}^{n}x_i^2 p_i-M^2(\xi) \qquad (17)$$

for a discrete random variable and in the form

$$D(\xi)=D_\xi=\int_{-\infty}^{\infty}x^2 f(x)dx-M^2(\xi) \qquad (18)$$

for that continuous. For this purpose it's sufficient to introduce the next function $\eta=\varphi(\xi)=\xi^2$ of the random variable ξ and then to apply the formulas (4), (5).

Ex. 10. The dispersion and root-mean-square deviation of the indicator of an event *A* (see Ex. 1), that is of the random variable ξ with possible values 1, 0 and corresponding probabilities $p_1=P(A), p_2=P(\overline{A})=1-P(A)=1-p_1$, are equal to

$$D(\xi)=P(A)\cdot(1-P(A)), \sigma(\xi)=\sqrt{D(\xi)}=\sqrt{P(A)\cdot(1-P(A))}.$$

■The distribution table of ξ and the value $M(\xi)=P(A)$ see in Ex. 1. The distribution table and the mathematical expectation of ξ^2 are respectively

$\eta=\varphi(\xi)=\xi^2$	$y_1=1$	$y_2=0$
$q_i=P(\eta=y_i)$	$q_1=P(A)$	$q_2=P(\overline{A})=1-P(A)$

$$M(\xi^2)=1\cdot P(A)+0\cdot P(\overline{A})=P(A).$$

Hence by the calculating formula (16)

$$D(\xi)=M(\xi^2)-M^2(\xi)=P(A)-P^2(A)=P(A)(1-P(A)),\ \sigma(\xi)=\sqrt{P(A)(1-P(A))}.$$ ■

Ex. 11. Calculate the dispersion and root-mean-square deviation of the random variable ξ of Ex. 2.

The distribution tables of ξ and its square (see Ex. 5) are

ξ	$x_1=-2$	$x_2=-1$	$x_3=0$	$x_4=1$	$x_5=3$
p_i	$p_1=0.1$	$p_2=0.3$	$p_3=0.2$	$p_4=0.1$	$p_5=0.3$

$\eta=\varphi(\xi)=\xi^2$	$y_1=0$	$y_2=1$	$y_3=4$	$y_5=9$
$q_i=P(\eta=y_i)$	$q_1=0.2$	$q_2=0.4$	$q_3=0.1$	$q_4=0.3$

Corresponding mathematical expectations equal

$$M(\xi)=0.5,\ M(\xi^2)=3.5,$$

and by the formulas (16) and (10) we obtain

$$D(\xi)=M(\xi^2)-M^2(\xi)=3.5-0.5^2=3.25$$

$$\sigma(\xi)=\sqrt{D(\xi)}=\sqrt{3.25}\approx 1.80.$$

Ex. 12. Calculate the dispersion and root-mean-square deviation of the random variable ξ of Ex. 4.

The distribution density of the random variable is

$$f(x)=\begin{cases}\frac{1}{2}\cos x & \text{if} \quad -\pi/2<x<\pi/2,\\ 0 & \text{if} \quad x<-\pi/2 \text{ or } x>\pi/2,\end{cases}$$

the mathematical expectation equals $\pi/2$. The integral of the formula (18) after two integrations by parts will be equal

$$\int_{-\infty}^{\infty} x^2 f(x)dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} x^2 \cos x dx = \begin{vmatrix} u = x^2 & dv = \cos x dx \\ du = 2x dx & v = \sin x \end{vmatrix} =$$

$$= \frac{1}{2}\left(\left(x^2 \sin x\right)\Big|_{-\pi/2}^{\pi/2} - 2\int_{-\pi/2}^{\pi/2} x \sin x dx \right) = \pi^2 - \int_{-\pi/2}^{\pi/2} x \sin x dx = \begin{vmatrix} u = x & dv = \sin x dx \\ du = dx & v = -\cos x \end{vmatrix} =$$

$$= \pi^2 - \left(-\left(x \cos x\right)\Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \cos x dx \right) = \pi^2 + 0 - \sin x\Big|_{-\pi/2}^{\pi/2} = \pi^2 - 2.$$

Therefore the dispersion and root-mean-square deviation of the random variable ξ equal $D(\xi) = \int_{-\infty}^{\infty} x^2 f(x)dx - M^2(\xi) = \pi^2 - 2 - 0^2 \approx 7.87$, $\sigma(\xi) = \sqrt{D(\xi)} \approx \sqrt{7.87} \approx 2.81$.

Properties of the dispersion

1. The dispersion of a constant quantity C equals zero,

$$D(C) = 0, C - const.$$

■Let's consider C as a random variable ξ which takes on unique possible value C with the probability 1 (see the proving of the first property of the mathematical expectation). In this case

$$M(\xi) = M(C) = C, \overset{0}{\xi} = \xi - M(\xi) = C - C = 0,$$

and by the definition of the dispersion

$$D(\xi) = D(C) = M\left(\left(\overset{0}{\xi}\right)^2\right) = M\left(0^2\right) = 0.■$$

Remark. The property can be proved with the help of the formula (16), namely

$$D(\xi) = D(C) = M\left(C^2\right) - M^2(C) = C^2 - C^2 = 0.$$

2. The square of a constant factor C can be taken outside the sign of the dispersion,

$$D(C\xi) = C^2 D(\xi).$$

■Let's consider a random variable

$$\eta = \varphi(\xi) = C\xi$$

By the second property of the mathematical expectation

$$M(\eta)=M(C\xi)=CM(\xi),\overset{0}{\eta}=\eta-M(\eta)=C\xi-CM(\xi)=C(\xi-M(\xi))=C\overset{0}{\xi},$$

and by virtue of the definition of the dispersion

$$D(\eta)=D(C\xi)=M\left(\left(\overset{0}{\eta}\right)^2\right)=M\left(\left(C\overset{0}{\xi}\right)^2\right)=M\left(C^2\left(\overset{0}{\xi}\right)^2\right)=C^2M\left(\left(\overset{0}{\xi}\right)^2\right)=C^2D(\xi).■$$

The other method of proving is based on the formula (16):

$$D(C\xi)=M\left((C\xi)^2\right)-M^2(C\xi)=C^2M\left(\xi^2\right)-C^2M^2(\xi)=C^2\left(M\left(\xi^2\right)-M^2(\xi)\right)=C^2D(\xi).$$

3. The dispersion of the sum of two random variables equals

$$D(\xi+\eta)=D(\xi)+D(\eta)+2M\left(\overset{0}{\xi}\cdot\overset{0}{\eta}\right). \qquad (19)$$

■Let

$$\zeta=\xi+\eta.$$

Then on the base of the definition of the dispersion we do as follows

$$M(\zeta)=M(\xi+\eta)=M(\xi)+M(\eta),\overset{0}{\zeta}=\zeta-M(\zeta)=\xi+\eta-(M(\xi)+M(\eta))=$$

$$=(\xi-M(\xi))+(\eta-M(\eta))=\overset{0}{\xi}+\overset{0}{\eta};$$

$$D(\zeta)=D(\xi+\eta)=M\left(\left(\overset{0}{\zeta}\right)^2\right)=M\left(\left(\overset{0}{\xi}+\overset{0}{\eta}\right)^2\right)=M\left(\left(\overset{0}{\xi}\right)^2+\left(\overset{0}{\eta}\right)^2+2\overset{0}{\xi}\overset{0}{\eta}\right)=$$

$$=M\left(\left(\overset{0}{\xi}\right)^2\right)+M\left(\left(\overset{0}{\eta}\right)^2\right)+2M\left(\overset{0}{\xi}\overset{0}{\eta}\right)=D(\xi)+D(\eta)+2M\left(\overset{0}{\xi}\overset{0}{\eta}\right).■$$

Corollary 1. The dispersion of the sum of two independent random variables equals the sum of their dispersions,

$$D(\xi+\eta)=D(\xi)+D(\eta). \qquad (20)$$

■For independent random variables ξ,η

$$M\left(\overset{0}{\xi}\overset{0}{\eta}\right)=M\left(\overset{0}{\xi}\right)M\left(\overset{0}{\eta}\right)=0,$$

and the formula (19) must be substituted by the formula (20).■

Corollary 2. The corollary 1 is extendable on the sum of arbitrary number of mutually independent random variables $\xi_1, \xi_2, ..., \xi_n$, that is

$$D(\xi_1 + \xi_2 + ... + \xi_n) = D(\xi_1) + D(\xi_2) + ... + D(\xi_n). \qquad (21)$$

Corollary 3. If $\xi_1, \xi_2, ..., \xi_n$ be mutually independent random variables, then the root-mean-square deviation of their sum equals

$$\sigma(\xi_1 + \xi_2 + ... + \xi_n) = \sqrt{\sigma^2(\xi_1) + \sigma^2(\xi_2) + ... + \sigma^2(\xi_n)}. \qquad (22)$$

■By the formulas (10) and (21)

$$\sigma(\xi_1 + \xi_2 + ... + \xi_n) = \sqrt{D(\xi_1 + \xi_2 + ... + \xi_n)} = \sqrt{D(\xi_1) + D(\xi_2) + ... + D(\xi_n)} =$$

$$= \sqrt{\sigma^2(\xi_1) + \sigma^2(\xi_2) + ... + \sigma^2(\xi_n)} ■$$

Corollary 4. For any constant quantity C

$$D(\xi + C) = D(\xi).$$

Corollary 5. The dispersion of the difference of two independent random variables ξ, η equals the sum of their dispersions,

$$D(\xi - \eta) = D(\xi) + D(\eta)$$

■ $D(\xi - \eta) = D(\xi) + D(-\eta) = D(\xi) + (-1)^2 D(\eta) = D(\xi) + D(\eta)$■

Ex. 13. Let a random variable

$$\eta = \frac{\xi_1 + \xi_2 + ... + \xi_n}{n} \qquad (23)$$

be the arithmetic mean [the arithmetic average] of n (mutualy) independent random variables. It's necessary to find the mathematical expectation and dispersion of η.

Making use of the properties of the mathematical expectation and dispersion we obtain

$$M(\eta) = M\left(\frac{\xi_1 + \xi_2 + ... + \xi_n}{n}\right) = \frac{M(\xi_1 + \xi_2 + ... + \xi_n)}{n} = \frac{M(\xi_1) + M(\xi_2) + ... + M(\xi_n)}{n}, \qquad (24)$$

that is the mathematical expectation of the arithmetic mean of n independent random variables equals the arithmetic mean of their mathematical expectations;

$$D(\eta) = D\left(\frac{\xi_1 + \xi_2 + ... + \xi_n}{n}\right) = \frac{D(\xi_1 + \xi_2 + ... + \xi_n)}{n^2} = \frac{D(\xi_1) + D(\xi_2) + ... + D(\xi_n)}{n^2}, \qquad (25)$$

the dispersion of the arithmetic mean of n independent random variables equals the sum of their dispersions divided by n^2 (or the arithmetic mean of their dispersions divided by n).

Ex. 14. Find the mathematical expectation, dispersion and root-mean-square deviation of Bernoulli and Poisson distributions (see PP. 2, 4 of the Lecture No. 3).

a) At first let's suppose that a random variable ξ has Bernoulli distribution (the number of successes A in n independent trials with a constant probability $p = P(A)$ of the success). One can represent ξ as the sum of n independent random variables

$$\xi = \xi_1 + \xi_2 + \ldots + \xi_i + \ldots + \xi_n$$

where ξ_i is the number of successes in the ith trial. As we know from Ex. 1 and 10

$$M(\xi_i) = P(A) = p,\ D(\xi_i) = P(A)(1 - P(A)) = p(1 - p) = pq\,.$$

Therefore by virtue of the property 3 (corollary 1) of the mathematical expectation and 3 (corollary 2) of the dispersion we obtain

$$M(\xi) = np, \quad D(\xi) = npq, \quad \sigma(\xi) = \sqrt{npq}\,. \qquad (26)$$

Remark. The formulas (26) can be obtained directly from the definition (1) of the mathematical expectation and Bernoulli formula (1) of the Lecture No. 3. It's sufficient to establish that

$$M(\xi) = \sum_{k=0}^{n} kP(\xi = k) = \sum_{k=0}^{n} kC_n^k p^k q^{n-k} = \sum_{k=1}^{n} kC_n^k p^k q^{n-k} = np,$$

$$M(\xi^2) = \sum_{k=0}^{n} k^2 P(\xi = k) = \sum_{k=0}^{n} k^2 C_n^k p^k q^{n-k} = \sum_{k=1}^{n} k^2 C_n^k p^k q^{n-k} = npq + (np)^2.$$

But our method of proving is the most simple.

b) Now let a random variable ξ have Poisson distribution with a parameter a. Its mathematical expectation, dispersion and root-mean-square deviation we can obtain passing in the formulas (26) to the limit when

$$n \to \infty,\ p \to 0,\ np = const = a\,.$$

We'll have

$$\lim np = \lim a = a,\ \lim npq = \lim aq = a \lim q = a \lim(1 - p) = a\,.$$

Thus, the mathematical expectation, dispersion and root-mean-square deviation of Poisson distribution with a parameter a are equal to

$$M(\xi)=D(\xi)=a, \quad \sigma(\xi)=\sqrt{a}\,. \qquad (27)$$

Remark. If we extend the definition (1) of the mathematical expectation on the case of a random variable with infinite number of non-negative integer values, we'll prove the formulas (27) making use of series. Indeed, by virtue of Poisson formula (12) of the Lecture No. 3 we'll have

$$M(\xi)=\sum_{k=0}^{\infty} kP(\xi=k)=\sum_{k=1}^{\infty} kP(\xi=k)=\sum_{k=1}^{\infty} k\frac{a^k}{k!}e^{-a}=ae^{-a}\sum_{k=1}^{\infty}\frac{a^{k-1}}{(k-1)!}=$$

$$=ae^{-a}\left(1+a+\frac{a^2}{2!}+\frac{a^3}{3!}+\ldots\right)=ae^{-a}\cdot e^{a}=a,$$

$$M(\xi^2)=\sum_{k=0}^{\infty} k^2P(\xi=k)=\sum_{k=1}^{\infty} k^2P(\xi=k)=\sum_{k=1}^{\infty} k^2\frac{a^k}{k!}e^{-a}=ae^{-a}\sum_{k=1}^{\infty} k\frac{a^{k-1}}{(k-1)!}=$$

$$=ae^{-a}\left(1+2a+3\frac{a^2}{2!}+4\frac{a^3}{3!}+\ldots\right)=ae^{-a}\left(a+a^2+\frac{a^3}{2!}+\frac{a^4}{3!}+\ldots\right)'_a=$$

$$=ae^{-a}\left(a\left(1+a+\frac{a^2}{2!}+\frac{a^3}{3!}+\ldots\right)\right)'_a=ae^{-a}\left(ae^{a}\right)'_a=ae^{-a}\left(e^{a}+ae^{a}\right)=a+a^2,$$

$$D(\xi)=M(\xi^2)-M^2(\xi)=a+a^2-a^2=a\,.$$

POINT 3. MOMENTS OF A RANDOM VARIABLE

Def. 10. The kth order **initial moment** of a random variable ξ is called the mathematical expectation of its kth power,

$$\alpha_k=\alpha_k(\xi)=M(\xi^k). \qquad (28)$$

Example 15.

$$\alpha_1(\xi)=M(\xi\ ),$$

that is the first order initial moment of a random variable ξ coincides with its mathematical expectation.

Def. 11. The *k*th order central moment of a random variable ξ is called the mathematical expectation of the kth power of its centered random variable,

$$\mu_k = \mu_k(\xi) = M\left(\left(\overset{0}{\xi}\right)^k\right) = M\left((\xi - M(\xi))^k\right) = M\left((\xi - m_\xi)^k\right). \qquad (29)$$

Ex. 16.

$$\mu_2(\xi) = D(\xi),$$

that is the second order central moment of a random variable ξ coincides with its dispersion.

For random variables with zero mathematical expectation (that is for centered random variables) the initial and central moments coincide.

The kth order initial moment can be evaluated with the help of the formula

$$\alpha_k(\xi) = \sum_{i=1}^{n} x_i^k p_i \qquad (30)$$

for a discrete random variable with *n* possible values, and the formula

$$\alpha_k(\xi) = \int_{-\infty}^{\infty} x^k f(x) dx \qquad (31)$$

for a continuous random variable. For the kth order central moment there are corresponding formulae

$$\mu_k(\xi) = \sum_{i=1}^{n} (x_i - M(\xi))^k p_i = \sum_{i=1}^{n} (x_i - m_\xi)^k p_i, \qquad (32)$$

$$\mu_k = \int_{-\infty}^{\infty} (x - M(\xi))^k f(x) dx = \int_{-\infty}^{\infty} (x - m_\xi)^k f(x) dx. \qquad (33)$$

Theorem. If the distribution of a random variable is symmetric about its mathematical expectation, then all its odd-order central moments are equal to zero

$$\mu_3 = \mu_5 = \mu_7 = ... = \mu_{2k-1} = ... = 0.$$

■For example let ξ be a continuous random variable with the distribution density $f(x)$. The symmetry of this variable about its mathematical expectation means that

$$f\left(m_\xi - x\right) = f\left(m_\xi + x\right).$$

In this case by virtue of the formula (33) after some simple changes of variables

$$\mu_{2k-1} = \int_{-\infty}^{\infty} \left(x - m_\xi\right)^{2k-1} f(x)dx = \int_{-\infty}^{m_\xi} \left(x - m_\xi\right)^{2k-1} f(x)dx + \int_{m_\xi}^{\infty} \left(x - m_\xi\right)^{2k-1} f(x)dx =$$

$$\left| \begin{matrix} x - m_\xi = z,\, dx = dz & x - m_\xi = y,\, dx = dy \\ \begin{array}{c|c|c} x & -\infty & m_\xi \\ \hline z & -\infty & 0 \end{array} & \begin{array}{c|c|c} x & m_\xi & \infty \\ \hline y & 0 & \infty \end{array} \end{matrix} \right| = \int_{-\infty}^{0} z^{2k-1} f\left(m_\xi + z\right)dz + \int_{0}^{\infty} y^{2k-1} f\left(m_\xi + y\right)dy =$$

$$\left| \begin{matrix} z = -y,\, dz = -dy \\ \begin{array}{c|c|c} z & -\infty & 0 \\ \hline y & \infty & 0 \end{array} \end{matrix} \right| = \int_{\infty}^{0} (-y)^{2k-1} f\left(m_\xi - y\right)(-dy) + \int_{0}^{\infty} y^{2k-1} f\left(m_\xi + y\right)dy =$$

$$= \int_{\infty}^{0} y^{2k-1} f\left(m_\xi - y\right)dy + \int_{0}^{\infty} y^{2k-1} f\left(m_\xi + y\right)dy = -\int_{0}^{\infty} y^{2k-1} f\left(m_\xi - y\right)dy + \int_{0}^{\infty} y^{2k-1} f\left(m_\xi + y\right)dy =$$

$$= -\int_{0}^{\infty} y^{2k-1} f\left(m_\xi + y\right)dy + \int_{0}^{\infty} y^{2k-1} f\left(m_\xi + y\right)dy = 0. \blacksquare$$

Cental moments can be expressed in terms of initial moments, for example those of the second, third and fourth orders are equal

$$\mu_2 = \alpha_2 - \alpha_1^2,$$

$$\mu_3 = \alpha_3 - 3\alpha_2\alpha_1 + 2\alpha_1^3,$$

$$\mu_4 = \alpha_4 - 4\alpha_3\alpha_1 + 6\alpha_2\alpha_1^2 - 3\alpha_1^3.$$

■Indeed,

$$\mu_2 = \mu_2(\xi) = D(\xi) = M\left(\xi^2\right) - M^2(\xi) = \alpha_2(\xi) - \alpha_1^2(\xi) = \alpha_2 - \alpha_1^2,$$

$$\mu_3 = \mu_3(\xi) = M\left((\xi - M(\xi))^3\right) = M\left((\xi - \alpha_1)^3\right) = M\left(\xi^3 - 3\xi^2\alpha_1 + 3\xi\alpha_1^2 - \alpha_1^3\right) =$$

$$= M\left(\xi^3\right) - 3M\left(\xi^2\right)\alpha_1 + 3M(\xi)\alpha_1^2 - \alpha_1^3 = \alpha_3 - 3\alpha_2\alpha_1 + 3\alpha_1\alpha_1^2 - \alpha_1^3 = \alpha_3 - 3\alpha_2\alpha_1 + 2\alpha_1\alpha_1^2,$$

$$\mu_4(\xi) = M\left((\xi - M(\xi))^4\right) = M\left((\xi - \alpha_1)^4\right) = M\left(\xi^4 - 4\xi^3\alpha_1 + 6\xi^2\alpha_1^2 - 4\xi\alpha_1^3 + \alpha_1^4\right) =$$

$$= M\left(\xi^4\right) - 4M\left(\xi^3\right)\alpha_1 + 6M\left(\xi^2\right)\alpha_1^2 - 4M(\xi)\alpha_1^3 + \alpha_1^4 = \alpha_4 - 4\alpha_3\alpha_1 + 6\alpha_2\alpha_1^2 - 4\alpha_1\alpha_1^3 + \alpha_1^4 =$$

$$= \alpha_4 - 4\alpha_3\alpha_1 + 6\alpha_2\alpha_1^2 - 3\alpha_1^4. \blacksquare$$

There are two quantities which one introduces side by side with the third and fourth central moments of a random variable, namely its asymmetry and excess. The asymmetry of a random variable ξ is defined by a quotient

$$As = As(\xi) = \frac{\mu_3(\xi)}{\sigma^3(\xi)} = \frac{\alpha_3(\xi) - 3\alpha_2(\xi)\alpha_1(\xi) + 2\alpha_1^3(\xi)}{\sigma^3(\xi)} \qquad (34)$$

and describes the symmetry or non-symmetry of its distribution law. The excess of ξ is given by a quotient

$$Ex = Ex(\xi) = \frac{\mu_4(\xi)}{\sigma^4(\xi)} - 3 = \frac{\alpha_4(\xi) - 4\alpha_3(\xi)\alpha_1(\xi) + 6\alpha_2(\xi)\alpha_1^2(\xi) - 3\alpha_1^4(\xi)}{\sigma^4(\xi)} - 3 \qquad (35)$$

and describes so-called disnormality of ξ that is the deviation of the distribution law of ξ from the normal distribution (see the next lecture).

It is obvious that the asymmetry and excess are dimensionless quantities, that is

$$[As(\xi)] = 0, \quad [Ex(\xi)] = 0.$$

Ex. 17. Evaluate the third-order and fourth-order initial and central moments, asymmetry and excess of the discrete random variable which were studied in Ex. 2, 10.

The mathematical expectation and root-mean-square deviation of the random variable in question are $m_\xi = 0.5, \sigma_\xi = 1.8$. By the formulas (30), (32) we'll get

№	x_i	p_i	$x_i^3 p_i$	$x_i^4 p_i$	$x_i - m_\xi$	$(x_i - m_\xi)^3 p_i$	$(x_i - m_\xi)^4 p_i$
1.	-2	0.1	-0.8	1.6	-2.5	-1.56	3.91
2.	-1	0.3	-0.3	0.3	-1.5	-1.01	1.52
3.	0	0.2	0.0	0.0	-0.5	-0.03	0.01
4.	1	0.1	0.1	0.1	0.5	0.01	0.01
5.	3	0.3	8.1	24.3	2.5	4.69	11.72
$\sum$			$\alpha_3(\xi) = 7.1$	$\alpha_4(\xi) = 26.3$		$\mu_3(\xi) = 2.10$	$\mu_4(\xi) = 17.17$

Thus, $\alpha_3(\xi) = 7.1$, $\alpha_4(\xi) = 26.3$, $\mu_3(\xi) = 2.10$, $\mu_4(\xi) = 17.17$.

Finally by the formulas (34), (35) we obtain

$$As(\xi) = \frac{\mu_3(\xi)}{\sigma^3(\xi)} = \frac{2.10}{1.8^3} = 0.36, \quad Ex(\xi) = \frac{\mu_4(\xi)}{\sigma^4(\xi)} - 3 = \frac{17.17}{1.8^4} - 3 = -1.36.$$

Remark. We can evaluate all moments and number characteristics of the random variable ξ with the help of a common table

		1	2	3	4	5	Σ
1	x_i	-2	-1	0	1	3	
2	p_i	0.1	0.3	0.2	0.1	0.3	1.0
3	$x_i p_i$	-0.2	-0.3	0.0	0.1	0.9	$\alpha_1 = M(\xi) = 0.5$
4	$x_i^2 p_i$	0.4	0.3	0.0	0.1	2.7	$\alpha_2 = 3.5$
5	$x_i^3 p_i$	-0.8	-0.3	0.0	0.1	8.1	$\alpha_3 = 7.1$
6	$x_i^4 p_i$	1.6	0.3	0.0	0.1	24.3	$\alpha_4 = 26.3$
7	$x_i - M(\xi)$	-2.5	-1.5	-0.5	0.5	2.5	
8	$(x_i - M(\xi))^2 p_i$	0.63	0.68	0.05	0.03	1.88	$\mu_2 = D(\xi) = 3.25$
9	$(x_i - M(\xi))^3 p_i$	-1.56	-1.01	-0.03	0.01	4.65	$\mu_3 = 2.10$
10	$(x_i - M(\xi))^4 p_i$	3.91	0.01	0.01	0.01	11.72	$\mu_4 = 17.17$

The rows 3 and 8 give $M(\xi) = 0.5$, $D(\xi) = 3.25, \sigma(\xi) = \sqrt{D(\xi)} = \sqrt{3.25} = 1.8$. The rows 3 and 4 give the same value of the dispersion with the help of initial moments $D(\xi) = \alpha_2 - \alpha_1^2 = 3.5 - 0.5^2 = 3.25$. The rows 9 and 10 give

$$As(\xi) = \frac{\mu_3(\xi)}{\sigma^3(\xi)} = \frac{2.10}{1.8^3} = 0.36, \quad Ex(\xi) = \frac{\mu_4(\xi)}{\sigma^4(\xi)} - 3 = \frac{17.17}{1.8^4} - 3 = -1.36,$$

and the rows 3-6 give the same result with the help of initial moments

$$As(\xi) = \frac{\alpha_3 - 3\alpha_2\alpha_1 + 2\alpha_1^3}{\sigma^3(\xi)} = \frac{7.1 - 3 \cdot 3.5 \cdot 0.5 + 2 \cdot 0.5^3}{1.8^3} = 0.36.$$

$$Ex(\xi) = \frac{\alpha_4 - 4\alpha_3\alpha_1 + 6\alpha_2\alpha_1^2 - 3\alpha_1^4}{\sigma^4(\xi)} - 3 = \frac{26.3 - 4 \cdot 7.1 \cdot 0.5 + 6 \cdot 3.5 \cdot 0.5^2 - 3 \cdot 0.5^4}{1.8^4} - 3 = -1.36.$$

POINT 4. CHEBYSHEV INEQUALITY. LAW OF LARGE NUMBERS

Chebyshev[1] inequality

[1] Chebyshev, P.L. (1821 - 1894), a prominent Russian mathematician

For any random variable ξ with the mathematical expectation $M(\xi)$ and the dispersion $D(\xi)$ the next inequality holds

$$P(|\xi - M(\xi)| \geq \varepsilon) \leq \frac{D(\xi)}{\varepsilon^2} \qquad (36)$$

or, as the corollary, the inequality

$$P(|\xi - M(\xi)| < \varepsilon) \geq 1 - \frac{D(\xi)}{\varepsilon^2}. \qquad (37)$$

■To prove the inequality (36) it's useful to introduce a set A_ε of all reals which satisfy the inequality

$$|x - M(\xi)| \geq \varepsilon,$$

$$A_\varepsilon = \{x : |x - M(\xi)| \geq \varepsilon\}.$$

In this condition we can interpret the inequality

$$|\xi - M(\xi)| \geq \varepsilon$$

as the result of hitting of a random variable ξ in the set A_ε,

$$\{|\xi - M(\xi)| \geq \varepsilon\} \Leftrightarrow \{\xi \in A_\varepsilon\}.$$

Starting the proof of the inequality (36) we'll consider two cases.

a) At first let ξ be a discrete random variable. It follows from the formula (14) that

$$D(\xi) = \sum_{i=1}^{n} (x_i - M(\xi))^2 p_i \geq \sum_{x_i \in A_\varepsilon} (x_i - M(\xi))^2 p_i \geq \varepsilon^2 \sum_{x_i \in A_\varepsilon} p_i = \varepsilon^2 P(\xi \in A_\varepsilon) =$$

$$= \varepsilon^2 P(|\xi - M(\xi)| \geq \varepsilon) \Rightarrow P(|\xi - M(\xi)| \geq \varepsilon) \leq \frac{D(\xi)}{\varepsilon^2}.$$

b) For the case of a continuous random variable ξ we use the formula (15), namely

$$D(\xi) = \int_{-\infty}^{\infty} (x - M(\xi))^2 f(x) dx \geq \int_{A_\varepsilon} (x - M(\xi))^2 f(x) dx \geq \varepsilon^2 \int_{A_\varepsilon} f(x) dx = \varepsilon^2 P(\xi \in A_\varepsilon) =$$

$$= \varepsilon^2 P(|\xi - M(\xi)| \geq \varepsilon) \Rightarrow P(|\xi - M(\xi)| \geq \varepsilon) \leq \frac{D(\xi)}{\varepsilon^2}.$$

The inequality (36) is proved. Validity of the inequality (37) follows from the fact that the events

$$|\xi - M(\xi)| < \varepsilon \text{ and } |\xi - M(\xi)| \geq \varepsilon$$

are those opposite.■

In practice Chebyshev inequality (37) is applied the most often.

Ex. 18. Estimate the probability of the inequality

$$|\xi - M(\xi)| < 3$$

for the random variable of Ex. 2.

By virtue of Ex. 10 the dispersion of the random variable in question equals $D(\xi) = 3.25$. Therefore Chebyshev inequality (37) (for $\varepsilon = 3$) gives

$$P(|\xi - M(\xi)| < 3) \geq 1 - \frac{3.25}{3^2} \approx 1 - 0.36 = 0.64, \; P(|\xi - M(\xi)| < 3) \geq 0.64.$$

In the case of **Bernoulli distribution** ξ (here $M(\xi) = np, D(\xi) = npq$; see the formulas (26) in Ex. 13) the inequality (37) takes on the form

$$P(|\xi - M(\xi)| < \varepsilon) = P(|\xi - np| < \varepsilon) \geq 1 - \frac{npq}{\varepsilon^2}, \tag{38}$$

and in the case of **Poisson distribution** ξ ($M(\xi) = D(\xi) = a$, ibid., (27)) the form

$$P(|\xi - a| < \varepsilon) \geq 1 - \frac{a}{\varepsilon^2}. \tag{39}$$

Ex. 19. A device consists of 200 independent working elements. The probability of the failure of each element during the time *T* equals 0.01. With the help of Chebyshev inequality estimate the probability that the number of failed elements during the time *T* differs from the mean number of failures: a) less than by 5; b) not less than by 5.

Let a random variable ξ be the number of failed elements during the time *T*. It is obvious that ξ distributed *B* with

$$n = 200, \; p = 0.01, \; q = 0.99, \; M(\xi) = np = 2, \; D(\xi) = npq = 1.98.$$

By the inequality (37) (in the form (38), with $\varepsilon = 5$)

$$P(|\xi - M(\xi)| < 5) \geq 1 - \frac{1.98}{5^2} \approx 0.92 .$$

On the base of the inequality (36)

$$P(|\xi - M(\xi)| \geq 5) \leq \frac{D(\xi)}{5^2} = \frac{1.98}{5^2} \approx 0.08 .$$

Let

$$\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots \qquad (40)$$

be a sequence of (mutually) independent random variables. The arithmetical average of n first its terms, that is

$$\eta = \frac{\xi_1 + \xi_2 + \ldots + \xi_n}{n} = \frac{1}{n}\sum_{i=1}^{n} \xi_i , \qquad (41)$$

has the next mathematical expectation and dispersion (see Ex. 12, (23), (24), (25))

$$M(\eta) = \frac{M(\xi_1) + M(\xi_2) + \ldots + M(\xi_n)}{n} = \frac{1}{n}\sum_{i=1}^{n} M(\xi_i),$$

$$D(\eta) = \frac{D(\xi_1) + D(\xi_2) + \ldots + D(\xi_n)}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n} D(\xi_i).$$

Chebyshev inequality (37) gives us the estimate of the deviation of the arithmetical average of n first independent random variables from the arithmetical average of their mathematical expectations, namely

$$P(|\eta - M(\eta)| < \varepsilon) \geq 1 - \frac{D(\eta)}{\varepsilon^2},$$

or in the complete form

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n} \xi_i - \frac{1}{n}\sum_{i=1}^{n} M(\xi_i)\right| < \varepsilon\right) \geq 1 - \frac{\sum_{i=1}^{n} D(\xi_i)}{n^2\varepsilon^2} . \qquad (42)$$

Further let the dispersions of all these random variables don't surpass some number C, that is

$$\forall i : D(\xi_i) \leq C .$$

In this case the inequality (42) passes to the next one:

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i-\frac{1}{n}\sum_{i=1}^{n}M(\xi_i)\right|<\varepsilon\right)\geq 1-\frac{C}{n\varepsilon^2}\quad (\forall i: D(\xi_i)\leq C). \qquad (43)$$

Let's finally suppose that all these random variables have the same mathematical expectation a. We'll get the next final result

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i-a\right|<\varepsilon\right)\geq 1-\frac{C}{n\varepsilon^2}\quad (\forall i: M(\xi_i)=a, D(\xi_i)\leq C). \qquad (44)$$

Ex. 20. How many measurements must one fulfil to assert that with the probability 0.99 the error of the arithmetic mean of results of these measurements is less than 0.01 if the root-mean-square deviation of each measurement equals 0.03?

We can consider results of measurements as independent random variables ξ_i with the same mathematical expectation a (the exact value of a quantity to be measured) and dispersion

$$D(\xi_i)=\sigma^2(\xi_i)=0.03^2=0.0009.$$

Supposing

$$C=D(\xi_i)==0.0009, \varepsilon=0.01$$

in the inequality (44) we have to find n from the next relation

$$P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i-a\right|<0.01\right)\geq 1-\frac{0.0009}{n\cdot 0.01^2}=0.99$$

or in point of fact

$$1-\frac{0.0009}{n\cdot 0.01^2}=0.99.$$

This last equation gives $n=900$, and therefore it's sufficient to fulfil not less than 900 measurements.

Chebyshev inequality permits us to estimate the **deviation of the relative frequency** $P_n^*(A)$ **of some event** A **from its probability** $p=P(A)$.

Let's suppose that we fulfil n independent trials with a constant probability $p=P(A)$ of an event A. If a random variable ξ is the number of occurrences of the event A, then the relative frequency of A equals

$$P_n^*(A) = \frac{\xi}{n}.$$

It is obvious that ξ has Bernoulli distribution for which $M(\xi) = np, D(\xi) = npq$. Therefore

$$P\left(\left|P_n^*(A) - P(A)\right| < \varepsilon\right) = P\left(\left|\frac{\xi}{n} - p\right| < \varepsilon\right) = P\left(\left|\xi - np\right| < n\varepsilon\right) = P\left(\left|\xi - M(\xi)\right| < n\varepsilon\right)..$$

Using the inequality (38) (with $n\varepsilon$ as ε) we'll get

$$P\left(\left|P_n^*(A) - P(A)\right| < \varepsilon\right) = P\left(\left|\xi - M(\xi)\right| < n\varepsilon\right) \geq 1 - \frac{npq}{(n\varepsilon)^2} = 1 - \frac{npq}{n^2\varepsilon^2} = 1 - \frac{pq}{n\varepsilon^2}.$$

Thus the deviation of the relative frequency of an event A from its probability is estimated by the next formula:

$$P\left(\left|P_n^*(A) - P(A)\right| < \varepsilon\right) = P\left(\left|\frac{\xi}{n} - p\right| < \varepsilon\right) \geq 1 - \frac{pq}{n\varepsilon^2}. \qquad (45)$$

Remark. The formula (45) can more directly be obtained from the formula (38). Indeed,

$$P\left(\left|\xi - np\right| < \varepsilon\right) = P\left(\left|\frac{\xi}{n} - p\right| < \frac{\varepsilon}{n}\right) = P\left(\left|P_n^*(A) - P(A)\right| < \frac{\varepsilon}{n}\right) \geq 1 - \frac{npq}{\varepsilon^2}.$$

If we substitute $\frac{\varepsilon}{n}$ by ε and so ε by $n\varepsilon$ in this inequality, we'll obtain (45).

Ex. 21 (see Ex. 15 of the Lecture No. 3). Probability for an item to be imperfect is 0.03. How many imperfect items are contained in a batch of 100 items with probability 0.9? Solve the problem with the help of Chebyshev inequality and compare the result with that obtained in the Lecture No. 3 on the base of the formula (10).

Let a random variable ξ be a number of imperfect items in a batch, and an event A means that an item, which is taken at random, isn't perfect one. By conditions of the problem $p = P(A) = 0.03$, $q = = 1 - p = 0.97$, and by virtue of the formula (45) we can write

$$P\left(\left|P_n^*(A) - P(A)\right| < \varepsilon\right) = P\left(\left|\frac{\xi}{100} - 0.03\right| < \varepsilon\right) \geq 1 - \frac{0.03 \cdot 0.97}{100 \cdot \varepsilon^2} = 0.9.$$

The equality $1-\frac{0.03\cdot 0.97}{100\cdot\varepsilon^2}=0.9$ gives $\varepsilon\approx 0.05$, and so we must find possible values of ξ from the relation

$$P\left(\left|\frac{\xi}{100}-0.03\right|<0.05\right)=0.9.$$

This latter gives that with probability 0.9

$$\left|\frac{\xi}{100}-0.03\right|<0,05,\ |\xi-3|<5,\ -5<\xi-3<5,\ -2<\xi<8,\ 0\le\xi<8.$$

Answer. Chebyshev inequality (45) gives that with the probability 0.9 there are less than 8 imperfect items in a batch ($0\le\xi<8$). The method of the Lecture 3 gives more exact result ($1\le\xi\le 5$).

Convergence in probability of random variables

Def. 12. One says that a sequence of random variables

$$\xi_1, \xi_2, \ldots, \xi_n, \ldots$$

converges to some value A **in probability**,

$$\xi_n \underset{prob}{\to} A,$$

if for any positive however small number ε the next limit takes place:

$$\lim_{n\to\infty} P(|\xi_n - A|<\varepsilon)=1. \qquad (46)$$

It follows from the definition that for any ε and sufficiently large n one can regard the event

$$|\xi_n - A|<\varepsilon$$

as that practically certain.

Law of large numbers

Law of large numbers consists of some important theorems. We'll study several of them.

Theorem 1 (Chebyshev small theorem). Let (40) be independent random variables with the same mathematical expectation a and whose dispersions don't surpass some number C. In this case the arithmetic mean (41) of these random variables converges in probability to their common mathematical expectation (if $n \to \infty$).

■Let ε be any positive however small number. From the conditions of the theorem it follows the inequality (44). Passing to limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i - a\right| < \varepsilon\right) \geq 1.$$

The sign ">" isn't possible. Therefore we'll have the equality

$$\lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i - a\right| < \varepsilon\right) = 1.$$

It means by virtue of the definition 12 that the arithmetic mean

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i$$

converges to a in probability,

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i \underset{prob}{\to} a. ■$$

Chebyshev small theorem is often applicable in measurement practice.

Let's we fulfil independent measurements of some quantity. We can regard results of measurements as independent random variables $\xi_i, i = 1, 2, ...,$ with the same mathematical expectation a (which is the exact value of a quantity to be measured). If systematic biases absent that is the measurement dispersions are uniformly bounded by some number C, $\forall i = 1, 2, ... \; D(\xi_i) \leq C$, then by Chebyshev small theorem the arithmetical mean of the results of measurements converges to a in probability.

Theorem 2 (Chebyshew large theorem). Let the random variables (40) have different mathematical expectations in conditions of Chebyshev small theorem. In this case the difference

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i - \frac{1}{n}\sum_{i=1}^{n}M(\xi_i),$$

that is the difference between the arithmetical average (41) of the first n random variables and the arithmetical average of their mathematical expectations, converges to zero in probability (if $n \to \infty$),

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i - \frac{1}{n}\sum_{i=1}^{n}M(\xi_i)\underset{prob}{\to} 0.$$

To prove the theorem it's sufficient to pass to the limit for $n \to \infty$ in the inequality (43). Do it yourselves.

Theorem 3 (Bernoulli[1]). A relative frequency of every event A converges in probability to its probability $P(A)$ (if $n \to \infty$),

$$P_n^*(A)\underset{prob}{\to} P(A).$$

It's sufficient to pass to the limit for $n \to \infty$ in the inequality (45). Verify!

Remark. Bernoulli theorem is the theoretical base of the probability theory.

It's sufficient to take into account the statistical definition of the probability of an event.

Theorem 4 (Markov[2]). Let (40) be dependent random variables and

$$\lim_{n\to\infty}\frac{D(\xi_1+\xi_2+\ldots+\xi_n)}{n^2} = \lim_{n\to\infty}\frac{1}{n^2}D\left(\sum_{i=1}^{n}\xi_i\right) = 0.$$

Then the difference between the arithmetical average (41) of the first n random variables of the sequence (40) and the arithmetical average of their mathematical expectations, that is the difference

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i - \frac{1}{n}\sum_{i=1}^{n}M(\xi_i),$$

converges to zero in probability (if $n \to \infty$),

$$\frac{1}{n}\sum_{i=1}^{n}\xi_i - \frac{1}{n}\sum_{i=1}^{n}M(\xi_i)\underset{prob}{\to} 0.$$

Try to prove this theorem yourselves!

[1] Bernoulli, Jacob (1654 - 1705), the famous Swiss mathematician
[2] Markov, A.A. (1856 - 1922), , a noted Russian mathematician

NUMBER CHARACTERISTICS OF RANDOM VARIABLES:

basic terminology RUEFD

1. асимметрия	асиметрія	Asýmmetry, skéwness	asymétrie *f*	Asymmetrie *f*
2. безразмерная величина	безрозмірна величина	diménsionless/ nóndiménsional mágnitude/ quántity	grandeur *f* [quantité *f*] sans dimension	dimensionslose Größe *f*
3. Бернулли теорема	Бернуллі теорема	Bernoulli théorem	théorème *m* de Bernoulli	Bernoullischer Satz *m*
4. дисперсия случайной величины	дисперсія випадкової величини	váriance [dispérsion] of a rándom váriable	variance *f* [dispersion *f*] d'une variable aléatoire	Varianz *f* [Dispersion *f* , Streuung *f*, Streuungsquadrat *n*] einer Zúfallsgröße
5. дисперсия суммы двух (не)зависимых случайных величин	дисперсія суми двох (не-) залежних випадкових величин	váriance [dispérsion] of a sum of two ìndepéndent [depéndent] rándom váriables	variance *f* [dispersion *f*] d'une somme de deux variables aléatoires indépendantes [dépendantes]	Varianz *f* [Dispersion *f* , Streuung *f*] der Súmme von zwei únabhängiger [abhängiger] Zúfallsgrößen
6. закон больших чисел	закон великих чисел	law of large númbers	loi des grands nombres	Gesetz *n* der großen Zahlen
7. Маркова теорема	Маркова теорема	Markov théorem	théorème *m* de Marcof	Markowscher Satz *m*
8. математическое ожидание	математичне сподівання	màthemátical èxpectátion [èxpectátion válue]	espérance *f* mathématique	Erwartungswert *m* [mathematische Erwartung *f*, mathematische Hoffnung *f*]
9. математическое ожидание произведения двух (не)зависимых случайных величин	математичне сподівання добутку двох (не)залежних випадкових величин	màthemátical èxpectátion of the product of two ìndepéndent [depéndent] rándom váriables	espérance *f* mathématique du produit de deux variables aléatoires indépendantes [dépendantes]	Erwartungswert *m* des Prodúkt(e) von zwei únabhängiger [abhängiger] Zúfallsgrößen
10.математи-	математичне	màthemátical	espérance *f*	Erwartungs-

ческое ожидание случайной величины	сподівання випадкової величини	èxpectátion of a rándom váriable	mathématique d'une variable aléatoire	wert *m* [mathematische Erwartung *f*, mathematische Hoffnung *f*] der Zúfallsgröße
11.математическое ожидание суммы двух случайных величин	математичне сподівання суми двох випадкових величин	màthemátical èxpectátion of the sum of two rándom váriables	espérance *f* mathématique d'une somme de deux variables aléatoires	Erwartungswert *m* der Súmme von zwei Zúfallsgrößen
12.медиана случайной величины	медіана випадкової величини	médian [médian válue] of a rándom váriable	médiane *f* d'une variable aléatoire	Medián *m* [Zentralwert *m*] einer Zúfalsgröße
13.мода случайной величины	мода випадкової величини	móde of a rándom váriable	mode *f* d'une variable aléatoire	Móde *f* [häufigster Wert *m*] einer Zúfallsgröße
14.момент высшего порядка	момент вищого порядку	hígh-órder móment (còefficient)	moment *m* d'ordre supérieur	höheres Moment
15.момент распределения	момент розподілу	móment of dìstribútion [dìstribútion móment]	moment *m* d'une distribution [d'une répartition]	Moment *n* von Vertéilung, Vertéilungsmoment *n*
16.момент случайной величины	момент випадкової величини	móment of a rándom váriable	moment *m* d'une variable aléatoire	Moment *n* einer Zúfallsgröße
17.начальный момент (первого, второго, третьего, четвертого, *n*-го порядка, порядка *n*)	початковий момент (першого, другого, третього, четвертого, *n*-го порядку, порядку *n*)	first-, second-, third-, fourth-órder, *n*-th órder) inítial [órdinary] móment	moment *m* initial (du premier, deuxième, troisième, quatrième, *n*-ième ordre, d'ordre *n*)	Ánfangsmoment *n* (erster, zweiter, dritter, vierter, *n*-ter Ordnung)
18.независимые случайные величины	незалежні випадкові величини	índepéndent rándom váriables	variables *f* aléatoires indépendantes	unabhängige Zúfallsgrößen
19.несимметричное распределение	несиметричний розподіл	àsymmétric(al) dìstribútion	distribution *f* [répartition *f*] asymétrique	únsymmetrische Vertéilung

20.отклонение (случайной величины от ее математического ожидания)	відхилення (випадкової величини від її математичного сподівання)	dèviátion of a rándom váriable from its màthemátical èxpectátion	déviation *f* [écart *m*](d'une variable aléatoire de son espérance mathématique	Ábweichung (*f* =,-en) einer Zúfallsgröße von ihm Erwartungwert(e) [von ihr mathematische Erwartung,von ihr mathematische Hoffnung]
21.первый момент	перший момент	first móment	le premier moment	erstes [das erste] Moment
22.последовательность случайных величин	послідовність випадкових величин	séquence of rándom váriables	suite *f* de variables aléatoires	Fólge *f* [Fólgenreihe *f*] der Zúfallsgrößen
23.Пуассона теорема	Пуассона теорема	Poisson théorem	théorème *m* de Poisson	Poissonscher Satz
24.размерная величина	розмірна величина	diménsional [denóminate] mágnitude [quántity]	variable *f* [quantité *f*, grandeur *f*] dimensionnelle	dimensionsbehaltete [dimensionale] Größe
25.размерность величины	розмірність величини	diménsion [dimènsionálity] of a mágnitude [of a quántity]	dimension *f* d'une variable [d'une quantité, d'une grandeur]	Dimension *f* einer Größe
26.рассеяние (случайной величины)	розсіяння (випадкової величини)	váriance [cònsentrátion, dìspérsion, dìssipátion, scátter, scáttering] of a rándom váriable	variance *f* [concentration *f*, dispersion *f*] (d'une variable aléatoire)	Varianz *f* [Dispersión *f*, Konzentratión *f*, Streuung *f*, Zerstreuung *f*] einer Zúfallsgröße
27.симметричное распределение	симетричний розподіл	symmétric(al) distribútion	distribution *f* [répartition *f*] symétrique	symmétrische Vertéilung
28.систематическая ошибка	систематична помилка	systemátic(al) [cónstant, fixed, hard, nonsámpling, sólid] érror	erreur *f* systématique	systematischer [regelmäßiger] Féhler
29.средне-	середнє квад-	root-méan-	ecart *f* quad-	durchschnitt-

квадратическое [среднее квадратическое] отклонение	ратичне [середньоквадратичне] відхилення	squáre [quadrátic méan, méan-squáre] dèviátion, stándard dèviátion	ratique moyen, écart type [écart-type *m*, dispersion *f* standard]	liche quadratische Abweichung, mittlere quadratische Abweichung *f*, Streuung *f* , Standardabweichung *f*
30.среднее арифметическое (наблюдавшихся значений случайной величины)	середнє арифметичне (спостережених значень випадкової величини)	àrithmétical average/méan [àrithmétical méan válue] (of obsérved values of a rándom váriable	moyenne *f* arithmétique (de valeurs observées d'une variable aléatoire)	arithmétisches Míttel (*n* –s,=) von Beobachtungswerten einer Zúfallsgröße
31.среднее взвешенное	середнє зважене	wéighted áverage	moyenne *f* pondérée	gewógenes Míttel *n* –s,=
32.среднее значение случайной величины	середнє значення випадкової величини	average [méan válue] of a rándom váriable	valeur *f* moyenne d'une variable aléatoire	Durchschnitt *m* [Durchschnittsgröße *f*, Durchschnittswert *m*, Mittelwert *m*, mittlerer Wert *m*] einer Zúfallsgröße
33.сходимость по вероятности	збіжність за ймовірністю	pròbabílity convérgence, convérgence in pròbabílity	convergence *f* en probabilité	Konvergénz in Wahrschéinlichkeit
34.сходиться по вероятности	збігатися за ймовірністю	convérge in pròbabílity	converger en probabilité	konvergíeren in Wahrschéinlichkeit
35.функция случайной величины	функція випадкової величини	fúnction of a rándom váriable	fonction *f* d'une variable aléatoire	Funktión *f* einer Zúfallsgröße
36.центр распределения [центр рассеяния] случайной величины	центр розподілу/розсіяння випадкової величини	dìstribútion centre, centre of dìstribútion [of còncentrátion, of dispérsion, of scátter, of scáttering] of a rándom	centre *m* de distribution [de répartition, de dispersion, de concentration de variance] d'une variable aléatoire	Vertéilungszentrum (*n*) [Zéntrum (*n*) von Vertéilung] einer Zúfallsgröße

		váriable		
37.центральный момент (первого, второго, третьего, четвертого, *n*-го порядка, порядка *n*)	центральний момент (першого, другого, третього, четвертого, *n*-го порядку, порядку *n*)	(first-, second-third-, fourth-órder, *n*-th órder) céntral móment	moment *m* central (du premier, deuxième, troisième, quatrième, *n*-ième ordre; d'ordre un, deux, trois, quatre, *n*)	zentrales Moment *n* (erster, zweiter, dritter, vierter, *n*-ter Ordnung)
38.центрированная случайная величина	центрована випадкова величина	céntered rándom váriable	variable *f* aléatoire centrée	zéntrische Zúfallsgröße
39.Чебышёва неравенство	Чебишова нерівність	Chebyshev's ìnequálity	inégalité *f* de Tchebichof	Tschebyschewsche Úngleichung
40.Чебышёва теорема	Чебишова теорема	Chebyshev théorem	théorème *m* Tchebichof	Tschebyschewscher Satz *m*
41.числовая характеристика (случайной величины)	числова характеристика (випадкової величини)	number [numérical] chàracterístic [k-] of a rándom váriable	caractéristique numérique d'une variable aléatoire	záhlenmäßige Charakteristik *f* einer Zúfallsgröße
42.эксцесс	ексцес	excéss	excès *m*	Exzeß *m*
43.энный (*n*-ый) момент	енний (*n*-й) момент	*n*-th móment	moment *m* *n*-ième	*n*-tes Moment

NUMBER CHARACTERISTICS OF RANDOM VARIABLES:

basic terminology EFDRU

1. àrithmétical average/méan [àrithmétical méan válue] (of obsérved values of a rándom vári-able	moyenne *f* arithmétique (de valeurs observées d'une variable aléatoire)	arithmétisches Míttel (*n* –s,=) von Beobachtungswerten einer Zúfallsgröße	среднее арифметическое (наблюдавшихся значений случайной величины)	середнє арифметичне (спостережених значень випадкової величини)
2. àsymmétric(al) dìstribútion	distribution *f* [répartition *f*] asymétrique	únsymmetrische Vertéilung	несимметричное распределение	несиметричний розподіл
3. asýmmetry, skéwness	asymétrie *f*	Asymmetrie *f*	асимметрия	асиметрія
4. average [méan válue] of a rándom váriable	valeur *f* moyenne d'une variable aléatoire	Durchschnitt *m* [Durchschnitsgröße *f*, Durchschnittswert *m*, Mittelwert *m*, mittlerer Wert *m*] einer Zúfallsgröße	среднее значение случайной величины	середнє значення випадкової величини
5. Bernoulli théorem	théorème *m* de Bernoulli	Bernoullischer Satz *m*	Бернулли теорема	Бернуллі теорема
6. céntered rándom vári-able	variable *f* aléatoire centrée	zéntrische Zúfallsgröße	центрированная случайная величина	центрована випадкова величина
7. céntral móment (first-, second-, third-, fourth-órder, *n*-th órder)	moment *m* central (du premier, deuxième, troisième, quatrième, *n*-ième ordre; d'ordre un, deux, trois, quatre, *n*)	zentrales Moment *n* (erster, zweiter, dritter, vierter, *n*-ter Ordnung)	центральный момент (первого, второго, третьего, четвертого, *n*-го порядка, порядка *n*)	центральний момент (першого, другого, третього, четвертого, *n*-го порядку, порядку *n*)
8. Chebyshev ìnequálity	inégalité *f* de Tchebichof	Tschebyschewsche Úngleichung	неравенство Чебышёва	нерівність Чебишова
9. Chebyshev	théorème *m*	Tscheby-	Чебышёва те-	Чебишова те-

théorem	Tchebichof	schewscher Satz *m*	орема	орема
10.convérge in pròbabílity	converger en probabilité	konvergíeren in Wahrschéinlichkeit	сходиться по вероятности	збігатися за ймовірністю
11.dèviátion of a rándom váriable from its màthemátical èxpectátion	déviation *f* [écart *m*](d'une variable aléatoire de son espérance mathématique	Ábweichung (*f* =,-en) einer Zúfallsgröße von ihm Erwartungwert(e) [von ihr mathematische Erwartung, von ihr mathematische Hoffnung]	отклонение (случайной величины от ее математического ожидания)	відхилення (випадкової величини від її математичного сподівання)
12.diménsion [dimènsionálity] of a mágnitude [of a quántity]	dimension *f* d'une variable [d'une quantité, d'une grandeur]	Dimension *f* einer Größe	размерность величины	розмірність величини
13.diménsional [denóminate] mágnitude [quántity]	variable *f* [quantité *f*, grandeur *f*] dimensionnelle	dimensionsbehaltete [dimensionale] Größe	размерная величина	розмірна величина
14.diménsionless/nóndiménsional mágnitude/quántity	grandeur *f* [quantité *f*] sans dimension	dimensionslose Größe	безразмерная величина	безрозмірна величина
15.dìstribútion centre, centre of dìstribútion [of còncentrátion, of dispérsion, of scátter, of scáttering] of a rándom váriable	centre *m* de distribution [de répartition, de dispersion, de concentration, de variance] d'une variable aléatoire	Vertéilungszentrum (*n*) [Zéntrum (*n*) von Vertéilung (*f*)] einer Zúfallsgröße	центр распределения [центр рассеяния] случайной величины	центр розподілу/розсіяння випадкової величини
16.excéss	excès *m*	Exzeß *m*	эксцесс	ексцес
17.first móment	le premier moment	erstes [das erste] Moment	первый момент	перший момент
18.first-, second-, third-, fourth-órder,	moment *m* initial (du premier, deuxième,	Ánfangsmoment (erster, zweiter, dritter,	начальный момент (первого, второго,	початковий момент (першого, друго-

n-th órder) inítial [órdinary] móment	troisième, quatrième, *n*-ième ordre, d'ordre *n*)	vierter, *n*-ter Ordnung)	третьего, четвертого, *n*-го порядка, порядка *n*)	го, третього, четвертого, *n*-го порядку, порядку *n*)
19.fúnction of a rándom váriable	fonction *f* d'une variable aléatoire	Funktión *f* einer Zúfallsgröße	функция случайной величины	функція випадкової величини
20.hígh-órder móment (còefficient)	moment *m* d'ordre supérieur	höheres Moment	момент высшего порядка	момент вищого порядку
21.índepéndent rándom váriables	variables *f* aléatoires indépendantes	unabhängige Zúfallsgrößen	независимые случайные величины	незалежні випадкові величини
22.law of large númbers	loi des grands nombres	Gesetz *n* der großen Zahlen	закон больших чисел	закон великих чисел
23.markov théorem	théorème *m* de marcof	Markowscher Satz	маркова теорема	маркова теорема
24.màthemátical èxpectátion [èxpectátion válue]	espérance *f* mathématique	Erwartungswert *m* [mathematische Erwartung *f*, mathematische Hoffnung *f*]	математическое ожидание	математичне сподівання
25.màthemátical èxpectátion of a rándom váriable	espérance *f* mathématique d'une variable aléatoire	Erwartungswert *m* [mathematische Erwartung *f*, mathematische Hoffnung *f*] der Zúfallsgröße	математическое ожидание случайной величины	математичне сподівання випадкової величини
26.màthemátical èxpectátion of the product of two èndepéndent [depéndent] rándom váriables	espérance *f* mathématique du produit de deux variables aléatoires indépendantes [dépendantes]	Erwartungswert *m* des Prodúkt(e) von zwei únabhängiger [abhängiger] Zúfalsgrößen	математическое ожидание произведения двух (не)зависимых случайных величин	математичне сподівання добутку двох (не)залежних випадкових величин
27.màthemátical èxpectátion of the sum of two rándom váriables	espérance *f* mathématique d'une somme de deux variables aléatoires	Erwartungswert *m* der Súmme von zwei Zúfallsgrößen	математическое ожидание суммы двух случайных величин	математичне сподівання суми двох випадкових величин

28.médian [médian válue] of a rándom váriable	médiane *f* d'une variable aléatoire	Medián *m* [Zentralwert *m*] einer Zúfal-lsgröße	медиана случайной величины	медіана випадкової величини
29.móde of a rándom váriable	mode *f* d'une variable aléatoire	Móde *f* [häufigster Wert *m*] einer Zúfallsgröße	мода случайной величины	мода випадкової величини
30.móment of a rándom váriable	moment *m* d'une variable aléatoire	Moment *n* einer Zúfallsgröße	момент случайной величины	момент випадкової величини
31.móment of dìstribútion [dìstribútion móment]	moment *m* d'une distribution [d'une répartition]	Moment *n* von Vertéilung, Vertéilungs-moment *n*	момент распределения	момент розподілу
32.*n*-th móment	moment *m* *n*-ième	*n*-tes Moment	энный (*n*-ый) момент	енний (*n*-й) момент
33.number [numérical] chàracterístic [k-] of a rándom váriable	caractéristique numérique d'une variable aléatoire	záhlenmäßige Charakteristik *f* einer Zúfallsgröße	числовая характеристика (случайной величины)	числова характеристика (випадкової величини)
34.poisson théorem	théorème *m* de poisson	Poissonscher Satz	пуассона теорема	пуассона теорема
35.pròbabílity convérgence, convérgence in pròbabílity	convergence *f* en probabilité	Konvergénz in Wahrschéin-lichkeit	сходимость по вероятности	збіжність за ймовірністю
36.root-méan-squáre [quadrátic méan, méan-squáre] dèviátion, stándard dèviátion	ecart *f* quadratique moyen, écart type [écart-type *m*, dispersion *f* standard]	durchschnitt-liche quadratische Abweichung *f*, mittlere quadratische Abweichung *f*, Streuung *f* , Standardab-weichung *f*	среднее квадратическое [среднеквад-ратическое] отклонение	середнє квадратичне [се-редньоквад-ратичне] відхилення
37.séquence of rándom váriables	suite *f* de variables aléatoires	Fólge *f* [Fólgenreihe *f*] der Zúfallsgrößen	последова-тельность случайных величин	послідовність випадкових величин
38.symmétric(al) dìstribútion	distribution *f* [répartition *f*] symétrique	symmétrische Vertéilung	симметрич-ное распределение	симетричний розподіл

39.systemátic(al) [cónstant, fixed, hard, non-sámpling, sólid] érror	erreur *f* systématique	systematischer [regelmäßiger] Féhler	систематическая ошибка	систематична помилка
40.váriance [cònsentrátion, dìspérsion, dìssipátion, scátter, scáttering] of a rándom váriable	variance *f* [concentration *f* , dispersion *f*] (d'une variable aléatoire)	Varianz *f* [Dispersión *f*, Konzentratión *f*, Streuung *f*, Zerstreuung *f*] einer Zúfallsgröße	рассеяние (случайной величины)	розсіяння (випадкової величини)
41.váriance [dispérsion] of a rándom váriable	variance *f* [dispersion *f*] d'une variable aléatoire	Varianz *f* [Dispersion *f* , Streuung *f*, Streuungsquadrat *n*] einer Zúfallsgröße	дисперсия случайной величины	дисперсія випадкової величини
42.váriance [dispérsion] of a sum of two ìndepéndent [depéndent] rándom váriables	variance *f* [dispersion *f*] d'une somme de deux variables aléatoires indépendantes [dépendantes]	Varianz *f* [Dispersion *f* , Streuung *f*] der Súmme *f* von zwei únabhängiger [abhängiger] Zúfallsgrößen	дисперсия суммы двух (не)зависимых случайных величин	дисперсія суми двох (не-) залежних випадкових величин
43.wéighted áverage	moyenne *f* pondérée	gewógenes Míttel *n* –s,=	среднее взвешенное	середнє зважене

LECTURE NO. 6. SOME REMARKABLE DISTRIBUTIONS

LE SIXIEME COURS. QUELQUES-UNS DISTRIBUTIONS REMARQUABLES.

SECHSTE VORLESUNG. EINIGE AUSGEZEICHNETE VERTEILUNGEN

POINT 1. THE UNIFORM DISTRIBUTION. Distribution uniforme. Gleichverteilung.

POINT 2. THE NORMAL DISTRIBUTION. Distribution normale. Normalverteilung.

POINT 3. THE EXPONENTIAL DISTRIBUTION. Distribution exponentielle. Exponentialverteilung.

POINT 4. GAMMA-DISTRIBUTION. Distribution gamma. Gammaverteilung.

POINT 5. SOME OTHER DISTRIBUTIONS. Quelques-uns autres d'entre distributions. Einige andere Verteilungen.

POINT 1. THE UNIFORM DISTRIBUTION

Def. 1. One says that a random variable ξ has the uniform distribution over an interval $< a, b >$ (ξ is uniformly distributed or simply ξ is the uniform distribution over an interval $< a, b >$), if its distribution density is constant inside and equals zero outside this interval,

$$f(x) = \begin{cases} C & if \quad x \in (a, b), \\ 0 & if \quad x \notin [a, b]. \end{cases} \tag{1}$$

We have to find the value of the constant C, the distribution function and number characteristics of the uniform distribution.

A. Finding the value of C.

On the base of the property 4 of the distribution density (see Lecture No. 4, point 2, the formula (16)) we must have

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

Dividing the interval $(-\infty, \infty)$ into three partes, namely

$$(-\infty, \infty) = (-\infty, a) \cup [a, b] \cup (b, \infty),$$

we'll get

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} 0dx + \int_{a}^{b} Cdx + \int_{b}^{\infty} 0dx = \int_{a}^{b} Cdx = C\int_{a}^{b} dx = C(b-a), C = \frac{1}{b-a},$$

and so the distribution density of the uniform distribution is the next one:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & if \quad x \in (a, b), \\ 0 & if \quad x \notin [a, b]. \end{cases} \qquad (2)$$

B. Finding the distribution function of the uniform distribution.

Using the property 3 of the distribution density (the formula (15) of the Lecture No. 4) we must study three cases.

a) The fist case: $x \le a$.

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} 0dx = 0.$$

b) The second case: $a < x \le b$.

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{a} 0dx + \int_{a}^{x} \frac{1}{b-a} dx = \frac{1}{b-a}\int_{a}^{x} dx = \frac{1}{b-a} \cdot (x-a) = \frac{x-a}{b-a}.$$

c) The third case: $b < x < \infty$.

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{a} 0dx + \int_{a}^{b} \frac{1}{b-a} dx + \int_{b}^{\infty} 0dx = \int_{a}^{b} \frac{1}{b-a} dx = \frac{1}{b-a}\int_{a}^{b} dx = \frac{b-a}{b-a} = 1.$$

Thus the distribution function of the uniform distribution is

$$F(x) = P(\xi < x) = \begin{cases} 0 & if \quad -\infty < x \le a, \\ \dfrac{x-a}{b-a} & if \quad a < x \le b, \\ 1 & if \quad b < x < \infty. \end{cases} \qquad (3)$$

It is continuous one for all values of x, and therefore the uniform distribution is a continuous random variable. The graphs of the distribution density and function of the uniform distribution are represented on the figures 1, 2.

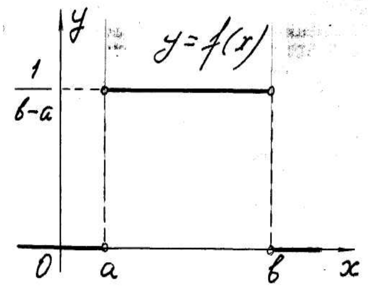


Fig. 1

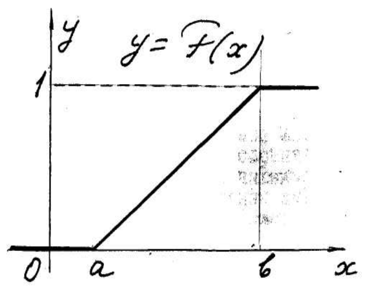


Fig. 2

C. Finding the number characteristics of the uniform distribution.

We'll limit ourselves to the mathematical expectation, dispersion and root-mean-square deviation. For this purpose we'll make use of the formulas (3), (6), (16), (10) of the Lecture No. 5.

$$m_\xi = M(\xi) = \int_a^b x f(x) dx = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \frac{x^2}{2} \Bigg|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}.$$

$$M(\xi^2) = \int_a^b x^2 f(x) dx = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a} \cdot \frac{x^3}{3} \Bigg|_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + ab + b^2}{3}.$$

$$D_\xi = D(\xi) = M(\xi^2) - M^2(\xi) = \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{a^2 + ab + b^2}{3} - \frac{a^2 + 2ab + b^2}{4} =$$

$$= \frac{4a^2 + 4ab + 4b^2 - 3a^2 - 6ab - 3b^2}{12} = \frac{a^2 - 2ab + b^2}{12} = \frac{(b-a)^2}{12},$$

$$\sigma_\xi = \sigma(\xi) = \sqrt{D(\xi)} = \sqrt{D_\xi} = \frac{b-a}{2\sqrt{3}}.$$

Thus for the uniform distribution on an interval $< a, b >$

$$m_\xi = M(\xi) = \frac{a+b}{2}, \quad D_\xi = D(\xi) = \frac{(b-a)^2}{12}, \quad \sigma_\xi = \sigma(\xi) = \frac{b-a}{2\sqrt{3}}. \qquad (4)$$

Ex. 1. The hitting probability of a random variable ξ, which is uniformly distributed over an interval $< a, b >$, on some part $< c, d >$ of $< a, b >$ is proportional to the length $d - c$ of that part (with a proportional coefficient $1/(b-a)$).

■On the base of the formula (9) of the Lecture No. 4 and the formula (3)

$$P(\xi \in < c, d > \subseteq < a, b >) = F(d) - F(c) = \frac{d-a}{b-a} - \frac{c-a}{b-a} = \frac{d-c}{b-a} = \frac{1}{b-a} \cdot (d-c).$$■

The preceding example gives us a useful formula for the uniform distribution

$$P(\xi \in < c, d > \subseteq < a, b >) = \frac{1}{b-a} \cdot (d-c). \qquad (5)$$

Ex. 2 (examples of application of the uniform distribution).

a) When we say, "Let's take a point ξ at random on an interval $< a, b >$", we keep in mind that ξ is a random variable which is uniformly distributed over the interval $< a, b >$.

b) Saying, "Let's at random choose a direction on a plane", we intend that an angle φ between the chosen direction and some axis on the plane is a random variable which is uniformly distributed over the interval $[0, 2\pi)$.

c) The rounding errors ξ in Bradis tables of (anti)logarithms, (co)sines, (co)-tangents etc. are uniformly distributed over the interval $(-0.00005, 0.00005)$.

d) The clock-face is marked every 5 minutes. The rounding error ξ to the nearest integer scale division is uniformly distributed over the interval $(-2.5, 2.5)$.

e) A trolleybus [ˊtrɒlɪbʌs] runs at intervals a minutes. One comes to a (trolleybus) stop at a random time moment. The waiting time T is a random variable uniformly distributed over the interval $(0, a)$.

Ex. 3. The time interval of the trolleybus service equals 5 minutes. Find the probability that one will wait a trolleybus no longer then 2 minutes.

Solution. By Ex. 2e the waiting time T is a random variable uniformly distributed over the interval $(0, 5)$, and we have to find the probability $P(T \in < 0, 2 >)$.

By virtue of the formula (5) (in which we must take respectively $0, 5, 0, 2$ for a, b, c, d)

$$P(T \in < 0, 2 >) = \frac{1}{5-0} \cdot (2-0) = 0.4 .$$

Ex. 4. A linear function of the uniform distribution.

Let a random variable ξ have the uniform distribution over an interval $< a, b >$, and a random variable $\eta = k\xi + l$ be a linear function of ξ. By the formula (20) of the Lecture No. 4 the distribution density of the random variable η equals

$$g(y) = \frac{1}{|k|} f\left(\frac{y-l}{k}\right).$$

Let's determine the form of the distribution law of the random variable η.

We must consider two cases, namely $k > 0$ and $k < 0$.

a) In the case $k > 0$ by the formula (2) for the distribution density $f(x)$ of the random variable ξ we have

$$g(y) = \begin{cases} 0 & \text{if} \quad \frac{y-l}{k} < a \quad \text{or} \quad y < ka + l, \\ \frac{1}{k} \cdot \frac{1}{b-a} = \frac{1}{(kb+l)-(ka+l)} & \text{if} \quad a < \frac{y-l}{k} < b \quad \text{or} \quad ka + l < y < kb + l, \\ 0 & \text{if} \quad \frac{y-l}{k} > b \quad \text{or} \quad y > kb + l. \end{cases}$$

It follows from the same formula (2) that the random variable $\eta = k\xi + l$ has the uniform distribution over the interval $< ka + l, kb + l >$.

b) In the case $k < 0$ the random variable $\eta = k\xi + l$ has the uniform distribution over the interval $< kb + l, ka + l >$. Prove this fact yourselves.

Thus a linear function of the uniform distribution also has the uniform distribution.

The mathematical expectation, dispersion and root-mean-square deviation of the random variable $\eta = k\xi + l$ we'll find on the base of corresponding properties of the mathematical expectation and dispersion (see Lecture No. 5, Points 1, 2). Indeed,

$$M(\eta) = M(k\xi + l) = M(k\xi) + l = kM(\xi) + l = k \cdot \frac{a+b}{2} + l,$$

$$D(\eta) = D(k\xi + l) = D(k\xi) = k^2 D(\xi) = \frac{k^2 (b-a)^2}{12},$$

$$\sigma(\eta) = \sqrt{D(\eta)} = \sqrt{k^2 D(\xi)} = |k|\sqrt{D(\xi)} = |k|\sigma(\xi) = \frac{|k|(b-a)}{2\sqrt{3}}.$$

POINT 2. THE NORMAL DISTRIBUTION

Def. 2. One says that a random variable ξ has the normal distribution with parameters $a, \sigma > 0$ (or that ξ is distributed $N(a, \sigma)$) if its distribution density is the next function:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}}. \quad (6)$$

The factor

$$\frac{1}{\sqrt{2\pi}\sigma}$$

provides satisfaction of the condition

$$\int_{-\infty}^{\infty} f(x)dx = 1. \quad (7)$$

Indeed, by virtue of the famous integral (Poisson integral)

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad (8)$$

we'll have

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{2\sigma^2}} dx = \left| \begin{matrix} \dfrac{x-a}{\sigma\sqrt{2}} = t, & x = a + \sigma t\sqrt{2} \\ dx = \sigma\sqrt{2}dt & \begin{array}{c|c|c} x & -\infty & \infty \\ \hline t & -\infty & \infty \end{array} \end{matrix} \right| =$$

$$= \frac{\sigma\sqrt{2}}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1.$$

Investigation the distribution density $f(x)$ and plotting its graph

1. The function $f(x)$ is determined for all values of x, $D(f) = (-\infty, \infty)$.

2. The function $f(x)$ is positive for any x, and so its graph lies above the Ox – axis.

3. The function $f(x)$ is even with respect to the $x = a$,

$$f(a-x)=f(a+x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}},$$

and therefore its graph is symmetric with respect to the straight line $x=a$.

4. The limit of the function for x tending to $\pm\infty$ equals zero, and so the $Ox-$ axis is the horizontal asymptote of the graph of the function.

5. The derivative of the function equals

$$f'(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-a)^2}{2\sigma^2}}\cdot\left(-\frac{2(x-a)}{2\sigma^2}\right)=-\frac{x-a}{\sigma^2}\cdot f(x).$$

It is positive for $x<a$, negative for $x>a$, and so the function $f(x)$ increases on the interval $(-\infty,a)$, decreases on the interval (a,∞) and has a maximum at the point $x=a$,

$$f_{\max}=f(a)=\frac{1}{\sqrt{2\pi}\sigma}.$$

6. The second order derivative of the function $f(x)$ equals

$$f''(x)=-\frac{1}{\sigma^2}((x-a)f(x))'=-\frac{1}{\sigma^2}(f(x)+(x-a)f'(x))=$$

$$=-\frac{1}{\sigma^2}\left(f(x)-(x-a)\frac{x-a}{\sigma^2}\cdot f(x)\right)=-\frac{f(x)}{\sigma^4}\left(\sigma^2-(x-a)^2\right)=\frac{f(x)\left((x-a)^2-\sigma^2\right)}{\sigma^4}.$$

It equals zero if

$$(x-a)^2-\sigma^2=0,\,(x-a)^2=\sigma^2,\,x-a=\pm\sigma,\,x=a\pm\sigma,$$

is positive for $x<a-\sigma,\ x>a+\sigma$ and negative for $a-\sigma<x<a+\sigma$. Hence the graph of the function $f(x)$ is concave over the intervals $(-\infty,a-\sigma),(a+\sigma,\infty)$, convex over the interval $(a-\sigma,a+\sigma)$ and has two inflexion points for $x=a\pm\sigma$ that is the next points:

$$(a\pm\sigma;\,f(a\pm\sigma))=\left(a\pm\sigma;\,\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{((a\pm\sigma)-a)^2}{2\sigma^2}}\right)=\left(a\pm\sigma;\,\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}}\right)=\left(a\pm\sigma;\,\frac{1}{\sqrt{2\pi e}\sigma}\right)$$

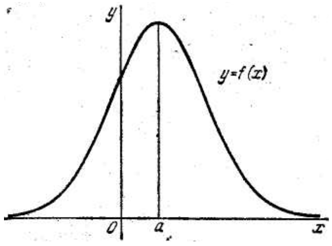


Fig. 3

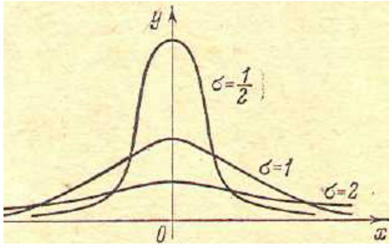


Fig. 4

The graph of the function $f(x)$, that is that of the distribution density of the normal distribution, is represented on the figure 3.

Def. 3. The graph of the distribution density of the normal distribution is called the normal curve.

The normal curve has the other and very fine name, namely the bell-like [or the bell-shaped] curve.

The normal distribution is often called Gauss distribution, and the corresponding normal curve is called Gauss curve.

Let's consider two important facts connected with the distribution density $f(x)$ of the normal distribution.

Let the parameter σ of the normal distribution tend to 0.

1. For $x = a$ we have

$$\lim_{\sigma\to 0} f_{\max} = \lim_{\sigma\to 0}\frac{1}{\sqrt{2\pi}\sigma} = \infty .$$

2. For $x \neq a$ we have

$$\lim_{\sigma\to 0} f(x) = 0 .$$

Indeed, with the help of L'Hospital rule one easy find

$$\lim_{\sigma\to 0} f(x) = \lim_{\sigma\to 0}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-a)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}}\lim_{\sigma\to 0}\sigma^{-1}e^{-\frac{(x-a)^2}{2}\sigma^{-2}} = \frac{1}{\sqrt{2\pi}}\lim_{\sigma\to 0}\frac{\sigma^{-1}}{e^{\frac{(x-a)^2}{2}\sigma^{-2}}} = \left|\begin{matrix}\sigma^{-1} = t\\ t\to +\infty\end{matrix}\right| =$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{t\to +\infty}\frac{t}{e^{\frac{(x-a)^2}{2}t^2}} = \frac{1}{\sqrt{2\pi}}\lim_{t\to +\infty}\frac{t'}{\left(e^{\frac{(x-a)^2}{2}t^2}\right)'} = \frac{1}{\sqrt{2\pi}}\lim_{t\to +\infty}\frac{1}{t(x-a)^2 e^{\frac{(x-a)^2}{2}t^2}} = 0 .$$

It follows from 1, 2 that for $\sigma \to 0$ the normal curve stretches along the straight line $x = a$ and simultaneously presses to the $Ox-$axis.

On the fig. 4 we've represented three normal curves for $a = 0$ and for decreasing values of σ, namely $\sigma = 2, \sigma = 1, \sigma = 1/2$.

Number characteristics of the normal distribution

Let a random variable ξ be normally distributed with parameters $a, \sigma\ (\sigma > 0)$ (ξ distributed $N(a,\sigma), \sigma > 0$). We assert that its number characteristics, namely the mathematical expectation, dispersion, root-mean-square deviation, asymmetry and excess, are represented by the next formulas:

$$M(\xi) = a, \quad D(\xi) = \sigma^2, \quad \sigma(\xi) = \sigma, \quad As(\xi) = 0, \quad Ex(\xi) = 0. \qquad (9)$$

■By virtue of the formulas (2), (5), (15), (10), (33), (34), (35) of the Lecture No. 5 and the formula (8) for Poisson integral we successively get

$$M(\xi) = \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x e^{-\frac{(x-a)^2}{2\sigma^2}} dx =$$

$$= \left| \begin{matrix} \frac{x-a}{\sqrt{2}\sigma} = t & x = a + \sqrt{2}\sigma t \\ dx = \sqrt{2}\sigma dt & \begin{array}{c|c|c} x & -\infty & \infty \\ \hline t & -\infty & \infty \end{array} \end{matrix} \right| = \frac{\sqrt{2}\sigma}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \left(a + \sqrt{2}\sigma t\right) e^{-t^2} dt = \frac{a}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt + \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} t e^{-t^2} dt =$$

$$= \left| \int_{-\infty}^{\infty} t e^{-t^2} dt = 0 \right| = \frac{a}{\sqrt{\pi}} \cdot \sqrt{\pi} + \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \cdot 0 = a;$$

$$D(\xi) = \int_{-\infty}^{\infty} (x-a)^2 f(x) dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-a)^2 e^{-\frac{(x-a)^2}{2\sigma^2}} dx = \left| \begin{matrix} \frac{x-a}{\sqrt{2}\sigma} = t & dx = \sqrt{2}\sigma dt \\ x - a = \sqrt{2}\sigma t & \end{matrix} \right| =$$

$$= \frac{\sqrt{2}\sigma}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \left(\sqrt{2}\sigma t\right)^2 e^{-t^2} dt = \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt = \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t \cdot t e^{-t^2} dt = \left| \begin{matrix} u = t & dv = t e^{-t^2} dt \\ du = dt & v = -\frac{1}{2} e^{-t^2} \end{matrix} \right| =$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left(-\frac{1}{2} t e^{-t^2} \Big|_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-t^2} dt \right) = \frac{2\sigma^2}{\sqrt{\pi}} \left(0 + \frac{1}{2}\sqrt{\pi} \right) = \sigma^2;$$

$$\sigma(\xi)=\sqrt{D(\xi)}=\sigma;$$

$$\mu_3(\xi)=0,\ As(\xi)=\frac{\mu_3(\xi)}{\sigma^3(\xi)}=0$$

because of symmetry of the normal distribution about its mathematical expectation (see the third property of the distribution density $f(x)$, namely $f(a-x)=f(a+x)$);

$$\mu_4(\xi)=\int_{-\infty}^{\infty}(x-a)^4 f(x)dx=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}(x-a)^4 e^{-\frac{(x-a)^2}{2\sigma^2}}dx=\left|\begin{matrix}\frac{x-a}{\sqrt{2}\sigma}=t & dx=\sqrt{2}\sigma dt\\ x-a=\sqrt{2}\sigma t & \end{matrix}\right|=$$

$$=\frac{\sqrt{2}\sigma}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\left(\sqrt{2}\sigma t\right)^4 e^{-t^2}dt=\frac{4\sigma^4}{\sqrt{\pi}}\int_{-\infty}^{\infty}t^4e^{-t^2}dt=\frac{4\sigma^4}{\sqrt{\pi}}\int_{-\infty}^{\infty}t^3\cdot te^{-t^2}dt=\left|\begin{matrix}u=t^3 & dv=te^{-t^2}dt\\ du=3t^2dt & v=-\frac{1}{2}e^{-t^2}\end{matrix}\right|=$$

$$=\frac{4\sigma^4}{\sqrt{\pi}}\left(-\frac{1}{2}t^3e^{-t^2}\Big|_{-\infty}^{\infty}+\frac{3}{2}\int_{-\infty}^{\infty}t^2e^{-t^2}dt\right)=\frac{4\sigma^4}{\sqrt{\pi}}\cdot 0+\frac{6\sigma^4}{\sqrt{\pi}}\cdot\int_{-\infty}^{\infty}t^2e^{-t^2}dt=\frac{6\sigma^4}{\sqrt{\pi}}\cdot\frac{1}{2}\sqrt{\pi}=3\sigma^4;$$

$$Ex(\xi)=\frac{\mu_4(\xi)}{\sigma^4(\xi)}-3=\frac{3\sigma^4}{\sigma^4}-3=0.\blacksquare$$

Ex. 5. A linear function of the normal distribution.

Let a random variable ξ be distributed $N(a,\sigma)$. Let's determine number characteristics (the mathematical expectation, dispersion and root-mean-square deviation) and the distribution law of a linear function of ξ that is of a random variable

$$\eta=k\xi+l.$$

1. As to number characteristics we are making use of their properties and the formulas (9), namely

$$M(\eta)=M(k\xi+l)=M(k\xi)+l=ka+l,$$

$$D(\eta)=D(k\xi+l)=D(k\xi)=k^2D(\xi)=k^2\sigma^2,$$

$$\sigma(\eta)=\sqrt{D(\eta)}=\sqrt{k^2\sigma^2}=|k|\sigma.$$

2. As for the law of the random variable $\eta=k\xi+l$ we'll find its distribution density $g(y)$ using the formula (20) of the Lecture No. 4 and the formula (6),

$$g(y)=\frac{1}{|k|}f\left(\frac{y-l}{k}\right)=\frac{1}{|k|}\cdot\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\left(\frac{y-l}{k}-a\right)^2}{2\sigma^2}}=\frac{1}{\sqrt{2\pi}\cdot|k|\sigma}e^{-\frac{(y-l-ka)^2}{2k^2\sigma^2}}=\frac{1}{\sqrt{2\pi}\cdot|k|\sigma}e^{-\frac{(y-(ka+l))^2}{2(|k|\sigma)^2}}.$$

We see by the formula (6) that the distribution density $g(y)$ of the random variable $\eta=k\xi+l$ is that of the normal distribution and therefore $\eta=k\xi+l$ is distributed $N(ka+l,|k|\sigma)$.

Thus a linear function of the normal distribution is also distributed normally.

The probability of hitting of the normal distribution on an interval

Let a random variable ξ be distributed $N(a,\sigma)$. The probability of its hitting on an interval $<\alpha,\beta>$ can be calculated by the next formula:

$$P(\xi\in<\alpha,\beta>)=\Phi(x_2)-\Phi(x_1), \qquad (10)$$

Where

$$x_1=\frac{\alpha-a}{\sigma},\quad x_2=\frac{\beta-a}{\sigma} \qquad (11)$$

and

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}}dt \qquad (12)$$

is known Laplace function.

■On the base of the formula (14) of the Lecture No. 4 the probability in question equals

$$P(\xi\in<\alpha,\beta>)=\int_\alpha^\beta f(x)dx=\frac{1}{\sqrt{2\pi}\sigma}\int_\alpha^\beta e^{-\frac{(x-a)^2}{2\sigma^2}}dx=\left|\begin{array}{c|c|c|c} \frac{x-a}{\sigma}=t & x & \alpha & \beta \\ x=a+\sigma t & t & \frac{\alpha-a}{\sigma}=x_1 & \frac{\beta-a}{\sigma}=x_2 \\ dx=\sigma dt & & & \end{array}\right|=$$

$$=\frac{\sigma}{\sqrt{2\pi}\sigma}\int_{x_1}^{x_2}e^{-\frac{t^2}{2}}dt=\frac{1}{\sqrt{2\pi}}\left(\int_{x_1}^{0}e^{-\frac{t^2}{2}}dt+\int_0^{x_2}e^{-\frac{t^2}{2}}dt\right)=\frac{1}{\sqrt{2\pi}}\int_0^{x_2}e^{-\frac{t^2}{2}}dt-\frac{1}{\sqrt{2\pi}}\int_0^{x_1}e^{-\frac{t^2}{2}}dt=$$

$$=\Phi(x_2)-\Phi(x_1)\ ■$$

Ex. 6. The probability of the deviation of a random variable ξ, which is distributed $N(a,\sigma)$, from its mathematical expectation a is given by the next formula:

$$P(|\xi - a| < \varepsilon) = 2\Phi\left(\frac{\varepsilon}{\sigma}\right). \qquad (13)$$

■Using the formula (10) and oddness of Laplace function we easy get

$$P(|\xi - a| < \varepsilon) = P(-\varepsilon < \xi - a < \varepsilon) = P(a - \varepsilon < \xi < a + \varepsilon) = \left| \begin{matrix} x_1 = \dfrac{a - \varepsilon - a}{\sigma} = -\dfrac{\varepsilon}{\sigma} \\ x_2 = \dfrac{a + \varepsilon - a}{\sigma} = \dfrac{\varepsilon}{\sigma} \end{matrix} \right| =$$

$$= \Phi\left(\frac{\varepsilon}{\sigma}\right) - \Phi\left(-\frac{\varepsilon}{\sigma}\right) = \Phi\left(\frac{\varepsilon}{\sigma}\right) + \Phi\left(\frac{\varepsilon}{\sigma}\right) = 2\Phi\left(\frac{\varepsilon}{\sigma}\right).$$

For example let $\varepsilon = 3\sigma$. The formula (13) gives

$$P(|\xi - a| < 3\sigma) = P(a - 3\sigma < \xi < a + 3\sigma) = 2\Phi\left(\frac{3\sigma}{\sigma}\right) = 2\Phi(3) = 2 \cdot 0.49865 = 0.9973.$$

We've got so-called **3σ – rule**: with the very large probability 0.9973 all values of the normal distribution are concentrated in the interval $(a - 3\sigma, a + 3\sigma)$.

The limit central theorem

The normal distribution is very widespread [is widely spread] because of the next theorem.

Theorem (Lyapunov[1] limit central theorem). Let a random variable ξ be a sum of a large number of independent random variables,

$$\xi = \xi_1 + \xi_2 + ... + \xi_n,$$

and influence of each of them on ξ isn't essential. In this case ξ has a distribution which approximately coincides with some normal distribution.

Ex. 7. Let a random variable ξ have Bernoulli distribution (be distributed B). It means that ξ is a number of occurrences of a success A in n independent trials with constant probability $p = P(A)$ of the success. We represent ξ as the sum of n inde-

[1] Lyapunov, A.M. (1857 - 1918), a noted Russian mathematician

pendent random variables

$$\xi = \xi_1 + \xi_2 + ... + \xi_i + ... + \xi_n$$

where ξ_i is the number of successes in the i-th trial (compare Ex. 13 in the Lecture No. 5). Let the number n of trials be large. Lyapunov limit central theorem permits us to consider ξ as a random variable with approximately the normal distribution, ξ distributed $\approx N\left(a = np, \sigma = \sqrt{npq}\right)$. As result we can use the formula (10) to find the approximate value of the probability $P(k_1 \le \xi \le k_2) = P_n(k_1, k_2)$. Indeed,

$$P(k_1 \le \xi \le k_2) = P_n(k_1, k_2) \approx \Phi(x_2) - \Phi(x_1),$$

where $x_1 = \dfrac{k_1 - np}{\sqrt{npq}}, \quad x_2 = \dfrac{k_2 - np}{\sqrt{npq}}$.

We've got Laplace integral theorem.

Ex. 8 (examples of normally distributed random variables). The normal distribution have:

a) errors of measurement of physical magnitudes;

b) errors of sizes of (industrial) products;

c) sizes of (industrial) products themselves under certain conditions (provided established process of production and absence of systematic biases changing in time);

d) a range of a missile [ˊmɪsaɪl];

e) human height and weight.

Ex. 9. An industrial product of some factory is considered as defective one if a deviation of its checking size from the set size surpasses 0.03. Root-mean-square deviation of the checking size equals 0.015 (in millimeters). Find a per cent of non-defective products of this factory. Find an average number of non-defective products in a batch of 200 products.

1) Let a random variable ξ be the checking size of a product. We can suppose that ξ distributed normally with parameters $a = M(\xi), \sigma = 0.015$.

At first let's find the probability of the inequality

$$|\xi - M(\xi)| < 0.03.$$

By the formula (13), in which it's necessary to take $\varepsilon = 0.03, \sigma = 0.015$, we'll obtain

$$P(|\xi - M(\xi)| < 0.03) = 2\Phi\left(\frac{0.03}{0.015}\right) = 2\Phi(2) = 0.95.$$

Therefore in average 95 per cent of products of the factory are non-defective.

2) Let's introduce a random variable η that is a number of non-defective products in a batch of 200 products. It is obvious that η has Bernoulli distribution (the number of independent trials $n = 200$, the probability of the success A (a product is non-defective) $p = 0.95$, and $q = 0.05$). Therefore an average number of non-defective products in a batch in question equals

$$M(\eta) = np = 200 \cdot 0.95 = 190.$$

Ex. 10. The probability of hitting of a normally distributed random variable in an interval $<2, 14>$, which is symmetric about a mathematical expectation, equals 0.9973. Find the mathematical expectation, root-mean-square deviation and the distribution density of the random variable.

On the base of 3σ - rule we have

$$\begin{cases} a - 3\sigma = 2, \\ a + 3\sigma = 14, \end{cases}$$

that is a system of linear equations in a, σ. Adding and subtracting its equations termwise we find $2a = 18, 6\sigma = 12$, hence $a = 9, \sigma = 2$. The distribution density in question is determined by the formula (6), and therefore

$$f(x) = \frac{1}{\sqrt{2\pi} \cdot 2} e^{-\frac{(x-9)^2}{2 \cdot 2^2}} = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-9)^2}{8}}.$$

Ex. 11. The average range of a missile is 1200 *m*, and the root-mean-square deviation of the range of a missile equals 50 *m*. Find the probability of the shot over the target from 40 *m* to 60 *m*. Find the probability of non greater then two such results after 10 shots.

1) In supposition of the normal distribution of the range ξ of a missile we have to calculate the probability $P(\xi \in <1240, 1260>)$. With the help of the formula (10)

(with $a = 1200, \sigma = 50$) we get

$$P(\xi \in < 1240, 1260) = \Phi\left(\frac{1260-1200}{50}\right) - \Phi\left(\frac{1240-1200}{50}\right) = \Phi(1.2) - \Phi(0.8) \approx 0.10\,.$$

2) Let a random variable η mean a number of the misses in question. η has Bernoulli distribution (why?), and it is a matter of determination of a probability that $\eta \le 2$. With the help of Bernoulli formula ($n = 10,\ p = 0.1, q = 0.9$) we'll get

$$P(\eta \le 2) = P(\eta = 0) + P(\eta = 1) + P(\eta = 2) = C_{10}^{0} \cdot 0.1^{0} \cdot 0.9^{10} + C_{10}^{1} \cdot 0.1^{1} \cdot 0.9^{9} + C_{10}^{2} \cdot 0.1^{2} \cdot 0.9^{8} =$$
$$= 0.9^{10} + 10 \cdot 0.1 \cdot 0.9^{9} + 45 \cdot 0.1^{2} \cdot 0.9^{8} = 0.9^{8}\left(0.9^{2} + 0.9 + 0.45\right) \approx 0.43 \cdot 2.16 \approx 0.93.$$

POINT 3. THE EXPONENTIAL DISTRIBUTION

We often deal with a call flow [a flow of calls] in a queuing system. Let's denote by λ an intensity of the flow that is a number of calls which take place (on the average) per unit of time. Let ξ be a number of calls during a time t. There are many flows (so-called poissonian flows) for which ξ has Poisson distribution with the parameter $a = \lambda t$. In particular the probability that ξ will take on a value k equals

$$P(\xi = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}\,.$$

Def. 4. Let a random variable T be the time interval between two successive calls of some poissonian call flow. One says that T has the exponential distribution. Our problem is to find the distribution function and density and number characteristics of the exponential distribution.

Remark. There is fuller variant of study. We state it below.

◊We often deal with flows of events.

A flow of events is called a sequence of events which take place at random moments of time.

Example. A flow of calls in every queuing [qju:ɪŋ] system.

A flow of events (in particular a flow of calls) is called that **simplest** (or that **poissonian**) if it possesses the next three properties:

a) **stàtionárity** [ˏsteɪʃˊnerətɪ]: probability of occúrrence of *n* evénts during a time interval (t_1, t_2) depends only on the length $\Delta t = t_2 - t_1$ of this interval;

b) **ábsence of áftereffècts** [ˊæbsəns, ˊɑ:fterɪˏfekts]: this probability doesn't depend on a number of events which have taken place before this time interval;

c) **órdinariness** [ˊɔ:dnərɪnəs]: for small $\Delta t = t_2 - t_1$ the probability of occúrrence of one evént (during a time interval (t_1, t_2)) is proportional to Δt, and the probability of occúrrence of $n \geq 2$ evénts can be negléctеd [nɪˊglektɪd].

We'll denote by λ an **intensity** of a flow of events that is a number of events which take place (on the average) per unit of time.

Let ξ be a number of events of a simplest (poissonian) flow which occur during a time *t*. It can be proved that ξ has Poisson distribution with the parameter $a = \lambda t$. It means that the probability of occúrrence of *k* events during a time *t* equals

$$P(\xi = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

Def. 4. Let a random variable *T* be the **time interval between two successive events of a simplest (poissonian) flow**. One says that *T* has the **exponential** distribu-tion.

Our problem is to find the distribution function and density and number characteristics of the exponential distribution.◊

For positive values of *t* the events $(T < t)$ and $(\xi \geq 1)$ coincide. They mean that during time *t* at least one call will occur. Hence the distribution function of the random variable *T* for $t > 0$ equals

$$F(t) = P(T < t) = P(\xi \geq 1) = 1 - P(\overline{\xi \geq 1}) = 1 - P(\xi = 0) = 1 - \frac{(\lambda t)^0}{0!} e^{-\lambda t} = 1 - e^{-\lambda t}.$$

The expression $1 - e^{-\lambda t}$ tends to zero with *t*, and therefore we can define the distribution function in question as follows

$$F(t) = \begin{cases} 1 - e^{-\lambda t} & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t \leq 0. \end{cases} \qquad (14)$$

It is continuous for all values of *t*, and therefore a random variable *T* which has the exponential distribution is continuous one.

Differentiating the distribution function (14) we'll obtain the distribution density of the exponential distribution,

$$f(t) = F'(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t < 0. \end{cases} \qquad (15)$$

To plot the functions $y = f(t)$, $y = F(t)$ on the interval $(0, \infty)$ we'll take into account that (for $t > 0$)

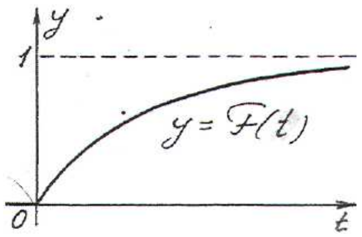


Fig. 5 a

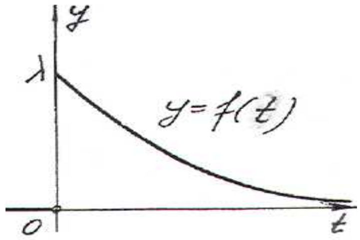


Fig. 5 b

$$F'(t) = f(t) > 0,\ F''(t) = f'(t) = -\lambda^2 e^{-\lambda t} < 0,\ f''(t) = \lambda^3 e^{-\lambda t} > 0.$$

It means that the distribution function $y = F(t)$ increases and has a convex graph, and the distribution density $y = f(t)$ decreases and has a concave graph. The right limit of the function $y = f(t)$ at the point $t = 0$ equals λ (one often supposes $f(0) = \lambda$). The graphs of both the functions $y = f(t)$, $y = F(t)$ are represented on fig. 5 a, 5 b.

Now we'll calculate the mathematical expectation, dispersion and root-mean-square deviation of the exponential distribution. For this purpose we use the formulas (2), (5), (16), (10) of the Lecture No. 5 and integrate by parts in improper integrals.

We successively obtain

$$M(T) = \int_{-\infty}^{\infty} tf(t)dt = \lambda \int_{0}^{\infty} te^{-\lambda t} dt = \left| \begin{array}{ll} u = t, & dv = e^{-\lambda t} dt, \\ du = dt, & v = \int e^{-\lambda t} dt = -\dfrac{1}{\lambda} e^{-\lambda t} \end{array} \right| =$$

$$= \lambda \left(-\frac{1}{\lambda} \left(te^{-\lambda t} \right) \Big|_0^{\infty} + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda t} dt \right) = \left| \begin{array}{l} \lim\limits_{t \to \infty} te^{-\lambda t} = \lim\limits_{t \to \infty} \dfrac{t}{e^{\lambda t}} = \\ = \lim\limits_{t \to \infty} \dfrac{(t)'}{\left(e^{\lambda t}\right)'} = \lim\limits_{t \to \infty} \dfrac{1}{\lambda e^{\lambda t}} = 0 \end{array} \right| = \int_0^{\infty} e^{-\lambda t} dt =$$

$$= -\frac{1}{\lambda}\left(e^{-\lambda t}\right)\Big|_0^\infty = -\frac{1}{\lambda}\left(\lim_{t\to\infty} e^{-\lambda t} - 1\right) = \frac{1}{\lambda}; \quad M(\xi) = \frac{1}{\lambda}.$$

$$M\left(T^2\right) = \lambda\int_0^\infty t^2 e^{-\lambda t}dt = \begin{vmatrix} u = t^2, & dv = e^{-\lambda t}dt, \\ du = 2tdt, & v = \int e^{-\lambda t}dt = -\frac{1}{\lambda}e^{-\lambda t} \end{vmatrix} =$$

$$= \lambda\left(-\frac{1}{\lambda}\left(t^2 e^{-\lambda t}\right)\Big|_0^\infty + \frac{2}{\lambda}\int_0^\infty te^{-\lambda t}dt\right) = \lambda\left(-\frac{1}{\lambda}\left(\lim_{t\to\infty} t^2 e^{-\lambda t} - 0\right) + \frac{2}{\lambda}\int_0^\infty te^{-\lambda t}dt\right) =$$

$$= \left|\lim_{t\to\infty} t^2 e^{-\lambda t} = \lim_{t\to\infty}\frac{t^2}{e^{\lambda t}} = \lim_{t\to\infty}\frac{\left(t^2\right)'}{\left(e^{\lambda t}\right)'} = \lim_{t\to\infty}\frac{2t}{\lambda e^{\lambda t}} = \frac{2}{\lambda}\lim_{t\to\infty}\frac{(t)'}{\left(e^{\lambda t}\right)'} = \frac{2}{\lambda}\lim_{t\to\infty}\frac{1}{\lambda e^{\lambda t}} = 0\right| =$$

$$= \lambda\left(0 + \frac{2}{\lambda}\int_0^\infty te^{-\lambda t}dt\right) = 2\int_0^\infty te^{-\lambda t}dt = \begin{vmatrix} u = t, & dv = e^{-\lambda t}dt, \\ du = dt, & v = \int e^{-\lambda t}dt = -\frac{1}{\lambda}e^{-\lambda t} \end{vmatrix} =$$

$$= 2\left(-\frac{1}{\lambda}\left(te^{-\lambda t}\right)\Big|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda t}dt\right) = 2\left(0 - \frac{1}{\lambda^2}e^{-\lambda t}\Big|_0^\infty\right) = \frac{2}{\lambda^2}.$$

$$D(T) = M\left(T^2\right) - M^2(T) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}; \quad \sigma(T) = \sqrt{D(T)} = \frac{1}{\lambda}.$$

Thus we've got

$$M(T) = \sigma(T) = \frac{1}{\lambda}, \tag{16}$$

that is the mathematical expectation and root-mean-square deviation of the exponential distribution coincide.

Ex. 12. Find the probability that during time t no one call of poissonian flow will occur.

It's necessary to find the next probability: $P(T > t)$. T is a continuous random variable, hence $P(T = t) = 0$ [1] and we get with the help of the formula (14) (for $t > 0$)

$$P(T > t) = P(T \geq t) - P(T = t) = P(T \geq t) - 0 = P(T \geq t) = 1 - P(T < t) = 1 - F(t) = e^{-\lambda t}.$$

Remark. A function determined (for $t > 0$) by the equality

$$R(t) = P(T > t) = e^{-\lambda t}$$

[1] The probability of a separate value of a continuous random variable equals 0.

is found in several applications and is called a reliability function. It quickly decreases with increase of t. For example $R(t)=0.001$ if $\lambda t=7$. Therefore large values of T are improbable.

Ex. 13. A time T of safe in service of an element is distributed exponentially,

$$f(t)=0.005e^{-0.005t} \text{ for } t>0, \quad f(t)=0 \text{ for } t<0.$$

Find the probability that the element will work during at least 50 hours.

On the base of the Ex. 12

$$P(T>50)=e^{-0.005\cdot 50}=e^{-0.25}=\frac{1}{\sqrt[4]{e}}\approx 0.78.$$

Ex. 14. Calculate the probability of hitting of exponentially distributed random variable T in a time interval $<t_1, t_2>$.

For example let $t_1>0, t_2>0$. Then on the base of the formula (14) for positive values of t

$$(T\in<t_1,t_2>)=F(t_2)-F(t_1)=\left(1-e^{-\lambda t_2}\right)-\left(1-e^{-\lambda t_1}\right)=e^{-\lambda t_1}-e^{-\lambda t_2}.$$

POINT 4. THE GAMMA-DISTRIBUTION

As we know from the Point 3 of the Lecture No. 23 the gamma-function is called the next improper integral

$$\Gamma(\alpha)=\int_0^{+\infty} x^{\alpha-1}e^{-x}dx. \qquad (17)$$

The integral (17) converges for any $\alpha>0$. The gamma-function is continuous and has continuous derivatives of all orders for $\alpha>0$.

Let's remember the properties of the gamma-function.

1) $\Gamma(1)=1$.

2) For any positive α

$$\Gamma(\alpha+1)=\alpha\cdot\Gamma(\alpha). \qquad (18)$$

3) For each natural number n

$$\Gamma(n+1)=n!$$

Def. 5. One says that a random variable ξ has the gamma-distribution if its distribution density is

$$f(x)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x} & \text{for} \quad x>0,\\ 0 & \text{for} \quad x<0\end{cases} \tag{19}$$

with two positive parameters α, λ. The first parameter, α, is called that of a form, and the second, λ, the parameter of a scale.

For $\alpha=1$ the gamma-distribution becomes that exponential. Indeed, $\Gamma(1)=1$, and therefore the distribution density

$$f(x)=\begin{cases}\lambda e^{-\lambda x} & \text{for} \quad x>0,\\ 0 & \text{for} \quad x<0\end{cases}$$

is that of the exponential distribution (compare with the formula (15)).

The mathematical expectation and dispersion of the gamma-distribution are equal to

$$M(\xi)=\frac{\alpha}{\lambda}, \quad D(\xi)=\frac{\alpha}{\lambda^2}. \tag{20}$$

■A. At first let $\lambda=1$ that is we are speaking about the distribution density

$$f(x)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x} & \text{for} \quad x>0,\\ 0 & \text{for} \quad x<0.\end{cases}$$

In this case by virtue of the formulas (17), (18)

$$M(\xi)=\int_0^{\infty}xf(x)dx=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}x\cdot x^{\alpha-1}e^{-x}dx=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}x^{(\alpha+1)-1}e^{-x}dx=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}=\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)}=\alpha,$$

$$M(\xi^2)=\int_0^{\infty}x^2f(x)dx=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}x^2\cdot x^{\alpha-1}e^{-x}dx=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}x^{(\alpha+2)-1}e^{-x}dx=\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}=\frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)}=$$

$$=\frac{(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)}=(\alpha+1)\alpha,$$

$$D(\xi)=M(\xi^2)-M^2(\xi)=\alpha^2+\alpha-(\alpha)^2=\alpha$$

B) In the general case $\lambda\neq1$ the reasonings are similar. Namely,

$$M(\xi)=\int_0^\infty xf(x)dx=\frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^\alpha e^{-\lambda x}dx=\left|\begin{matrix}\lambda x=t,\ x=\dfrac{t}{\lambda} \\ dx=\dfrac{dt}{\lambda}\end{matrix}\ \begin{array}{c|c|c} x & 0 & \infty \\ \hline t & 0 & \infty\end{array}\right|=\frac{\lambda^\alpha}{\Gamma(\alpha)\lambda^\alpha\lambda}\int_0^\infty t^\alpha e^{-t}dt=$$

$$=\frac{1}{\Gamma(\alpha)\lambda}\int_0^\infty t^{(\alpha+1)-1}e^{-t}dt=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\lambda}=\frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)\lambda}=\frac{\alpha}{\lambda};\quad M(\xi)=\frac{\alpha}{\lambda}.$$

$$M(\xi^2)=\int_0^\infty x^2f(x)dx=\frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^{\alpha+1}e^{-\lambda x}dx=|\lambda x=t|=\frac{\lambda^\alpha}{\Gamma(\alpha)\lambda^{\alpha+1}\lambda}\int_0^\infty t^{\alpha+1}e^{-t}dt=$$

$$=\frac{1}{\Gamma(\alpha)\lambda^2}\int_0^\infty t^{(\alpha+2)-1}e^{-t}dt=\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)\lambda^2}=\frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)\lambda^2}=\frac{(\alpha+1)\alpha\Gamma(\alpha)}{\Gamma(\alpha)\lambda^2}=\frac{(\alpha+1)\alpha}{\lambda^2},$$

$$D(\xi)=M(\xi^2)-M^2(\xi)=\frac{(\alpha+1)\alpha}{\lambda^2}-\left(\frac{\alpha}{\lambda}\right)^2=\frac{\alpha}{\lambda^2};\quad D(\xi)=\frac{\alpha}{\lambda^2}.\blacksquare$$

The gamma-distribution has a lot of applications in the mathematic statistics.

With the help of the gamma-distribution one can study distributions of interests and, in certain situations, of population saving.

POINT 5. SOME OTHER DISTRIBUTIONS

χ^2**- distribution**

Let

$$\xi_1,\ \xi_2,\ \xi_3,\ldots,\ \xi_n$$

be normally distributed random variables with the same zero mathematical expectations and unit root-mean-square deviations,

$$\forall i=\overline{1,n}: M(\xi_i)=0,\ \sigma(\xi_i)=1.$$

A random variable

$$\chi^2=\xi_1^2+\xi_2^2+\xi_3^2+\ldots+\xi_n^2 \tag{21}$$

is called χ^2- distribution with k degrees of freedom, where

$$k=\begin{cases} n & \text{if}\quad \text{the random variables } \xi_1,\xi_2,\xi_3,\ldots,\xi_n \text{ are independent,} \\ n-1 & \text{if}\quad \xi_1,\xi_2,\xi_3,\ldots,\xi_n \text{ are connected by a linear relation.}\end{cases} \tag{22}$$

A linear relation between the random variables can have the next form:

$$\xi_1 + \xi_2 + \xi_3 + \ldots + \xi_n = n\overline{\xi}$$

where $\overline{\xi}$ is the arithmetical mean of these random variables.

The distribution density of χ^2 - distribution equals

$$f(x) = \begin{cases} 0 & \text{if} \quad x \le 0, \\ \dfrac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} e^{-\frac{x}{2}} x^{\frac{k}{2}-1} & \text{if} \quad x > 0. \end{cases} \qquad (23)$$

The formula (23) means that χ^2 - distribution depends only on unique parameter, namely on the number *k* of degrees of freedom.

There is a table which allows finding so-called critical value χ^2_{crit} of χ^2 to have

$$P\left(\chi^2 > \chi^2_{crit}\right) = \alpha, \qquad (24)$$

where α is some small probability, if one knows the number *k* of degrees of freedom.

If $k \to \infty$, χ^2 - distribution slowly approaches the normal distribution.

***Student*[1] *distribution* [*t - distribution*]**

Let a random variable *Z* be distributed $N(0, 1)$, a random variable *V* be χ^2 - distributed with *k* degrees of freedom, and *Z*, *V* are independent. The random variable

$$T = \frac{Z}{\sqrt{\frac{1}{k}V}} \qquad (25)$$

is said to have Student distribution (or the *t*-distribution) with *k* degrees of freedom, where *k* is determined by the formula (22). One can call *T* simply Student distribution (the *t*-distribution).

The distribution density of Student distribution for the case $k = n - 1$ (when random variables ξ_1, ξ_2, ξ_3,..., ξ_n are connected by a linear dependence) equals

$$f(x) = S(x, n) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi(n-1)}}\left(1 + \frac{x^2}{n-1}\right)^{-\frac{n}{2}}. \qquad (26)$$

[1] Gosset W.S. (pseudonym: Student W. S. (1907)), an English statistician

It is an even function that is $f(-x)=f(x)$.

Let γ be some great probability, $k=n-1$, and t_γ is a number such that

$$P(|T|<t_\gamma)=\gamma . \qquad (27)$$

By the formula (26) we can write

$$P(|T|<t_\gamma)=P(-t_\gamma<T<t_\gamma)=\int_{-t_\gamma}^{t_\gamma} S(x,n)dx=2\int_0^{t_\gamma} S(x,n)dx=\gamma .$$

There is a table which permits to find the number t_γ if one sets a great probability γ ($\gamma=0.9$, $\gamma=0.95$, $\gamma=0.99$) and knows the value of n.

If $k\to\infty$, Student distribution quickly approaches that normal. For $k>30$ we can consider it as the normal distribution.

As can be proved,

$$M(T)=0,\ D(T)=k/(k-2) \quad (k>2) \qquad (28)$$

REMARKABLE DISTRIBUTIONS: basic terminology RUEFD

1. гамма-распределение	гама-розподіл	gamma distribution	distribution *f* [répartition *f*] gamma	Gammaverteilung *f*
2. гамма-распределенная случайная величина	гама-розподілена випадкова величина	gamma random variable	variable *f* aléatoire distribuée gamma	gammaverteilte Zufallsgröße *f*
3. интенсивность	інтенсивність	inténsity	intensité *f*	Intensität *f*
4. колоколообразная кривая	дзвоноподібна [дзвонувата] крива	bell-like [bell-shaped] curve	courbe *f* en cloche	Glockenkurve *f*
5. кривая Гаусса [Лапласа - Гаусса]	крива Ґаусса [Лапласа-Ґаусса]	Gaussian [Laplace-Gauss] curve	courbe *f* gaussienne [de Gausse, de Laplace-Gausse]	Gaußsche [Laplace-Gauß] Kurve *f*
6. кривая плотности нормального распределения	крива щільності нормального розподілу	nórmal distribution curve	courbe de densité de la distribution normale	Normalverteilungskurve *f*
7. нормальная кривая	нормальна крива	nórmal cúrve	courbe *f* normale	Normalkurve *f*
8. нормально распределенная случайная величина	нормально розподілена випадкова величина	nórmally distríbuted rándom váriable	variable *f* aléatoire distribuée normalement	normalverteilte Zufallsgröße *f*
9. нормальное распределение	нормальний розподіл	nórmal dìstribútion, gaussian distribution	distribution *f* [répartition *f*] normale [gaussienne]	Normalverteilung *f*, normale Verteilung *f*
10.нормальный закон распределения	нормальний закон розподілу	nórmal [gaussian] distribútion law, gaussian law, nórmal law	loi *f* normale de distribution [de répartition]	Normalverteilungsgesetz, normales Verteilungsgesetz
11.показательное [экспоненциальное] распределение	показниковий [експоненційний] розподіл	èxponéntial dìstribútion, exponential	distribution *f* [répartition *f*] exponentielle	Exponentiálverteilung *f*
12.показате-	показниковий	èxponéntial	loi *f* exponen-	exponentielles

льный [экспоненциальный] закон распределения	[експоненційний] закон розподілу	law	tielle de distribution [de répartition]	Verteilungsgesetz *n*
13.поток вызовов	потік викликів	flow of calls	flot *m* d'unités, flot *m* d'appels	Strom *m* von Áufrufen *m*, Áufrufstrom *m*
14.поток требований	потік вимог	flow of demands	flot *m* de demandes, arrivée *f*	Forderungsstrom *m*
15.поток событий	потік подій	flow of evénts	flot *m* d'événements	Strom *m* von Ereignissen
16.правило трех сигм	правило трьох сигм	three sigma rule	règle *f* de trois sigmas	Régel *f* von drei Sigma
17.пуассоновский поток	пуассонівський потік	poissonian flow	flot *m* poissonien	Poissonscher Strom *m*
18.равномерно распределенная случайная величина	рівномірно розподілена випадкова величина	uniform random variable, rándom váriable distríbuted únifòrmly	variable *f* aléatoire distribuée uniformément	gleichverteilte Zufallsgröße *f*
19.равномерное распределение	рівномірний розподіл	úniform dìstribútion	distribution *f* uniforme	Gleichverteilung *f*, gleichmäßige Verteilung
20.распределение «хи-квадрат»	розподіл «хі-квадрат»	chi-squáred [chi-squáre] dìstribútion	distribution *f* [répartition *f*] de khi carré	Chiquadratverteilung *f*
21.распределение Гаусса, гауссово распределение	розподіл Ґаусса, ґауссів розподіл	gaussian dìstribútion	distribution *f* [répartition *f*] gaussienne [de Gausse]	Gaußverteilung *f*, Gaußsche Verteilung
22.распределение Стьюдента [*t*-распределение]	розподіл Стьюдента [*t* -розподіл]	Student distribution, *t*-distribution	distribution *f* [répartition *f* , loi *f*] de Student, de *t*	Student-Verteilung *f* [Studentsche Verteilung, *t*-Verteilung]
23.распределение Фишера-Снедекора, F-распределение	розподіл Фішера-Снедекора, F-розподіл	Fisher-Snedecor distribution, F-distribution	distribution *f* [répartition *f* , loi *f*] de Fisher-Snedecor [de f]	Fisher-Snedecorsche Verteilung, F-Verteilung
24.распределение хи-ква-	розподіл хі-квадрат	chi-squáred [chi-squáre]	distribution *f* [répartition *f*,	Chi-quadrat-

драт		distribution	loi *f*] khi carré	verteilung, χ^2-Verteilung
25.система массового обслуживания	система масового обслуговування	qúeuing sýstem	système *m* d'attente	Massenbedienungssystem *n*, System *n* der Massenbedienung
26.случайная величина, распределенная по Стьюденту	розподілена за Стьюдентом випадкова величина	Student random variable	variable *f* aléatoire distribuée de Student	Studentisierte Zufallsgröße *f*
27.хи-квадрат распределение	хі-квадрат розподіл	chi-squáred [chi-squáre] distribution, χ^2- distribution	distribution *f* [répartition *f*, loi *f*] khi carré	Chi-quadrat-verteilung, χ^2-Verteilung
28.хи-квадрат распределенная случайная величина	хі-квадрат розподілена випадкова величина	chi-squáred [chi-squáre] rándom váriable	variable *f* aléatoire distribuée suivant [en,par, conformément] la loi khi carré	chi-quadrat-verteilte Zufallsgröße *f*
29.центральная предельная теорема ляпунова	центральна гранична теорема ляпунова	lyapunov céntral límit théorem	théorème *m* limite central de liapounof	zentraler Grenzverteilungssatz *m* von Ljapunow
30.число степеней свободы	число степенів вільності	númber of degrées of freedom	nombre *m* de degrés de liberté	Anzahl *f* der Freiheitsgrade
31.экспоненциально распределенная случайная величина	експоненційно розподілена випадкова величина	èxponéntial random variable	variable *f* aléatoire distribuée exponentiellement	exponential-verteilte Zufallsgröße *f*
32.экспоненциальное (показательное распределение)	експоненційий (показниковий) розподіл	èxponéntial dìstribútion	distribution *f* [répartition *f* , loi *f*] exponentielle	Exponential-verteilung *f*

REMARKABLE DISTRIBUTIONS: basic terminology EFDRU

1. bell-like [bell-shaped] curve	courbe *f* en cloche	Glockenkurve *f*	колоколообразная кривая	дзвоноподібна [дзвонувата] крива
2. chi-squáred [chi-squáre] dìstribútion	distribution *f* [répartition *f*] de khi carré	Chiquadratverteilung *f*	распределение «хи-квадрат»	розподіл «хі-квадрат»
3. chi-squáred [chi-squáre] distribution, χ^2- distribution	distribution *f* [répartition *f*, loi *f*] khi carré	Chi-quadrat-verteilung *f* , χ^2-Verteilung	хи-квадрат распределение	хі-квадрат розподіл
4. chi-squáred [chi-squáre] rándom váriable	variable *f* aléatoire distribuée suivant [en,par, conformément] la loi khi carré	Chi-quadrat-verteilte Zufallsgröße *f*	хи-квадрат распределенная случайная величина	хі-квадрат розподілена випадкова величина
5. èxponéntial dìstribútion	distribution *f* [répartition *f* , loi *f*] exponentielle	Exponential-verteilung *f*	экспоненциальное (показательное) распределение	експоненційний (показниковий) розподіл
6. èxponéntial dìstribútion, exponential	distribution *f* [répartition *f*] exponentielle	Exponentiál-verteilung *f*	показательное [экспоненциальное] распределение	показниковий [експоненційний] розподіл
7. èxponéntial law	loi *f* exponentielle de distribution [de répartition]	exponentielles Verteilungs-gesetz	показательный [экспоненциальный] закон распределения	показниковий [експоненційний] закон розподілу
8. èxponéntial random variable	variable *f* aléatoire distribuée exponentiellement	exponential-verteilte Zufallsgröße	экспоненциально распределенная случайная величина	експоненційно розподілена випадкова величина
9. Fisher-Snedecor distribution, F-distribution	distribution *f* [répartition *f* , loi *f*] de Fisher-Snedecor [de F]	Fisher-Snedecorsche Verteilung, F-Verteilung	распределение Фишера-Снедекора, F-распределение	розподіл Фішера-Снедекора, F-розподіл
10.flow of	flot *m* d'unités,	Strom *m* von	поток вызо-	потік викли-

calls	flot *m* d'appels	Áufrufen, Áufrufstrom *m*	вов	ків
11.flow of demands	flot *m* de demandes, arrivée *f*	Forderungsstrom *m*	поток требований	потік вимог
12.flow of evénts	flot *m* d'événements	Strom *m* von Ereignissen	поток событий	потік подій
13.gamma distribution	distribution *f* [répartition *f*] gamma	Gammaverteilung *f*	гамма-распределение	гама-розподіл
14.gamma random variable	variable *f* aléatoire distribuée gamma	Gammaverteilte Zufallsgröße *f*	гамма-распределенная случайная величина	гама-розподілена випадкова величина
15.Gaussian [Laplace-Gauss] curve	courbe *f* gaussienne [de Gausse, de Laplace-Gausse]	Gaußsche [Laplace-Gauß] Kurve	кривая Гаусса [Лапласа - Гаусса]	крива Ґаусса [Лапласа-Ґаусса]
16.gaussian dìstribútion	distribution *f* [répartition *f*] gaussienne [de Gausse]	Gaußverteilung *f*, Gaußsche Verteilung	распределение Гаусса, гауссово распределение	розподіл ґаусса, ґауссів розподіл
17.inténsity	intensité *f*	Intensität *f*	интенсивность	інтенсивність
18.Lyapunov céntral límit théorem	théorème *m* limite central de Liapounof	zentraler Grenzverteilungssatz *m* von Ljapunow	центральная предельная теорема Ляпунова	центральна гранична теорема Ляпунова
19.nórmal [gaussian] dìstribútion law, gaussian law, nórmal law	loi *f* normale de distribution [de répartition]	Normalverteilungsgesetz *n*, normales Verteilungsgesetz	нормальный закон распределения	нормальний закон розподілу
20.nórmal cúrve	courbe *f* normale	Normalkurve *f*	нормальная кривая	нормальна крива
21.nórmal distribution curve	courbe de densité *f* de la distribution normale	Normalverteilungskurve *f*	кривая плотности нормального распределения	крива щільності нормального розподілу
22.nórmal dìstribútion,gaussian distribution	distribution *f* [répartition *f*] normale [gaussienne]	Normalverteilung *f*, normale Verteilung	нормальное распределение	нормальний розподіл

23.nórmally distríbuted rándom váriable	variable *f* aléatoire distribuée normalement	normalverteilte Zufallsgröße	нормально распределенная случайная величина	нормально розподілена випадкова величина
24.númber of degrées of freedom	nombre *m* de degrés de liberté	Anzahl *f* der Freiheitsgrade	число степеней свободы	число степенів вільності
25.poissonian flow	flot *m* poissonien	Poissonscher Strom	пуассоновский поток	пуассонівський потік
26.qúeuing sýstem	système *m* d'attente	Massenbedienungssystem *n*, System *n* der Massenbedienung	система массового обслуживания	система масового обслуговування
27.Student distribution, *t*-distribution	distribution *f* [répartition *f* , loi *f*] de Student, de *t*	Student-Verteilung [Studentsche Verteilung, *t*-Verteilung]	распределение Стьюдента [*t*-распределение]	розподіл Стьюдента [*t* -розподіл]
28.Student random variable	variable *f* aléatoire distribuée de Student	Studentisierte Zufallsgröße	случайная величина, распределенная по Стьюденту	розподілена за Стьюдентом випадкова величина
29.three sigma rule	règle *f* de trois sigmas	Régel *f* von drei Sigma	правило трех сигм	правило трьох сигм
30.úniform dìstribútion	distribution *f* uniforme	Gleichverteilung *f*, gleichmäßige Verteilung	равномерное распределение	рівномірний розподіл
31.uniform radom variable, rándom váriable distríbuted únifòrmly	variable *f* aléatoire distribuée uniformément	gleichverteilte Zufallsgröße	равномерно распределенная случайная величина	рівномірно розподілена випадкова величина

Отформатировано: Расстояние от края до верхнего колонтитула: 1,7 см, Различать колонтитулы: первой страницы

LECTURE NO. 7.

TWO DIMENSIONAL RANDOM VARAIABLES

LE SEPTIEME COURS. VARIABLES ALEATOIRES BIDIMENSIONNELLES

SIEBTE VORLESUNG. ZWEIDIMENSIONALE ZUFALLSGRÖSSEN

POINT 1. GENERAL NOTIONS. Notions générales. Allgemeine Begriffe.

POINT 2. THE DISTRIBUTION FUNCTION OF A TWO-DIMENSIONAL RANDOM VARIABLE. La fonction de distribution d'une variable aléatoire bidimensionnelle. Die Verteilungsfunktion von zweidimensionalen Zufallsgrösse.

POINT 3. THE DISTRIBUTION DENSITY OF A TWO-DIMENSIONAL CONTINUOUS RANDOM VARIABLE. La densité d'une distribution d'une variable aléatoire bidimensionnelle. Die Verteilungsdichte von zweidimensionalen Zufallsgrösse.

POINT 4. FUNCTIONS OF TWO RANDOM VARIABLES. Fonctions de variables aléatoires bidimensionnelles. Funktionen zweidimensionaler Zufallsgrössen.

POINT 5. CORRELATION MOMENT AND COEFFICIENT. Moment corrélatif et coefficient de corrélation. Korrelationsmoment und Korrelationskoeffizient.

POINT 1. GENERAL NOTIONS

Def. 1. A system of *n* random variables

$$\xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n)$$

is called the *n*-dimensional random variable or *n*-dimensional random vector.

For example, the system of *n* sizes of some item generates the *n*-dimensional random variable.

A random variable $\xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n)$ is called discrete [continuous] one if all its components ξ_i are discrete [*resp.* continuous] random variables.

We'll consider for simplicity two-dimensional random variables $\zeta = (\xi, \eta)$.

Distribution law of a discrete two-dimensional random variable

Let $\zeta = (\xi, \eta)$ be a discrete two-dimensional random variable. It means that its components ξ, η are those discrete. Let the random variable ξ have *n* possible values $x_1, x_2, x_3,..., x_n$, and η has *m* possible values $y_1, y_2, y_3,..., y_m$. In this case we can represent the distribution law of $\zeta = (\xi, \eta)$ by the next **distribution table** (table 1).

Table 1

η \ ξ	x_1	x_2	$\cdots$	x_i	$\cdots$	x_n	p_y
y_1	p_{11}	p_{21}	$\cdots$	p_{i1}	$\cdots$	p_{n1}	p_{y_1}
y_2	p_{12}	p_{22}	$\cdots$	p_{i2}	$\cdots$	p_{n2}	p_{y_2}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
y_j	p_{1j}	p_{2j}	$\cdots$	p_{ij}	$\cdots$	p_{nj}	p_{y_j}
$\cdots$	$\cdots.$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
y_m	p_{1m}	p_{2m}	$\cdots$	p_{im}	$\cdots$	p_{nm}	p_{y_m}
p_x	p_{x_1}	p_{x_2}	$\cdots$	p_{x_i}	$\cdots$	p_{x_n}	1

Here

$$p_{ij} = P(\xi = x_i, \eta = y_j) = P((\xi = x_i)(\eta = y_j))$$

is the probability of occurrence of the pair $\xi = x_i, \eta = y_j$. The sum of all probabilities of the *i*-th column of the table gives the probability of occurrence of the value $\xi = x_i$ of the random variable ξ,

$$p_{i1} + p_{i2} + ... + p_{ij} + ... + p_{im} = \sum_{j=1}^{m} p_{ij} = P(\xi = x_i) = p_{x_i}, \qquad (1)$$

and the sum of all probabilities of the *j*-th row equals the probability of occurrence of the value $\eta = y_j$ of the random variable η,

$$p_{1j} + p_{2j} + \ldots + p_{ij} + \ldots + p_{nj} = \sum_{i=1}^{n} p_{ij} = P(\eta = y_j) = p_{yj}. \qquad (2)$$

■Using the formula of total probability we'll have

$$\sum_{j=1}^{m} p_{ij} = \sum_{j=1}^{m} P((\xi = x_i)(\eta = y_j)) = \sum_{j=1}^{m} P(\eta = y_j) P((\xi = x_i)/(\eta = y_j)) = P(\xi = x_i) = p_{x_i};$$

$$\sum_{i=1}^{n} p_{ij} = \sum_{i=1}^{n} P((\xi = x_i)(\eta = y_j)) = \sum_{i=1}^{n} P(\xi = x_i) P((\eta = y_j)/(\xi = x_i)) = P(\eta = y_j) = p_{y_j}.■$$

It is obvious that

$$p_{x_1} + p_{x_2} + p_{x_3} + \ldots + p_{x_n} = \sum_{i=1}^{n} p_{x_i} = 1, \quad p_{y_1} + p_{y_2} + p_{y_3} + \ldots + p_{y_m} = \sum_{j=1}^{m} p_{y_j} = 1$$

(see right lower square in the table 1).

Let's remark that the distribution table 1 of the random variable $\zeta = (\xi, \eta)$ gives us the distribution tables of both the random variables ξ and η (table 2). It means

Table 2

ξ	x_1	x_2	$\cdots$	x_i	$\cdots$	x_n
p_x	p_{x_1}	p_{x_2}	$\cdots$	p_{x_i}	$\cdots$	p_{x_n}
η	y_1	y_1	$\cdots$	y_1		y_1
p_y	p_{y_1}	p_{y_2}	$\cdots$	p_{y_j}	$\cdots$	p_{y_m}

that knowing the distribution law of a discrete two-dimensional random variable we always can find the distribution laws of its components.

The inverse statement isn't true in general.

POINT 2. THE DISTRIBUTION FUNCTION OF A TWO-DIMENSIONAL RANDOM VARIABLE

Def. 2. The distribution function of a two-dimensional random variable (ξ, η) is called the next function:

$$F(x, y) = P(\xi < x, \eta < y) = P((\xi < x)(\eta < y)) = P((\xi, \eta) \in R_{xy}). \qquad (3)$$

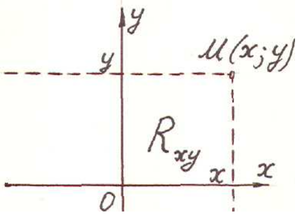


Fig. 1

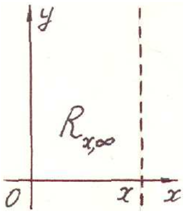


Fig. 2

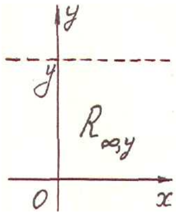


Fig. 3

Here R_{xy} is the infinite open rectangle generated by two rays starting from the point $(x;\, y)$ to the left and downwards in parallel to the coordinate axes (fig. 1). It is the infinite open quadrant with the right upper vertex at the point $(x;\, y)$. The distribution function in question is the probability of hitting of the random variable (ξ, η) in the rectangle (quadrant) R_{xy}.

Properties of the distribution function

1. If $y \to +\infty$, the rectangle R_{xy} transforms into the left half-plane $R_{x,\infty}$ (with respect to the straight line passing through the point $(x;\, y)$ in parallel to the *Oy*-axis, fig. 2). So we can take by definition

$$F(x, +\infty) = P(\xi < x) = F_\xi(x),$$

where $F_\xi(x)$ is the distribution function of the random variable ξ.

If now $x \to +\infty$ we obtain by analogous way (see $R_{\infty,y}$, fig. 3)

$$F(+\infty, y) = P(\eta < y) = F_\eta(y)$$

where $F_\eta(y)$ is the distribution function of the random variable η.

2. Directing x and y to $+\infty$ we rush the rectangle R_{xy} to the *xOy*-plane and the event $(\xi, \eta) \in R_{xy}$ to certain one. As result we arrive at the property

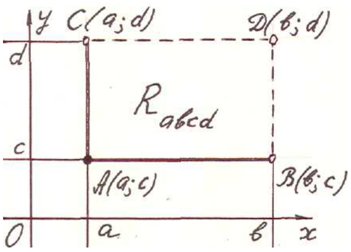


Fig. 4

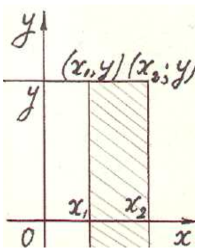


Fig. 5

$$F(+\infty, +\infty) = F_\xi(+\infty) = F_\eta(+\infty) = 1.$$

3. If $x \to -\infty$ or $y \to -\infty$, or $x \to -\infty$ and $y \to -\infty$, the rectangle R_{xy} vanishes, the event $(\xi, \eta) \in R_{xy}$ rushes to impossible one, and we can accept

$$F(x, -\infty) = 0, \quad F(-\infty, y) = 0, \quad F(-\infty, -\infty) = 0.$$

4. Probability of hitting of a two-dimensional random variable (ξ, η) in a finite half-open rectangle

$$R_{abcd} = \{(x, y): a \le x < b, c \le y < d\}$$

with vertices A, B, C, D (fig. 4) is determined by the formula

$$P(a \le x < b, c \le y < d) = P((\xi, \eta) \in R_{abcd}) = F(b, d) - F(b, c) - F(a, d) + F(a, c). \quad (4)$$

■ In reality we must prove the next equality (see the formula (3) and fig. 4):

$$P((\xi,\eta) \in R_{abcd}) = P((\xi,\eta) \in R_{bd}) - P((\xi,\eta) \in R_{bc}) - P((\xi,\eta) \in R_{ad}) + P((\xi,\eta) \in R_{ac}).$$

But it is evident (see fig. 4) that

$$R_{bd} = R_{abcd} \cup R_{ad} \cup R_{bc},\ R_{ad} \cap R_{bc} = R_{ac},\ R_{abcd} \cap (R_{ad} \cup R_{bc}) = \varnothing,$$

and on the base of the axiom 4 and the corollary 1 of the lecture No. 2 (Point 3)

$$P((\xi,\eta) \in R_{bd}) = P((\xi,\eta) \in R_{abcd}) + P((\xi,\eta) \in (R_{ad} \cup R_{bc})) =$$

$$= P((\xi,\eta) \in R_{abcd}) + P((\xi,\eta) \in R_{ad}) + P((\xi,\eta) \in R_{bc}) - P((\xi,\eta) \in R_{ac}),$$

$$P((\xi,\eta) \in R_{abcd}) = P((\xi,\eta) \in R_{bd}) - P((\xi,\eta) \in R_{bc}) - P((\xi,\eta) \in R_{ad}) + P((\xi,\eta) \in R_{ac}).■$$

5. The distribution function never decreases with respect to each its argument: if $x_1 < x_2$, $y_1 < y_2$, then

$$F(x_1, y) \le F(x_2, y), \quad F(x, y_1) \le F(x, y_2).$$

■ For example let $x_1 < x_2$. In this case by virtue of the formula (3) the difference

$$F(x_2, y) - F(x_1, y) = P((\xi,\eta) \in R_{x_2 y}) - P((\xi,\eta) \in R_{x_1 y})$$

is the probability of hitting of the random variable (ξ,η) in the hatched strip which is shown on the fig. 5. Being a probability this difference must be non-negative, hence

$$F(x_2, y) - F(x_1, y) \ge 0,\ F(x_2, y) \ge F(x_1, y) \text{ and } F(x_1, y) \le F(x_2, y).■$$

6. The components ξ, η of a two-dimensional random variable (ξ, η) are independent if and only if the distribution function $F(x, y)$ of (ξ, η) can be represented in the form of a product of the distribution functions of these components,

$$F(x, y) = F_\xi(x) \cdot F_\eta(y).$$

■1. At first let the random variables ξ, η be independent. In this case

$$F(x, y) = P(\xi < x, \eta < y) = P((\xi < x)(\eta < y)) = P(\xi < x)\cdot P(\eta < y) = F_\xi(x)\cdot F_\eta(y).$$

2. Now let

$$F(x, y) = F_\xi(x)\cdot F_\eta(y).$$

It means that

$$P((\xi < x)(\eta < y)) = P(\xi < x)\cdot P(\eta < y),$$

and therefore the random variables ξ, η are independent.

Remark. Knowing the distribution function of a two-dimensional random variable (ξ, η) we always can find the distribution functions of its components ξ and η (see the property 1). The inverse fact isn't true in general. But the property 6 indicates that knowing the distribution functions of two independent random variables we can find the distribution function of their system (ξ, η).

POINT 3. THE DISTRIBUTION DENSITY OF A TWO-DIMENSIONAL CONTINUOUS RANDOM VARIABLE

Let (ξ, η) be a continouos two-dimensional random variable, and R_Δ (fig. 6) is an infinitely small rectangle with vertices

$$M(x; y), N(x+\Delta x; y), Q(x; y+\Delta y), P(x+\Delta x; y+\Delta y).$$

The probability of hitting of (ξ, η) in this rectangle by the property 4 of the point 2 equals

$$P((\xi,\eta)\in R_\Delta) = F(P) - F(N) - F(Q) + F(M) =$$
$$= F(x+\Delta x; y+\Delta y) - F(x+\Delta x; y) - F(x; y+\Delta y) + F(x; y).$$

If we divide it by the area $\Delta x\Delta y$ of the rectangle R_Δ, we'll obtain the average density [or the mean density] of the probability on R_Δ that is

$$f_{av} = \frac{P((\xi,\eta)\in R_\Delta)}{\Delta x\Delta y}.$$

Now let $\Delta x \to 0, \Delta y \to 0$. We'll get the density $f(x, y)$ of the probability at the point $M(x; y)$

$$f(x, y) = \lim_{\Delta x \to 0, \Delta y \to 0} \frac{P((\xi, \eta) \in R_\Delta)}{\Delta x \Delta y}. \quad (5)$$

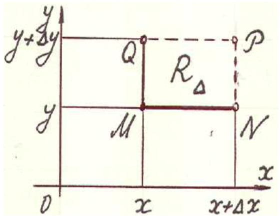


Fig. 6

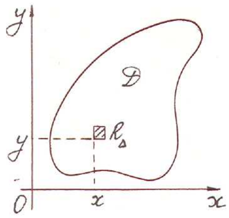


Fig. 7

As can be proved the density of the probability is equal to

$$f(x, y) = F''_{xy}(x, y) \quad (6)$$

that is to the second mixed partial derivative of the distribution function of the random variable (ξ, η).

Def. 3. The distribution density (or the probability density) of a two-dimensional continuous random variable (ξ, η) is called the function $f(x, y)$ defined by the formula (6), that is the second mixed partial derivative of the distribution function $F(x, y)$ of this random variable.

It follows from the formula (5) that the probability of hitting of a two-dimensional random variable (ξ, η) in the infinitely small rectangle R_Δ equals (with an accuracy to infinitely small of higher order)

$$P((\xi, \eta) \in R_\Delta) = f(x, y)dxdy \quad (dx = \Delta x, dy = \Delta y). \quad (7)$$

The formula (7) defines so-called **probability element**.

Properties of the distribution density

1. The distribution density is non-negative function,

$$f(x, y) \geq 0.$$

■Indeed,

$$f(x, y) = \lim_{\Delta x \to 0, \Delta y \to 0} \frac{P((\xi, \eta) \in R_\Delta)}{\Delta x \Delta y} \geq 0$$

as the limit of the ratio of a non-negative probability to positive area $\Delta x \Delta y$.■

2. The probability of hitting of a random variable (ξ, η) in some domain D of the xOy-plane (fig. 7) equals the double integral of its distribution density over the

domain *D*,

$$P((\xi,\eta)\in D)=\iint_D f(x,y)dxdy\,. \qquad (8)$$

■One can obtain the formula (8) by adding the probability elements (7) over the domain *D*.■

3. As corollary we can find the distribution function of a random variable (ξ, η) that is the double integral of its distribution density over the rectangle R_{xy},

$$F(x,y)=P((\xi,\eta)\in R_{xy})=\iint_{R_{xy}} f(x,y)dxdy\,. \qquad (9)$$

The formula (9) of the Lecture No. 24 permits us to represent $F(x,\,y)$ in the form of repeated integrals

$$F(x,y)=\int_{-\infty}^{x} dx\int_{-\infty}^{y} f(x,y)dy\,, \qquad (9\text{ a})$$

$$F(x,y)=\int_{-\infty}^{y} dy\int_{-\infty}^{x} f(x,y)dx\,. \qquad (9\text{ b})$$

4. On the base of the property (1) of the distribution function of a random variable (ξ,η) we find the distribution functions of its components ξ and η,

$$F_\xi(x)=F(x,+\infty)=\int_{-\infty}^{x} dx\int_{-\infty}^{\infty} f(x,y)dy\,, \qquad (10\text{ a})$$

$$F_\eta(y)=F(+\infty,y)=\int_{-\infty}^{y} dy\int_{-\infty}^{\infty} f(x,y)dx\,. \qquad (10\text{ b})$$

5. On the supposition of satisfaction of corresponding conditions we find the distribution densities of the random variables ξ and η by differentiation the integrals (10 a), (10 b) with respect to their variable upper bounds [limits] *x* and *y*,

$$f_\xi(x)=F'_\xi(x)=\int_{-\infty}^{\infty} f(x,y)dy\,, \qquad (11\text{ a})$$

$$f_\eta(y)=F'_\eta(y)=\int_{-\infty}^{\infty} f(x,y)dx\,. \qquad (11\text{ b})$$

6. On the base of the property 2 of the distribution function $F(x, y)$ we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dx = F_{\xi}(+\infty) = F_{\eta}(+\infty) = 1 \qquad (12)$$

7. The components ξ, η of a two-dimensional continuous random variable (ξ, η) are independent if and only if the distribution density $f(x, y)$ of (ξ, η) can be represented in the form of a product of the distribution densities of these components,

$$f(x, y) = f_{\xi}(x) \cdot f_{\eta}(y).$$

■We'll prove the first part of the property (prove yourselves the second one).

Let the random variables ξ, η be independent. By the property 6 of the distribution function we have

$$F(x, y) = F_{\xi}(x) \cdot F_{\eta}(y),$$

and therefore by (6)

$$f(x, y) = F''_{xy}(x, y) = F'_{\xi}(x) \cdot F'_{\eta}(y) = f_{\xi}(x) \cdot f_{\eta}(y). \blacksquare$$

By virtue of this last property the distribution density of a two-dimensional random variable (ξ, η) with independent components equals the product of the dis-tribu-tion densities of the components.

Ex. 1. The simplest normal distribution on the plane. Let ξ, η be two independent normally distributed random variables with the distribution densities

$$f_{\xi}(x) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-a_1)^2}{2\sigma_1^2}}, \quad f_{\eta}(y) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-a_2)^2}{2\sigma_2^2}}.$$

The two-dimensional random variable (ξ, η) with the distribution density

$$f(x, y) = f_{\xi}(x) f_{\eta}(y) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\left(\frac{(x-a_1)^2}{2\sigma_1^2} + \frac{(y-a_2)^2}{2\sigma_2^2}\right)}$$

is said to have the simplest normal distribution on the plane xOy.

The function $f(x, y)$ retains a constant value along an ellipse (the concentration ellipse)

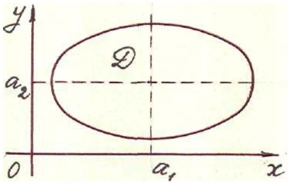


Fig. 8

$$\frac{(x-a_1)^2}{2\sigma_1^2}+\frac{(y-a_2)^2}{2\sigma_2^2}=\lambda^2$$

with the center (a_1, a_2) and semi-axes $\lambda\sigma_1, \lambda\sigma_2$ (fig. 8). The probability of hitting of the random variable (ξ, η) inside the concentration ellipse equals

$$1-e^{-\frac{\lambda^2}{2}};$$

it takes on the value 0.989 for $\lambda = 3$ (analogue of the 3σ-rule for the one-dimensional normal distribution).

Ex. 2. A two-dimensional continuous random variable (ξ, η) is given by the distribution density

$$f(x, y)=\begin{cases} Axy & \text{if} \quad (x, y)\in D, \\ 0 & \text{if} \quad (x, y)\notin D, \end{cases}$$

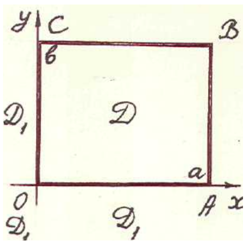


Fig. 9

where D is a rectangle: $D=\{(x, y): 0\le x\le a, 0\le y\le b\}$ (fig. 9). Find the value of the parameter A, the distribution densities of the components ξ, η and the distribution function of the given random variable (ξ, η).

1. The value of A we find on the base of the property 12 of the distribution density, namely

$$1=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dx=\iint_D Axydxdy=A\iint_D xydxdy=A\int_0^a xdx\int_0^b ydy=\frac{A}{4}a^2b^2, \quad A=\frac{4}{a^2b^2},$$

$$f(x, y)=\begin{cases} \frac{4}{a^2b^2}xy & \text{if} \quad (x, y)\in D, \\ 0 & \text{if} \quad (x, y)\notin D. \end{cases}$$

2. To find the distribution densities of the components ξ, η we'll make use of the formulas (11 a), (11 b).

$$f_\xi(x)=\int_{-\infty}^{\infty} f(x, y)dy=\begin{cases} \frac{4}{a^2b^2}\int_0^b xydy=\frac{4}{a^2b^2}x\int_0^b ydy=\frac{2}{a^2}x & \text{if} \quad 0\le x\le a, \\ 0 & \text{if} \quad x\notin[0, a]; \end{cases}$$

$$f_\eta(y)=\int_{-\infty}^{\infty} f(x,y)dx=\begin{cases}\dfrac{4}{a^2b^2}\displaystyle\int_0^a xydx=\dfrac{4}{a^2b^2}y\int_0^a xdx=\dfrac{2}{b^2}y & \text{if} \quad 0\le y\le b,\\ 0 & \text{if} \quad y\notin[0,b].\end{cases}$$

3. Finding the distribution function with the help of the formula (9) we must take into account five cases of disposition of the point (x, y).

a) If the point (x, y) lies in the domain D_1 (the union of the second, third and fourth quadrants, fig. 9), the distribution density $f(x, y)=0$ in R_{xy}, and $F(x, y)=0$.

In the next four cases the distribution density $f(x, y)$ doesn't equal zero in corresponding intersections (in common parts) of the rectangle R_{xy} and the domain D.

These intersections are represented on figures 10 a-d.

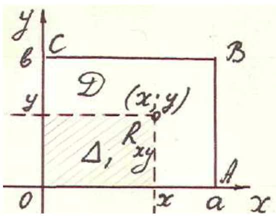


Fig. 10 a

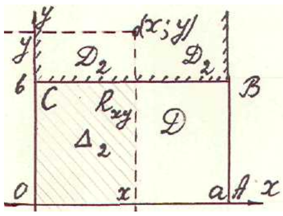


Fig. 10 b

b) If the point (x, y) lies in the rectangle D, the distribution density $f(x, y)$ doesn't equal zero in the inersection Δ_1 of R_{xy} and D (which is the rectangle, fig. 10 a), hence

$$F(x,y)=\frac{4}{a^2b^2}\iint_{\Delta_1} xydxdy=\frac{4}{a^2b^2}\int_0^x xdx\int_0^y ydy=\frac{x^2y^2}{a^2b^2}.$$

c) If the point (x, y) lies in the domain D_2 defined by the inequalities $0\le x\le a$, $y>b$, $f(x, y)\ne 0$ in the rectangle $\Delta_2=R_{xy}\cap D$ (fig. 10 b), and so

$$F(x,y)=\frac{4}{a^2b^2}\iint_{\Delta_2} xydxdy=\frac{4}{a^2b^2}\int_0^x xdx\int_0^b ydy=\frac{x^2}{a^2}.$$

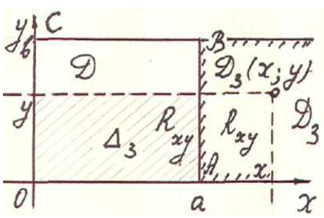


Fig. 10 c

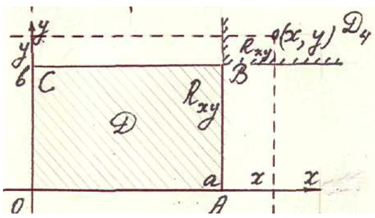


Fig. 10 d

d) If the point (x, y) lies in the domain D_3 defined by the inequalities

$x>a$, $0\le y\le b$,

$f(x, y)\ne 0$ in the rectangle Δ_3 (fig. 10 c), so

$$F(x,y)=\frac{4}{a^2b^2}\iint_{\Delta_3} xydxdy=\frac{4}{a^2b^2}\int_0^a xdx\int_0^y ydy=\frac{y^2}{b^2}.$$

e) At last if the point (x, y) lies in the domain D_4 defined by the inequalities $x>a$, $y>b$, we have $R_{xy}\cap D=D$ (fig. 10 d), and therefore

$$F(x,y)=\iint_{R_{xy}} f(x,y)dxdy=\iint_D f(x,y)dxdy=\frac{4}{a^2b^2}\iint_D xydxdy=\frac{4}{a^2b^2}\int_0^a xdx\int_0^b ydy=1.$$

Thus the distribution function of our two-dimension random variable (ξ,η) equals 0 in the domain D_1, $(x^2y^2)/(a^2b^2)$ in the domain D, x^2/a^2 in the domain D_2 and y^2/b^2 in the domain D_3.

POINT 4. FUNCTIONS OF TWO RANDOM VARIABLES

The distribution law of a function of two random variable

Let a random variable ζ be a function of two random variables ξ,η namely

$$\zeta=\varphi(\xi,\eta).$$

We'll find the distribution law of ζ for the case of continuous random variables ξ,η with known distribution density $f(x,y)$ of two-dimensional random variable (ξ,η).

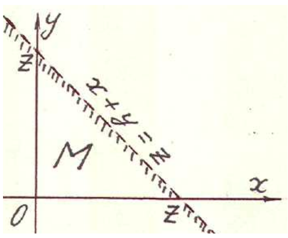


Fig. 11

Denoting by *M* the next point set on the *xOy*-plane

$$M=\{(x,y):\varphi(x,y)<z\},$$

we obtain for the distribution density $F_\zeta(z)$ of the random variable $\zeta=\varphi(\xi,\eta)$

$$F_\zeta(z)=P(\zeta<z)=P(\varphi(\xi,\eta)<z)=\iint_M f(x,y)dxdy=\iint_{\varphi(x,y)<z} f(x,y)dxdy \qquad (13)$$

We'll dwell on the important case namely the sum of two continuous random variables

$$\zeta=\varphi(\xi,\eta)=\xi+\eta.$$

The formula (13), in which it's necessary to put

$$M = \{(x, y): x + y < z\} \text{ (fig. 11)},$$

gives

$$F_\zeta(z) = P(\zeta < z) = P(\xi + \eta < z) = \iint_M f(x, y)dxdy = \iint_{x+y<z} f(x, y)dxdy = \int_{-\infty}^{\infty} dx \int_{-\infty}^{z-x} f(x, y)dy$$

because of $-\infty < x < \infty$, and for each $x \in (-\infty, \infty)$ $-\infty < y < z - x$.

To find the distribution density of the sum $\zeta = \xi + \eta$ we must differentiate the distribution function. If necessary conditions for this operation are fulfilled we obtain

$$f_\zeta(z) = F'_\zeta(z) = \int_{-\infty}^{\infty} dx \left(\int_{-\infty}^{z-x} f(x, y)dy \right)' = \int_{-\infty}^{\infty} f(x, z - x)dx.$$

If the random variables ξ, η are independent, we have

$$f(x, y) = f_\xi(x) f_\eta(y),$$

and therefore

$$f_\zeta(z) = \int_{-\infty}^{\infty} f_\xi(x) f_\eta(z - x)dx. \qquad (14)$$

Ex. 3 (the distribution law of the sum of two independent normal distributions).

Let independent random variables ξ, η be normally distributed with parameters (a_1, σ_1), (a_2, σ_2) and the distribution densities

$$f_\xi(x) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-a_1)^2}{2\sigma_1^2}}, \quad f_\eta(y) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-a_2)^2}{2\sigma_2^2}}$$

correspondingly. We can at once find the mathematical expectation, dispersion and root-mean-square deviation of the sum $\zeta = \xi + \eta$ of these random variables, namely

$$M(\zeta) = M(\xi + \eta) = M(\xi) + M(\eta) = a_1 + a_2,\ D(\zeta) = D(\xi + \eta) = D(\xi) + D(\eta) = \sigma_1^2 + \sigma_2^2,$$
$$\sigma(\zeta) = \sigma(\xi + \eta) = \sqrt{D(\xi) + D(\eta)} = \sqrt{\sigma_1^2 + \sigma_2^2}$$

but we don't know its distribution law and must seek it.

On the base of the formula (14) the distribution density of the sum $\zeta = \xi + \eta$ is represented by the next integral:

$$f_\zeta(z)=\int_{-\infty}^{\infty} f_\xi(x)f_\eta(z-x)dx=\frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{(x-a_1)^2}{2\sigma_1^2}}e^{-\frac{(z-x-a_2)^2}{2\sigma_2^2}}dx=$$

$$=\frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{(x-a_1)^2}{2\sigma_1^2}-\frac{(z-x-a_2)^2}{2\sigma_2^2}}dx=\frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}}e^{-\frac{(z-(a_1+a_2))^2}{2(\sigma_1^2+\sigma_2^2)}}.$$

This result[1] shows that the random variable $\zeta=\xi+\eta$ has the normal distribution with the mathematical expectation $a=a_1+a_2$ and root-mean-square deviation $\sigma=\sqrt{\sigma_1^2+\sigma_2^2}$.

Thus the sum of two independent normally distributed random variables has the normal distribution.

This result is valid for every number of summands.

The mathematical expectation of a function of two random variables

Suppose that we have a function of two random variables

$$\zeta=\varphi(\xi,\eta),$$

and the question is to find its mathematical expectation.

If ζ is a discrete random variable (it means that ξ,η are discrete random variables), then its mathematical expectation is given by the next sum

$$M(\zeta)=M(\varphi(\xi,\eta))=\sum_{i,j}\varphi(x_i,y_j)p_{ij}=\sum_{i,j}\varphi(x_i,y_j)P(\xi=x_i,\eta=y_j), \qquad (15)$$

which is extended on all pairs of values $(\xi=x_i,\eta=y_j)$.

If ζ is a continuous random variable (that is ξ,η are continuous random variables), then the mathematical expectation is represented by the double integral over the whole plane xOy

$$M(\zeta)=M(\varphi(\xi,\eta))=\iint_{R^2}\varphi(x,y)f(x,y)dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x,y)f(x,y)dxdy, \qquad (16)$$

[1] It can be obtained with the help of the known integral $\int_{-\infty}^{\infty} e^{-Ax^2+Bx+C}dx=\sqrt{\frac{\pi}{A}}e^{\frac{B^2+4AC}{4A}}$ $(A>0)$

where $f(x, y)$ is the distribution density of the two-dimensional random variable (ξ, η).

The formulas (15), (16) allow us to prove the properties of the mathematical expectation for a sum and product of two random variables. For discrete random variables they were proved in the Lecture 5. Now we'll consider the case of those continuous.

For the mathematical expectation of a sum the formulas (16), (11 a), (11 b) give

$$M(\xi+\eta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f(x, y)dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xf(x, y)dxdy+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}yf(x, y)dxdy=$$

$$=\int_{-\infty}^{\infty}xdx\int_{-\infty}^{\infty}f(x, y)dy+\int_{-\infty}^{\infty}ydy\int_{-\infty}^{\infty}f(x, y)dx=\int_{-\infty}^{\infty}xf_{\xi}(x)dx+\int_{-\infty}^{\infty}yf_{\eta}(y)dy=M(\xi)+M(\eta).$$

For the mathematical expectation of a product of two independent random variables one gets with the help of the same formulas and the property 7 of the distribution density

$$M(\xi\eta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x\,yf(x, y)dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x\,yf_{\xi}(x)f_{\eta}(y)dxdy=$$

$$=\int_{-\infty}^{\infty}xf_{\xi}(x)dx\int_{-\infty}^{\infty}yf_{\eta}(y)dy=M(\xi)M(\eta).$$

POINT 5. CORRELATION MOMENT AND COEFFICIENT OF TWO RANDOM VARIABLES

In the furute we'll widely applicate so-called correlation moment and correlation coefficient of two random variables.

Def. 4. The correlation moment $K(\xi,\eta)=K_{\xi\eta}$ of two random variables ξ, η is called the mathematical expectation of a product of the centered random variables $\overset{0}{\xi}$ and $\overset{0}{\eta}$,

$$K(\xi,\eta)=K_{\xi\eta}=M\left(\overset{0}{\xi}\overset{0}{\eta}\right)=M((\xi-M(\xi))(\eta-M(\eta))). \qquad (17)$$

It follows from the formula (17) the next representation for the correlation moment:

$$K(\xi,\eta)=K_{\xi\eta}=M(\xi\eta)-M(\xi)M(\eta) \qquad (18)$$

By virtue of the formulas (15), (16) we can write

$$K(\xi,\eta)=\sum_{i,j}x_i y_j p_{ij}-M(\xi)M(\eta)=\sum_{i,j}x_i y_j P(\xi=x_i,\eta=y_j)-M(\xi)M(\eta)$$

for discrete random variables and

$$K(\xi,\eta)=\iint_{R^2}xyf(x,y)dxdy-M(\xi)M(\eta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf(x,y)dxdy-M(\xi)M(\eta)$$

for those continuous.

Def. 5. The correlation coefficient of two random variables ξ,η is called their correlation moment divided by the product of their root-mean-squares deviations,

$$r(\xi,\eta)=r_{\xi\eta}=\frac{K_{\xi\eta}}{\sigma_\xi\sigma_\eta}=\frac{K(\xi,\eta)}{\sigma(\xi)\sigma(\eta)}. \qquad (19)$$

The correlation coefficient is dimensionless [non-dimensional] quantity.

To ascertain probabilistic sense of the correlation moment and especially the correlation coefficient we'll study some their properties.

Cauchy[1]-Bunjakowsky[2] inequality

For two arbitrary random variables ξ and η the inequality (Cauchy-Bunjakowsky inequality) holds

$$|M(\xi\cdot\eta)|\le\sqrt{M(\xi^2)}\cdot\sqrt{M(\eta^2)}. \qquad (20)$$

■For any real number λ the next mathematical expectation $M\left((\xi-\lambda\eta)^2\right)$ isn't negative. Hence,

$$M(\xi-\lambda\eta)^2=M\left(\xi^2-2\lambda\xi\eta+\lambda^2\eta^2\right)=M\left(\xi^2\right)-2\lambda M(\xi\eta)+\lambda^2M\left(\eta^2\right)\ge 0.$$

[1] Cauchy, A.L. (1780 - 1859), an eminent French mathematician
[2] Bunyakowsky, V.J. (1804 - 1889), a Ukrainian mathematician

We've got a quadratic trinomial in λ which can take on only non-negative values. Therefore its discriminant D must be non-positive, and so

$$\frac{1}{4}D = M^2(\xi\eta) - M(\xi^2)M(\eta^2) \le 0,$$

$$M^2(\xi\eta) \le M(\xi^2)M(\eta^2), \quad |M(\xi\eta)| \le \sqrt{M(\xi^2)}\sqrt{M(\eta^2)}. \blacksquare$$

Properties of the correlation moment and coefficient

1. The correlation moment of two random variables ξ, η doesn't surpass in modulus the product of their root-mean-square deviations,

$$|K(\xi, \eta)| \le \sigma(\xi)\sigma(\eta). \quad (21)$$

■By virtue of Cauchy-Bunjakowsky inequality (20) and the definition of the dispersion and root-mean-square deviation

$$|K(\xi, \eta)| = \left|M\left(\overset{0}{\xi}\overset{0}{\eta}\right)\right| \le \sqrt{M\left(\left(\overset{0}{\xi}\right)^2\right)}\sqrt{M\left(\left(\overset{0}{\eta}\right)^2\right)} = \sqrt{D(\xi)}\sqrt{D(\eta)} = \sigma(\xi)\sigma(\eta). \blacksquare$$

Corollary. The correlation coefficient doesn't surpass in modulus the unity,

$$|r(\xi, \eta)| \le 1$$

because of the formula (19) and the inequality (21).

2. If random variables ξ, η are independent, then their correlation moment and therefore the correlation coefficient are equal to zero.

■In the case of independency of ξ, η one has

$$K(\xi, \eta) = M\left(\overset{0}{\xi}\overset{0}{\eta}\right) = M\left(\overset{0}{\xi}\right)M\left(\overset{0}{\eta}\right) = 0 \cdot 0 = 0. \blacksquare$$

3. Let random variables ξ, η be connected by a linear dependence

$$\eta = k\xi + l.$$

In this case

$$|r(\xi, \eta)| = 1,$$

or more exactly

$$r(\xi,\eta)=\begin{cases}1 & \text{if} \quad k>0,\\ -1 & \text{if} \quad k<0.\end{cases}$$

■It's sufficient to find the correlation coefficient under the condition of a linear dependence $\eta = k\xi + l$ between ξ, η. One obtains successively:

$$M(\eta)=M(k\xi+l)=kM(\xi)+l, \quad \overset{0}{\eta}=\eta-M(\eta)=k\xi-kM(\xi)=k(\xi-M(\xi))=k\overset{0}{\xi},$$

$$K(\xi,\eta)=M\left(\overset{0}{\xi}\cdot\overset{0}{\eta}\right)=M\left(\overset{0}{\xi}\cdot k\overset{0}{\xi}\right)=kM\left(\left(\overset{0}{\xi}\right)^2\right)=kD(\xi)=k\sigma^2(\xi),$$

$$\sigma(\eta)=\sqrt{D(\eta)}=\sqrt{M\left(\left(\overset{0}{\eta}\right)^2\right)}=\sqrt{M\left(\left(k\overset{0}{\xi}\right)^2\right)}=\sqrt{M\left(k^2\left(\overset{0}{\xi}\right)^2\right)}=\sqrt{k^2M\left(\left(\overset{0}{\xi}\right)^2\right)}=|k|\sigma(\xi),$$

$$r(\xi,\eta)=\frac{K(\xi,\eta)}{\sigma(\xi)\sigma(\eta)}=\frac{k\sigma^2(\xi)}{\sigma(\xi)\cdot|k|\sigma(\xi)}=\frac{k}{|k|}=\begin{cases}1 & \text{if} \quad k>0,\\ -1 & \text{if} \quad k<0.\end{cases}$$ ■

It follows from the properties 2 and 3 that the correlation coefficient is the measure of a linear dependence between random variables.

The equality to zero of the correlation coefficient of two random variables means absence of a linear dependence between them but it doesn't mean that they are independent. A dependence of the other kind can exist between the random variables.

Def. 6. Two random variables are called those correlated if their correlation coefficient doesn't equal zero and non-correlated otherwise.

TWO DIMENSIAL RANDOM VARIABLES : basic terminology RUEFD

1. векторная случайная величина	векторна випадкова величина	véctor rándom váriable	variable *f* aléatoire vectorielle	vektorielle Zúfallsgröße *f*
2. вероятность попадания двумерной случайной величины в прямоугольник, в область	імовірність попадання двовимірної випадкової величини в прямокутник, в область	pròbabílity of hit/hitting of a two-diménsional rándom váriable in a rectangle, in a domain	probabilité *f* du coup [d'impact [-kt] *m*, d'atteinte *f*] d'une variable aléatoire bidimensionnelle dans un rectangle, dans un domaine	Wahrschéinlichkeit (*f*) von Tréffer einer zweidimensionalen Zúfallsgröße in ein Réchteck, in ein Gebiet
3. взаимно независимые случайные величины	взаємно незалежні випадкові величини	mútually ìndepéndent rándom váriables	variables *f* aléatoires mutuellement indépendantes	einánder únabhängige Zúfallsgrößen *f*
4. двумерная плотность распределения	двовимірна щільність розподілу	two-diménsional dìstribútion dénsity	densité *f* de répartition *f* [de distribution *f*] bidimensionnelle	zweidimensionale Vertéilungsdichte (*f*)
5. двумерная случайная величина	двовимірна випадкова величина	two-diménsional rándom váriable	variable *f* aléatoire bidimensionnelle	zweidimensionale Zúfallsgröße (*f*)
6. двумерная случайная величина, распределенная по нормальному закону	двовимірна випадкова величина, розподілена за нормальним законом	two-diménsional rándom váriable distríbuted by [accórding to, accórdingly, correspóndingly] nórmal law	variable *f* aléatoire bidimensionnelle distribuée/répartie suivant [en,par, conformément] la loi *f* normale	zweidimensionale Zúfallsgröße die nach dem Normalgesetz(e) vertéilt sein
7. двумерная функция распределения	двовимірна функція розподілу	two-diménsional dìstribútion [partítion] fúnction	fonction *f* de répartition *f* [de distribution *f*] bidimensionnelle	zweidimensionale Verteilungsfunktion *f*
8. двумерное нормальное распределение	двовимірний нормальний розподіл	two-diménsional nórmal dìstribútion	distribution *f* [répartition *f* , loi *f*] normale bidimension-	zweidimensionale Normalverteilung *f*

			nelle	
9. двумерное распределение	двовимірний розподіл	two-diménsional dìstribútion	distribution *f* [répartition *f* , loi *f*] bidimensionnelle	zweidimensionale Verteilung *f*
10.двумерный закон распределения	двовимірний закон розподілу	two-diménsional dìstribútion law	loi *f* de distribution *f* [de répartition *f*] bidimensionnelle	zweidimensionales Verteilungsgesetz *n*
11.дискретная двумерная [трехмерная, *n*-мерная] случайная величина	дискретна двовимірна [тривимірна, *n*-вимірна] випадкова величина	discréte two-diménsional [three-diménsional, *n*-diménsional] rándom váriable	variable *f* aléatoire bidimensionnelle [tridimensionnelle, *n*-dimensionnelle] discrète	diskréte zweidimensionale [dreidimensionale, *n*-dimensionale] Zúfallsgröße (*f* =,-n)
12.композиция законов распределения	композиція законів розподілу	còmposítion [súperposítion] of laws of dìstribútion	composée *f* [composition *f*] de lois de distribution *f*	Komposition *f* von Verteilungsgesetzen *n*
13.композиция нормальных законов	композиція нормальних законів	còmposítion [súperposítion] of nórmal laws	composée *f* des lois normales	Komposition *f* von Normalverteilungsgesetzen
14.коррелированные случайные величины	корельовані випадкові величини	córrelàted rándom/álea-tory váriables	variables aléatoires corrélées	korrelíerte Zúfallsgrößen *f*
15.корреляционный момент	кореляційний момент	còrrelátion [correlátive] móment [còefficient]	moment *m* corrélatif	Korrelationsmoment *n*
16.Коши-Буняковского неравенство	Коші – Буняковського нерівність	Cauchy – Bunjakowsky ìnequálity	inégalité *f* de Cauchy–Bouniacovsky	Úngleichung *f* von Cauchy-Bunjakowski
17.коэффициент корреляции	коефіцієнт кореляції	còrrelátion còefficient, còefficient of còrrelátion	coefficient *m* de corrélation	Korrelationskoeffizi¦ent *m*
18.мера линейной зависимости между двумя случай-	міра лінійної залежності між двома випадковими ве-	méasure of línear dependénce between two rándom	mesure *f* de dépendence *f* linéaire entre deux variables	Maß *n* von linéarer Abhängigkeit *f* zwischen zwei

ными величинами	личинами	váriables	aléatoires	Zúfallsgrößen
19.многомерная случайная величина	багатовимірна випадкова величина	múltidiménsial rándom váriable	variable *f* aléatoire multidimensionnelle [à plusieurs dimensions]	mehrdimensionale Zúfallsgröße *f*
20.многомерное распределение	багатовимірний розподіл	múltidiménsial [manydiménsional] dìstribútion	distribution *f* [répartition *f* , loi *f*] multidimensionnelle [à plusieurs dimensions]	mehrdimensionale Verteilung *f*
21.некоррелированные случайные величины	некорельовані випадкові величини	uncórrelàted [nón-córrelàted] rándom váriables	variables *f* aléatoires non-corrélées	únkorrelierte Zúfallsgrößen *f*
22.непрерывная двумерная [трехмерная, *n*-мерная] случайная величина	неперерервна двовимірна [тривимірна, *n*-вимірна] випадкова величина	contínuous two-diménsional [three-diménsional, *n*-diménsional] rándom váriable	variable *f* aléatoire bidimensionnelle [tridimensionnelle, *n*-dimensionnelle] continue	stétige [kontinuíerliche] zweidimensionale [dreidimensionale, *n*-dimensionale] Zúfallsgröße (*f* =,-n)
23.нормальное распределение на плоскости [в пространстве]	нормальний розподіл на площині (в просторі)	nórmal dìstribútion on the plane (in the space)	distribution *f* [répartition *f*] normale sur le plan (dans l'espace)	Normálverteilung auf der Ebene *f*, im Raum(e) *m*)
24.нормальный закон распределения на плоскости (в пространстве)	нормальний закон розподілу на площині (в просторі)	nórmal law of dìstribútion on the plane (in the space)	loi *f* normale de distribution *f* [répartition *f*], loi *f* de laplace-gauss sur le plan (dans l'espace)	normáles Verteilungsgesetz *n* auf der Ebene *f*, im Raum(e) *m*)
25.плотность вероятности	щільність ймовірності	probabílity dénsity	densité *f* de probabilité	Wahrschéinlichkeits‖dichte *f* =
26.плотность распределения	щільність розподілу	distribution dénsity	densité *f* d'une distribution	Verteilungsdichte *f*
27.система	система двох	sýstem of two	système *m* de	System von

двух [трех, *n*] случайных величин	[трьох, *n*] випадкових величин	[three, *n*] rándom váriables	deux/trois/*n* variables *f* aléatoires	zwei [drei, *n*] Zúfallsgrößen
28.случайный вектор	випадковий вектор	rándom véctor	vecteur *m* aléatoire	Zúfallsvektor *m*
29.совместное распределение	сумісний розподіл	joint [sìmultáneous] dìstribútion	distribution *f* jointe [loi *f* conjointe] (de probabilité)	gemeinsame Verteilung *f*
30.составляющая двумерной [трехмерной, *n*-мерной] случайной величины	складова двовимірної [тривимірної, багатовимірної, *n*-вимірної] випадкової величини	compónent of a two-diménsional [three-diménsional, múltidiménsial, *n*-diménsional] rándom váriable	composante *f* [composant *m*] d'une variable aléatoire bidimensionnelle [tridimensionnelle, multidimensionnelle, *n*-dimensionnelle]	Anteil *m*[Komponente *f*] einer zweidimensionalen [dreidimensionalen] Zúfallsgröße (*f*)
31.таблица распределения дискретной двумерной случайной величины	таблиця розподілу дискретної двовимірної випадкової величини	table of the dìstribútion [dìstribútion table] (of a discréte two-diménsional rándom váriable	table *f* [tableau *m*] de distribution *f* [de répartition *f*] d'une variable aléatoire discrète bidimensionnelle	Vertéilungstabelle (*f*) einer diskréte zweidimensionale Zúfallsgröße (*f* =,-n)
32.трехмерная случайная величина	тривимірна випадкова величина	three-diménsional rándom váriable	variable *f* aléatoire tridimensionnelle [à trois dimensions]	dreidimensionale Zúfallsgröße *f*
33.трехмерное распределение	тривимірний розподіл	three-diménsional dìstribútion	distribution *f* [répartition *f* , loi *f*] tridimensionnelle	dreidimensionale Verteilung *f*
34.функция двух [трех, *n*] случайных величин	функція двох [трьох, *n*] випадкових величин	fúnction of two [three, *n*] rándom váriab-les	fonction *f* de deux [trois, *n*] variables aléatoires	Funktión von zwei [von drei, von *n*] Zúfallsgrößen
35.функция распределения	функція розподілу	dìstribútion fúnction	fonction *f* de distribution *f* [de répartition	Verteilungsfunktion *f*

			f]	
36.эллипс рассеяния	еліпс розсіяння	ellípse of còncentrátion	ellipse *f* de concentration	Streuungsellipse *f*
37.эн-мерная (*n*-мерная) случайная величина	ен-вимірна (*n* вимірна) випадкова величина	*n*-diménsional rándom váriable	variable *f* aléatoire de dimension *n* [*n*-dimensionnelle]	*n*-dimensionale Zúfallsgröße *f*
38.эн-мерное (*n*-мерное) распределение	ен-вимірний (*n*-вимірний) розподіл	*n*-diménsional dìstribútion	distribution *f* [répartition *f*] de dimension *n* [*n*-dimensionnelle]	*n*-dimensionale Vertéilung (*f*)
39.эн-мерный (*n*-мерный) случайный вектор	ен-вимірний (*n*-вимірний) випадковий вектор	*n*-diménsional rándom véctor	vecteur *m* aléatoire de dimension *n* [*n*-dimensionnelle]	*n*-dimensionaler Zúfallsvektor *m*

TWO DIMENSIAL RANDOM VARIABLES : basic terminology EFDRU

1. Cauchy – Bunjakowsky ìnequálity	inégalité *f* (de Cauchy–Bouniacovsky)	Úngleichung *f* von Cauchy-Bunjakowski	неравенство Коши - Буняковского	нерівність Коші – Буняковського
2. compónent of a two-diménsional [three-diménsional, múltidiménsial, *n*-diménsional] rándom váriable	composante *f* [composant *m*] d'une variable aléatoire bidimensionnelle [tridimensionnelle, multidimensionnelle, *n*-dimensionnelle]	Anteil *m*[Komponente *f*] einer zweidimensionalen [dreidimensionalen] Zúfallsgröße (*f*)	составляющая двумерной [трехмерной, *n*-мерной] случайной величины	складова двовимірної [тривимірної, багатовимірної, *n*-вимірної] випадкової величини
3. còmposítion [súperposítion] of laws of dìstribútion	composée *f* [composition *f*] de lois de distribution *f*	Komposition *f* von Verteilungsgesetzen	композиция законов распределения	композиція законів розподілу
4. còmposítion [súperposítion] of nórmal laws	composée *f* des lois normales	Komposition *f* von Normalverteilungsgesetzen	композиция нормальных законов	композиція нормальних законів
5. contínuous two-diménsional [three-diménsional, *n*-diménsional] rándom váriable	variable *f* aléatoire bidimensionnelle [tridimensionnelle, *n*-dimensionnelle] continue	stétige [kontinuíerliche] zweidimensionale [dreidimensionale, *n*-dimensionale] Zúfallsgröße	непрерывная двумерная [трехмерная, *n*-мерная] случайная величина	непererервна двовимірна [тривимірна, *n*-вимірна] випадкова величина
6. córrelàted rándom/áleatory váriables	variables aléatoires corrélées	korrelíerte Zúfallsgrößen *f*	коррелированные случайные величины	корельовані випадкові величини
7. còrrelátion [correlátive] móment [còefficient]	moment *m* corrélatif	Korrelationsmoment *n*	корреляционный момент	кореляційний момент
8. còrrelátion còefficient, còefficient of còrrelátion	coefficient *m* de corrélation	Korrelationskoeffizi¦ent *m*	коэффициент корреляции	коефіцієнт кореляції
9. discréte	variable *f* aléa-	diskréte zwei-	дискретная	дискретна

two-diménsional [three-diménsional, *n*-diménsional] rándom váriable	toire bidimensionnelle [tridimensionnelle, *n*-dimensionnelle] discrète	dimensionale [dreidimensionale, *n*-dimensionale] Zúfallsgröße	двумерная [трехмерная, *n*-мерная] случайная величина	двовимірна [тривимірна, *n*-вимірна] випадкова величина
10.distribution dénsity	densité *f* d'une distribution	Verteilungsdichte *f*	плотность распределения	щільність розподілу
11.dìstribútion fúnction	fonction *f* de distribution *f* [de répartition *f*]	Verteilungsfunktion *f*	функция распределения	функція розподілу
12.ellípse of còncentrátion	ellipse *f* de concentration	Streuungsellipse *f*	эллипс рассеяния	еліпс розсіяння
13.fúnction of two [three, *n*] rándom váriables	fonction *f* de deux [trois, *n*] variables aléatoires	Funktión von zwei [von drei, von *n*] Zúfallsgrößen	функция двух [трех, *n*] случайных величин	функція двох [трьох, *n*] випадкових величин
14.joint [sìmultáneous] dìstribútion	distribution *f* jointe [loi *f* conjointe] (de probabilité)	gemeinsame Verteilung *f*	совместное распределение	сумісний розподіл
15.méasure of línear depéndence between two rándom váriables	mesure *f* de dépendence *f* linéaire entre deux variables aléatoires	Maß *n* von lineárer Abhängigkeit zwischen zwei Zúfallsgrößen	мера линейной зависимости между двумя случайными величинами	міра лінійної залежності між двома випадковими величинами
16.múltidiménsial [many-diménsional] dìstribútion	distribution *f* [répartition *f*, loi *f*] multidimensionnelle [à plusieurs dimensions]	Mehrdimensionale Verteilung *f*	многомерное распределение	багатовимірний розподіл
17.múltidiménsial rándom váriable	variable *f* aléatoire multidimensionnelle [à plusieurs dimensions]	mehrdimensionale Zúfallsgröße	многомерная случайная величина	багатовимірна випадкова величина
18.mútually ìndepéndent rándom váriables	variables *f* aléatoires mutuellement indépendantes	einánder únabhängige Zúfallsgrößen *f*	взаимно независимые случайные величины	взаємно незалежні випадкові величини

19.n-dimén-sional dìstribú-tion	distribution *f* [répartition *f*] de dimension *n* [n-dimension-nelle]	n-dimensionale Vertéilung (*f*)	эн-мерное (n-мерное) рас-пределение	ен-вимірний (n-вимірний) розподіл
20.n-dimén-sional rándom váriable	variable *f* aléa-toire de dimen-sion *n* [n-di-mensionnelle]	n-dimensionale Zúfallsgröße *f*	эн-мерная (n-мерная) слу-чайная вели-чина	ен-вимірна (n вимірна) ви-падкова вели-чина
21.n-dimén-sional rándom véctor	vecteur *m* alé-atoire de di-mension *n* [n-dimensionnel-le]	n-dimensiona-ler Zúfallsvek-tor *m*	эн-мерный (n-мерный) слу-чайный век-тор	ен-вимірний (n-вимірний) випадковий вектор
22.nórmal dì-stribútion on the plane (in the space)	distribution *f* [répartition *f*] normale sur le plan (dans l'es-pace)	Normálvertei-lung auf der Ebene *f*, im Raum(e) *m*)	нормальное распределе-ние на пло-скости [в про-странстве]	нормальний розподіл на площині (в просторі)
23.nórmal law of dìstribútion on the plane (in the space)	loi *f* normale de distribution *f* [répartition *f*], loi *f* de Lap-lace-Gauss sur le plan (dans l'espace)	normáles Ver-teilungsgesetz *n* auf der Ebe-ne *f*, im Raum-(e) *m*)	нормальный закон распре-деления на плоскости (в пространстве)	нормальний закон розпо-ділу на пло-щині (в прос-торі)
24.probabílity dénsity	densité *f* de probabilité	Wahrschéin-lichkeits‖dich-te *f* =	плотность ве-роятности	щільність ймовірності
25.pròbabílity of hit/hitting of a two-dimén-sional rándom váriable in a rectangle, in a domain	probabilité *f* du coup [d'impact [-kt] *m*, d'at-teinte *f*] d'une variable aléa-toire bidimen-sionnelle dans un rectangle, dans un domai-ne	Wahrschéin-lichkeit (*f*) von Tréffer einer zweidimensio-nalen Zúfalls-größe (*f*) in ein Réchteck *n*, in ein Gebiet *n*	вероятность попадания двумерной случайной ве-личины в пря-моугольник, в область	імовірність попадання двовимірної випадкової величини в прямокутник, в область
26.rándom véctor	vecteur *m* alé-atoire	Zúfallsvektor *m*	случайный вектор	випадковий вектор
27.sýstem of two [three, *n*] rándom vári-	système *m* de deux/trois/*n* variables *f* alé-	System von zwei [drei, *n*] Zúfallsgrößen	система двух [трех, *n*] слу-чайных ве-	система двох [трьох, *n*] ви-падкових ве-

ables	atoires		личин	личин
28.table of the dìstribútion [dìstribútion table] (of a discréte two-diménsional rándom váriable	table *f* [tableau *m*] de distribution *f* [de répartition *f*] d'une variable aléatoire discrète bidimensionnelle	Vertéilungstabelle (*f*) einer diskréte zweidimensionale Zúfallsgröße (*f* =,-n)	таблица распределения дискретной двумерной случайной величины	таблиця розподілу дискретної двовимірної випадкової величини
29.three-diménsional dìstribútion	distribution *f* [répartition *f* , loi *f*] tridimensionnelle	dreidimensionale Verteilung *f*	трехмерное распределение	тривимірний розподіл
30.three-diménsional rándom váriable	variable *f* aléatoire tridimensionnelle [à trois dimensions]	dreidimensionale Zúfallsgröße *f*	трехмерная случайная величина	тривимірна випадкова величина
31.two-diménsional dìstribútion	distribution *f* [répartition *f* , loi *f*] bidimensionnelle	zweidimensionale Verteilung	двумерное распределение	двовимірний розподіл
32.two-diménsional dìstribútion [partítion] fúnction	fonction *f* de répartition *f* [de distribution *f*] bidimensionnelle	zweidimensionale Verteilungsfunktion	двумерная функция распределения	двовимірна функція розподілу
33.two-diménsional dìstribútion dénsity	densité *f* de répartition *f* [de distribution *f*] bidimensionnelle	zweidimensionale Vertéilungsdichte (*f*)	двумерная плотность распределения	двовимірна щільність розподілу
34.two-diménsional dìstribútion law	loi *f* de distribution *f* [de répartition *f*] bidimensionnelle	zweidimensionales Verteilungsgesetz	двумерный закон распределения	двовимірний закон розподілу
35.two-diménsional nórmal dìstribútion	distribution *f* [répartition *f* , loi *f*] normale bidimensionnelle	zweidimensionale Normalverteilung	двумерное нормальное распределение	двовимірний нормальний розподіл
36.two-diménsional rándom	variable *f* aléatoire bidimen-	zweidimensionale Zúfalls-	двумерная случайная ве-	двовимірна випадкова ве-

váriable	sionnelle	größe (*f*)	личина	личина
37.two-diménsional rándom váriable distríbuted by [acórding to, acórdingly, corrержespóndingly] nórmal law	variable *f* aléatoire bidimensionnelle distribuée/répartie suivant [en,par, conformément] la loi *f* normale	zweidimensionale Zúfallsgröße die nach dem Normalgesetz(e) vertéilt sein	двумерная случайная величина, распределенная по нормальному закону	двовимірна випадкова величина, розподілена за нормальним законом
38.uncórrelàted [nón-córrelàted] rándom váriables	variables *f* aléatoires non-corrélées	únkorrelierte Zúfallsgrößen	некоррелированные случайные величины	некорельовані випадкові величини
39.véctor rándom váriable	variable *f* aléatoire vectorielle	vektorielle Zúfallsgröße *f*	векторная случайная величина	векторна випадкова величина

LECTURE NO. 8.

ELEMENTS OF MATHEMATICAL STATISTICS

LE HUITIEME COURS. ELEMENTS DE STATISTIQUE MATHÉMATIQUE

ACHTE VORLESUNG. ELEMENTE MATHEMATISCHER STATISTIK

POINT 1. GENERAL REMARKS. SAMPLING METHOD. Notes générales. Echantillonage. Allgemeine Bemerkungen. Stichprobenmethode.

POINT 2. VARIATION SERIES. Séries variationnelles. Variationsreihen.

POINT 3. APPROXIMATE DETERMINATION OF THE DISTRIBUTION LAW OF A RANDOM VARIABLE. Détermination approchée de la loi de distribution d'une variable aléatoire. Naherungsweise Bestímmung des Vertéilungsgesetzes einer Zúfallsgröße.

POINT 4. ESTIMATION OF PARAMETERS OF THE DISTRIBUTION LAW OF A RANDOM VARIABLE. Estimation de paramètres de la loi de distribution d'une variable aléatoire. Parameterschätzung des Vertéilungsgesetzes von Zúfallsgröße.

POINT 5. TESTING STATISTIC HYPOTHESES. Test d'hypothèses en statistique. Statistischer Hypothesenprüfung.

POINT 1. GENERAL REMARKS. SAMPLING METHOD

Let's suppose that we study some random variable ξ.

We'll dwell upon three typical problems of the mathematical statistics.

1. Exact or approximate determination of the distribution law of a random variable (for example it can be stated or hypothesized that a random variable ξ is distributed normally).

2. Estimation (approximate calculation) of parameters of the distribution law of a random variable (for example estimation of $m_\xi, D_\xi, \sigma_\xi, As_\xi, Ex_\xi$ of a random variable ξ).

3. Testing statistical hypotheses (for example testing a hypothesis that a given random variable ξ is distributed normally).

There are various methods of solving such the problems. One of the most widespread is the sampling method.

Suppose that we have some population[1] consisting a great number N of things (the population of the size N) which must be studied with respect to some random variable ξ. We take at random n things ($n<<N$) from the population, fulfil their all-round testing with respect to ξ and extend obtained results on the whole population.

On the language of the mathematical statistics we do a sampling of the size n ($n << N$) getting the sample (of the size n) which is subjected to thoroughly investigation with respect to a random variable in question.

A sample must be representative, that is it must certainly represent the population. To be representative the sample must be random one.

POINT 2. VARIATION SERIES

A sample of the size n, which we obtain by a random sampling from the population, we study with the help of so-called variation (or statistical) series. There are variation series of two types namely those discrete and interval.

Table 1. A discrete variation series

ξ, x_i	x_1	x_2	$\dots$	x_l
n_i	n_1	n_2	$\dots$	n_l
$p_i^* = n_i/n$	$p_1^* = n_1/n$	$p_2^* = n_2/n$	$\dots$	$p_l^* = n_l/n$

1. A discrete variation series contains the row of observed values x_i of a random variable ξ to be the investigated (as the rule in increasing order), the row of numbers[2] n_i of occurrences of these values and the row or their relative frequencies

[1] Or: a) parent [general, grand, total] population; b) parent universe.
[2] These numbers can be called frequencies.

$p_i^* = n_i/n$ (the table 1)[1].

It must be

$$n_1 + n_2 + \ldots + n_l = n, \quad p_1^* + p_k^* + \ldots + p_l^* = 1.$$

Such the discrete variation series can be represented geometrically with the help of a polygon of frequencies or a polygon of relative frequencies. The polygon of relative frequencies is a broken line which joins successively the next points:

$$A_1(x_1;\, p_1^*), A_2(x_2;\, p_2^*), \ldots, A_l(x_l;\, p_l^*)$$

(see fig. 1). It's useful to compare it with a distribution polygon of a random variable (see Lecture No. 3, fig. 1).

The polygon of frequencies is a broken line joining successively the other points namely those with ordinates (frequencies) $n_1, n_2, \ldots, n_l$,

$$B_1(x_1;\, n_1), B_2(x_2;\, n_2), \ldots, B_l(x_l;\, n_l).$$

If there are a lot of distinct observed values of a random variable ξ, they can be united in intervals which generate so-called interval variation series.

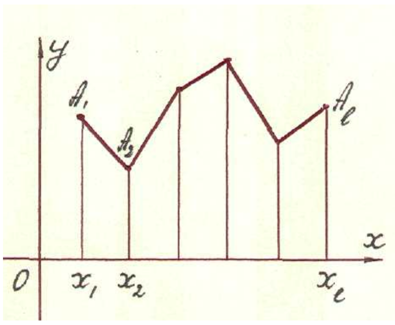


Fig. 1

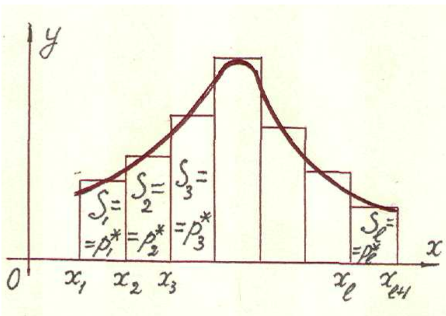


Fig. 2

2. An interval variation series contains the row of intervals with all observed values of a random variable ξ in question, the row of numbers (frequencies) n_i of hitting of these values in the ith interval and the row of corresponding relative frequencies $p_i^* = n_i/n$ (the table 2).

[1] One applies sometimes only one of two last rows.

Table 2. An interval variation series

Intervals	$[x_1, x_2]$	$(x_2, x_3]$	$(x_3, x_4]$	…	$(x_l, x_{l+1}]$
n_i	n_1	n_2	n_3	…	n_l
$p_i^* = n_i/n$	$p_1^* = n_1/n$	$p_2^* = n_2/n$	$p_3^* = n_3/n$	…	$p_l^* = n_l/n$

The first interval is that close and the other l - 1 intervals are semi-closed [1]. Just as in the case of a discrete variation series it must be

$$n_1 + n_2 + ... + n_l = n, \quad p_1^* + p_2^* + ... + p_l^* = 1.$$

The interval variation series can be represented geometrically by a histogram[2] of frequencies or relative frequencies. The histogram of relative frequencies is the set of rectangles with the bases $(x_i, x_{i+1}]$ of the lengths $\Delta x_i = x_{i+1} - x_i$, $i = \overline{1, l}$, and the areas $S_i = p_i^*$ (fig. 2). The altitude of the ith rectangle of such the histogram equals

$$h_i = S_i / \Delta x_i = p_i^* / \Delta x_i .$$

The histogram of frequencies contains rectangles of the areas $S_i = n_i$ with the same bases.

Often we form intervals of the same length Δx (in particular on the fig. 2).

If one wants to generate l intervals, he can put

$$\Delta x = \frac{x_{\max} - x_{\min}}{l}, \qquad (1)$$

where $x_{\max}$ is the greatest and $x_{\min}$ is the least of observed values of a random variable ξ. In practice it's useful sometimes to take instead $x_{\max}$ some greater number and instead $x_{\min}$ some less number in the formula (1).

One can preset the length of intervals instead their number. For this purpose he can use the next known approximate formula:

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[1] One can take the other intervals: $[x_1, x_2), [x_2, x_3), [x_3, x_4), ..., [x_l, x_{l+1}]$.

[2] Or a bar chart, bar diagram, block diagram, column diagram

$$\Delta x \approx \frac{x_{\max} - x_{\min}}{1 + 3.322 \cdot \lg n}. \qquad (2)$$

As the left point of the first interval he can take

$$x_1 \approx x_{\min} - \Delta x/2 \qquad (3)$$

and get then the other points as follows

$$x_2 = x_1 + \Delta x,\ x_3 = x_2 + \Delta x, ...,\ x_{l+1} = x_l + \Delta x. \qquad (4)$$

The number l of intervals is determined by the condition

$$x_{l+1} \geq x_{\max} \qquad (5)$$

which means that the last lth interval must contain the value $x_{\max}$ of the random variable ξ in question.

Having an interval variation series we often must compile a corresponding discrete variation series by taking some inner point x_i^* in the ith interval $\left(i = \overline{1, l}\right)$. For example we can take

$$x_1^* \approx \frac{x_1 + x_2}{2},\ x_2^* = x_1^* + \Delta x,\ x_3^* = x_2^* + \Delta x, ...,\ x_l^* = x_{l-1}^* + \Delta x \qquad (4)$$

and obtain the discrete variation series which is given by the table 3. We suppose conditionally that the point x_i^* represents the ith interval and therefore that the value $\xi = x_i^*$ were observed n_i times (with the relative frequency $p_i^* = n_i/n$). It means that the tables 2 and 3 contain the same second and third rows.

Table 3. The discrete variation series which corresponds to that interval

x_i^*	x_1^*	x_2^*	$\cdots$	x_l^*
n_i	n_1	n_2	$\cdots$	n_l
$p_i^* = n_i/n$	$p_1^* = n_1/n$	$p_2^* = n_2/n$	$\cdots$	$p_l^* = n_k/n$

Ex. 1 (the ***Basic example***). To study a random variable ξ a sampling is fulfilled and the sample of the size $n = 200$ is obtained (the table 4). Study the random variable.

Table 4. The sample with respect to the random variable ξ of the ***Basic example***

1.	79	75	81	71	80	75	87	80	74	88
2.	80	84	81	80	78	80	73	81	78	79
3.	83	76	77	82	80	85	86	85	86	81
4.	70	83	82	77	83	87	75	80	82	88
5.	82	75	84	71	72	84	81	79	87	83
6.	78	81	75	77	89	76	85	82	77	84
7.	81	84	88	82	86	85	90	87	83	86
8.	76	71	84	76	80	79	82	80	78	83
9.	76	81	78	75	84	86	78	91	84	77
10.	82	78	83	85	79	82	85	93	79	73
11.	76	84	83	83	87	83	82	92	81	85
12.	77	88	76	72	89	77	80	94	86	76
13.	82	88	87	89	83	86	95	90	84	88
14.	74	78	84	76	80	75	81	72	89	84
15.	83	89	82	81	73	91	80	92	83	77
16.	86	77	75	80	89	81	93	83	89	94
17.	75	85	84	72	76	88	74	82	81	78
18.	87	76	82	78	81	88	76	89	75	95
19.	83	81	71	84	80	87	91	93	92	94
20.	77	70	79	82	75	90	70	83	73	84

We have $x_{\max} = 95, x_{\min} = 70$ here. Let's compile $l = 5$ intervals of the length

$$\Delta x = \frac{x_{\max} - x_{\min}}{l} = \frac{95 - 70}{5} = 5 \text{ (see the formula (1)).}$$

A table 5 represents the interval variation series, inner points x_i^* for corresponding discrete variation series and altitudes h_i of rectangles for plotting the histogram of relative frequencies.

Table 5. The variation series for the ***Basic example***

Intervals	$[70, 75]$	$(75, 80]$	$(80, 85]$	$(85, 90]$	$(90, 95]$
Inner points x_i^*	72	77	82	87	92
Frequencies n_i	18	50	76	40	16
Relative frequencies $p_i^* = n_i / n$	0.09	0.25	0.38	0.20	0.08
Altitudes $h_i = p_i^* / \Delta x$	0.018	0.050	0.076	0.040	0.016

The histogram and polygon of relative frequencies for the interval and corresponding discrete variation series are represented on fig. 3, 4. For the sake of geometric visualization we draw to different scales along the axes.

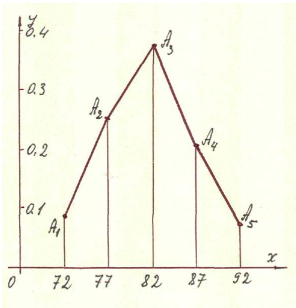


Fig. 3

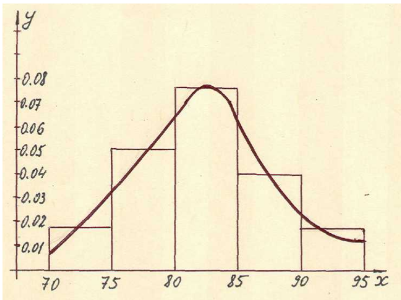


Fig. 4

POINT 3. APPROXIMATE DETERMINATION OF THE DISTRIBUTION LAW OF A RANDOM VARIABLE

The statistic distribution function

Def. 1. The statistic distribution function of a random variable ξ is called the relative frequency of hitting of its observed values in the interval $(-\infty, x)$,

$$F^*(x) = P_n^*(\xi < x). \qquad (5)$$

It's obvious that the statistic distribution function $F^*(x)$ equals the sum of relative frequencies of those observed values of ξ which are less than *x*,

$$F^*(x) = \sum_{x_i < x} p_i^* \qquad (6)$$

To find the statistic distribution function we must distinguish the cases of discrete and interval variation series.

1. In the first case (for a discrete variation series) we have immediately by definition

$$F^*(x) = \begin{cases} 0 & \text{if} \quad x \le x_1, \\ p_1^* & \text{if} \quad x_1 < x \le x_2, \\ p_1^* + p_2^* & \text{if} \quad x_2 < x \le x_3, \\ p_1^* + p_2^* + p_3^* & \text{if} \quad x_3 < x \le x_4, \\ \dots\dots\dots & \dots \quad \dots\dots\dots \\ 1 & \text{if} \quad x > x_l. \end{cases}$$

2. In the second case (for an interval variation series) we put successively

$$F^*(x_1) = 0, F^*(x_2) = p_1^*, F^*(x_3) = p_1^* + p_2^*, F^*(x_4) = p_1^* + p_2^* + p_3^*, \dots, F^*(x_{l+1}) = 1,$$

then represent the points

$$A_1(x_1; 0), A_2(x_2; p_1^*), A_3(x_3; p_1^* + p_2^*), A_4(x_4; p_1^* + p_2^* + p_3^*), \dots, A_{l+1}(x_{l+1}; 1)$$

on the xOy-plane and plot the approximate graph of the function $F^*(x)$, namely in the form of semi-segments or (if ξ is a continuous random variable) in the form of a continuous line.

On the base of Bernoulli theorem the statistic distribution function $F^*(x)$ tends in probability to usual (or theoretical) distribution function $F(x) = P(\xi < x)$ for n tending to infinity,

$$F^*(x) = P_n^*(\xi < x) \xrightarrow{prob} P(\xi < x) = F(x).$$

Ex. 2. Find the statistic distribution function of the random variable ξ which is studied in the ***Basic example***.

Proceeding from the discrete variation series (see inner points x_i^* and corresponding relative frequencies $p_i^* = n_i/n$ in the table 5) we obtain

$$F^*(x)=\begin{cases} 0.00 & \text{if} \quad x\le 72,\\ 0.09 & \text{if} \quad 72<x\le 77,\\ 0.09+0.25=0.34 & \text{if} \quad 77<x\le 82,\\ 0.34+0.38=0.72 & \text{if} \quad 82<x\le 87,\\ 0.72+0.20=0.92 & \text{if} \quad 87<x\le 92,\\ 0.92+0.08=1.00 & \text{if} \quad x>92, \end{cases}$$

(see fig. 5).

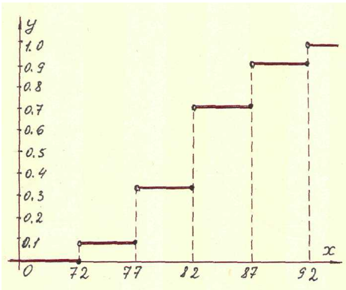


Fig. 5

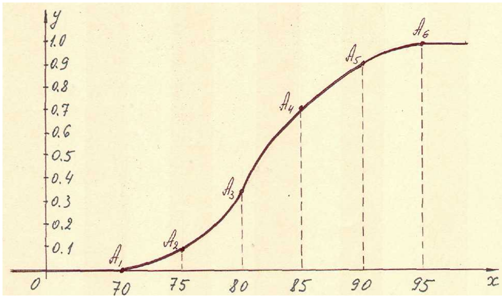


Fig. 6

Using the interval variation series (the first and fourth rows of the table 5) we get approximate values of the statistic distribution function

$$F^*(70)=0,\ F^*(75)=0.09,\ F^*(80)=0.09+0.25=0.34,\ F^*(85)=0.34+0.38=0.72,$$

$$F^*(90)=0.72+0.20=0.92,\ F^*(95)=0.92+0.08=1.00,$$

represent corresponding points

$$A_1(70;0),\ A_2(75;0.09),\ A_3(80;0.34),\ A_4(85;0.72),\ A_5(90;0.92),\ A_6(95;1)$$

on the xOy-plane and plot the approximate graph of the statistic distribution function in the form of a continuous line (fig. 6).

Idea of the distribution density

If a random variable ξ is continuous one and the number l of intervals (of the same length Δx) tends to infinity,

$$l \to \infty \, (\Delta x \to 0),$$

then the upper part of its histogram of relative frequencies tends to some curve which is the graph of the distribution density $f(x)$ for ξ. On fig. 4 we've represented the approximate graph of the distribution density.

Obvious symmetry of the approximate graph of the distribution density of the random variable ξ which we study in the ***Basic example*** (fig. 6) permits us to state the next hypothesis: ξ is normally distributed.

POINT 4. ESTIMATION OF PARAMETERS OF THE DISTRIBUTION LAW OF A RANDOM VARIABLE

Let θ be a parameter to be estimated and θ^* be its estimate [or estimator]. This latter is a function of results of *n* trials on ξ.

In theory we consider results of trials as random variables $\xi_1, \xi_2, \xi_3, ..., \xi_n$ with the same distribution law as ξ, in particular with the same mathematical expectation and dispersion

$$\forall i : M(\xi_i) = M(\xi), D(\xi_i) = D(\xi), \quad i = \overline{1, n}.$$

Respectively we suppose the estimate θ^* to be a function of these random variables:

$$\theta^* = f(\xi_1, \xi_2, \xi_3, ..., \xi_n).$$

In practice we express the estimate θ^* with the help of results of trials on the random variable ξ represented by variation series.

For discrete variation series (with *l* observed various values x_i of the random variable ξ) we can write

$$\theta^* = f(x_1, x_2, ..., x_l).$$

For interval variation series (with l intervals) we use the inner points x_i^* of the intervals and so

$$\theta^* = f(x_1^*, x_2^*, ..., x_l^*)$$

There exist three necessary requirements which must be laid to every estimate: consistency, unbiasedness and efficiency [effectiveness].

1. An estimate θ^* is called a consistent estimator of the parameter θ if it converges to θ in probability ($\theta^* \xrightarrow{prob.} \theta$).

2. This estimate θ^* is called an unbiased estimator of the parameter θ if its mathematical expectation equals θ ($M(\theta^*) = \theta$).

3. The estimate θ^* is called an efficient [effective] estimator of θ if it has minimal dispersion in comparison with all other estimates.

There are pointwise and interval estimates of the parameters of distribution laws of random variables.

Pointwise [point] estimates [estimators]

Let's remember the formulas for number characteristics of a discrete random variable ξ (the mathematical expectation, dispersion, root-mean-square deviation, asymmetry and excess)

$$m_\xi = M(\xi) = \sum_i x_i p_i;$$

$$D_\xi = D(\xi) = \sum_i (x_i - m_\xi)^2 p_i, \qquad D(\xi) = D_\xi = m_{\xi^2} - (m_\xi)^2 = M(\xi^2) - M^2(\xi);$$

$$\sigma(\xi) = \sigma_\xi = \sqrt{D_\xi} = \sqrt{D(\xi)};$$

$$As(\xi) = \frac{\mu_3(\xi)}{\sigma^3(\xi)} = \frac{\sum_i (x_i - m_\xi)^3 p_i}{\sigma^3(\xi)}; \qquad Ex(\xi) = \frac{\mu_4(\xi)}{\sigma^4(\xi)} = \frac{\sum_i (x_i - m_\xi)^4 p_i}{\sigma^4(\xi)} - 3.$$

In mathematical statistics we try at first to estimate corresponding parameters by analogous formulas.

1. Estimate [estimator] of the mathematical expectation [estimation of expectation].

As the estimate [estimator] of the mathematical expectation of a random variable ξ we take so-called **sample mean** [sample average, sample mathematical expectation], that is the arithmetical average of the results of n independent trials on ξ,

$$\overline{\xi_s} = \frac{\xi_1 + \xi_2 + \ldots + \xi_n}{n}. \tag{2}$$

Let's write the calculating formulas based on known variation series.

a) For a discrete variation series

$$\overline{\xi_s} = \frac{1}{n}\sum_{i=1}^{l} x_i n_i = \sum_{i=1}^{l} x_i p_i^* . \tag{2a}$$

b) For an interval variation series

$$\overline{\xi_s} = \frac{1}{n}\sum_{i=1}^{l} x_i^* n_i = \sum_{i=1}^{l} x_i^* p_i^* . \tag{2b}$$

Testing of requirements to the sample mean that is to the estimate (2).

a) By small Chebyshev theorem

$$\overline{\xi_s} = \frac{\xi_1 + \xi_2 + \ldots + \xi_n}{n} \xrightarrow{prob} M(\xi) = m_\xi ,$$

and therefore the sample mean $\overline{\xi_s}$ is a consistent estimate.

b) The mathematical expectation of the sample mean equals

$$M\left(\overline{\xi_s}\right) = M\left(\frac{\xi_1 + \xi_2 + \ldots + \xi_n}{n}\right) = \frac{1}{n} M(\xi_1 + \xi_2 + \ldots + \xi_n) = \frac{1}{n}\left(M(\xi_1) + M(\xi_2) + \ldots + M(\xi_n)\right) =$$

$$= \frac{1}{n}\left(M(\xi) + M(\xi) + \ldots + M(\xi)\right) = \frac{1}{n} \cdot nM(\xi) = M(\xi) = m_\xi ,$$

and so $\overline{\xi_s}$ is an unbiased estimate.

c) If ξ distributed normally then, as can be proved, the sample mean $\overline{\xi_s}$ is an effective estimate.

2. Estimate [estimator] of the dispersion [variance estimate].

Let's take at first so-called **sample dispersion**

$$D_s = \frac{(\xi_1 - \overline{\xi_s})^2 + (\xi_2 - \overline{\xi_s})^2 + \ldots + (\xi_n - \overline{\xi_s})^2}{n} \qquad (3)$$

as the estimate [estimator] of the dispersion of a random variable ξ.

Corresponding calculating formulas

a) For a discrete variation series

$$D_s = \frac{1}{n}\sum_{i=1}^{l}\left(x_i - \overline{\xi_s}\right)^2 n_i = \sum_{i=1}^{l}\left(x_i - \overline{\xi_s}\right)^2 p_i^* . \qquad (3\text{ a})$$

b) For an interval variation series

$$D_s = \frac{1}{n}\sum_{i=1}^{l}\left(x_i^* - \overline{\xi_s}\right)^2 n_i = \sum_{i=1}^{l}\left(x_i^* - \overline{\xi_s}\right)^2 p_i^* . \qquad (3\text{ b})$$

c) It follows from the formula (3) the next useful general formula

$$D_s = \overline{\xi_s^2} - \left(\overline{\xi_s}\right)^2 . \qquad (3\text{ c})$$

$$\blacksquare\ D_s = \frac{(\xi_1 - \overline{\xi_s})^2 + (\xi_2 - \overline{\xi_s})^2 + \ldots + (\xi_n - \overline{\xi_s})^2}{n} =$$

$$= \frac{\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 + \left((\overline{\xi_s})^2 + (\overline{\xi_s})^2 + \ldots + (\overline{\xi_s})^2\right) - 2\left(\xi_1\overline{\xi_s} + \xi_2\overline{\xi_s} + \ldots + \xi_n\overline{\xi_s}\right)}{n} =$$

$$= \frac{\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 + n\cdot(\overline{\xi_s})^2 - 2\left(\xi_1 + \xi_2 + \ldots + \xi_n\right)\overline{\xi_s}}{n} =$$

$$= \frac{\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2}{n} + (\overline{\xi_s})^2 - 2\cdot\frac{\xi_1 + \xi_2 + \ldots + \xi_n}{n}\cdot\overline{\xi_s} = \overline{\xi_s^2} + (\overline{\xi_s})^2 - 2(\overline{\xi_s})^2 = \overline{\xi_s^2} - (\overline{\xi_s})^2 .\ \blacksquare$$

As to necessary requirements to the sample dispersion (3) one can prove that it converges to the dispersion $D(\xi)$ in probability but its mathematical expectation is less that $D(\xi)$ namely

$$M(D_s) = \frac{n-1}{n} D(\xi) < D(\xi).$$

Therefore the sample dispersion D_s is consistent but biased [shifted] estimate of the dispersion. To have an unbiased consistent estimator one introduces so-called **corrected dispersion**

$$\widetilde{D} = \frac{n}{n-1} D_s. \qquad (4)$$

3. The root-mean-square deviation of the random variable ξ is estimated by the **sample root-mean-square deviation**

$$\sigma_s = \sqrt{D_s} \qquad (5)$$

and the **corrected root-mean-square deviation** (or the **standard**)

$$\widetilde{\sigma} = s = \sqrt{\widetilde{D}}. \qquad (6)$$

4. To estimate the asymmetry and excess we'll mention so-called sample asymmetry and sample excess.

a) For discrete variation series they are

$$As_s = \frac{\frac{1}{n}\sum_{i=1}^{l}\left(x_i - \overline{\xi_s}\right)^3 n_i}{\sigma_s^3} = \frac{\sum_{i=1}^{l}\left(x_i - \overline{\xi_s}\right)^3 p_i^*}{\sigma_s^3}, \qquad (7\,a)$$

$$Ex_s = \frac{\frac{1}{n}\sum_{i=1}^{l}\left(x_i - \overline{\xi_s}\right)^4 n_i}{\sigma_s^4} - 3 = \frac{\sum_{i=1}^{l}\left(x_i - \overline{\xi_s}\right)^4 p_i^*}{\sigma_s^4} - 3. \qquad (8\,a)$$

b) For interval variation series

$$As_s = \frac{\frac{1}{n}\sum_{i=1}^{l}\left(x_i^* - \overline{\xi_s}\right)^3 n_i}{\sigma_s^3} = \frac{\sum_{i=1}^{l}\left(x_i^* - \overline{\xi_s}\right)^3 p_i^*}{\sigma_s^3}, \qquad (7\,b)$$

$$Ex_s = \frac{\frac{1}{n}\sum_{i=1}^{l}\left(x_i^* - \overline{\xi_s}\right)^3 n_i}{\sigma_s^4} - 3 = \frac{\sum_{i=1}^{l}\left(x_i^* - \overline{\xi_s}\right)^4 p_i^*}{\sigma_s^4} - 3. \qquad (8\,b)$$

Ex. 3. Estimate the mathematical expectation, dispersion, root-mean-square deviation, asymmetry and excess of the random variable ξ of the ***Basic example***.

Corresponding calculations are represented in the table 6.

Table 6

Interval	[70, 75]	(75, 80]	(80, 85]	(85, 90]	(90, 95]	Σ
n_i	18	50	76	40	16	n=200
$p_i^* = n_i/n$	0.09	0.25	0.38	0.20	0.08	1
x_i^*	72	77	82	87	92	
$x_i^* p_i^*$	6.48	19.25	31.16	17.40	7.36	$\overline{\xi_s} = 81.651$
$\left(x_i^*\right)^2$	5184	5929	6724	7569	8464	
$\left(x_i^*\right)^2 p_i^*$	466.56	1482.25	2555.12	1513.80	677.12	$\overline{\xi_s^2} = 6694.25$
$x_i^* - \overline{\xi_s}$	-9.65	-4.65	0.35	5.35	10.35	
$\left(x_i^* - \overline{\xi_s}\right)^2$	93.12	21.62	0.12	28.62	107.12	
$\left(x_i^* - \overline{\xi_s}\right)^2 p_i^*$	8.38	5.41	0.05	5.72	8.57	$D_s = 28.13$
$\left(x_i^* - \overline{\xi_s}\right)^3$	-898.63	-100.54	0.04	153.13	1108.72	
$\left(x_i^* - \overline{\xi_s}\right)^3 p_i^*$	-80.88	-25.14	0.02	30.63	88.70	$\mu_{3s} = 13.33$
$\left(x_i^* - \overline{\xi_s}\right)^4$	8671.80	467.53	0.02	819.25		
$\left(x_i^* - \overline{\xi_s}\right)^4 p_i^*$	780.46	115.88	0.01	163.85	918.02	$\mu_{4s} = 1979.22$

Thus the sample means of the random variable ξ and its square are equal to

$$\overline{\xi_s} = \sum_{i=1}^{l} x_i^* p_i^* = 81.65, \quad \overline{\xi_s^2} = \sum_{i=1}^{l}\left(x_i^*\right)^2 p_i^* = 6694.85,$$

and therefore by (3 c) and (5) the sample dispersion and root-mean-square deviation are equal to

$$D_s = \overline{\xi_s^2} - \left(\overline{\xi_s}\right)^2 = 6694.85 - (81.65)^2 \approx 28.13, \quad \sigma_s = \sqrt{D_s} \approx 5.30.$$

The corrected dispersion and root-mean-square deviation equal by virtue of the formulas (4), (6)

$$\widetilde{D} = \frac{n}{n-1} D_s = \frac{200}{199} \cdot 28.13 \approx 28.27, \quad \widetilde{\sigma} = s = \sqrt{\widetilde{D}} = \sqrt{28.27} \approx 5.32.$$

The sample symmetry and excess of the random variable ξ are equal by the formulas (7 b) (8 b)

$$As_s = \frac{\sum_{i=1}^{l} \left(x_i^* - \overline{\xi_s}\right)^3 p_i^*}{\sigma_s^3} = \frac{13.33}{5.30^3} \approx 0.09; Ex_s = \frac{\sum_{i=1}^{l} \left(x_i^* - \overline{\xi_s}\right)^4 p_i^*}{\sigma_s^4} - 3 = \frac{1979.22}{5.30^4} - 3 \approx -0.49.$$

Interval estimates [estimation by confidence interval]

Let θ be a parameter to be estimated, θ^* its estimate, ε some small positive number (so-called **accuracy**) and γ is some large probability (the **reliability**).

The next relation

$$P(|\theta - \theta^*| < \varepsilon) = \gamma \qquad (9)$$

means that with the reliability γ

$$|\theta - \theta^*| < \varepsilon, \quad -\varepsilon < \theta - \theta^* < \varepsilon, \quad \theta^* - \varepsilon < \theta < \varepsilon + \theta^*, \quad \theta \in (\theta^* - \varepsilon, \theta^* + \varepsilon);$$

therefore it is equivalent to the next one:

$$P(\theta^* - \varepsilon < \theta < \varepsilon + \theta^*) = P(\theta \in (\theta^* - \varepsilon, \theta^* + \varepsilon)) = \gamma. \qquad (10)$$

This last equality shows that with the reliability γ the estimating parameter θ is covered by a random interval $(\theta^* - \varepsilon, \theta^* + \varepsilon)$.

Def. An interval $(\theta^* - \varepsilon, \theta^* + \varepsilon)$ which covers an estimated parameter θ with the reliability γ is called **confidence** one.

Let a random variable ξ is normally distributed, and it's necessary to find the confidence interval for its mathematical expectation a.

We can suppose that the sample mean $\overline{\xi_s}$ and corrected root-mean-square deviation $\tilde{\sigma}$ of ξ are found.

Let's study an auxiliary random variable

$$t = \sqrt{n}\,\frac{\overline{\xi_s} - a}{\tilde{\sigma}}. \qquad (11)$$

It can be proved that t has Student distribution (or t-distribution) with $k = n-1$ degrees of freedom (see Lecture No. 6, Point 5). Therefore for the given reliability γ we can find a number t_γ such that

$$P(|t| < t_\gamma) = \gamma \qquad (12)$$

(see the formula (28) in indicated Lecture). There's corresponding table for finding t_γ by known γ and n (see the table 7 hear; the more detailed table see in [4, 11]).

Table 7

$n = \infty\ (n > 120)$	$\gamma = 0.9$	$t_\gamma = 1.645$	$n = 100$	$\gamma = 0.9$	$t_\gamma = 1.662$
	$\gamma = 0.95$	$t_\gamma = 2.576$		$\gamma = 0.95$	$t_\gamma = 1.984$
	$\gamma = 0.99$	$t_\gamma = 3.291$		$\gamma = 0.99$	$t_\gamma = 2.627$

The formulas (11) and (12) determine the confidence interval in question. Indeed

$$\gamma = P(|t| < t_\gamma) = P\left(\left|\sqrt{n}\,\frac{\overline{\xi_s} - a}{\tilde{\sigma}}\right| < t_\gamma\right) = P\left(\left|\overline{\xi_s} - a\right| < t_\gamma \cdot \frac{\tilde{\sigma}}{\sqrt{n}}\right) = P\left(\left|a - \overline{\xi_s}\right| < t_\gamma \cdot \frac{\tilde{\sigma}}{\sqrt{n}}\right) =$$

$$= P\left(-t_\gamma \cdot \frac{\tilde{\sigma}}{\sqrt{n}} < a - \overline{\xi_s} < t_\gamma \cdot \frac{\tilde{\sigma}}{\sqrt{n}}\right) = P\left(\overline{\xi_s} - t_\gamma \cdot \frac{\tilde{\sigma}}{\sqrt{n}} < a < \overline{\xi_s} + t_\gamma \cdot \frac{\tilde{\sigma}}{\sqrt{n}}\right),$$

and with the reliability γ the mathematical expectation a is covered by the interval

$$\left(\overline{\xi_s} - \varepsilon, \overline{\xi_s} + \varepsilon\right), \qquad (13)$$

where the accuracy

$$\varepsilon = t_\gamma \cdot \frac{\tilde{\sigma}}{\sqrt{n}}. \qquad (14)$$

Ex. 4. Supposing that the random variable ξ of the ***Basic example*** has the normal distribution find the confidence intervals for its mathematical expectation a with the reliabilities $\gamma = 0.9, \gamma = 0.95$.

a) For the reliability $\gamma = 0.9$ and $n = 200$ $(\text{or } n = \infty)$ we have $t_\gamma = 1.645$ from the table 7, hence the corresponding accuracy equals

$$\varepsilon = t_\gamma \cdot \frac{\tilde{\sigma}}{\sqrt{n}} = 1.645 \cdot \frac{5.32}{\sqrt{200}} \approx 0.62,$$

and the confidence interval is

$$\left(\overline{\xi_s} - \varepsilon, \overline{\xi_s} + \varepsilon\right) = (81.65 - 0.62, 81.65 + 0.62) = (81.03, 82.27).$$

b) In the case $\gamma = 0.95$ the corresponding values of t_γ and ε are greater then those for $\gamma = 0.9$, namely $t_\gamma = 2.576$,

$$\varepsilon = t_\gamma \cdot \frac{\tilde{\sigma}}{\sqrt{n}} = 2.576 \cdot \frac{5.32}{\sqrt{200}} \approx 0.97,$$

and we obtain the confidence interval

$$\left(\overline{\xi_s} - \varepsilon, \overline{\xi_s} + \varepsilon\right) = (81.65 - 0.97, 81.65 + 0.97) = (80.68, 82.62)$$

which is wider than that preceding.

Results of Ex. 4 demonstrate so-called ***Principle of indeterminacy*****:** if the reliability γ of estimation increases, then its precision [correctness, exactness] decreases and vice versa.

POINT 5. TESTING STATISTIC HYPOTHESES

We'll limit ourselves to hypotheses about distribution law of a random variable which is investigated. For example our ***Basic example*** has generated the hypothesis that ξ has the normal distribution.

Let we test a hypothesis that a random variable ξ has certain distribution law. For this purpose we introduce some non-negative random variable (**goodness-of-fit test**) K which is the measure of deviation of the theoretical assumption (based on the hypothesis) and the results of trials on the random variable. It's supposed that we know exact or approximate distribution law of K. On the base of the result of trials on

ξ (for example on the base of the fulfilled sample) we find so-called calculated value K_{calc} of K and compare it with some well defined critical value K_{crit} of the same K.

Let for example the critical value K_{crit} of the goodness-of-fit test K is defined by the relation

$$P(K > K_{crit}) = \alpha, \qquad (15)$$

where α is some small probability $(0.05, 0.025, 0.01 \text{ and so on})$. This probability is often called the **significance level** in such the sense that the event $K > K_{crit}$ can be considered as highly improbable or even practically impossible. Correspondingly we consider the occurrence of the result $K_{calc} > K_{crit}$ as low-probability [unlikely] outcome.

Comparison the calculated value K_{calc} of the test K with its critical value K_{crit} can give rise to two cases: $K_{calc} > K_{crit}$ or $K_{calc} \le K_{crit}$.

In the first case we say that the results of trials **contradict the hypothesis** because of occurrence of the low-probability event. Therefore we can reject the hypothesis in question.

In the second case we say that the results of trials **don't contradict the hypothesis**, and we can accept it.

Remark 1. It's necessary to understand that we ascertain contradiction or non-contradiction of the results of trials to the hypothesis but **we don't state its validity or invalidity**.

Henceforth we'll study some frequently used goodness-of-fit tests.

Pearson*[1] χ^2 *goodness-of-fit test

Let's introduce the next random variable (so-called Pearson distribution; it's often called Pearson χ^2 goodness-of-fit test or simply χ^2 goodness-of-fit test)

$$\chi^2 = \sum_{i=1}^{l} \frac{(n_i - np_i)^2}{np_i} = \sum_{i=1}^{l} \frac{(n_i - n_i')^2}{n_i'} \quad (n_i' = np_i), \qquad (16)$$

[1] Pearson K. (Ch.) (1857 - 1936) - an English mathematician-stàtistícian, bìólogist and philosopher-pósitivist

where n is the number of trials on the random variable ξ (for example the size of a sample if the trials consist in sampling); the sense of the other quantities, namely the sense of l, n_i, p_i, depends on the form of a variation series which represents the results of trials.

a) In the case of a **discrete variation series** l is the number of different observed values of the random variable ξ, n_i is the number of occurrences of the value x_i of ξ, and p_i is the probability of occurrence of this value, $p_i = P(\xi = x_i)$. This latter is calculated on the base of the advanced hypothesis.

If for example we've hypothesized that a random variable ξ has Poisson distribution, then

$$p_i = P(\xi = x_i) = \frac{a_s^{x_i}}{(x_i)!} e^{-a_s},$$

where a_s is the point estimate of the parameter a (sample a).

b) In the case of an **interval variation series** l is the number of intervals, n_i is the number of values of ξ which have hit in the ith interval (that is the frequency of hittig of observed values of ξ in this interval), and p_i is the probability of hitting of ξ in this interval, $p_i = P(\xi \in [x_i, x_{i+1}))$, calculated by virtue of the hypothesis.

For example in the hypothesis of the normal distribution of a random variable ξ we can write

$$p_i = P(x_i < \xi < x_{i+1}) = \Phi(z_{i+1}) - \Phi(z_i), \qquad (17)$$

where

$$z_i = \frac{x_i - \overline{\xi_s}}{\sigma_s}, \; z_{i+1} = \frac{x_{i+1} - \overline{\xi_s}}{\sigma_s}. \qquad (18)$$

The law of Pearson distribution (16) is known: it approximately coincides with the χ^2 - distribution (see the Lecture No. 6, Point 5). The number k of degrees of freedom of this distribution is proved to be the next:

$$k = l - (s + 1), \qquad (19)$$

where s is the number of independent parameters which we estimate as to our random variable ξ.

Values of the parameter s for some known distributions are represented in the table 8.

For known number of degrees of freedom k and a small probability (a significance level) α there exists the **critical value** χ^2_{crit} of χ^2 such that

$$P(\chi^2 > \chi^2_{crit}) = \alpha . \qquad (20)$$

It can be found with the help of a corresponding table if we'll preset yourselves the significance level (see [4, 11]; the extract see in the table 9 hear).

Table 8. Values of the parameter s for some distributions

№	Hypothesis	Parameters to be estimated	The corresponding value of the parameter s
1	ξ has the normal distribution	The mathematical expectation and dispersion	2
2	ξ has Bernoulli distribution	The mathematical expectation $\Rightarrow D(\xi) = M(\xi)q$	1
3	ξ has Poisson distribution with the parameter a	The mathematical expectation $\Rightarrow D(\xi) = M(\xi) = a$	1
4	ξ has the exponential distribution with the parameter λ	The mathematical expectation $\Rightarrow \sigma(\xi) = M(\xi) = \frac{1}{\lambda}$	1
5	ξ has the uniform distribution on an interval with endpoints a and b	There aren't such the parameters because of $M(\xi) = \frac{a+b}{2}, \sigma(\xi) = \frac{b-a}{2\sqrt{3}}$ are known beforehand	0

Table 9. The critical values of the χ^2- distribution

A number of degrees of freedom		1	2	3	4	5	6	7	8	9	10
A significance level α	0.01	6.6	9.2	11.3	13.3	15.1	16.8	18.5	20.1	21.7	23.2
	0.025	5.0	7.4	9.4	11.1	12.8	14.4	16.0	17.5	19.0	20.5
	0.05	3.8	6.0	7.8	9.5	11.1	12.6	14.1	15.5	16.9	18.3

Application of the goodness-of-fit test χ^2 is fulfilled by the next plan.

1. At first on the base of advanced hypothesis we find approximate values of all the probabilities $p_i = P(\xi \in [x_i, x_{i+1}))$ and define the calculated value χ^2_{calc} of χ^2.

2. Knowing the number of degrees of freedom k and specifying some significance level (a small probability) α we find the critical value χ^2_{crit} of χ^2 from the table.

3. Now we compare χ^2_{calc} with χ^2_{crit}.

a) If $\chi^2_{\text{calc}} \le \chi^2_{crit}$, we say that results of trials don't contradict the hypothesis.

b) If $\chi^2_{\text{calc}} > \chi^2_{crit}$, we say that results of trials contradict the hypothesis (because of the event $\chi^2 > \chi^2_{crit}$, which we regarded as practically impossible, has occurred)

In the case a) we can accept the hypothesis, and in the case b) we can reject it.

Ex. 5. Test the hypothesis that the random variable ξ of the ***Basic example*** has the normal distribution.

1. By virtue of the hypothesis we find approximate values of the probabilities of hitting of the random variable in all the intervals of the interval variation series using the formulas (17), (18). Corresponding evaluations up to finding the calculated value $\chi^2_{\text{calc}} =$ 2,3 of χ^2 are represented in the table 10.

Table 10

Numbers of intervals	End-points of intervals		$z_i = \frac{x_i - \xi_s}{\sigma_s}$	$z_{i+1} = \frac{x_{i+1} - \xi_s}{\sigma_s}$	Values of Laplace function		$p_i = \Phi(z_{i+1}) - \Phi(z_i)$	$n'_i = np_i$	n_i	$\frac{(n_i - n'_i)^2}{n'_i}$
	x_i	x_{i+1}			$\Phi(z_i)$	$\Phi(z_{i+1})$				
1	70	75	-2.22	-1.27	-0.487	-0.398	0.089	18	18	0.00
2	75	80	-1.27	-0.31	-0. 398	-0.122	0.276	55	50	0.45
3	80	85	-0.31	0.64	-0.122	0.239	0.361	72	76	0.22
4	85	90	0.64	1.59	0.239	0.444	0.205	41	40	0.02
5	90	95	1.59	2.54	0.444	0.495	0.051	10	6	1.60
							$\sum$	196	200	$\chi^2_{calc} \approx 2.3$

2. Let's choose the significance level (a small probability) $\alpha = 0.05$. Finding by the formula (19) the number of degrees of freedom ($l = 5, s = 2$)

$$k = l - (s+1) = 5 - (2+1) = 2$$

we find the critical value $\chi^2_{crit} = 6.0$ of χ^2 from the table 9.

3. We've obtained χ^2_{calc} = 2,3 < 6.0 = χ^2_{crit}. It means that the results of trials on the random variable ξ in question don't contradict the hypothesis that its distribution law is normal one.

Romanovsky*[1] *goodness-of-fit test

Let's consider a random variable (Romanovsky goodness-of-fit test)

$$R = \frac{\chi^2_{calc} - k}{\sqrt{2k}} \qquad (21)$$

where χ^2_{calc} is the calculated value of Pearson χ^2 goodness-of-fit test (16) and k is the number (19) of degrees of freedom. If

$$R < 3, \qquad (22)$$

then the results of trials on a random variable ξ don't contradict the hypothesis to be tested.

Ex. 6. Test the hypothesis about the normal distribution of the random variable ξ of the ***Basic example.***

We have

$$\chi^2_{calc} = 2.3, k = 2$$

hence

$$R = \frac{2.3 - 2}{\sqrt{2 \cdot 2}} = 0.15 < 3,$$

and the results of trials don't contradict the hypothesis.

Kolmogorov*[2] *goodness-of-fit test

Let ξ is a continuous random variable, and we advance the next hypothesis: the given function $F(x)$ is the distribution function of this random variable.

Let a number D is the greatest value of the absolute value of a difference of the function $F(x)$ and statistical distribution function $F^*(x)$ that is

[1] Romanovsky, V.I. (1879 - 1954), a noted Soviet mathematician
[2] Kolmogorov, A.N. (1903 - 1987), an eminent Russian mathematician.

$$D = \max \left| F(x) - F^*(x) \right|. \quad (23)$$

Kolmogorov has proved that the probability of the inequality

$$D\sqrt{n} \geq \lambda$$

has a finite limit as $n \to \infty$. This limit is quite definite function $P(\lambda)$ of a parameter λ namely

$$\lim_{n\to\infty} P\left(D\sqrt{n} \geq \lambda\right) = P(\lambda). \quad (24)$$

The table 11 gives some values of the function $P(\lambda)$.

Table 11. The values of the function $P(\lambda)$

λ	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
$P(\lambda)$	0.270	0.172	0.112	0.068	0.040	0.022	0.012	0.006	0.003	0.002

The table 11 indicates decrease of the function $P(\lambda)$.

On the base of the formula (24) we can suppose that for sufficiently large *n* the next approximate equality

$$P\left(D\sqrt{n} \geq \lambda\right) = P(\lambda) \quad (25)$$

holds. It gives us the goodness-of-fit test in question.

We act in the next order.

1. At first we seek the calculated value D_{calc} of D and so the calculated value $\left(D\sqrt{n}\right)_{calc}$ of $D\sqrt{n}$. It possible to search out D_{calc} by simultaneous plotting the functions $F(x)$ and $F^*(x)$ if the function $F(x)$ is completely determined. But in practice there exist more simple approximate methods (see Ex. 7).

2. Prescribing [assigning] a significance level (a small probability) α we search out the critical value λ_{crit} of λ such that

$$P(\lambda_{crit}) = \alpha$$

or

$$P\left(D\sqrt{n} \geq \lambda_{crit}\right) = P(\lambda_{crit}) = \alpha. \quad (26)$$

3. Finally we compare $\left(D\sqrt{n}\right)_{calc}$ and λ_{crit}.

a) If $\left(D\sqrt{n}\right)_{calc} \geq \lambda_{crit}$, then the results of trials contradict the hypothesis because of occurrence of a highly improbable event.

b) In the case of $\left(D\sqrt{n}\right)_{calc} < \lambda_{crit}$ the results of trials don't contradict the hypothesis.

Remark 2. The table 11 doesn't contain values of λ for the values 0.05, 0.025, 0.01 of the significance level $\alpha = P(\lambda_{crit})$. They can be found approximately with the help of so-called linear interpolation method [or the method of proportional parts].

The method consists in approximate substitution of the function $P(\lambda)$ by that linear on a segment $[\lambda_1, \lambda_2]$. Compiling the equation of a straight line through two points $(\lambda_1; P(\lambda_1)), (\lambda_2; P(\lambda_2))$, namely

$$\frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} = \frac{P(\lambda) - P(\lambda_1)}{P(\lambda_2) - P(\lambda_1)},$$

we get

$$\frac{\lambda_{crit} - \lambda_1}{\lambda_2 - \lambda_1} = \frac{P(\lambda_{crit}) - P(\lambda_1)}{P(\lambda_2) - P(\lambda_1)},$$

hence

$$\lambda_{crit} = \lambda_1 + \frac{P(\lambda_{crit}) - P(\lambda_1)}{P(\lambda_2) - P(\lambda_1)} \cdot (\lambda_2 - \lambda_1) = \lambda_1 + \frac{\alpha - P(\lambda_1)}{P(\lambda_2) - P(\lambda_1)} \cdot (\lambda_2 - \lambda_1). \qquad (27)$$

Let for example $\alpha = P(\lambda_{crit}) = 0.05$. If we presume $\lambda_1 = 1.3,\ \lambda_2 = 1.4$, we'll obtain $P(\lambda_1) = P(1.3) = 0.068,\ P(\lambda_2) = P(1.4) = 0.040$ from the table 11 and then λ_{crit} by the formula (27)

$$\lambda_{crit} = 1.3 + \frac{0.05 - 0.068}{0.040 - 0.068} \cdot (1.4 - 1.3) = 1.36.$$

Analogously we get

$\lambda_{crit} = 1.48$ for $\alpha = P(\lambda_{crit}) = 0.025$ and $\lambda_{crit} = 1.63$ for $\alpha = P(\lambda_{crit}) = 0.01$.

Ex. 7. Test the hypothesis of the normal distribution of the random variable of the ***Basic example*** with the help of Kolmogorov goodness-of-fit test.

All necessary calculations for finding the calculated value D_{calc} of D we represent in the table 12 using the results obtained above (tables 6 and 10, Ex. 2). The values of the statistical distribution function $F^*(x)$ at the right end-points of the intervals were found in the example 2. By the same way we find approximate values of the distribution function $F(x)$. The value $D_{calc} = 0.03$ is situated in the seventh column of the table, and so $\left(D\sqrt{n}\right)_{calc} = 0.03 \cdot \sqrt{200} \approx 0.24$.

Table 12

Numbers of intervals	Intervals	p_i^*	$F^*(x)$	p_i	$F(x)$	$\lvert F(x) - F^*(x)\rvert$
1	[70, 75]	0.09	0.09	0.09	0.09	0.00
2	(75, 80]	0.25	0.09+0.25=0.34	0.28	0.09+0.28=0.37	0.03=D_{calc}
3	(80, 85]	0.38	0.34+0.38=0.72	0.36	0.37+0.36=0.73	0.01
4	(85, 90]	0.20	0.72+0.20=0.92	0.21	0.73+0.21=0.94	0.02
5	(90, 95]	0.08	0.92+0.08=1.00	0.05	0.94+0.05=0.99	0.01

1. If we choose the significance level $\alpha = P(\lambda_{crit}) = 0.04$, we'll get $\lambda_{crit} = 1.4$ from the table 11. Choosing $\alpha = P(\lambda_{crit}) = 0.05$ we obtain $\lambda_{crit} = 1.36$ by remark 2.
2. Comparing $\left(D\sqrt{n}\right)_{calc} = 0.24$ and $\lambda_{crit} = 1.4$ (or $\lambda_{crit} = 1.36$) we see that

$$\left(D\sqrt{n}\right)_{calc} = 0.24 < \lambda_{crit} = 1.4 \quad or \quad \left(D\sqrt{n}\right)_{calc} = 0.24 < \lambda_{crit} = 1.36.$$

Therefore the results of trials don't contradict the stated hypothesis.

MATHEMATICAL STATISTICS: basic terminology RUEFD

1. Варианта	Варіанта	Váriant	Variante	Variante *f*
2. Вариационный ряд	Варіаційний ряд	Vàriátion séries	Séries *f* variationnelle	Variationsreihe *f*
3. Выборка объема *n* (процесс)	Вибірка об"єму *n* (процес)	Sámpling of the size *n*	Echantillonnage *m* [sondage *m*, tirage *m*] de la taille *n*	Stichprobe *f* [Auswahl *f*, Stichprobenauswahl *f*, des Úmfang(e)s (*m*) *n*
4. Выборка объема *n* (результат)	Вибірка об"єму *n* (результат)	Sample of the size *n*	Echantillon *m* de la taille *n*	Stichprobe *f* [Auswahl *f*] des Úmfang(e)s (*m*) *n*
5. Выборочная асимметрия	Вибіркова асиметрія	Sample asýmmetry [skewness]	Asymétrie *f* d'échantillonnage	Stichprobenartige Asymmetrie *f*
6. Выборочная дисперсия	Вибіркова дисперсія	Sample váriance [dispérsion]	Variance *f* [dispersion *f*] d'échantillonnage	Stichprobensstreuung *f*, Stichprobenvarianz *f*
7. Выборочное среднее	Вибіркове середнє	Sample mean [áverage]	Moyenne *f* d'échantillonnage	Stichprobendurchschnitt *m*, Stichprobenmittel *n* Durchschnitt *m*, Mittel *n*)
8. Выборочное среднее квадратическое [среднеквадратическое] отклонение	Вибіркове середнє квадратичне відхилення	Sample root-méan-squáre [quadrátic méan, méan-squáre] dèviátion	Ecart *f* [é-] quadratique moyen, écart type [écart-type, écart quadratique] d'échantillonnage	Stichprobenartige durchschnittliche/mittlere quadratische Abweichung *f*, stichprobenartige Streuung *f*
9. Выборочный метод	Вибірковий метод	Sámpling (méthod)	Echantillonage *m*, sondage *m*	Stichprobenmethode *f*
10.Выборочный эксцесс	Вибірковий ексцес	Sample excéss	Excès *m* d'échantillonnage	Stichprobenartiger Exzeß *m*
11.Генеральная совокупность (объема *N*)	Генеральна сукупність (об"єму *N*)	Párent úniverse, (párent, géneral, grand, tótal) pòpulá-	Population *f* générale, univers *m* (de la taille *N*)	Grundgesamtheit *f* (des Úmfang(e)s (*m*) *N*)

		tion (of the size *N*)		
12.Гистограмма (относительных) частот	Гістограма (відносних) частот	Hístogram of (rélative) fréquencies	Histogramme *m* des fréquences *f* (relatives)	Histogramm *n* [Säulendiagramm *n*, Staffelbild *n*, Treppenpolygon *n*] von (relatíven) Frequénzen
13.Дискретный вариационный ряд	Дискретний варіаційний ряд	Discréte vàriátion séries	Séries *f* variationnelle discrète	Diskrete Variationsreihe *f* Reihe *f*
14.Доверительная вероятность (надежность)	Довірча ймовірність (надійність)	Cónfidence pròbabílity	Probabilité *f* de confiance	Konfidenzwahrscheinlichkeit *f*
15.Доверительный интервал	Довірчий інтервал	Cónfidence ínterval	Intervalle *m* de confiance	Konfidenzintervall *n*
16.Интервальная оценка	Інтервальна оцінка	Ínterval éstimate	Intervalle d´estimation *f*	Intervallschätzung *f*
17.Интервальный вариационный ряд	Інтервальний варіаційний ряд	Ínterval vàriátion séries	Séries *f* variationnelle aux intervalles	Intervallvariationsreihe *f* Intervallreihe *f*)
18.Исправленная дисперсия	Виправлена дисперсія	Corrécted váriance [dispérsion]	Variance *f* [dispersion *f*] corrigée	Korrigíerte Varianz *f* [Dispersion *f*, Streuung *f*]
19.Исправленное среднее квадратическое [среднеквадратическое] отклонение	Виправлене середнє квадратичне відхилення	Corrécted root-méan-squáre [quadrátic méan, méan-squáre] dèviátion	Ecart *f* [é-] quadratique moyen [écart type [écart-type, écart quadratique]] corrigé(e)	Korrigíerte durchschnittliche/mittlere quadratische Abweichung *f*, korrigíerte Streuung *f*
20.Количество степеней свободы	Кількість степенів свободи	Númber of degrées of fréedom	Nombre *m* de degrés de liberté	Ánzahl *f* der Freiheitsgrade
21.Критерий согласия	Критерій згоди	Goodness-of-fit test, goodness measure, goodness of fit, test for con-	Test d´accord	Anpassungstest *m*, Gute *f* der Obereinstimmung,Verträglichkeits-

		cordance, test of fit		kriterium *n*
22.Критерий согласия Колмогорова	Критерій згоди Колмогорова	Kolmogorov goodness-of-fit test	Test d´accord de Kolmogoroff	Anpassungs-test *m* von Kolmogorov
23.Критерий согласия Пирсона	Критерій згоди Пірсона	Pearson goodness-of-fit test	Test d´accord de Pearson	Anpassungs-test *m* von Pearson
24.Критерий согласия хи-квадрат	Критерій згоди хі-квадрат	chi-squared goodness-of-fit test	Test d´accord χ^2 (khi carré); critère *m* [test *m*] χ^2 (khi carré)	Chiquadrattest *m,* Chi-quadratverfahren *n*
25.Критическое значение критерия	Критичне значення критерія	Crítical válue of a test	Valeur *f* critique d'un test	Krítischer Wert *m* eines Test(e)s
26.Математическая статистика	Математична статистика	Mathematical statistics	Statistique *f* mathématique	Mathematische Statistik *f*
27.Метод обработки (данных, результатов)	Метод обробки (даних, результатів)	Méthod of data procéssing, of tréatment the resúlts	Méthode *f* de traitement *m* (des données, des résultats)	Auswertungs-methode *f*
28.Надежность [доверительная вероятность] оценки	Надійність [довірча ймовірність] оцінки	Reliabílity [depèndabílity] of an éstimate	Fiabilité *f* [sécurité *f*, confiance *f*] d´un estimateur	Sicherheit *f* [Verläßlichkeit *f*, Zuverlässigkeit *f*] einer Schätzung *f*
29.Не противоречить гипотезе (о результатах испытаний)	Не протирічити гіпотезі (про результати випробувать)	Do not cóntradict the hypothesis (about the resúlts of tríals)	Ne pas contredire l'hypothèse (sur les résultats d'expérience)	Der Hypothése nicht widerspréchen (über die Resultáte [Ergébnisse] von Experiménten)
30.Несмещенная оценка	Незміщена оцінка	Unbíased [bías-free] éstimate	Estimateur *m* sans biais [absolument correct, bien centré, sans distorsion]	Erwartungs-treue [bias-freie, nichtverzerrte, unverzerrte] Schätzung *f*, unverzerrte Abschätzung *f*

31.Несмещенность оценки	Незміщеність оцінки	Nòn-bías [ùnbíasedness] of an éstimate		
32.Определение закона распределения случайной величины	Визначення закону розподілу випадкової величини	Detèrminátion the dìstribútion [partítion] law of a rándom váriable	Détermination de la loi de répartition *f* [de distribution *f*] d'une variable aléatoire	Bestímmung *f* des Vertéilungsgesetzes (*n*) einer Zúfallsgröße
33.Отбросить [отклонить] гипотезу	Відкинути [відхилити] гіпотезу	Rejéct a hypóthesis	Rejeter une hypothèse	Eine Hypothése áblehnen
34.Относительная частота варианты	Відносна частота варіанти	Váriant rélative fréquency	Fréquence *f* relative d´une variante	Relatíve Frequénz *f* =,-en einer Variante *f*
35.Относительная частота попадания вариант в интервал	Відносна частота попадання варіант в інтервал	Rélative fréquency of hitting of váriants in an ínterval	Fréquence *f* relative du coup [d'impact *m*, d'atteinte *f*] des variantes dans un intervalle	Relatíve Frequénz *f* =, -en von Tréffer der Varianten *f* in ein Intervάll (*n*)
36.Оценивать, находить оценку	Оцінювати, знаходити оцінку	Éstimate	Estimer	Einschätzen, eine Schätzung *f* finden*
37.Оценка [нахождение оценки, оценивание] параметра закона распределения случайной величины (как процесс)	Оцінка [знаходження оцінки, оцінювання] параметра закону розподілу випадкової величини (*як процес*)	Èstimátion of a parámeter of a dìstríbutive law of a rándom váriable	Estimation *f* d´un paramètres de la loi de ré-partition *f* [de distribution *f*] d'une variable aléatoire	Parameter-schätzung *f* des Vertéilungsgesetzes (*n*) einer Zúfallsgröße
38.Оценка параметра закона распределения случайной величины (как результат)	Оцінка параметра закону розподілу випадкової величини (*як результат*)	Éstimate [éstimator; asséssed/éstimated válue] of a parámeter of the dìstríbutive law of a rándom váriable	Estimateur *m* [estimation *f*, valeur estimée] d´un paramètre de la loi de répartition *f* [de distribution *f*] d'une variable	Parameter-schätzung *f* des Vertéilungsgesetzes (*n*) einer Zúfallsgröße

			aléatoire	
39.Полигон (относительных) частот	Полігон (відносних) частот	Pólygon of (rélative) fréquencies	Polygone *m* des fréquences *f* (relatives)	Polygon *n* von (relatíven) Frequénzen
40.Приближенный график плотности распределения	Наближений графік щільності розподілу	Appróximate graph of the dístribútion density	Graph *m* [graphique *m*] approximatif de la densité de distribution	Angenäherte Gráphik *f* [angenähertes Bild] der Vertéilungsdichte (*f*)
41.Приближенный закон распределения случайной величины	Наближений закон розподілу випадкової величини	Appróximate distríbution law of a rándom váriable	Loi *f* approximative de distribution *f* [de répartition *f*] d'une variable aléatoire	Angenähertes Vertéilungsgesetz (*n*) einer Zúfallsgröße (*f*)
42.Принять гипотезу	Прийняти гіпотезу	Accépt a hypóthesis	Accepter une hypothèse	Eine Hypothése akzeptieren
43.Проверить статистическую гипотезу	Перевірити статистичну гіпотезу	Test [check, vérify] a statistical hypothesis, réalize statístical tésting	Tester [vérifier, contrôler] une hypothèse en statistique	Statistische Hypothese *f* kontrollieren [prüfen, testen, überprüfen, überwachen]
44.Проверка статистической гипотезы	Перевірка статистичної гіпотези	statistical hypothesis checking, statistical testing, statistical hypothesis test, proof of statistical hypothesis	test *m* d'hypothèse en statistique	Statistischer Hypothesenprüfung *f*, Prüfung *f* von statistischer Hypothese
45.Производить выборку объема *n*	Робити вибірку об"єму *n*	Sample, carry out [fulfíll, accómplish] a sámpling of the size *n*	Effectuer echantillonnage *m* [sondage *m*, tirage *m*] de la taille *n*	Stichprobe *f* [Auswahl *f*] des Úmfang(e)s *n* erzeugen
46.Противоречить гипотезе (о результатах испытаний)	Протирічити гіпотезі (про результати випробувать)	Cóntradict the hypóthesis (about the resúlts of tríals)	Contredire l'hypothèse (sur les résultats d'expérience)	Der Hypothése widerspréchen (über die Resultáte [Ergébnisse] von Experiménten)
47.Распреде-	Розподіл Сть-	Student´s dìs-	Loi *f* de Stu-	Studentvertei-

ление Стьюдента (*t*-распределение)	юдента [*t*-розподіл]	tribútion, (Student´s) t-dìstribútion	dent	lung *f*, Studentsche Verteilung, *t*–Verteilung
48.Распределение хи-квадрат	Розподіл хі-квадрат	χ^2-dìstribútion, chi-squared [chi-square] dìstribútion	Distribution *f* de χ^2 (khi carré)	χ^2-Verteilung *f* , Chi-quadratverteilung *f*
49.Распространить результаты исследования выборки на генеральную совокупность	Поширити [перенести] результати дослідження вибірки на генеральну сукупність	Tránsfer [carry over, exténd] resúlts of invéstigation of the sample on the párent úniverse	Etendre (il étends, n. atendons) les résultats d'étude d´échantillon sur l'univers	Resultáte [Ergébnisse] der Stichprobenuntersuchung auf die Grundgesamtheit *f* dilatieren
50.Расчетное значение критерия	Розрахункове значення критерія	Ráted [cálculated] válue of a test	Valeur *f* calculée d'un test	Réchnungsgröße *f*
51.Репрезентативная выборка	Репрезентативна вибірка	Rèpreséntative sample	Echantillon *m* représentatif	Repräsentative Stichprobe *f* [Auswahl *f*]
52.Система прямоугольников с основаниями… и площадями…	Система прямокутників з основами … і площами …	Sýstem of réctangles with bases … and the areas …	Système *m* des rectangles des bases … et des aires …	Rechteckssystem mit den Basisen *f pl*… und den Flächeninhalten *m pl* …
53.Случайная выборка	Випадкова вибірка	Rándom sample	Echantillon *m* aléatoire [probabiliste]	Zufällige Stichprobe *f* [Auswahl *f*]
54.Смещенная оценка	Зміщена оцінка	Bíased [shifted] éstimate	Estimateur *m* avec biais [mal centré, biaisé]	Nichterwartungstreue Schätzung *f*
55.Состоятельная оценка	Обгрунтована оцінка	Consístent éstimate	Estimateur *m* convergent [correct]	Konsistente Schätzung *f*
56.Состоятельность оценки	Обгрунтованість оцінки	Consístency of an éstimate	Consistance *f* d´un estimateur	Konsisténz *f* = einer Schätzung *f*
57.Сплошное исследование выборки	Суцільне дослідження вибірки	Tótal invéstigation of a sample	Etude *f* totale d´un échantillon *m*	Gesamte Stichprobenuntersuchung *f*
58.Среднее	Середнє ари-	Àrithmétical	Moyenne *f*	Arithmétisches

арифметическое (полученных значений случайной величины)	фметичне (спостережених значень випадкової величини)	áverage/méan [àrithmétical méan válue] (of obsérved values of a rándom váriable	arithmétique (de valeurs *f* observées d'une variable *f* aléatoire)	Míttel (*n* –s,=) von Beobachtungswerten (*m*) einer Zúfallsgröße (*f* =,-n)
59.Стандарт	Стандарт	Stándard	Standard *m*	Stándard *m* –s,-s
60.Статистическая (эмпирическая) функция распределения	Статистична (емпірична) функція розподілу	Statistic(al)/ empíric(al) dìstribútion fúnction	Fonction statistique/empirique de distribution	Statistische [empirische] Verteilungsfunktion *f*
61.Статистическая оценка параметра (как результат)	Статистична оцінка параметра (*як результат*)	Statístic(al) éstimate [éstimator; asséssed/éstimated válue] of a parámeter	Estimateur *m* [estimation *f*, valeur estimée] statistique d´un paramètre	Statistische Parameterschätzung *f*
62.Статистический ряд	Статистичний ряд	Statístic(al) séries	Séries *f* statistique	Statistische Reihe *f*
63.Стремиться по вероятности к точному значению (параметра)	Прямувати за ймовірністю до точного значення (параметра)	Tend in pròbabílity to exact válue of a parámeter	Tendre (il/elle tend) en probabilité vers une valeur *f* exacte (d´un paramètre)	Gegen genauen Wert *m* (eines Parámeters (*m*)) in Wahrschéinlichkeit konvergíeren [streben]
64.Сходиться по вероятности к теоретической функции распределения (о статистической функции распределения)	Збігатися за ймовірністю до теоретичної функції розподілу (про статистичну [емпіричну] функцію розподілу)	Convérge in pròbabílity to the theoretic(al) dìstribútion fúnction (of a statistic(al)/empíric(al) dìstribútion fúnction)	Converger en probabilité vers la fonction théorique de distribution (sur une fonction statistique/ empirique de distribution)	Gegen theorétische Ver-teilungsfunktion *f* in Wahrschéinlichkeit konvergíeren (über eine statistische/empirische Verteilungsfunktion *f*)
65.Теоретическая функция распределения	Теоретична функція розподілу	Thèorétic(al) distribútion fúnction	Fonction théorique de distribution	Theorétische Verteilungsfunktion *f*
66.Типичные	Типові задачі	Týpical prób-	Problèmes ty-	Typische Áuf-

задачи математической статистики	математичної статистики	lems of mathematical statistics	piques de statistique *f* mathématique	gaben (*f*) [Probléme (*n*)] der mathematischen Statistik *f*
67.Точечная оценка	Точкова оцінка	Póint éstimate	Estimation *f* ponctuelle	Punktschätzung *f*
68.Точное значение (параметра)	Точне значення (параметра)	Exact válue (of a parámeter)	Valeur *f* exacte (d´un paramètre)	Genauer Wert *m* (eines Parámeters (*m*))
69.Точность оценки	Точність оцінки	Exáctness [áccuracy, pricísion] of an éstimate	Exactitude *f* [précision *f*] d´un estimateur	Exáctheit *f* [Genauigkeit *f*, Präzision *f*] einer Schätzung *f*
70.Требования к оценке	Вимоги до оцінки	Requíerments [demánds] to an éstimate	Demandes *f pl* [réquisitions *f pl*] à un estimateur *m*	(An)forderungen *f pl* gégen eine Schätzung *f*
71.Уровень значимости	Рівень значущості	Signíficance level	Niveau *m* [seuil *m*] de signification	Signifikanzniveau *n*
72.Частота варианты	Частота варіанти	Váriant fréquency	Fréquence *f* d´une variante	Frequénz *f* =,-en einer Variante *f*
73.Частота попадания вариант на интервал	Частота попадання варіант на інтервал	Fréquency of hitting of váriants on an ínterval	Fréquence *f* du coup [d'impact *m*, d'atteinte *f*] des variantes sur un intervalle	Frequénz *f* =,-en von Tréffer der Varianten *f* in ein Interváll (*n*)
74.Эффективная оценка	Ефективна оцінка	Efficient éstimate [éstimator]	Estimateur *m* efficient [efficace] [efisjã, efikas]	Effiziente [effektive, leistungsfähige, wirksame, wirksamste] Schätzung *f*
75.Эффективность оценки	Ефективність оцінки	Efficiency [efféctiveness] of an èstimáte	Effectivité *f* [efficacité *f*] [efɛ-, efi-] d´un estimateur	Effektivität *f* [Leistungsfähigkeit *f*, Wírksamkeit *f*] der Schätzung

MATHEMATICAL STATISTICS: basic terminology EFDRU

1. accépt a hypóthesis	accepter une hypothèse	eine Hypothése akzeptieren	принять гипотезу	прийняти гіпотезу
2. appróximate dìstríbution law of a rándom váriable	loi *f* approximative de distribution *f* [de répartition *f*] d'une variable aléatoire	angenähertes Vertéilungsgesetz *n* einer Zúfallsgröße	приближенный закон распределения случайной величины	наближений закон розподілу випадкової величини
3. appróximate graph of the dístribútion density	graph *m* [graphique *m*] approximatif de la densité de distribution	angenäherte Gráphik *f* [angenähertes Bild *n*] der Vertéilungsdichte	приближенный график плотности распределения	наближений графік щільності розподілу
4. àrithmétical áverage/ méan [àrithmétical méan válue] (of obsérved values of a rándom váriable	moyenne *f* arithmétique (de valeurs *f* observées d'une variable *f* aléatoire)	arithmétisches Míttel *n* von Beobachtungswerten einer Zúfallsgröße	среднее арифметическое (полученных значений случайной величины)	середнє арифметичне (спостережених значень випадкової величини)
5. bíased [shifted] éstimate	estimateur *m* avec biais [mal centré, biaisé]	nichterwartungstreue Schätzung *f*	смещенная оценка	зміщена оцінка
6. chi-squared goodness-of-fit test	test d´accord χ^2 (khi carré); critère *m* [test *m*] χ^2 (khi carré)	Chiquadrattest *m,* Chi-quadratverfahren *n*	критерий согласия хи-квадрат	критерій згоди хі-квадрат
7. cónfidence ínterval	intervalle *m* de confiance	Konfidenzintervall *n*	доверительный интервал	довірчий інтервал
8. cónfidence pròbabílity	probabilité *f* de confiance	Konfidenzwahrscheinlichkeit *f*	доверительная вероятность (надежность)	довірча ймовірність (надійність)
9. consístency of an éstimate	consistance *f* d´un estimateur	Konsisténz *f* einer Schätzung	состоятельность оценки	обґрунтованість оцінки
10.consístent éstimate	estimateur *m* convergent [correct]	konsistente Schätzung *f*	состоятельная оценка	обґрунтована оцінка

11.cóntradict the hypóthesis (about the resúlts of tríals)	contredire l'hypothèse (sur les résultats d'expérience)	der Hypothése widerspréchen (über die Resultáte [Ergébnisse] von Experiménten)	противоречить гипотезе (о результатах испытаний)	протирічити гіпотезі (про результати випробувать)
12.convérge in pròbabílity to the theoretic(al) dìstribútion fúnction (of a statistic(al)/empíric(al) dìstribútion fúnction)	converger en probabilité vers la fonction théorique de distribution (sur une fonction statistique/ empirique de distribution)	gegen theorétische Verteilungsfunktion in Wahrschéinlichkeit konvergíeren (über eine statistische/empirische Verteilungsfunktion)	сходиться по вероятности к теоретической функции распределения (о статистической функции распределения)	збігатися за ймовірністю до теоретичної функції розподілу (про статистичну [емпіричну] функцію розподілу)
13.corréctеd root-méan-squáre [quadrátic méan, méan-squáre] dèviátion	ecart *f* [é-] quadratique moyen [écart type [écart-type, écart quadratique]] corrigé(e)	korrigíerte durchschnittliche/mittlere quadratische Abweichung *f*, korrigíerte Streuung *f*	исправленное среднее квадратическое [среднеквадратическое] отклонение	виправлене середнє квадратичне відхилення
14.corréctеd váriance [dispérsion]	variance *f* [dispersion *f*] corrigée	korrigíerte Varianz *f* [Dispersion *f*, Streuung *f*]	исправленная дисперсия	виправлена дисперсія
15.crítical válue of a test	valeur *f* critique d'un test	krítischer Wert eines Test(e)s	критическое значение критерия	критичне значення критерія
16.detèrminátion the dìstribútion [partítion] law of a rándom váriable	détermination de la loi de répartition *f* [de distribution *f*] d'une variable aléatoire	Bestímmung *f* des Vertéilungsgesetzes einer Zúfallsgröße	определение закона распределения случайной величины	визначення закону розподілу випадкової величини
17.discréte vàriátion séries	séries *f* variationnelle discrète	diskrete Variationsreihe *f*	дискретный вариационный ряд	дискретний варіаційний ряд
18. χ^2-dìstribútion, chi-squared [chi-square] dìstribútion	distribution *f* de χ^2 (khi carré)	χ^2-Verteilung *f* , Chi-quadratverteilung *f*	распределение хи-квадрат	розподіл хі-квадрат

19.do not cóntradict the hypothesis (about the resúlts of tríals)	ne pas contredire l'hypothèse (sur les résultats d'expérience)	der Hypothése nicht widerspréchen (über die Resultáte [Ergébnisse] von Experiménten)	не противоречить гипотезе (о результатах испытаний)	не протирічити гіпотезі (про результати випробувать)
20.efficiency [efféctiveness] of an èstimáte	effectivité *f* [efɛ-] [efficacité *f*] d´un estimateur	Effektivität *f* [Leistungsfähigkeit *f*, Wírksamkeit *f*] der Schätzung	эффективность оценки	ефективність оцінки
21.efficient éstimate [éstimator]	estimateur *m* efficient [efisjã] [efficace [efikas]]	effiziente [effektive, leistungsfähige, wirksame, wirksamste] Schätzung *f*	эффективная оценка	ефективна оцінка
22.éstimate	estimer	Einschätzen, eine Schätzung *f* finden*	оценивать, находить оценку	оцінювати, знаходити оцінку
23.éstimate [éstimator; asséssed/éstimated válue] of a parámeter of the dìstríbutive law of a rándom váriable	estimateur *m* [estimation *f*, valeur estimée] d´un paramètre de la loi de répartition *f* [de distribution *f*] d'une variable aléatoire	Parameterschätzung *f* des Vertéilungsgesetzes einer Zúfallsgröße	оценка параметра закона распределения случайной величины (как результат)	оцінка параметра закону розподілу випадкової величини (*як результат*)
24.èstimátion of a parámeter of a dìstríbutive law of a rándom váriable	estimation *f* d´un paramètres de la loi de répartition *f* [de distribution *f*] d'une variable aléatoire	Parameterschätzung *f* des Vertéilungsgesetzes (*n*) einer Zúfallsgröße	оценка [нахождение оценки, оценивание] параметра закона распределения случайной величины (как процесс)	оцінка [знаходження оцінки, оцінювання] параметра закону розподілу випадкової величини (*як процес*)
25.exact válue (of a parámeter)	valeur *f* exacte (d´un paramètre)	genauer Wert *m* (eines Parámeters (*m*))	точное значение (параметра)	точне значення (параметра)
26.exáctness	exactitude *f*	Exáctheit *f*	точность	точність оцін-

[áccuracy, pricísion] of an éstimate	[précision *f*] d´un estimateur	[Genauigkeit *f*, Präzision *f*] einer Schätzung	оценки	ки
27.fréquency of hitting of váriants on an ínterval	fréquence *f* du coup [d'impact *m*, d'atteinte *f*] des variantes sur un intervalle	Frequénz *f* von Tréffer der Varianten in ein Interváll	частота попадания вариант на интервал	частота попадання варіант на інтервал
28.goodness-of-fit test, goodness measure, goodness of fit, test for concordance, test of fit	test d´accord	Anpassungstest *m*, Gute *f* der Obereinstimmung,Verträglichkeitskriterium *n*	критерий согласия	критерій згоди
29.hístogram of (rélative) fréquencies	histogramme *m* des fréquences *f* (relatives)	Histogramm *n* [Säulendiagramm *n*, Staffelbild *n*, Treppenpolygon *n*] von (relatíven) Frequénzen	гистограмма (относительных) частот	гістограма (відносних) частот
30.ínterval éstimate	intervalle d´estimation *f*	Intervallschätzung *f*	интервальная оценка	інтервальна оцінка
31.ínterval vàriátion séries	séries *f* variationnelle aux intervalles	Intervallvariationsreihe *f* Intervallreihe *f*)	интервальный вариационный ряд	інтервальний варіаційний ряд
32.Kolmogorov goodness-of-fit test	test d´accord de Kolmogoroff	Anpassungstest *m* von Kolmogorov	критерий согласия Колмогорова	критерій згоди Колмогорова
33.mathematical statistics	statistique *f* mathématique	mathematische Statistik *f*	математическая статистика	математична статистика
34.méthod of data procéssing, of tréatment the resúlts	méthode *f* de traitement *m* (des données, des résultats)	Auswertungsmethode *f*	метод обработки (данных, результатов)	метод обробки (даних, результатів)
35.nòn-bías [ùnbíasedness] of an éstimate			несмещенность оценки	незміщеність оцінки
36.númber of	nombre *m* de	Ánzahl *f* der	количество	кількість сте-

degrées of fréedom	degrés de liberté	Freiheitsgrade	степеней свободы	пенів свободи
37.párent úniverse, (párent, géneral, grand, tótal) pòpulátion (of the size *n*)	population *f* générale, univers *m* (de la taille *n*)	Grundgesamtheit *f* (des Úmfang(e)s *N*)	генеральная совокупность (объема *n*)	генеральна сукупність (об"єму *n*)
38.Pearson goodness-of-fit test	test d´accord de Pearson	Anpassungstest *m* von Pearson	критерий согласия Пирсона	критерій згоди Пірсона
39.póint éstimate	estimation *f* ponctuelle	Punktschätzung *f*	точечная оценка	точкова оцінка
40.pólygon of (rélative) fréquencies	polygone *m* des fréquences *f* (relatives)	Polygon *n* von (relatíven) Frequénzen	полигон (относительных) частот	полігон (відносних) частот
41.rándom sample	echantillon *m* aléatoire [probabiliste]	zufällige Stichprobe *f* [Auswahl *f*]	случайная выборка	випадкова вибірка
42.rated [cálculated] válue of a test	valeur *f* calculée d'un test	Réchnungsgrößе *f*	расчетное значение критерия	розрахункове значення критерія
43.rejéct a hypóthesis	rejeter une hypothèse	eine Hypothése áblehnen	отбросить [отклонить] гипотезу	відкинути [відхилити] гіпотезу
44.rélative fréquency of hitting of váriants in an ínterval	fréquence *f* relative du coup [d'impact *m*, d'atteinte *f*] des variantes dans un intervalle	relatíve Frequénz *f* von Tréffer der Varianten in ein Intervåll	относительная частота попадания вариант в интервал	відносна частота попадання варіант в інтервал
45.relìabílity [depèndabílity] of an éstimate	fiabilité *f* [sécurité *f*, confiance *f*] d´un estimateur	Sicherheit *f* [Verläßlichkeit *f*, Zuverlässigkeit *f*] einer Schätzung	надежность [доверительная вероятность] оценки	надійність [довірча ймовірність] оцінки
46.rèpreséntative sample	echantillon *m* représentatif	repräsentative Stichprobe *f* [Auswahl *f*]	репрезентативная выборка	репрезентативна вибірка
47.requírements [demánds] to an éstimate	demandes *f pl* [réquisitions *f pl*] à un estimateur *m*	(An)forderungen *f pl* gégen eine Schätzung	требования к оценке	вимоги до оцінки

48.sample root-méan-squáre [quad-rátic méan, méan-squáre] dèviátion	ecart *f* [é-] quadratique moyen, écart type [écart-type, écart quadratique] d'échantillonnage	stichprobenartige durchschnittliche/mittlere quadratische Abweichung *f*, stichprobenartige Streuung *f*	выборочное среднее квадратическое [среднеквадратическое] отклонение	вибіркове середнє квадратичне відхилення
49.sample asýmmetry [skewness]	asymétrie *f* d'échantillonnage	stichprobenartige Asymmetrie *f*	выборочная асимметрия	вибіркова асиметрія
50.sample excéss	excès *m* d'échantillonnage	stichprobenartiger Exzeß *m*	выборочный эксцесс	вибірковий ексцес
51.sample mean [áverage]	moyenne *f* d'échantillonnage	stichprobendurchschnitt *m*, Stichprobenmittel *n* Durchschnitt *m*, Mittel *n*)	выборочное среднее	вибіркове середнє
*52.*sample of the size *n*	echantillon *m* de la taille *n*	Stichprobe *f* [Auswahl *f*] des Úmfang(e)s *n*	выборка объема *n* (результат)	вибірка об"єму *n* (результат)
53.sample váriance [dispérsion]	variance *f* [dispersion *f*] d'échantillonnage	Stichprobensstreuung *f*, Stichprobenvarianz *f*	выборочная дисперсия	вибіркова дисперсія
54.sample, carry out [fulfill, accómplish] a sámpling of the size *n*	effectuer echantillonnage *m* [sondage *m*, tirage *m*] de la taille *n*	Stichprobe *f* [Auswahl *f*] des Úmfang(e)s *n* erzeugen	производить выборку объема *n*	робити вибірку об"єму *n*
55.sámpling (méthod)	echantillonage *m*, sondage *m*	Stichprobenmethode *f*	выборочный метод	вибірковий метод
56.sámpling of the size *n*	echantillonnage *m* [sondage *m*, tirage *m*] de la taille *n*	Stichprobe *f* [Auswahl *f*, Stichprobenauswahl *f*] des Úmfang(e)s *n*	выборка объема *n* (процесс)	вибірка об"єму *n* (процес)
57.signíficance level	niveau *m* [seuil *m*] de signification	Signifikanzniveau *n*	уровень значимости	рівень значущості
58.stándard	standard *m*	Stándard *m*	стандарт	стандарт
59.statístic(al)	estimateur *m*	statistische Pa-	статистиче-	статистична

éstimate [éstimator; assésed/éstimated válue] of a parámeter	[estimation *f*, valeur estimée] statistique d´un paramètre	rameterschätzung *f*	ская оценка параметра (как результат)	оцінка параметра (*як результат*)
60.statístic(al) séries	séries *f* statistique	statistische Reihe *f*	статистический ряд	статистичний ряд
61.statistic(al)/ empíric(al) dìstribútion fúnction	fonction statistique/empirique de distribution	statistische [empirische] Verteilungsfunktion	статистическая (эмпирическая) функция распределения	статистична (емпірична) функція розподілу
62.statistical hypothesis checking, statistical testing, statistical hypothesis test, proof of statistical hypothesis	test *m* d'hypothèse en statistique	statistischer Hypothesenprüfung *f*, Prüfung *f* von statistischer Hypothese	проверка статистической гипотезы	перевірка статистичної гіпотези
63.Student´s dìstribútion, (Student´s) t-dìstribútion	loi *f* de Student	Studentverteilung *f*, Studentsche Verteilung, *t*–Verteilung	распределение Стьюдента (*t*-распределение)	розподіл Стьюдента [*t*-розподіл]
64.sýstem of réctangles with bases … and the areas …	système *m* des rectangles des bases … et des aires …	Rechteckssystem mit den Basisen *f pl*… und den Flächeninhalten…	система прямоугольников с основаниями… и площадями…	система прямокутників з основами … і площами …
65.tend in pròbabílity to exact válue of a parámeter	tendre (il/elle tend) en probabilité vers une valeur *f* exacte (d´un paramètre)	gegen genauen Wert *m* (eines Parámeters) in Wahrschéinlichkeit konvergíeren [streben]	стремиться по вероятности к точному значению (параметра)	прямувати за ймовірністю до точного значення (параметра)
66.test [check, vérify] a statistical hypothesis, réalize statístical tésting	tester [vérifier, contrôler] une hypothèse en statistique	statistische Hypothese *f* kontrollieren [prüfen, testen, überprüfen, überwachen]	проверить статистическую гипотезу	перевірити статистичну гіпотезу

67.thèorétic(al) dìstribútion fúnction	fonction théorique de distribution	theorétische Verteilungsfunktion *f*	теоретическая функция распределения	теоретична функція розподілу
68.tótal invéstigation of a sample	etude *f* totale d´un échantillon *m*	gesamte Stichprobenuntersuchung *f*	сплошное исследование выборки	суцільне дослідження вибірки
69.tránsfer [carry over, exténd] resúlts of invéstigation of the sample on the párent úniverse	etendre (il étends, n. atendons) les résultats d'étude d´échantillon sur l'univers	Resultáte [Ergébnisse] der Stichprobenuntersuchung auf die Grundgesamtheit *f* dilatieren	распространить результаты исследования выборки на генеральную совокупность	поширити [перенести] результати дослідження вибірки на генеральну сукупність
70.týpical próblems of mathematical statistics	problèmes typiques de statistique *f* mathématique	typische Áufgaben (*f*) [Probléme (*n*)] der mathematischen Statistik	типичные задачи математической статистики	типові задачі математичної статистики
71.unbíased [bías-free] éstimate	estimateur *m* sans biais [absolument correct, bien centré, sans distorsion]	erwartungstreue [biasfreie, nichtverzerrte, unverzerrte] Schätzung *f*, unverzerrte Abschätzung *f*	несмещенная оценка	незміщена оцінка
72.váriant	variante	Variante *f*	варианта	варіанта
73.váriant fréquency	fréquence *f* d´une variante	Frequénz *f* einer Variante *f*	частота варианты	частота варіанти
74.váriant rélative fréquency	fréquence *f* relative d´une variante	relatíve Frequénz *f* einer Variante	относительная частота варианты	відносна частота варіанти
75.vàriátion séries	séries *f* variationnelle	Variationsreihe *f*	вариационный ряд	варіаційний ряд

Tasks for individual work on mathematical statistics

PROBLEM. 100 independent trials are fulfilled on a random variable ξ, and the results of trials are represented by corresponding sample of the size n = 100.

1. Compile the interval variation series for the random variable ξ, plot the **histogram of relative frequencies** and approximate graph of the **distribution density**. Find the **statistical distribution function** for the interval variation series and construct its approximate graph.

2. On the base of the interval variation series form the **discreet variation series** by taking inner points in each interval. Construct the **polygon of relative frequencies**. Form the **statistical distribution function** for the discreet variation series and plot its graph.

3. Calculate the sample mean [the sample average], dispersion, root-mean-square deviation, asymmetry and excess of the random variable. Draw a conclusion as to symmetry of its distribution law and deviation of this law from the normal distribution.

4. Find the corrected dispersion and root-mean-square deviation of the random variable. Compare their values with corresponding sample estimators.

5. Search out the confidence intervals for the mathematical expectation of the random variable ξ with reliabilities 0.95 and 0.99, basing on the hypothesis of its normal distribution (see a corresponding table in the books [4, 11], appendices 3).

6. Find approximate values of the probabilities of hitting of the random variable ξ on all the intervals of its variation series proceeding from the hypothesis of the normal distribution of ξ.

7. Test the hypothesis on the normal distribution of the random variable ξ with the significance levels $\alpha = 0.01, \alpha = 0.025, \alpha = 0.05$ making use of Pearson χ^2 (see a corresponding table in the books [4, 11], appendices 5), Romanovsky and Kolmogorov goodness-of-fit tests.

Variations of tasks

Variant 1

0,89	1,33	2,81	0,64	2,67	2,13	4,27	5,48	2,22	4,11
2,54	0,33	3,69	0,84	2,96	3,63	2,44	4,83	4,12	1,89
1,80	1,13	1,28	1,98	2,33	3,89	2,53	1,67	3,70	3,22
1.17	0,71	0,71	2,67	2,97	3,48	2,87	3,61	2,20	0,35
0,81	0,50	1,41	1,72	2,46	2,09	3,47	2,94	4,38	1,82
1,43	1,74	1,28	1,50	5,16	3,13	4,92	3,18	2,41	1,79
1,98	1,69	2,14	1,80	2,58	3,06	2,07	3,83	2,56	3,65
0,71	1,83	0,48	3,72	5,84	2,72	2,10	3,39	2.19	1,47
0,62	1,28	0,96	0,81	2,93	2,38	2,12	3,77	2,84	1,61
3.10	0,89	3,84	1,61	2,75	3,01	0,45	0,48	2,95	2,77

Variant 2

5,48	6,24	10,05	11,18	4,29	6,28	12,23	6,22	13,90	8,57
1.18	3,05	7,28	12,70	9,07	5,05	6,83	10,70	4,88	10,10
5,77	4,21	6,57	11,54	5,24	11,30	8,06	8,07	11,40	9,45
2,91	1,07	6,54	8,18	4,37	7,94	6,29	6,18	6,01	6,29
3,77	2,01	7,21	6,92	6,01	11,97	4,64	9,39	6,98	13,70
2,54	4,10	8,02	6,72	8,21	5,11	6,21	7,39	10,15	14,10
1,98	2,40	5,11	9,74	5,97	4,49	6,12	5,17	6,30	8,29
3,83	2,21	6,09	8,23	10,20	9,11	8,99	8,50	4,94	8,62
2,23	2,97	1,57	2,89	1,32	3,86	4,11	6,23	4,51	5,18
5,58	0,98	3,17	1,05	1,69	6,38	4,00	3,47	4,23	6,03

Variant 3

53,9	82,6	56,6	20,4	52,8	23,8	37,4	36,0	31,3	31,9
71,4	70,5	45,2	18,5	48,2	29,0	20.0	31,8	31,8	33,4
72.8	77,5	45,6	17,0	32,8	23,2	32,3	24,2	32,1	31,0
76,6	71,8	42,3	37,2	23,9	33,2	35,0	45,3	28,9	61,5
47,6	47,6	49,5	49,5	45,0	28,5	47,9	27,7	27,6	43,2
41.7	35,4	51,6	41,8	53,1	44,9	16,5	30,9	38,3	41,5
69.9	63,0	64,6	46,1	57,5	25,1	37,3	31,5	34,1	57,4
74,1	53,5	44,3	28,3	56,0	32,1	43,8	29,3	36,5	53,1
68,1	59,4	41,1	51,0	19,8	30,7	58,0	16,9	34,9	59,3
79,3	34,2	69,3	49,0	44,7	22,5	35,0	33,2	35,9	63,6

Variant 4

12,8	12,0	12,4	2,2	12,5	13,9	15,4	11,1	18,5	16,6
4.8	10,8	13,5	16,8	23,4	5,2	6,0	16,4	10,9	8,0
9.0	13,1	7,5	15,6	8,8	18,3	24,3	10,7	15,1	16,2
10,2	14,3	3,2	9,6	14,2	6,5	7,0	17,9	9,4	11,4
19,4	10,6	20,5	14,1	12,2	8,1	11,0	12,7	16,6	15,8
7.1	19,0	9,8	13,0	11,6	3,5	8,5	12,3	9,3	12,4
12.4	6,3	12,9	12,3	18,7	10,7	12,8	8,0	11,7	19,1
12,6	18,8	16,2	9,3	22,2	11,9	14,0	10,4	6,5	10,7
7.9	15,8	12,2	20,5	16,1	8,1	12,5	9,5	14,8	4,7
17,8	9,2	12,7	20,8	14,7	13,1	11,8	7,6	18,4	12,2

Variant 5

13,13	15,35	12,49	17,01	15,33	16,59	12,05	15,93	13,75	17,32
13,63	13,84	10,98	17,39	16,58	10,65	9,35	19,51	14,18	11,98
19.05	14,28	14,08	11,00	10,84	10,55	16,01	12,62	14,55	15,79
11,68	14,56	12,33	16,19	14,90	11,26	10,78	14,87	16,15	12,11
12,74	14,70	14,83	9,02	12,57	16,18	12,90	9,05	13,68	16,54
14,79	14,77	9,18	16,03	19,62	11,17	10,66	13,63	14,97	17,91
18,30	16,07	18,43	14,57	14,67	16,91	10,02	14,95	15,13	12,13
11,95	18,59	13,74	11,05	16,05	13,87	10,74	14,99	12,04	17,23
18,13	13,96	18,89	16,12	14,79	17,43	14,73	17,17	14,89	19,40
12,88	12,45	12,89	12,79	12,85	14,51	12,89	12,91	17,51	10,16

Variant 6

4,00	2,45	1,83	1,15	2,91	1,91	1,84	1,79	1,95	1,36
1,17	2,48	3,60	1,54	3,61	1,53	1,54	2,04	1,04	1,34
1,16	2,35	1,36	1,78	2,68	1,37	2,16	2,02	2,48	1,21
1,08	1,93	3,50	1,69	2,89	1,29	1,88	2,37	1,87	1,34
1,13	1,99	2,08	1,94	3,02	1,34	1,76	2,60	1,81	1,30
0,99	3,30	2,13	1,74	3,03	1,36	1,77	1,91	1,73	1,22
1,00	3,80	4,10	2,56	3,08	1,48	1,86	1,78	4,08	0,99
0,98	1,15	2,15	2,22	3,78	2,54	1,80	2,47	2,67	1,10
2,87	3,70	2,30	2,56	1,53	3,05	1,72	2,79	1,80	1,38
2,72	2,26	1,70	2,16.	1,72	1,94	2,05	2,01	1,51	1,86

Variant 7

2,61	1,56	1,84	2,00	2,11	2,63	2,98	3,29	2,15	2,15
1,58	1,56	2,03	2,80	2,34	2,93	2,73	3,40	2,16	2,16
1,54	1,81	3,60	2,04	3,07	2,95	2,66	3,43	2,20	2,37
2,35	1,55	3,61	2,46	3,54	2,74	2,59	2,50	2,30	2,27
2,68	2,05	1,79	2,38	2,34	2,81	3,52	2,59	2,03	2,24
2,88	1,47	2,47	2,64	2,32	2,50	3,15	2,56	2,15	2,34
1,68	1,86	1,81	1,81	2,06	2,98	3,05	2,66	2,23	2,42
1.78	2,06	3,31	2,51	2,88	2,56	2,57	2,72	2,21	2,11
1,75	2,51	3,15	1,87	3,05	2,96	2,85	3,12	3,24	2,27
1,58	2,85	2,11	1,86	3,57	2,93	3,28	2,22	2,23	2,35

Variant 8

1,15	1,90	1,69	1,43	1,75	1,38	1,96	1,79	1,70	1,51
1,53	1,93	1,95	1,76	1,69	1,47	1,64	1,22	1,86	1,41
1,05	1,78	1,87	1,34	1,90	1,73	1,83	1,42	1,80	1,91
1,06	1,35	1,26	1,56	1,55	1,13	1,58	1,19	1,31	1,77
1,21	1,11	1,51	1,08	1,45	1,85	1,55	1,65	1,59	1,26
1,32	1,52	1,44	1,68	1,36	1,90	1,46	1,32	1,42	1,87
1,98	1,45	1,18	1,42	1,94	1,57	1,27	1,46	1,62	1,99
1.41	1,24	1,53	1,16	1,56	1,43	1,74	1,09	1,75	1,39
1,87	1,53	1,30	1,54	1,64	1,55	1,39	1,60	1,52	1,64
1.79	1,67	1,78	1,25	1,49	1,70	1,69	1,53	1,29	1,60

Variant 9

1,98	3,73	8,63	3,68	4,79	8,53	3,89	9,59	7,05	1,06
2,57	5,28	2,22	9,14	7,88	8,36	2,88	9,13	11,87	5,32
5,70	4,52	3,42	4,03	6,37	1,13	7,26	10,85	3,87	1,21
3,30	2,04	1,84	6,89	7,81	10,82	11,63	4,85	9,28	1,27
2,93	7,81	7,13	7,85	8,01	1,08	1,14	5,76	10,01	6,26
3,24	4,93	2,18	8,37	4,76	8,60	8,47	1,28	3,65	11,51
1,85	1,99	3,60	4,75	6,90	5,32	6,34	6,00	9,23	4,02
4,68	1,47	1,61	3,17	11,34	9,37	4,10	6,10	4,44	10,50
3.42	3,20	1,90	1,01	8,26	6,11	4,91	9,39	5,88	4,68
1,61	2,33	11,44	11,62	6,07	9,54	4,96	9,18	6,11	6,98

Variant 10

1,49	1,44	1,68	2,48	3,93	2,34	7,00	1,56	0,46	1,14
0,60	3,57	1,34	1,59	4,30	0,74	1,62	2,13	2,22	3,63
1,26	0,86	2,39	3,05	2,38	0,80	1,24	0,76	1,65	0,92
2,34	0,64	1,26	4,32	0,68	0,92	0,69	1,77	1,02	1,07
2,69	2,02	3,42	1,19	2,66	1,00	1,82	3,04	3,92	1,44
1,63	3,41	1,16	1,44	0,45	2,41	0,87	0,81	2,85	1,94
1,25	1,90	3,72	2,05	2,38	1,80	2,88	2,02	1,26	1,11
0,54	0,94	1,71	1,52	1,38	1,32	1,01	0,79	1,71	0,99
0,78	0,99	1,60	2,07	2,17	1,47	0,82	1,95	0,28	2,36
2,01	1,51	0,95	3,17	1,08	1,09	2,43	1,88	2,64	4,80

Variant 11

3,89	1,33	2,81	1,04	0,67	2,13	4,97	5,48	2,22	2,71
2,54	1,13	3,69	0,84	0,98	1,63	1,44	4,83	2,82	1,89
1,80	1,13	1,28	0,98	1,33	0,89	1,53	1,67	2,71	3,22
1,17	1,11	3,71	2,67	4,97	3,48	4,80	4,61	2,20	4,25
0,41	1,50	1,41	1,72	1,46	2,09	3,17	4,14	1,38	1,82
1,43	1,74	1,26	1,50	2,16	1,13	4,12	4,18	2,41	1,79
1,98	1,69	2,14	1,80	1,58	1,06	2,07	3,33	2,56	3,05
3,71	1,83	0,81	2,72	0,84	0,70	2,10	3,39	2,19	1,47
0,62	3,78	0,66.	0,54	1,53	2,38	2,12	4,77	2,95	1,61
2,10	0,89	3,84	1,61	1,45	1,01	0,45	0,48	0,84	2,77

Variant 12

37,91	37,82	37,86	37,80	37,84	37,85	37,90	37,88	37,90	37,79
37.90	37,87	37,86	37,87	37,88	37,81	37,91	37,87	37,93	37,83
37,82	37,89	37,84	37,80	37,85	37,85	37,81	37,91	37,77	37,88
37,80	37,90	37,87	37,78	37,86	37,90	37,87	37,86	37,91	37,89
37.87	37,85	37,81	37,83	37,90	37,84	37,88	37,88	37,94	37,90
37.88	37,80	37,88	37,83	37,84	37,78	37,76	37,82	37,81	37,78
37,82	37,87	37,78	37,85	37,87	37,81	37,87	37,83	37,82	37,95
37,87	37,85	37,86	37,86	37,92	37,86	37,91	37,88	37,86	37,90
37.91	37,89	37,83	37,83	37,93	37,85	37,80	37,87	37,85	37,88
37,90	37,87	37,82	37,81	37,86	37,84	37,87	37,85	37,80	37,84

Variant 13

14,4	6,9	18,8	18,9	20,1	20,6	22,0	24,4	26,9	10,2
16.8	16,9	7,0	19,0	21,4	21,7	22,9	25,5	6,7	30,7
2.9	3,0	11,0	7,1	19,5	22,8	23,0	6,2	27,2	30,9
8,8	9,0	3,3	1,1	7,9	20,9	6,1	26,6	28,3	31,4
10.9	11,0	10,1	3,3	1,2	6,0	23,3	23,7	29,4	26,3
12,7	12,9	11,7	9,5	3,5	1,4	8,0	24,8	30,5	30,9
14,9	15,0	13,0	5,9	9,6	3,6	1,7	8,3	29,6	30,4
15,3	15,5	12,6	12,3	11,9	9,9	3,9	1,9	8,5	30,9
16,0	2,4	16,3	16,7	14,0	12,0	10,3	4,3	2,0	8,7
22,3	16,9	17,0	17,1	17,2	14,1	12,3	10,7	5,7	22,1

Variant 14

11,7	13,2	75,5	10,8	10,2	56,3	50,3	57,6	48,2	11,8
32.4	16,5	25,2'	8,9	15,7	63,5	19,3	63,2	63,8	36,8
17,7	25,8	12,5	21,7	12,2	23,9	41,3	26,6	22,8	77,5
7,8	13,9	12,3	15,6	39,0	83,5	65,0	32,3	39,1	33,7
27.7	12,9	61,5	15,3	83,6	33,1	16,1	14,8	44,4	26,4
8,5	21,3	75,6	47,4	46,0	12,4	23,6	27,6	40,7	39,1
24,8	17,9	27,3	60,7	37,4	60,7	22,5	87,8	22,8	34,2
58,4	78,8	11,7	45,6	28,1	73,2	53,0	41,6	24,9	48,1
15,9	10,5	12,0	11,1	42,0	50,2	22,3	49,5	94,2	21,6
84,7	74,5	14,1	48,5	93,8	49,0	67,8	37,1	39,1	6,6

Variant 15

1,55	2,15	0,80	2,40	1,35	1,60	1,15	1,50	2,35	1,65
0,95	1,25	1,00	1,50	1,75	2,10	1,35	0,70	1,15	1,95
0,75	1,60	1,50	0,95	1,00	1,10	1,10	1,90	1,40	1,15
2.10	1,40	2,10	1,15	0,70	1,05	0,35	2,25	1,70	1,40
1,05	2,05	1,30	1,30	1,95	1,75	1,20	1,50	0,95	1,75
1,30	1,50	1,20	0,60	1,55	2,15	0,90	1,45	1,50	1,90
1,10	1,10	2,35	1,20	0,70	1,20	2,40	2,10	1,95	1,20
1,45	2,10	0,90	1,45	1,35	1,50	1,70	1,95	1,55	1,85
0,75	1,10	1,75	0,80	1,90	1,80	2,00	1,35	0,65	1,15
0,90	1,55	1,35	1,75	1,70	1,40	1,30	1,55	0,10	1,35

Variant 16

10,09	9,64	11,19	9,23	9,68	11,58	10,68	11,16	11,47	12,16
9,27	9,19	13,76	12,33	11,75	13,38	9,77	11,71	9,91	11,47
10,36	11,15	10,23	9,76	12,69	10,58	11,72	10,73	10 ,98	12,56
8,38	12,18	11,08	10,35	9,95	9,56	10,52	8,74	13,88	11,71
9,18	9,15	10,49	9,55	10,49	12,33	9,82	12,65	8,26	11,92
10,49	10,10	12,42	10,51	10,71	10,50	9,37	9,57	10,12	9,28
11,28	11,48	7,73	9.23	10,64	9,76	9,31	10,05	13,31	9,75
9,96	8,75	11,86	10,28	10,31	10,42	11,85	11,22	10,34	10,21
11,23	11,43	10,05	10,22	10,45	10,22	9,16	11,76	10,36	10,47
11,13	10,75	10,95	10,79	11,24	13,74	11,13	10,52	10,69	11,57

Variant 17

31,8	54,8	46,4	28,0	35,8	27,7	23,6	84,3	29,3	58,7
33,8	52,8	42,9	38,1	43,5	18,0	22,6	58,7	18,3	67,5
24.7	41,1	42,6	39,4	55,3	35,3	22,7	68,0	20,2	73,1
26.8	44,0	29,5	46,1	40,5	29,3	31,7	58,2	86,8	21,9
18,3	59,0	49,2	42,3	42,6	29,7	59,0	80,1	67,9	77,0
36,5	27,7	31,5	33,2	60,4	36,2	26,3	$7,0	83,2	86,0
39,6	54,2	46,2	36,2	85,4	35,1	26,5	24,2	28,9	34,5
46.8	51,3	47,0	86,9	37,7	36,6	86,6	65,3	75,1	80,5
45,2	56,9	52,4	22,2,	50,7	36,3	67,8	27,7	59,5	21,5
46.9	55,2	35,5	45,1	68,1	32,0	29,6	60,5	74,7	38,4

Variant 18

2,15	2,32	1,82	2,91	2,60	3,77	3,95	2,44	2,86	2,98
2.00	1,99	1,93	2,63	2,44	4,06	3,75	3,53	2,92	2,73
2,37	2,05	2,03	2,67	2,59	2,22	3,02	3,33	3,92	2,96
2,31	1,82	3,12	2,61	2,79	2,31	3,24	2,80	2,22	2 93
2,40	3,81	2,45	2,28	2,48	2,19	3,50	3,23	2,15	3,04
3,60	1,76	3,13	2,50	2,33	2,17	3,04	3,40	2,16	3,15
2,58	1,77	2,96	2,46	2,36	2,14	3,20	3,43	2,32	3,05
2,47	1,80	2,81	2,10	2,64	2,11	2,69	2,50	2,27	2,87
2,40	1,68	3,01	2,70	3,36	2,14	2,42	2,89	2,24	2,85
2.01	1,74	2,63	2,64	2,51	3,83	2,87	2,36	2,34	3,28

Variant 19

9,02	38,26	24,66	50,06	28,45	60,37	33,22	31,38	19,96	50,36
18,16	30,44	30,42	39,10	51,19	23,78	44,77	35,94	71,14	24,50
39,99	40,05	47,92	33,72	30,24	26,13	53,32	59,10	36,03	43,29
45,23	61,22	43,14	10,40	31,20	29,94	58,14	18,09	49,32	59,95
54,90	35,20	63,00	36,02	40,51	42,18	34,33	39,02	32,52	35,34
26,89	63,22	66,47	53,36	56,13	42,55	15,00	84,02	36,33	53,14
37,12	45,88	73,36	36,33	37,40	76,58	19,23	40,38	62,93	70,36;
32,36	20,30	34,10	62,39	62,90	49,90	34,95	71,38	43,18	25,28
28,22	28,56	35,55	46,50	22,76	38,35	79,19	89,37	26,30	80,86
26,98	46,95	44,36	38,27	37,94	32,50	44,10	35,72	47,84	44,35

Variant 20

1,50	2,35	0,40	1,15	1,50	1,30	1,30	1,10	1,10	0,40
0,70	1,15	1,65	1,35	1,10	0,75	1,60	0,35	1,05	1,55
1,90	0,80	1,95	0,70	1,00	1,50	0,90	1,20	1,75	1,00
2,25	1,40	1,15	1,80	0,20	1,05'	2,15	0,90	2,15	0,85
1,50	1,70	1,40	2,35	1,10	1,10	1,15	2,40	1,20	2,30
1,45	0,95	0,65	1,15	2,20	2,40	1,65	1,70	1,50	1,25
2.10	1,50	1,75	1,20	1,70	0,30	1,10	2,00	1,80	1,05
1,95	1,95	1,90	1,55	0,85	1,50	1,85	1,30	1,40	1,45
1,35	1,55	1,20	2,40	1,20	0,50	1,35	0,95	1,15	2,05
1,55	0,65	1,85	0,45	2,30	0,50	1,00	2,10	2,00	0,95

Variant 21

50,1	21,6	13,2	8,7	38,8	19,2	20,7	35,3	19,7	12,3
52,2	42,9	9,4	13,7	19,3	20,3	26,9	35,4	18,9	12,0
54,3	11,0	12,9	19,1	20,1	24,1	27,0	36,5	18,4	4,7
19,9	12,5	19,0	19,9	25,3	34,0	27,8	49,6	18,7	4,2
21,4	18,0	19,8	25,2	33,9	43,4	27,9	48,5	15,1	4,1
23,6	19,9	26,2	32,8	42,3	5,9	28,0	54,4	16,7	4,9
26,1	26,1	30,3	41,3	6,3	4,8	34,1	20,9	17,3	10,3
16,3	30,1	40,9	6,1	5,0	11,9	44,4	25,1	17,0	10,1
16,2	40,1	12,5	5,1	9,0	14,0	35,2	25,3	11,1	5,7
16,0	12,3	4,3	8,5	14,1	19,4	42,5	25,9	12,4	5,3

Variant 22

15,60	2,46	5,06	4,32	6,01	9,02	7,20	11,20	11,24	13,21
2,11	4,14	5,16	3,68	6,12	9,12	7,25	11,06	11,34	13,40
2,12	4,18	4,14	3,33	6,44	9,14	7,93	11,01	11,66	13,80
2,14	4,42	5,18	4,41	6,90	9,16	7,98	10,01	11,69	6,92
2,13	3,10	5,20	5,17	6,92	9,18	8,01	10,22	11,80	6,99
4,90	2,14	4,10	5,21	6,96	8,13	8,24	10,27	11,86	6,01
2,99	2,18	2,44	2,41	7,00	9,22	8,15	10,12	12,01	6,90
2,64	3,11	5,00	5,90	7,02	9,28	8,45	10,16	12,12	15,01
2,73	2,44	5,08	5,92	7,14	9,44	9,00	10,14	13,01	15,20
4,10	5,08	3,64	5,98	7,19	9,98	8,36	10,99	13,20	15,21

Variant 23

14,13	14,17	14,20	14,10	14,21	14,16	14,18	14,18	14,12	14,17
14.13	14,18	14,18	14,21	14,17	14,23	14,18	14,19	14,19	14,11
14,18	14,18	14,15	14,18	14,16	14,13	14,14	14,24	14,29	14,25
14,20	14,16	14,17	14,14	14,12	14,17	14,19	14,18	14,26	14,20
14.20	14,20	14,21	14,11	14,15	14,23	14,16	14,17	14,20	14,11
14.14	14,21	14,23	14,21	14,20	14,20	14,15	14,16	14,19	14,12
14.21	14,12	14,16	14,19	14,15	14,18	14,21	14,12	14,23	14,12
14,12	14,18	14,27	14,22	14,19	14,17	14,17	14,10	14,18	14,21
14,18	14,19	14,22	14,21	14,18	14,19	14,18	14,25	14,26	14,28
14,19	14,13	14,12	14,15	14,16	14,12	14,19	14,17	14,24	14,09

Variant 24

21,9	20,2	24,0	21,0	19,7	21,0	20,8	19,8	22,4	21,0
20,5	19,5	23,5	22,7	19,6	22,4	20,0	20,0	20,7	19,0
20,0	22,5	23,7	21,3	22,4	20,7	24,6	22,5	24,6	20,6
19,5	18,2	24,2	21,5	21,7	18,6	21,6	21,8	19,6	23,5
21.5	20,3	23,6	24,9	23,0	21,0	19,5	22,3	21,2	19,5
23,0	20,4	21,5	19,8	23,5	21,6	20,1	20,6	23,0	22,5
22.3	21,5	21,2	19,6	22,3	20,5	20,8	23,0	21,5	20,4
20.6	21,4	19,8	19,7	24,5	22,5	21,5	21,5	22,0	20,8
20.4	20,8	20,0	18,0	23,8	20,4	20,5	19,7	20,8	21,8
18,3	22,1	23,0	20,1	23,0	24,0	23,0	21,0	22,3	22,0

Variant 25

1225,0	609,6	346,1	886,0	575,4	323,1	664,2	551,0	281,2	984,7
1062,7	605,6	213,9	735,4	582,0	305,9	533,2	524,0	313,0	1145,4
686,5	492,1	224,4	528,9	426,7	321,2	805,9	445,5	286,7	749,5
696,4	413,1	395,8	583,0	458,1	363,4	705,1	446,8	357,2	801,0
762,2	438,4	333,7	626,5	519,0	212,5	640,4	496,8	384,2	772,2
629,1	415,7	367,6	539,6	483,0	921,3	557,5	402,2	1215,9	688,6
798,0	432,4	212,0	519,4	451,8	899,0	501,9	157,0	1100,0	752,9
677,0	402,3	1064,9	460,8	131,8	626,5	748,5	144,2	913,1	743,2
710,5	331,5	680,6	585,6	215,0	734,7	582,0	132,9	882,1	743,5
576,2	361,9	924,0	732,6	348,2	752,4	511,6	215,9	1083,3	814,9

Variant 26

19	21	13	19	21	29	30	27	12	13
13	12	35	15	22	23	28	21	14	20
25	18	16	20	30	18	15	22	20	25
32	20	19	21	23	12	24	18	18	27
23	24	33	13	20	19	24	20	11	16
17	28	16	22	25	16	16	22	26	21
20	15	24	16	18	27	24	12	21	14
20	17	18	24	31	22	17	14	14	23
24	18	26	24	24	22	20	19	23	24
22	13	22	18	16	14	18	21	24	22

Variant 27

4,24	4,25	4,24	4,25	4,40	4,05	4,00	4,17	4,12	4,08
4,20	4,23	4,28	4,24	4,38	4,10	4,47	4,35	4,14	4,50
4,03	4,06	4,30	4,23	4,42	4,12	4,46	4,07	4,15	4,12
4,25	4,16	4,32	4,25	4,18	4,14	4,40	4,27	4,19	4,45
4,07	4,18	4,17	4,26	3,97	4,08	4,20	4,17	4,31	4,34
4,27	4,17	4,16	4,28	4,03	4,03	4,18	4,32	4,19	4,32
4,23	4,12	4,18	4,17	4,20	4,17	4,15	4,12	4,23	4,30
4,44	4,10	4,40	4,19	4,22	4,15	4,21	4,22	4,18	4,17
4,25	4,37	4,42	4,15	4,18	4,13	4,32	4,20	4,24	4,35
4,18	4,35	4,38	4,12	4,24	4,16	4,10	4,30	4,30	4,40

LECTURE NO. 9.

ELEMENTS OF CORRELATION THEORY

LE NEUVIÈME COURS. ELEMENTS DE THEORIE DE CORRELATION

NOUNTE VORLESUNG. ELEMENTE DER KORRELATIONSTHEORIE

POINT 1. CORRELATION DEPENDENCE BETWEEN TWO RAN-DOM VARIABLES. Dépendance corrélative de deux variables aléatoires. Korrelationsabhängigkeit zwischen zwei Zufallsgrößen.

POINT 2. STATISTICAL ESTIMATES OF PARAMETERS OF CORRELATION DEPENDENCE. Estimations statistiques de paramètres de dépendance corrélative. Statistische Parameterschätzungen von Korrelationsabhängigkeit.

POINT 3. LEAST-SQUARES METHOD. Méthode des moindres carrés. Methode der kleinsten Quadrate.

POINT 4. MULTIPLE CORRELATION. Corrélation multiple. Mehrfache [multiple] Korrelation.

POINT 1. CORRELATION DEPENDENCE BETWEEN TWO RANDOM VARIABLES

We study correlation dependence between two random variables in Points 1- 3 and that between three random variables in the Point 4.

There is deterministic [functional, stiff] dependence between random variables ξ, η (for example linear dependence $\eta = a\xi + b$).

And there is undetermined [non-functional, non-stiff, statistic, correlation] dependence between ξ, η, for example dependence between labour productivity and living standard, between a state of health and a productivity of a worker, between height and weight of a man.

Correlation dependence between random variables ξ and η is that between values of one variable and corresponding mean value [average value, distribution centre, mathematical expectation] of the other. Such dependence is defined by introducing conditional distributions of random variables and conditional mathematical expectations.

Conditional distributions

Let ξ, η be discrete random variables. The probability p_{ij} of ocurrence of a pair $(\xi = x_i, \eta = y_j)$, $i = \overline{1, k}$, $j = \overline{1, k}$ can be represented in two ways

$$p_{ij} = P(\xi = x_i, \eta = y_j) = P((\xi = x_i)(\eta = y_j)) =$$
$$= P(\xi = x_i) P((\eta = y_j)/(\xi = x_i)) = P(\eta = y_j) P((\xi = x_i)/(\eta = y_j)),$$

whence it follows that the probability for ξ to take on a value $\xi = x_i$ in the condition that η takes on a value $\eta = y_j$ (the conditional probability of the event $\xi = x_i$) equals

$$P((\xi = x_i)/(\eta = y_j)) = \frac{P((\xi = x_i)(\eta = y_j))}{P(\eta = y_j)} = \frac{p_{ij}}{p_{y_j}}, \qquad (1\text{ a})$$

where it's denoted

$$p_{y_j} = P(\eta = y_j),$$

and the probability for η to take on a value $\eta = y_j$ under the condition that ξ takes on a value $\xi = x_i$ (the conditional probability of the event $\eta = y_j$) equals

$$P((\eta = y_j)/(\xi = x_i)) = \frac{P((\xi = x_i)(\eta = y_j))}{P(\xi = x_i)} = \frac{p_{ij}}{p_{x_i}}, \qquad (1\text{ b})$$

where

$$p_{x_i} = P(\xi = x_i),$$

Let now ξ, η be continuous random variables. In this case one introduces the conditional distribution densities. Namely, the distribution density of the random variable η under the condition that the random variable ξ takes on a value x is given by the formula

$$f(y/x)=\frac{f(x,y)}{f_\xi(x)}=\frac{f(x,y)}{\int\limits_{-\infty}^{\infty}f(x,y)dy}, \qquad (2\text{ a})$$

Similarly the distribution density of ξ under the condition that η takes on a value y is

$$f(x/y)=\frac{f(x,y)}{f_\eta(y)}=\frac{f(x,y)}{\int\limits_{-\infty}^{\infty}f(x,y)dx}. \qquad (2\text{ b})$$

If it's known the distribution density $f(x,y)$ of the two-dimensional random variable (ξ,η), one can find distribution densities of its components ξ,η, because of

$$f_\xi(x)=\int\limits_{-\infty}^{\infty}f(x,y)dy, \quad f_\eta(y)=\int\limits_{-\infty}^{\infty}f(x,y)dx. \qquad (3)$$

Conditional mathematical expectations. Regression functions

Let ξ, η be discrete random variables. The mathematical expectation of η in the condition that ξ takes on a value $\xi=x_i$ (the conditional mathematical expectation of η) equals (with utilization of the formula (1 a))

$$\begin{aligned}m_\eta(x_i)&=M(\eta/(\xi=x_i))=\sum_{j=1}^{l}y_jP((\eta=y_j)/(\xi=x_i))=\frac{1}{p_{x_i}}\sum_{j=1}^{l}y_jp_{ij}=\\&=\frac{1}{p_{x_i}}(y_1p_{i1}+y_2p_{i2}+\ldots+y_lp_{il})\end{aligned} \qquad (4\text{ a})$$

Analogously the mathematical expectation of ξ under the condition that η takes on a value $\eta=y_j$ (the conditional mathematical expectation of ξ) equals (with utilization of the formula (1 b))

$$\begin{aligned}m_\xi(y_j)&=M(\xi/(\eta=y_\xi))=\sum_{i=1}^{k}x_iP((\xi=x_i)/(\eta=y_j))=\frac{1}{p_{y_j}}\sum_{i=1}^{k}x_ip_{ij}=\\&=\frac{1}{p_{y_j}}(x_1p_{1j}+x_2p_{2j}+\ldots+x_kp_{kj}).\end{aligned} \qquad (4\text{ b})$$

Let now ξ, η be continuous random variables. The mathematical expectation of η in the condition that ξ takes on a value $\xi = x$ (the conditional mathematical expectation of η) equals (with utilization of the formula (2 a))

$$m_\eta(x) = M(\eta/(\xi = x)) = \int_{-\infty}^{\infty} yf(y/x)dy = \frac{1}{f_\xi(x)}\int_{-\infty}^{\infty} yf(x,y)dy. \qquad (5\text{ a})$$

Analogously the mathematical expectation of ξ under condition that η takes on a value $\eta = y$ (the conditional mathematical expectation of ξ) equals (with utilization of the formula (2 b))

$$m_\xi(y) = M(\xi/(\eta = y)) = \int_{-\infty}^{\infty} xf(x/y)dx = \frac{1}{f_\eta(y)}\int_{-\infty}^{\infty} xf(x,y)dx. \qquad (5\text{ b})$$

Def.1. The conditional mathematical expectation of the random variable η, that is $m_\eta(x)$, is called the **regression function** of η on ξ, its graph is called the **regression curve** [regression line] of η on ξ (fig. 1), and the equation

$$y = m_\eta(x) \qquad (6\text{ a})$$

is called the **regression equation** of η on ξ.

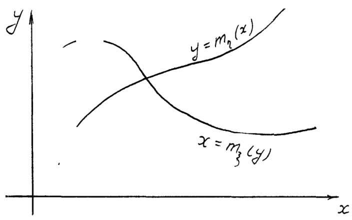


Fig. 1

By analogy is defined the regression function $m_\xi(y)$, the regression curve [regression line] (fig. 1) and the regression equation

$$x = m_\xi(y) \qquad (6\text{ b})$$

of ξ on η.

It can be proved that in the case when there is a functional dependence between the random variables ξ and η, both regression lines coincide.

Ex. 1. The general normal distribution on the plane that is two-dimensional random variable (ξ, η) with the distribution density

$$f(x,y) = \frac{1}{2\pi\sigma_\xi\sigma_\eta\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}\left(\frac{(x-m_\xi)^2}{\sigma_\xi^2} - \frac{2x(x-m_\xi)(y-m_\eta)}{\sigma_\xi\sigma_\eta} + \frac{(y-m_\eta)^2}{\sigma_\eta^2}\right)}. \qquad (7)$$

If we find the distribution densities $f_\xi(x)$, $f_\eta(y)$ of the components ξ,η making use of the formula (3), we'll get (see [1, p. 188])

$$f_\xi(x)=\int_{-\infty}^{\infty} f(x,y)dy=\frac{1}{\sigma_\xi\sqrt{2\pi}}e^{-\frac{(x-m_\xi)^2}{2\sigma_\xi^2}},\ f_\eta(y)=\int_{-\infty}^{\infty} f(x,y)dx=\frac{1}{\sigma_\eta\sqrt{2\pi}}e^{-\frac{(y-m_\eta)^2}{2\sigma_\eta^2}}.$$

Therefore the components ξ,η are distributed normally ($N(m_\xi,\sigma_\xi)$, $N(m_\eta,\sigma_\eta)$).

For the conditional distribution density $f(y/x)$ the formula (2 a) gives

$$f(y/x)=\frac{1}{\sqrt{2\pi}\sigma_\eta\sqrt{1-r^2}}e^{-\frac{1}{2(1-r^2)\sigma_\eta^2}\left(y-\left(m_\eta+r\frac{\sigma_\eta}{\sigma_\xi}(x-m_\xi)\right)\right)^2}.$$

Hence $f(y/x)$ is the density of a normal distribution with the mathematical expectation (conditional mathematical expectation, regression function of η on ξ)

$$m_\eta(x)=r\frac{\sigma_\eta}{\sigma_\xi}(x-m_\xi)+m_\eta$$

and the regression equation

$$y=r\frac{\sigma_\eta}{\sigma_\xi}(x-m_\xi)+m_\eta .$$

By analogous way with the help of the formula (2 b) we find the conditional distribution density $f(x/y)$, the regression function and the regression equation of ξ on η

$$x=m_\xi(y)=r\frac{\sigma_\xi}{\sigma_\eta}(y-m_\eta)+m_\xi .$$

If $r=0$ in the formula (7), we have so-called simplest normal distribution on the plane with the distribution density

$$f(x,y)=f_\xi(x)f_\eta(y)=\frac{1}{\sqrt{2\pi}\sigma_\xi}e^{-\frac{(x-m_\xi)^2}{2\sigma_\xi^2}}\cdot\frac{1}{\sqrt{2\pi}\sigma_\eta}e^{-\frac{(y-m_\eta)^2}{2\sigma_\eta^2}}$$

from which it follows that its components are independent and normally distributed random variables.

Regression functions possess so-called principle property. For the regression function $m_\eta(x)$ the mathematical expectation of the square of the deviation of the random variable η from $m_\eta(x)$ isn't greater then from any other function $h(x)$, that is

$$M\left((\eta - m_\eta(x))^2\right) \le M\left((\eta - h(x))^2\right).$$

In other words, the function

$$M\left((\eta - h(x))^2\right)$$

attains [reaches, comes, obtains] its least value on the regression function $m_\eta(x)$.

The principle property of the regression function lies in the basis of many propositions of correlation theory, in particular the justification of the least-squares method for finding of parameters of correlation dependence.

The regression function $m_\eta(x)$ (correspondingly $m_\xi(y)$) is a function which assigns the mathematical expectation of the random variable η to every possible value x of the random variable ξ (corr. the mathematical expectation of the random variable ξ to every possible value y of the random variable η).

Def. 2. A correlation dependence between random variables ξ, η is called a functional dependence between possible values of one random variable and corresponding regression function (average [mean] value) of the other.

Main problems of correlation theory (for the case of two random variables)

1. Determine the **form of correlation dependence** between random variables.

If regression functions of random variables ξ, η are those linear, then one says about a linear correlation (or a linear correlation dependence) between these random variables. Otherwise one says about non-linear (or curvilinear) correlation.

2. Determine the **closeness of relation between** random variables ξ, η.

Closeness problem is resolved with the help of the correlation coefficient $r_{\xi\eta}$, which is the measure of a linear dependence between ξ, η, and the correlation ratios

$\rho_{\eta\xi}$, $\rho_{\xi\eta}$ which are the measure of a functional (not necessary linear) dependence between ξ, η.

The **correlation coefficient** of random variables ξ, η is defined by the formula

$$r_{\xi\eta} = \frac{K_{\xi\eta}}{\sigma_\xi \sigma_\eta}, \qquad (8)$$

where

$$K_{\xi\eta} = M\left(\overset{0}{\xi}\overset{0}{\eta}\right) = M\left((\xi - m_\xi)(\eta - m_\eta)\right) = M(\xi\eta) - m_\xi m_\eta \qquad (9)$$

is the correlation moment of ξ, η. For discrete random variables

$$K_{\xi\eta} = \sum_{i,j}(x_i - m_\xi)(y_j - m_\eta)p_{ij} = \sum_{i,j} x_i y_j p_{ij} - m_\xi m_\eta, \quad p_{ij} = P((\xi = x_i)(\eta = y_j)), \qquad (10)$$

and for continuous random variables

$$K_{\xi\eta} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - m_\xi)(y - m_\eta)f(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy - m_\xi m_\eta. \qquad (11)$$

If random variables are independent, then $r_{\xi\eta} = 0$. The converse isn't true in general: there are dependent random variables with $r_{\xi\eta} = 0$.

Def. 3. Random variables ξ, η are called those **correlated** if their correlation coefficient doesn't equal zero ($r_{\xi\eta} \neq 0$), and **non-correlated** otherwise ($r_{\xi\eta} = 0$).

It’s known that

$$|r_{\xi\eta}| \leq 1, \quad \text{and} \quad |r_{\xi\eta}| = 1$$

if ξ, η are connected by linear (functional) dependence $\eta = k\xi + b$,

$$r_{\xi\eta} = \begin{cases} 1 \; if \; k > 0, \\ -1 \; if \; k < 0. \end{cases}$$

Ex. 2. For the general normal distribution on the plane with the distribution density (7) the formulas (8), (11) give

$$K_{\xi\eta} = r\sigma_\xi\sigma_\eta, \; r_{\xi\eta} = r,$$

and so the parameter r is the correlation coefficient of the components ξ, η of the distribution. Theses components are independent if and only if $r = 0$. It means that for the general normal distribution on the plane the notions independence and uncorrelatedness coincide.

The **correlation ratio** of a random variable η on a random variable ξ is called the next quantity

$$\rho_{\eta\xi} = \frac{\sigma_{m_\eta(\xi)}}{\sigma_\eta}, \qquad (12\text{ a})$$

and the correlation ratio of ξ on η is a quantity

$$\rho_{\xi\eta} = \frac{\sigma_{m_\xi(\eta)}}{\sigma_\xi}. \qquad (12\text{ b})$$

It's known that

$$0 \le \rho_{\eta\xi} \le 1, \quad 0 \le \rho_{\xi\eta} \le 1,$$

and

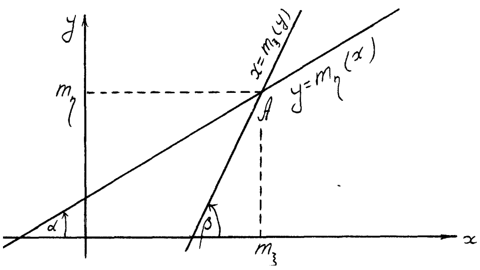
$$\rho_{\eta\xi} = 0, \quad \rho_{\xi\eta} = 0$$

if the random variables ξ, η are independent,

$$\rho_{\eta\xi} = 1, \quad \rho_{\xi\eta} = 1$$

if there is a functional (not necessary linear) dependence between them.

Linear correlation



Let the regression functions of random variables ξ, η are linear, that is there is a linear correlation between ξ, η. It can be proved that the regression functions are given by the next formulas

Fig. 2 $$m_\eta(x) = \frac{\sigma_\eta}{\sigma_\xi} r_{\xi\eta}(x - m_\xi) + m_\eta, \quad m_\xi(y) = \frac{\sigma_\xi}{\sigma_\eta} r_{\xi\eta}(y - m_\eta) + m_\xi \;. \qquad (13)$$

Corresponding regression straight lines $y = m_\eta(x), \quad x = m_\xi(y)$ (see fig. 2): a) intersect at the point $A(m_\xi, m_\eta)$; b) have the slopes

$$k_1 = tg\alpha = \frac{\sigma_\eta}{\sigma_\xi} r_{\xi\eta},\ k_2 = tg\beta = \frac{\sigma_\eta}{\sigma_\xi} \cdot \frac{1}{r_{\xi\eta}}; \qquad (14)$$

c) coincide if $r_{\xi\eta} = \pm 1$ that is if there is a linear functional dependence between the random variables ξ and η; d) are perpendicular respectively to the *Ox*-axis and *Oy*-axis if $r_{\xi\eta} = 0$ that is if ξ and η aren't correlated.

Ex. 3. Components ξ, η of the general normal distribution on the plane are connected by linear correlation dependence.

POINT 2. STATISTICAL ESTIMATES OF PARAMETERS OF CORRELATION DEPENDENCE

Let *n* independent trials on two-dimensional random variable (ξ, η) are fulfilled. We study correlative dependence between random variables ξ, η with the help of the next table, which is called the **correlation table (table 1)**.

Table 1

η \ ξ	x_1	x_2	...	x_i	...	x_k	n_y
y_1	n_{11}	n_{21}	...	n_{i1}	...	n_{k1}	n_{y_1}
y_2	n_{11}	n_{22}	...	n_{i2}	...	n_{k2}	n_{y_2}
...	...	...	...	...	...	...	...
y_j	n_{1j}	n_{2j}	...	n_{ij}	...	n_{kj}	n_{y_j}
...	...	...	...	...	...	...	...
y_l	n_{1l}	n_{2l}	...	n_{il}	...	n_{kl}	n_{y_l}
n_x	n_{x_1}	n_{x_2}	...	n_{x_i}	...	n_{x_k}	n

Here n_{ij} is the number of occurrences [appearances] of the pair ($\xi = x_i, \eta = y_j$); the sum

$$n_{i1} + n_{i2} + \ldots + n_{ij} + \ldots + n_{il} = \sum_{j=1}^{l} n_{ij} = n_{x_i}$$

is the number of occurrences of the value $\xi = x_i$ of the random variable ξ; the sum

$$n_{1j} + n_{2j} + \ldots + n_{ij} + \ldots + n_{kj} = \sum_{i=1}^{k} n_{ij} = n_{y_j}$$

is the number of occurrences of the value $\eta = y_j$ of the random variable η. It's evident that $n_{x_1} + n_{x_2} + \ldots + n_{x_m} = n_{y_1} + n_{y_2} + \ldots + n_{y_l} = n$.

At first we get the variation series for ξ,η from the correlation table (table 2) and estimate the parameters of their laws $\overline{\xi}_s, D_{\xi s}, \sigma_{\xi s}, \widetilde{D}_\xi, \widetilde{\sigma}_\xi; \overline{\eta}_s, D_{\eta s}, \sigma_{\eta s}, \widetilde{D}_\eta, \widetilde{\sigma}_\eta$:

Table 2

ξ	x_1	x_2	...	x_k	η	y_1	y_2	...	y_l
n_x	n_{x_1}	n_{x_2}	...	n_{x_k}	n_y	n_{y_1}	n_{y_2}	...	n_{y_l}
$p_x^* = n_x/n$	$p_{x_1}^*$	$p_{x_2}^*$	...	$p_{x_k}^*$	$p_y^* = n_y/n$	$p_{y_1}^*$	$p_{y_2}^*$	...	$p_{y_l}^*$

a) the sample (the statistical) means [averages]

$$\overline{\xi}_s = m_\xi^* = \frac{1}{n}\sum_{i=1}^{k} x_i n_{x_i} = \sum_{i=1}^{k} x_i p_{x_i}^*, \qquad \overline{\eta}_s = m_\eta^* = \frac{1}{n}\sum_{j=1}^{l} y_j n_{y_j} = \sum_{j=1}^{l} y_j p_{y_j}^*; \quad (15)$$

b) the sample (the statistical) dispersions

$$D_{\xi s} = D_\xi^* = \overline{\xi_s^2} - \left(\overline{\xi}_s\right)^2 = m_{\xi^2}^* - \left(m_\xi^*\right)^2, \quad D_{\eta s} = D_\eta^* = \overline{\eta_s^2} - \left(\overline{\eta}_s\right)^2 = m_{\eta^2}^* - \left(m_\eta^*\right)^2; \quad (16)$$

c) the sample (the statistical) root-mean-square deviations

$$\sigma_{\xi s} = \sigma_\xi^* = \sqrt{D_{\xi s}} = \sqrt{D_\xi^*}, \qquad \sigma_{\eta s} = \sigma_\eta^* = \sqrt{D_{\eta s}} = \sqrt{D_\eta^*}; \quad (17)$$

d) corrected dispersions and root-mean-square deviations

$$\widetilde{D}_\xi = \frac{n}{n-1} D_{\xi s} = \frac{n}{n-1} D_\xi^*, \widetilde{D}_\eta = \frac{n}{n-1} D_{\eta s} = \frac{n}{n-1} D_\eta^*, \widetilde{\sigma}_\xi = \sqrt{\widetilde{D}_\xi}, \widetilde{\sigma}_\eta = \sqrt{\widetilde{D}_\eta}. \quad (18)$$

Then we estimate correlation moment and correlation coefficient of ξ,η.

The sample (the statistical) **correlation moment** on the base of the formula (10) is

$$K_s = K_{\xi\eta}^* = \sum_{i,j} x_i y_j p_{ij}^* - \overline{\xi}_s \overline{\eta}_s = \left| p_{ij}^* = \frac{n_{ij}}{n} \right| = \frac{1}{n}\sum_{ij} x_i y_j n_{ij} - \overline{\xi}_s \overline{\eta}_s, \quad (19)$$

where $p_{ij}^* = n_{ij}/n$ is the relative frequency of occurrences of a pair ($\xi = x_i, \eta = y_j$).

The formula (19) can be written more simple if we'll denote pairs of ξ, η by unique index,

$$K_s = K_{\xi\eta}^* = \sum_i x_i y_i p_i^* - \overline{\xi}_s \overline{\eta}_s = \frac{1}{n} \sum_i x_i y_i n_i - \overline{\xi}_s \overline{\eta}_s , \qquad (19\text{ a})$$

where $p_i^* = n_i / n$ and n_i are correspondingly the relative frequency and the number of occurrences of the pair $(\xi = x_i, \eta = y_i)$ and the sum is taken over all different pairs.

If we suppose that every pair occurs only one time[1], we'll get the simplest formula

$$K_s = K_{\xi\eta}^* = \frac{1}{n} \sum_i x_i y_i - \overline{\xi}_s \overline{\eta}_s = \sum_i x_i y_i - \overline{\xi}_s \overline{\eta}_s . \qquad (19\text{ b})$$

As can be proved the sample correlation moment is a biased estimate of the correlation moment $K_{\xi\eta}$ because of

$$M(K_s) = \frac{n-1}{n} K_{\xi\eta} < K_{\xi\eta} ,$$

and therefore one can introduce the **corrected correlation moment**

$$\widetilde{K} = \frac{n}{n-1} K_s .$$

The sample (the statistical) **correlation coefficient** equals

$$r_s = r_{\xi\eta}^* = \frac{K_s}{\sigma_{\xi s} \sigma_{\eta s}} = \frac{K_{\xi\eta}^*}{\sigma_\xi^* \sigma_\eta^*} . \qquad (20)$$

It coincides with the corrected correlation coefficient, for

$$\widetilde{r} = \frac{\widetilde{K}}{\widetilde{\sigma}_\xi \widetilde{\sigma}_\eta} = \frac{K_s}{\sigma_{\xi s} \sigma_{\eta s}} = r_s .$$

The sample (the statistical) **correlation ratios** are given by the next expressions (with $\overline{\eta}_{x=x_i} = m_\eta^*(x_i), \overline{\xi}_{y=y_j} = m_\xi^*(y_j)$ defined below by formulas (22 a), (22 b))

$$\rho_{\eta\xi s} = \rho_{\eta\xi}^* = \frac{1}{\sigma_{\eta s}} \sqrt{\frac{1}{n} \sum_i \left(\overline{\eta}_{x=x_i} - \overline{\eta}_s\right)^2 n_{x_i}} = \frac{1}{\sigma_{\eta s}} \sqrt{\frac{1}{n} \sum_i \left(m_\eta^*(x_i) - \overline{\eta}_s\right)^2 n_{x_i}} \qquad (21\text{ a})$$

[1] In this case we say to have so-called **non-integrated** data. In the formulas (19), (19a) data are **integrated**.

$$\rho_{\xi\eta s} = \rho^*_{\xi\eta} = \frac{1}{\sigma_{\xi s}} \sqrt{\frac{1}{n} \sum_j \left(\bar{\xi}_{y=y_j} - \bar{\xi}_s\right)^2 n_{y_j}} = \frac{1}{\sigma_{\xi s}} \sqrt{\frac{1}{n} \sum_j \left(m^*_\xi(y_j) - \bar{\xi}_s\right)^2 n_{y_j}} \quad (21\ б)$$

Estimation of regression functions

Regression functions are estimated by so called **conditional means** which one gets by replacing probabilities p_{x_i}, p_{y_j}, p_{ij} in the formulas (4 a), (4 b) by corresponding relative frequencies

$$p^*_{x_i} = \frac{n_{x_i}}{n},\ p^*_{y_j} = \frac{n_{y_j}}{n},\ p^*_{ij} = \frac{n_{ij}}{n},$$

namely

$$\bar{\eta}_{x=x_i} = m^*_\eta(x_i) = \frac{1}{p^*_{x_i}} \sum_j y_j p^*_{ij} = \frac{1}{n_{x_i}} \sum_j y_j n_{ij} = \frac{1}{n_{x_i}} \left(y_1 n_{i1} + y_1 n_{i2} + ... + y_l n_{il}\right), \quad (22\ a)$$

$$\bar{\xi}_{y=y_j} = m^*_\xi(y_j) = \frac{1}{p^*_{y_j}} \sum_i x_i p^*_{ij} = \frac{1}{n_{y_j}} \sum_i x_i n_{ij} = \frac{1}{n_{y_j}} \left(x_1 n_{1j} + x_2 n_{2j} + ... + x_k n_{kj}\right) \quad , \quad (22\ b)$$

Table 3

η \ ξ	10	30	50	70	n_y	$\bar{\xi}_{y=y_j} = m^*_\xi(y_j)$
10	9	1			10	12.00
12	4	10	2		16	27.50
14	1	9	6	1	17	38.24
16		3	14	10	27	55.19
18			6	18	24	65.00
20				6	6	70.00
n_x	14	23	28	35	$n = 100$	
$\bar{\eta}_{x=x_i} = m^*_\eta(x_i)$	10.85	13.22	15.71	17.66		

Ex. 4. Results of n = 100 trials on random variables ξ and η are represented by the table (table 3). Estimate number characteristics of ξ , η, regression functions of η

on ξ and ξ on η, correlation coefficient and correlation ratio of η on ξ.

Estimates of number characteristics of the random variable ξ (table 4)

$$\overline{\xi}_s = m_\xi^* = \frac{1}{n}\sum_i x_i n_{x_i} = \frac{4680}{100} = 46.80; \quad \overline{\xi_s^2} = m_{\xi^2}^* = \frac{1}{n}\sum_i x_i^2 n_{x_i} = 2636;$$

$$D_{\xi s} = D_\xi^* = \overline{\xi_s^2} - \left(\overline{\xi}_s\right)^2 = 2636 - (46.80)^2 = 445.76; \sigma_{\xi s} = \sigma_\xi^* = \sqrt{D_{\xi s}} = \sqrt{D_\xi^*} = 21.11.$$

Table 4

№	x_i	n_{x_i}	$x_i n_{x_i}$	x_i^2	$x_i^2 n_{x_i}$
1	10	14	140	100	1400
2	30	23	690	900	20700
3	50	28	1400	2500	70000
4	70	35	2450	4900	171500
Σ		100	4680		263600

Table 5

№	y_j	n_{y_j}	$y_j n_{y_j}$	y_j^2	$y_j^2 n_{y_j}$
1	10	10	100	100	1000
2	12	16	192	144	2304
3	14	17	238	196	3332
4	16	27	432	256	6912
5	18	24	432	324	7776
6	20	6	120	400	2400
Σ		100	1514		23724

Estimates of number characteristics of the random variable η (table 5)

$$\overline{\eta}_s = m_\eta^* = \frac{1}{n}\sum_j y_j n_{y_j} = \frac{1514}{100} = 15.14; \quad \overline{\eta_s^2} = m_{\eta^2}^* = \frac{1}{n}\sum_j y_j^2 n_{y_j} = 237.24;$$

$$D_{\eta s} = D_\eta^* = \overline{\eta_s^2} - \left(\overline{\eta}_s\right)^2 = 237.24 - (15.14)^2 = 8.02; \quad \sigma_{\eta s} = \sigma_\eta^* = \sqrt{D_{\eta s}} = \sqrt{D_\eta^*} = 2.83.$$

Estimates of values of the regression function η on ξ, namely the values of conditional means $\overline{\eta}_{x=x_i} = m_\eta^*(x_i)$, we give by the formula (22 a)

$$\overline{\eta}_{x=10} = m_\eta^*(10) = \frac{1}{14}(10\cdot 9 + 12\cdot 4 + 14\cdot 1) = 10.85;$$

$$\overline{\eta}_{x=30} = m_\eta^*(30) = \frac{1}{23}(10\cdot 1 + 12\cdot 10 + 14\cdot 9 + 16\cdot 3) = 13.22;$$

$$\overline{\eta}_{x=50} = m_\eta^*(50) = \frac{1}{28}(12\cdot 2 + 14\cdot 6 + 16\cdot 14 + 18\cdot 6) = 15.71;$$

$$\overline{\eta}_{x=70} = m_\eta^*(70) = \frac{1}{35}(14\cdot 1 + 16\cdot 10 + 18\cdot 18 + 20\cdot 6) = 17.66.$$

All these values are situated in the ninth row of the table 3.

By the formula (22 b) we find the estimates of values of the regression function

ξ on η, that is the values of conditional means $\overline{\xi}_{y=y_j} = m_\xi^*(y_j)$

$$\overline{\xi}_{y=10} = m_\xi^*(10) = \frac{1}{10}(10\cdot 9 + 30\cdot 1) = 12.00;$$

$$\overline{\xi}_{y=12} = m_\xi^*(12) = \frac{1}{16}(10\cdot 4 + 30\cdot 10 + 50\cdot 2) = 27.50;$$

$$\overline{\xi}_{y=14} = m_\xi^*(14) = \frac{1}{17}(10\cdot 1 + 30\cdot 9 + 50\cdot 6 + 70\cdot 1) = 38.24;$$

$$\overline{\xi}_{y=16} = m_\xi^*(16) = \frac{1}{27}(30\cdot 3 + 50\cdot 14 + 70\cdot 10) = 55.19;$$

$$\overline{\xi}_{y=18} = m_\xi^*(18) = \frac{1}{24}(50\cdot 6 + 70\cdot 18) = 65.00;$$

$$\overline{\xi}_{y=20} = m_\xi^*(20) = \frac{1}{6}\cdot 70\cdot 6 = 70.00.$$

All these values are in the seventh row of the table 3.

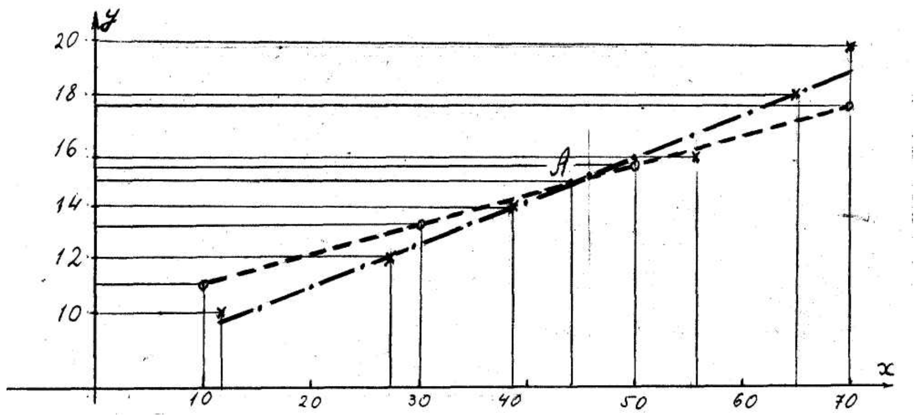


Fig. 3

Let's denote (see fig. 3) by circles the points

$$\left(x_i; \overline{\eta}_{x=x_i}\right) \equiv \left(x_i; m_\eta^*(x_i)\right),$$

that is the points

$$(10; 10.85), (30; 13.22), (50; 15.71), (70; 17.66),$$

and by crosses the points

$$\left(\overline{\xi}_{y=y_j}; y_j\right) \equiv \left(m_\xi^*(y_j); y_j\right),$$

that is

$$(12.00;10),(27.50;12),(38.34;14),(55.19;16),\ (65.00;18),(70.00;20).$$

We represent then the approximate graphs of the regression functions (statistic regression lines) $y = m_\eta(x)$ and $x = m_\xi(y)$ by a broken line and dash-and-dot line respectively.

Nearness [aboutness, closeness, proximity] of the statistic regression lines suggests [indicates] about essential correlation dependence between the random variables ξ, η. We can suppose that this dependence is a linear one.

Let's pass to estimation of the correlation coefficient and correlation ratio of the random variables ξ, η.

We estimate the Correlation moment by the formula (19)

$$K_s = K^*_{\xi\eta} = \frac{1}{100}(10\cdot10\cdot9 + 30\cdot10\cdot1 + 10\cdot12\cdot4 + 30\cdot12\cdot10 + 50\cdot12\cdot2 +$$

$$+10\cdot14\cdot1+30\cdot14\cdot9+50\cdot14\cdot6+70\cdot14\cdot1+30\cdot16\cdot3+50\cdot16\cdot14+70\cdot16\cdot10+50\cdot18\cdot6+70\cdot18\cdot18+$$

$$+70\cdot20\cdot6) - 46.80\cdot15.14 = \frac{1}{100}\cdot75900 - 46.80\cdot15.14 = 50.448,$$

and the correlation coefficient by the formula (20)

$$r_s = r^*_{\xi\eta} = \frac{50.448}{21.11\cdot2.83} = 0.84.$$

This latter is sufficiently large, so we can say that ξ and η are connected by essential linear dependence.

The sample correlation ratio $\rho_{\eta\xi s}$ of η on ξ we'll find on the base of the formula (21 a). The radicand equals

$$\frac{1}{100}((10.85-15.14)^2\cdot14 + (13.22-15.14)^2\cdot23 + (15.71-15.14)^2\cdot28 +$$

$$(17.66-15.14)^2\cdot35) = \frac{1}{100}(18.40\cdot14 + 3.69\cdot23 + 0.32\cdot28 + 6.35\cdot35) =$$

$$= \frac{573.68}{100} \approx 5.74.$$

Remark. It's better to mount computings of the radicand with the help of the next table

x_i	n_{x_i}	$\overline{\eta}_{x=x_i} = m_\eta^*(x_i)$	$\overline{\eta}_{x=x_i} - \overline{\eta}_s = \overline{\eta}_{x=x_i} - 15.14$	$\left(\overline{\eta}_{x=x_i} - \overline{\eta}_s\right)^2$	$\left(\overline{\eta}_{x=x_i} - \overline{\eta}_s\right)^2 n_{x_i}$
10	14	10.85	- 4.29	18.40	257.60
30	23	13.22	- 1.92	3.69	84.87
50	28	15.71	0.57	0.32	8.96
70	35	17.66	2.52	6.35	222.25
				Σ	573.68

Thus

$$\frac{1}{n}\left(\overline{\eta}_{x=x_i} - \overline{\eta}_s\right)^2 n_{x_i} = \frac{573.68}{100} \approx 5.74,$$

and therefore

$$\rho_{\eta\xi s} = \rho_{\eta\xi}^* = \frac{1}{\sigma_{\eta s}}\sqrt{\frac{1}{n}\sum_i \left(\overline{\eta}_{x=x_i} - \overline{\eta}_s\right)^2 n_{x_i}} = \frac{\sqrt{5.74}}{2.83} = 0.85.$$

By the same way we calculate the sample correlation ratio $\rho_{\xi\eta s}$ of ξ on η. We'll fulfil calculations with the help of the next table.

y_j	n_{y_j}	$\overline{\xi}_{y=y_j} = m_\xi^*(y_j)$	$\overline{\xi}_{y=y_j} - \overline{\xi}_s = \overline{\xi}_{y=y_j} - 46.80$	$\left(\overline{\xi}_{y=y_j} - \overline{\xi}_s\right)^2$	$\left(\overline{\xi}_{y=y_j} - \overline{\xi}_s\right)^2 n_{y_j}$
10	10	12.00	- 34.80	1211.04	12110.40
12	16	27.50	- 19.60	384.16	6146.56
14	17	38.24	- 8.56	73.27	1245.59
16	27	55.19	8.39	70.39	1900.53
18	24	65.00	18.20	331.24	7949.76
20	6	70.00	23.20	538.24	3229.44
				Σ	32582.28

Thus

$$\frac{1}{n}\sum_j \left(\overline{\xi}_{y=y_j} - \overline{\xi}_s\right)^2 n_{y_j} = \frac{32582.28}{100} \approx 325.82,$$

and so

$$\rho_{\xi\eta s} = \rho_{\xi\eta}^* = \frac{1}{\sigma_{\xi s}}\sqrt{\frac{1}{n}\sum_j \left(\overline{\xi}_{y=y_j} - \overline{\xi}_s\right)^2 n_{y_j}} = \frac{\sqrt{325.82}}{21.11} \approx 0.86$$

Large values of the sample correlation ratios mean that there is essential functional dependence between η and ξ, which is (by virtue of the preceding result $r_s = r_{\xi\eta}^* = 0.84$) a linear one.

POINT 3. LEAST-SQUARES METHOD

Let's suppose that results of trials on two random variables ξ and η lead to a hypothesis of existing of correlation dependence of the form $m_\eta(x) = f(x, a, b, ...)$ between ξ and η with unknown parameters a, b,… . It's necessary to find a, b,… by the best way. This way is the least-squares method which was devises by Legendre[1] and Gauss[2] and jus-tified by Gauss.

The simplest case. Every pair of random variables was observed only one time[3].

Let each pair $(\xi = x_i, \eta = y_i)$ of the random variables ξ, η was occurred only one time.

To each value x_i of the random variable $\xi = x_i$ there correspond: a) a theoretical value $m_\eta(x_i) = f(x_i, a, b, ...)$ because of the hypothesis; b) the value y_i of η as the result of trials on ξ, η. The difference

$$\varepsilon_i = f(x_i, a, b, ...) - y_i \qquad (23)$$

is called the error, or the deficiency [the residual]. The substance [the essentiality] of the least-squares method is to minimize the sum of squares of the errors [deficiencies, residuals], that is to minimize the next function of the parameters a, b,…

$$\Phi(a, b, ...) = \sum_i \varepsilon_i^2 = \sum_i \left(f(x_i, a, b, ...) - y_i\right)^2. \qquad (24)$$

In accordance with the theory of functions of several variables we have to solve a system of equations

$$\begin{cases} \Phi'_a = 0, \\ \Phi'_b = 0, \\ .... \end{cases} \qquad (25)$$

a) Let's suppose that we have hypothesized $m_\eta(x) = ax + b$. Therefore we must

[1] Legendre, A.M. (1752 - 1833), a French mathematician
[2] Gauss, K.F. (1777 - 1855), a great German mathematician, astronomer, physicist, and land-surveyor
[3] The case of non-integrated data.

minimize the function

$$\Phi(a,b)=\sum_i (ax_i+b-y_i)^2 \qquad (26)$$

Its partial derivatives with respect to *a* and *b* are equal to

$$\Phi'_a=\sum_i 2(ax_i+b-y_i)x_i=2\left(a\sum_i x_i^2+b\sum_i x_i-\sum_i x_i y_i\right),$$

$$\Phi'_b=\sum_i 2(ax_i+b-y_i)\cdot 1=2\left(a\sum_i x_i+nb-\sum_i y_i\right).$$

Equating them to zero gives the system of linear equations in *a* and *b*

$$\begin{cases} a\sum_i x_i^2+b\sum_i x_i=\sum_i x_i y_i, \\ a\sum_i x_i+nb=\sum_i y_i, \end{cases} \qquad (27)$$

which is called the normal system of the least-squares method.

Ex. 5. In the result of $n = 5$ trials one has obtained the next pairs of values of random variables ξ,η:

$$(0.0;\,2.9),(1.0;\,6.3),(1.5;\,7.9),(2.1;\,10.0),(3.0;\,13.2).$$

Disposition of points

$$M_1(0.0;2.9), M_2(1.0;6.3),\ M_3(1.5;7.9), M_4(2.1;10.0), M_5(3.0;13.2)$$

(construct them yourselves!) suggests the hypothesis that

$$m_\eta(x)=ax+b.$$

To find *a*, *b* we'll make use of the least-squares method (see the table 6)

By virtue of (27) we compile the system of equations

$$\begin{cases} 16.66a+7.60b=78.75, \\ 7.60a+5.00b=40.30, \end{cases}$$

which solving gives $a=3.42, b=2.86,\quad m_\eta(x)=3.42x+2.86$.

The sixth column of the table 6 contains the values y_i^* of $m_\eta(x)$ calculated by the found formula, and the seventh column the moduli of differences

$$\Delta y_i=y_i-y_i^*.$$

Small values of these differences indicate the well coincidence of the experimental and theoretical results.

Table 6

№	x_i	y_i	x_i^2	$x_i y_i$	y_i^*	$\lvert\Delta y_i\rvert$
1	0.0	2.9	0.00	0.00	2.86	0.04
2	1.0	6.3	1.00	6.30	6.28	0.02
3	1.5	7.9	2.25	11.85	7.99	0.09
4	2.1	10.0	4.41	21.00	10.04	0.04
5	3.0	13.2	9.00	39.60	13.12	0.08
Σ	7.6	40.3	16.66	78.75		

Let's study the normal system (27) theoretically. Dividing both its equations by n, we get

$$\begin{cases} a\frac{1}{n}\sum_i x_i^2 + b\frac{1}{n}\sum_i x_i = \frac{1}{n}\sum_i x_i y_i, \\ a\frac{1}{n}\sum_i x_i + b = \frac{1}{n}\sum_i y_i; \end{cases} \quad \begin{cases} a\overline{\xi_s^2} + b\overline{\xi}_s = \frac{1}{n}\sum_i x_i y_i, \\ a\overline{\xi}_s + b = \overline{\eta}_s. \end{cases}$$

Using the formulas (16), (19 b) we have

$$\Delta = \begin{vmatrix} \overline{\xi_s^2} & \overline{\xi}_s \\ \overline{\xi}_s & 1 \end{vmatrix} = \overline{\xi_s^2} - \left(\overline{\xi}_s\right)^2 = D_{\xi s} = \sigma_{\xi s}^2,$$

$$\Delta_1 = \begin{vmatrix} \frac{1}{n}\sum_i x_i y_i & \overline{\xi}_s \\ \overline{\eta}_s & 1 \end{vmatrix} = \frac{1}{n}\sum_i x_i y_i - \overline{\xi}_s\overline{\eta}_s = K_s = r_s\sigma_{\xi s}\sigma_{\eta s},$$

and by Cramer's rule

$$a = \frac{\Delta_1}{\Delta} = \frac{r_s\sigma_{\xi s}\sigma_{\eta s}}{\sigma_{\xi s}^2} = \frac{\sigma_{\eta s}}{\sigma_{\xi s}} r_s,\ b = \overline{\eta}_s - a\overline{\xi}_s = \overline{\eta}_s - \frac{\sigma_{\eta s}}{\sigma_{\xi s}} r_s\overline{\xi}_s.$$

Therefore,

$$m_\eta(x) = ax + b = \frac{\sigma_{\eta s}}{\sigma_{\xi s}} r_s x + \overline{\eta}_s - \frac{\sigma_{\eta s}}{\sigma_{\xi s}} r_s\overline{\xi}_s; \quad m_\eta(x) = \frac{\sigma_{\eta s}}{\sigma_{\xi s}} r_s\left(x - \overline{\xi}_s\right) + \overline{\eta}_s.$$

Obtained regression function (the sample, or the statistical regression function) completely corresponds to theoretical regression function given by the formula (13).

If we apply the least-squares method to find the sample regression function of the form

$$m_\xi(y) = cx + d\,,$$

we'll obtain the next result

$$m_\xi(y) = \frac{\sigma_{\xi s}}{\sigma_{\eta s}} r_s\left(y - \overline{\eta}_{s\,s}\right) + \overline{\xi}\,,$$

which corresponds to the second of the formulas (13).

b) Now let's state a hypothesis of curvilinear, namely parabolic, correlation

$$m_\eta(x) = ax^2 + bx + c\,.$$

In this case we have to minimize the function

$$\Phi(a,b,c) = \sum_i \left(ax_i^2 + bx_i + c + y_i\right)^2, \tag{28}$$

and the same method leads to the next system of equations in a, b, c:

$$\begin{cases} a\sum_i x_i^4 + b\sum_i x^3 + c\sum_i x_i^2 = \sum_i x_i^2 y_i \\ a\sum_i x_i^3 + b\sum_i x_i^2 + c\sum_i x_i = \sum_i x_i y_i \\ a\sum_i x_i^2 + b\sum_i x_i + cn = \sum_i y_i \end{cases} \tag{29}$$

General case

Now we'll consider the **general case** when to arbitrary value of a random variable ξ it corresponds as the rule several values of a random variable η, or well an arbitrary pair of values ($\xi = x_i, \eta = y_j$) of random variables ξ, η appears n_{ij} times[1] (see the table 1).

In conditions of a hypothesis $m_\eta(x) = f(x, a, b, ...)$ to every of n_{x_i} values $\xi = x_i$ there corresponds: a) a theoretical value $m_\eta(x_i) = f(x_i, a, b, ...)$ because of the hypothesis; b) the conditional mean $\overline{\eta}_{x=x_i} = m_\eta^*(x_i)$. Hence there are n_{x_i} errors [deficiencies, residuals]

[1] The case of integrated data.

$$\varepsilon_i = f(x_i, a, b, ...) - \overline{\eta}_{x=x_i} = f(x_i, a, b, ...) - m_\eta^*(x_i),$$

and the question is to minimize a function

$$\Phi(a, b, ...) = \sum_i \left(f(x_i, a, b, ...) - \overline{\eta}_{x=x_i}\right)^2 n_{x_i} = \sum_i \left(f(x_i, a, b, ...) - m_\eta^*(x_i)\right)^2 n_{x_i}, \quad (30)$$

the problem which is reduced to solving the system of the form (25).

a) Finding parameters of a **linear correlation**.

If we hypothesize that

$$m_\eta(x) = ax + b,$$

we'll have to minimize the function

$$\Phi(a, b) = \sum_i \left(ax_i + b - \overline{\eta}_{x=x_i}\right)^2 n_{x_i} = \sum_i \left(ax_i + b - m_\eta^*(x_i)\right)^2 n_{x_i}, \quad (31)$$

whence it follows the next system of equations:

$$\begin{cases} a\sum_i x_i^2 n_{x_i} + b\sum_i x_i n_{x_i} = \sum_i x_i m_\eta^*(x_i) n_{x_i}, \\ a\sum_i x_i n_{x_i} + bn = \sum_i m_\eta^*(x_i) n_{x_i}. \end{cases} \quad (32)$$

After dividing by n and corresponding transformations we obtain the same result

$$m_\eta(x) = \frac{\sigma_{\eta s}}{\sigma_{\xi s}} r_s (x - \overline{\xi}_s) + \overline{\eta}_s. \quad (33)$$

By analogous way one gets

$$m_\xi(y) = \frac{\sigma_{\xi s}}{\sigma_{\eta s}} r_s (y - \overline{\eta}_s) + \overline{\xi}_s, \quad (34)$$

proceeding from the hypothesis

$$m_\xi(y) = cy + d.$$

Ex. 6. Let's apply the least-squares method to find parameters a, b of a linear regression function $m_\eta(x) = ax + b$ of the example 4 (see the table 7).

By virtue of the system (32) we get the next system of linear equations in a, b:

$$\begin{cases} 263600a + 4680b = 75901.8 \\ 4680a + 100b = 1513.94, \end{cases}$$

whence $a = 0.113, b = 9.852$. Therefore the sample (statistical) regression function (and corresponding regression equation) of η on ξ

$$y = m_\eta(x) = 0.113x + 9.852.$$

Table 7

№	x_i	n_{x_i}	$x_i n_{x_i}$	x_i^2	$x_i^2 n_{x_i}$	$m_\eta^*(x_i)$	$m_\eta^*(x_i) n_{x_i}$	$x_i m_\eta^*(x_i) n_{x_i}$	$m_\eta(x_i)$	Δ_i
1	10	14	140	100	1400	10.85	151.90	1519.0	10.98	0.13
2	30	23	690	900	20700	13.22	304.06	9121.8	13.24	0.02
3	50	28	1400	2500	70000	15.71	439.88	21994.0	15.50	0.21
4	70	35	2450	4900	171500	17.66	618.10	43267.0	17.76	0.10
Σ		100	4680		263600		1513.94	75901.8		

Let's remark that we can get this result from the formula (33), namely

$$y = m_\eta(x) = \frac{2.83}{21.11} \cdot 0.84(x - 46.80) + 15.14, \; y = m_\eta(x) = 0.113x + 9.852.$$

The sample (statistical) regression function (and corresponding regression equation) of ξ on η we obtain from the formula (34),

$$x = m_\xi(y) = \frac{21.11}{2.83} \cdot 0.84(y - 15.14) + 46.80, \quad x = m_\xi(y) = 6.27y - 48.07.$$

b) Finding parameters of a curvilinear, namely parabolic, correlation.

For a hypothesis of a parabolic correlation $m_\eta(x) = ax^2 + bx + c$ we have to minimize a function $\Phi(a,b,c) = \sum_i \left(ax_i^2 + bx_i + c - m_\eta^*(x_i)\right)^2 n_{x_i}$ whence it follows

$$\begin{cases} a\sum_i x_i^4 n_{x_i} + b\sum_i x_i^3 n_{x_i} + c\sum_i x_i^2 n_{x_i} = \sum_i x_i^2 m_\eta^*(x_i) n_{x_i}, \\ a\sum_i x_i^3 n_{x_i} + b\sum_i x_i^2 n_{x_i} + c\sum_i x_i n_{x_i} = \sum_i x_i m_\eta^*(x_i) n_{x_i}, \\ a\sum_i x_i^2 n_{x_i} + b\sum_i x_i n_{x_i} + cn = \sum_i m_\eta^*(x_i) n_{x_i}. \end{cases}$$

POINT 4. MULTIPLE CORRELATION.

If there is correlation dependence between several (more then two) random variables, one says about multiple correlation. We'll confine ourselves to study correlation dependence between three random variables ξ, η, ζ. With this purpose we introduce corresponding conditional mathematical expectations (regressions).

Mathematical expectation of a random variable ζ under condition that a random variable ξ takes on a value x and a random variable η takes on a value y, that is

$$m_\zeta(x,y) = M(\zeta / (\xi = x)(\eta = y)),$$

is called the regression function of ζ on ξ, η. Analogously the regression functions $m_\xi(y,z)$, $m_\eta(x,z)$ of ξ on η, ζ and η on ξ, ζ are introduced. All these functions determine dependence between values of two random variables and corresponding means values of the third one.

At the same time one considers conditional mathematical expectations of one random variable in conditions that the other takes on some value and the value of the third one is fixed. For example, $m_\zeta(x/y)$ is the mathematical expectations of ξ on conditions that ξ takes on a value x, and a value of η is fixed and equals y,

$$m_\zeta(x/y) = M(\zeta / (\xi = x)(\eta = y \; is \; fixed)), \; m_\zeta(y/x) = M(\zeta / (\eta = y)(\xi = x \; is \; fixed)).$$

Main problems of multiple correlation

1. Determination the form of correlation dependence between ξ, η, ζ.
2. Ascertaining the closeness of connection between ξ, η, ζ.
3. Finding out the closeness of relation between ξ, ζ for fixed $\eta = y$ or between η, ζ for fixed $\xi = x$.

Results of n independent trials on three random variables ξ, η, ζ we represent by a system of tables. Every table settles, for example, the number of observed pairs $\xi = x_i, \eta = y_j$ for a value $\zeta = z_k$. For sufficiently general example we'll consider the next table system

ζ	$\zeta = z_1$				$\zeta = z_2$				$\zeta = z_3$			
ξ / η	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y
y_1	3			3	3	1		4	1	3		4
y_2	5	1	4	10	5	2	1	8	4	3		7
y_3			3	3			4	4		2	5	7
	8	1	7	16	8	3	5	16	5	8	5	18

From the table system we at first compile the variation series for each component

ξ	x_1	x_2	x_3	η	y_1	y_2	y_3	ζ	z_1	z_2	z_3
n_x	21	12	17	n_y	11	25	14	n_z	16	16	18

and find statistic estimates of their number characteristics (sample means [averages], dispersions and root-mean-square deviations)

$$\bar{\xi}_s, \bar{\eta}_s, \bar{\zeta}_s; D_{\xi s}, D_{\eta s}, D_{\zeta s}, \ \sigma_{\xi s}, \sigma_{\eta s}, \sigma_{\zeta s}$$

or omitting s for the sake of simplicity

$$\bar{\xi}, \bar{\eta}, \bar{\zeta}; D_{\xi}, D_{\eta}, D_{\zeta}, \ \sigma_{\xi}, \sigma_{\eta}, \sigma_{\zeta}.$$

Then we form the correlation tables for each pair of components and obtain the

η \ ξ	x_1	x_2	x_3	n_y
y_1	7	4		*11*
y_2	14	6	5	*25*
y_3		2	12	*14*
n_x	*21*	*12*	*17*	*n=50*

η \ ζ	z_1	z_2	z_3	n_y
y_1	3	4	4	11
y_2	10	8	7	25
y_3	3	4	7	14
n_z	16	16	18	*n=* 50

ζ \ ξ	x_1	x_2	x_3	n_z
z_1	8	1	7	*16*
z_2	8	3	5	*16*
z_3	5	8	5	*18*
n_x	*21*	*12*	*17*	*n=50*

sample correlation coefficients (without index s)

$$r_{\xi\eta} = \frac{K_{\xi\eta}}{\sigma_\xi \sigma_\eta}, r_{\xi\zeta} = \frac{K_{\xi\zeta}}{\sigma_\xi \sigma_\zeta}, r_{\eta\zeta} = \frac{K_{\eta\zeta}}{\sigma_\eta \sigma_\zeta}. \tag{35}$$

Here (if we denote n_{ij} the number of occurrences of a pair $(\xi = x_i, \eta = y_j)$, n_{ik} that of a pair $(\xi = x_i, \zeta = z_k)$, n_{jk} that of a pair $(\eta = y_j, \zeta = z_k)$)

$$K_{\xi\eta} = \frac{1}{n}\sum_{i,j}(x_i - \bar{\xi})(y_j - \bar{\eta})n_{ij} = \frac{1}{n}\sum_{i,j} x_i y_j n_{ij} - \bar{\xi}\bar{\eta}, \tag{36 a}$$

$$K_{\xi\zeta} = \frac{1}{n}\sum_{i,k}(x_i - \bar{\xi})(z_k - \bar{\zeta})n_{ik} = \frac{1}{n}\sum_{i,k} x_i z_k n_{ik} - \bar{\xi}\bar{\zeta}, \tag{36 b}$$

$$K_{\eta\zeta} = \frac{1}{n}\sum_{j,k}(y_j - \bar{\eta})(z_k - \bar{\zeta})n_{jk} = \frac{1}{n}\sum_{j,k} y_j z_k n_{jk} - \bar{\eta}\bar{\zeta} \tag{36 c}$$

are the sample correlation moments of the pairs $\xi\eta, \xi\zeta, \eta\zeta$ of the random variables respectively.

For the case of a linear correlation dependence between ξ, η, ζ we'll find a linear regression function of ζ on ξ, η by the least-squares method.

The closeness of a linear correlation dependence between ξ, η, ζ is estimated by so-called **total correlation coefficient**

$$R = \sqrt{\frac{r_{\xi\zeta}^2 - 2r_{\xi\eta} r_{\eta\zeta} r_{\xi\zeta} + r_{\eta\zeta}^2}{1 - r_{\xi\eta}^2}}. \qquad (37)$$

It's known that

$$0 \le R \le 1;$$

if the random variables ξ, η, ζ are independent, then $R = 0$; if between ξ, η and ζ there is a functional linear dependence, then $R = 1$.

The closeness of linear correlation dependence between two random variables ξ and ζ for fixed η ($\eta = y$) or between η and ζ for fixed ξ ($\xi = x$) are estimated by **partial correlation coefficients**, namely

$$r_{\xi\zeta(y)} = \frac{r_{\xi\zeta} - r_{\xi\eta} r_{\eta\zeta}}{\sqrt{\left(1 - r_{\xi\eta}^2\right)\left(1 - r_{\eta\zeta}^2\right)}}, \quad r_{\eta\zeta(x)} = \frac{r_{\eta\zeta} - r_{\xi\eta} r_{\xi\zeta}}{\sqrt{\left(1 - r_{\xi\eta}^2\right)\left(1 - r_{\xi\zeta}^2\right)}}. \qquad (38)$$

Least-squares method in multiple correlation

Let's now find a linear regression function (more precisely: a sample linear regression function) of ζ on ξ, η, which we'll seek in the next form

$$z = m_\zeta(x, y) = a\left(x - \overline{\xi}\right) + b\left(y - \overline{\eta}\right) + \overline{\zeta} \qquad (39)$$

with unknown parameters a, b. For the sake of simplicity we suppose the data not to be integrated that in result of n trials on the random variables ξ, η, ζ n different triplets of values were appeared, namely

$$\xi = x_i, \eta = y_i, \zeta = z_i, \quad i = \overline{1, n}.$$

In this case to the values $\xi = x_i, \eta = y_i$ there correspond: a) the theoretical value

$$m_\zeta(x_i, y_i) = a\left(x_i - \overline{\xi}\right) + b\left(y_i - \overline{\eta}\right) + \overline{\zeta} \qquad (40)$$

and b) the value $\zeta = z_i$ which is obtained as the result of trials. The corresponding error [deficiency, residual] is

$$\varepsilon_i = a\left(x_i - \overline{\xi}\right) + b\left(y_i - \overline{\eta}\right) + \overline{\zeta} - z_i,$$

and it's necessary to minimize the sum of squares of the errors [deficiencies, residuals], that is to minimize the next function of two variables *a* and *b*:

$$\Phi(a, b) = \sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \left(a\left(x_i - \overline{\xi}\right) + b\left(y_i - \overline{\eta}\right) + \overline{\zeta} - z_i\right)^2. \qquad (41)$$

Let's write formulas which give sample dispersions D_ξ, D_η and correlation moments $K_{\xi\eta}, K_{\xi\zeta}, K_{\eta\zeta}$ for the case of non-integrated data, namely:

$$D_\xi = \sigma_\xi^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \overline{\xi_s}\right)^2, D_\eta = \sigma_\eta^2 = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \overline{\eta_s}\right)^2,$$

$$K_{\xi\eta} = \frac{1}{n}\sum_i\left(x_i - \overline{\xi}\right)\left(y_i - \overline{\eta}\right), K_{\xi\zeta} = \frac{1}{n}\sum_i\left(x_i - \overline{\xi}\right)\left(z_i - \overline{\zeta}\right), K_{\eta\zeta} = \frac{1}{n}\sum_i\left(y_i - \overline{\eta}\right)\left(z_i - \overline{\zeta}\right).$$

$$\Phi'_a = \sum_{i=1}^{n} 2\left(a\left(x_i - \overline{\xi}\right) + b\left(y_i - \overline{\eta}\right) + \overline{\zeta} - z_i\right)\cdot\left(x_i - \overline{\xi}\right) =$$
$$= 2\left(a\sum_{i=1}^{n}\left(x_i - \overline{\xi}\right)^2 + b\sum_{i=1}^{n}\left(x_i - \overline{\xi}\right)\left(y_i - \overline{\eta}\right) - \sum_{i=1}^{n}\left(x_i - \overline{\xi}\right)\left(z_i - \overline{\zeta}\right)\right),$$

$$\Phi'_b = \sum_{i=1}^{n} 2\left(a\left(x_i - \overline{\xi}\right) + b\left(y_i - \overline{\eta}\right) + \overline{\zeta} - z_i\right)\cdot\left(y_i - \overline{\eta}\right) =$$
$$= 2\left(a\sum_{i=1}^{n}\left(x_i - \overline{\xi}\right)\left(y_i - \overline{\eta}\right) + b\sum_{i=1}^{n}\left(y_i - \overline{\eta}\right)^2 - \sum_{i=1}^{n}\left(y_i - \overline{\eta}\right)\left(z_i - \overline{\zeta}\right)\right),$$

Equating of partial derivatives Φ'_a, Φ'_b to zero leads to a system of linear equations in *a*, *b*

$$\begin{cases} a\sum_{i=1}^{n}\left(x_i - \overline{\xi}\right)^2 + b\sum_{i=1}^{n}\left(x_i - \overline{\xi}\right)\left(y_i - \overline{\eta}\right) = \sum_{i=1}^{n}\left(x_i - \overline{\xi}\right)\left(z_i - \overline{\zeta}\right), \\ a\sum_{i=1}^{n}\left(x_i - \overline{\xi}\right)\left(y_i - \overline{\eta}\right) + b\sum_{i=1}^{n}\left(y_i - \overline{\eta}\right)^2 = \sum_{i=1}^{n}\left(y_i - \overline{\eta}\right)\left(z_i - \overline{\zeta}\right). \end{cases} \qquad (42)$$

Dividing by *n* we get

$$\begin{cases} a\cdot\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{\xi})^2+b\cdot\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{\xi})(y_i-\bar{\eta})=\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{\xi})(z_i-\bar{\zeta}),\\ a\cdot\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{\xi})(y_i-\bar{\eta})+b\cdot\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{\eta})^2=\frac{1}{n}\sum_{i=1}^{n}(y_i-\bar{\eta})(z_i-\bar{\zeta}),\end{cases}$$

or by preceding formulas

$$\begin{cases} aD_\xi+bK_{\xi\eta}=K_{\xi\zeta},\\ aK_{\xi\eta}+bD_\eta=K_{\eta\zeta};\end{cases}\quad \begin{cases} a\sigma_\xi^2+br_{\xi\eta}\sigma_\xi\sigma_\eta=r_{\xi\zeta}\sigma_\xi\sigma_\zeta,\\ ar_{\xi\eta}\sigma_\xi\sigma_\eta+b\sigma_\eta^2=r_{\eta\zeta}\sigma_\eta\sigma_\zeta;\end{cases}\quad \begin{cases} a\sigma_\xi+br_{\xi\eta}\sigma_\eta=r_{\xi\zeta}\sigma_\zeta,\\ ar_{\xi\eta}\sigma_\xi+b\sigma_\eta=r_{\eta\zeta}\sigma_\zeta.\end{cases}$$

Finally on the base of Cramer rule we find corresponding main and auxilliary determinants

$$\Delta=\begin{vmatrix}\sigma_\xi & r_{\xi\eta}\sigma_\eta\\ r_{\xi\eta}\sigma_\xi & \sigma_\eta\end{vmatrix}=\sigma_\xi\sigma_\eta\left(1-r_{\zeta\eta}^2\right),$$

$$\Delta_1=\begin{vmatrix}r_{\xi\zeta}\sigma_\zeta & r_{\xi\eta}\sigma_\eta\\ r_{\eta\zeta}\sigma_\zeta & \sigma_\eta\end{vmatrix}=\sigma_\eta\sigma_\zeta\left(r_{\xi\zeta}-r_{\xi\eta}r_{\eta\zeta}\right),\ \Delta_2=\begin{vmatrix}\sigma_\xi & r_{\xi\zeta}\sigma_\zeta\\ r_{\xi\eta}\sigma_\xi & r_{\eta\zeta}\sigma_\zeta\end{vmatrix}=\sigma_\xi\sigma_\zeta\left(r_{\eta\zeta}-r_{\xi\eta}r_{\xi\zeta}\right)$$

and therefore

$$a=\frac{\sigma_\eta\sigma_\zeta\left(r_{\xi\zeta}-r_{\xi\eta}r_{\eta\zeta}\right)}{\sigma_\xi\sigma_\eta\left(1-r_{\xi\eta}^2\right)}=\frac{\sigma_\zeta}{\sigma_\xi}\cdot\frac{r_{\xi\zeta}-r_{\xi\eta}r_{\eta\zeta}}{1-r_{\xi\eta}^2},b=\frac{\sigma_\xi\sigma_\zeta\left(r_{\eta\zeta}-r_{\xi\eta}r_{\xi\zeta}\right)}{\sigma_\xi\sigma_\eta\left(1-r_{\xi\eta}^2\right)}=\frac{\sigma_\zeta}{\sigma_\eta}\cdot\frac{r_{\eta\zeta}-r_{\xi\eta}r_{\xi\zeta}}{1-r_{\xi\eta}^2}.\quad(43)$$

Thus, the (sample) linear regression function of ζ on ξ,η is

$$z=m_\zeta(x,y)=\frac{\sigma_\zeta}{\sigma_\xi}\cdot\frac{r_{\xi\zeta}-r_{\xi\eta}r_{\eta\zeta}}{1-r_{\xi\eta}^2}\cdot\left(x-\bar{\xi}\right)+\frac{\sigma_\zeta}{\sigma_\eta}\cdot\frac{r_{\eta\zeta}-r_{\xi\eta}r_{\xi\zeta}}{1-r_{\xi\eta}^2}\cdot\left(y-\bar{\eta}\right)+\bar{\zeta}.\quad(44)$$

Ex. Statistical analysis of results of trials on three random variables ξ,η,ζ have given the next data:

a) sample means $\bar{\xi}=0.05,\quad \bar{\eta}=0.025,\quad \bar{\zeta}=0.02$;

b) sample root-mean-square deviations $\sigma_\xi=0.015,\quad \sigma_\eta=0.06,\quad \sigma_\zeta=0.01$;

c) paired sample correlation coefficients $r_{\xi\eta}=0.5,\quad r_{\xi\zeta}=0.6,\quad r_{\eta\zeta}=0.4$.

By the formulas (43) $a=0.36, b=0.022$, and the sample equation of regression of ζ on ξ,η is

$$z=m_\zeta(x,y)=0.36(x-0.05)+0.022(y-0.025)+0.02,\ z=0.36x+0.022y+0.0015.$$

The total correlation coefficient on the base of the formula (37) equals

$$R = 0.61.$$

It's large enough and testifies to essential linear dependence between ξ, η and ζ.

The sample partial correlation coefficients by virtue of (38) equal

$$r_{\xi\zeta(y)} = 0.5, \quad r_{\eta\zeta(x)} = 0.14,$$

whence it follows that a linear dependence between ξ, ζ is more essential than between η, ζ.

CORRELATION THEORY: basic terminology RUEFD

1. выборочное корреляционное отношение	вибіркове кореляційне відношення	sample còrrelátion rátio	rapport *m* de corrélation d'échantillonnage	stichprobenartiges Korrelationsverhältnis *n*
2. выборочное уравнение регрессии	вибіркове рівняння регресії	sample regréssion equation	équation *f* de régression d'échantillonnage	stichprobenartige Regressionsgleichung *f*
3. выборочный корреляционный момент	вибірковий кореляційний момент	sample correlátive [còrrelátion] móment	moment *m* corrélatif d'échantillonnage	stichprobenartiges Korrelationsmoment *n*
4. выборочный коэффициент корреляции	вибірковий коефіцієнт кореляції	sample còrrelátion còefficient	coefficient de corrélation d'échantillonnage	stichprobenartiger Korrelationskoeffizient *m*
5. детерминированная зависимость между двумя случайными величинами	детермінована залежність між двома випадковими величинами	deterministic depéndence between two rándom váriables	dépendence *f* déterminée entre deux variables aléatoires	determinierte Äbhängigkeit *f* zwischen zwei Zúfallsgrößen
6. исправленный корреляционный момент	виправлений кореляційний момент	corréсted correlátive [còrrelátion] móment	moment *m* corrélatif corrigé	korrigíertes Korrelationsmoment *n*
7. исправленный коэффициент корреляции	виправлений коефіцієнт кореляції	corréсted còrrelátion còefficient	coefficient *m* de corrélation corrigé	korrigíerter Korrelationskoeffizi¦ent *m*
8. коррелированные случайные величины	корельовані випадкові величини	correlated random variables	variables *f* aléatoires corréléеs	korrelierte Zufallsvariablen *f pl*
9. коррелированный	корельований	correlated	corrélé	korreliert
10. корреляционная зависимость	кореляційна залежність	correlation [correlátive] depéndence	dépendance *f* corrélative	Korrelationsabhängigkeit *f*
11. корреляционная таблица	кореляційна таблиця	còrrelátion table	table *f* corrélative, tableau *m* corrélatif	Korrelationstabelle *f*

12.корреляционное отношение	кореляційне відношення	correlation rátio	rapport *m* de corrélation	Korrelationsverhältnis *n*
13.корреляционный момент	кореляційний момент	còrrelátion [correlátive] móment	moment *m* corrélatif	Korrelationsmoment *n*
14.корреляция	кореляція	correlation	corrélation *f*	Korrelation *f*
15.коэффициент корреляции	коефіцієнт кореляції	correlation còefficient	coefficient *m* de corrélation	Korrelationskoeffizi¦ent *m*
16.кривая регрессии, линия регрессии	крива [лінія] реґресії	regression curve, regréssion line	courbe *f* de régression, ligne *f* de régression	Regressionskurve *f* , Regressionslinie *f*, Beziehungslinie *f*
17.криволинейная кореляция	криволінійна кореляція	curvilinear [non-linear] còrrelátion	corrélation *f* curviligne	krummlinige [nichtlinere] Korrelation *f*
18.линейная корреляция	лінійна кореляція	linear còrrelátion	corrélation *f* linéaire	lineare Korrelation *f*
19.линия регрессии	лінія реґресії	regréssion line	ligne *f* de régression	Regressionslinie *f* , Beziehungslinie *f*
20.мера функциональной зависимости между двумя случайными величинами	міра функціональної залежності між двома випадковими величинами	méasure of fúnctional depéndence between two rándom váriables	mesure *f* de dépendence *f* fonctionnelle entre deux variables aléatoires	Maß *n* von Funktionalabhängigkeit *f* zwischen zwei Zúfallsgrößen
21.мера линейной зависимости между двумя случайными величинами	міра лінійної залежності між двома випадковими величинами	méasure of línear depéndence between two rándom váriables	mesure *f* de dépendence *f* linéaire entre deux variables aléatoires	Maß *n* von linéarer Abhängigkeit *f* zwischen zwei Zúfallsgrößen
22.метод наименьших квадратов	метод найменших квадратів	least-squares method,method of least squares	méthode *f* des moindres carrés	Methóde *f* der kleinsten Quadrate
23.минимизировать	мінімізувати	minimize	minimiser	minimieren, minimisieren
24.множественная кореляция	множинна коряція	multiple [multivariable] correlation	corrélation *f* multiple	mehrfache [multiple] Korrelation *f*

25.некоррелированные случайные величины	некорельовані випадкові величини	uncorrelated random variables	variables *f* aléatoires non-corrélées	nichtkorrelierte [unkorrelierte] Zufallsgrößen [Zufallsvariablen]
26.некоррелированный	некорельований	uncorrelated	non-corrélé	nichtkorreliert, unkorreliert
27.нормальная система метода наименьших квадратов	нормальна система методу найменших квадратів	normal system of the least-squares method	système *m* normal de la méthode *f* des moindres carrés	normales System *n* der Methóde der kleinsten Quadrate
28.ошибка	помилка	érror	erreur *f*	Féhler *m* –s, =
29.прямая регрессии	пряма реґресії	regréssion straight line	ligne droite de régression	Regressionsgerade *f* , Ausgleichende Gerade *f*
30.регрессия	реґресія	regréssion	régression *f*	Regression *f*
31.сумма квадратов	сума квадратів	sum of squares	somme *f* des carrés	Quadrátsumme *f*
32.сумма квадратов ошибок	сума квадратів помилок	sum of squares of the érrors	somme des carrés des erreurs	Quadrátsumme von Féhler
33.теория корреляции	теорія кореляції	còrrelátion theory	théorie *f* de corrélation *f*	Korrelationstheorie *f*
34.уравнение линейной регрессии	рівняння лінійної реґресії	equation of linear regréssion	équation *f* de régression linéaire	linearer Regressionsgleichung *f*
35.уравнение регрессии	рівняння регресії	regréssion equation	équation *f* de régression	Regressionsgleichung *f*
36.условная вероятность	умовна ймовірність	conditional probability	probabilité *f* conditionée, conditionnelle	bedingte Wahrscheinlicheit *f*
37.условная плотность распределения	умовна щільність розподілу	condítional dìstribútion dénsity	densité *f* conditionnelle de distribution *f* [de répartition *f*]	bedingte Verteilungsdichte *f*
38.условная функция распределения	умовна функція розподілу	condítional dìstribútion fúnction	fonction *f* conditionnelle de distribution *f* [de répartition *f*]	bedingte Verteilungsfunktion *f*
39.условное	умовне мате-	condítional	espérance *f*	Bedingter Er-

математичес-кое ожидание	матичне спо-дівання	màthemátical èxpectátion	mathématique conditionnelle	wartungswert *m* [bedingte mathematische Erwartung *f*]
40.условное распределе-ние	умовний роз-поділ	condítional dìs-tribútion	distribution *f* conditionnelle	bedingte Ver-teilung *f*
41.условное среднее	умовна серед-ня	conditional mean, condi-tional average	moyenne con-ditionnée [con-ditionnelle]	bedingtes Mit-tel *n*, beding-ter Mittelwert *m*
42.условный закон распре-деления	умовний за-кон розподілу	condítional law of dìstribútion	loi *f* condition-nelle de distri-bution *f* [de ré-partition *f*]	bedingtes Ver-teilungsgesetz *n*
43.функцио-нальная зави-симость меж-ду двумя слу-чайными ве-личинами	функціональ-на залежність між двома ви-падковими ве-личинами	fúnctional de-péndence bet-ween two rán-dom váriables	dépendence *f* fonctionnelle entre deux va-riables aléatoi-res	Funktionalab-hängigkeit *f* zwischen zwei Zúfallsgrößen
44.функция регрессии	функція рег-ресії	regréssion fun-ction	fonction de ré-gression	Regressions-funktion *f*

CORRELATION THEORY: basic terminology EFDRU

1. condítional dìstribútion	distribution *f* conditionnelle	bedingte Verteilung *f*	условное распределение	умовний розподіл
2. condítional dìstribútion dénsity	densité *f* conditionnelle de distribution *f* [de répartition *f*]	bedingte Verteilungsdichte *f*	условная плотность распределения	умовна щільність розподілу
3. condítional dìstribútion fúnction	fonction *f* conditionnelle de distribution *f* [de répartition *f*]	bedingte Verteilungsfunktion *f*	условная функция распределения	умовна функція розподілу
4. condítional law of dìstribútion	loi *f* conditionnelle de distribution *f* [de répartition *f*]	bedingtes Verteilungsgesetz *n*	условный закон распределения	умовний закон розподілу
5. condítional màthemátical èxpectátion	espérance *f* mathématique conditionnelle	bedingter Erwartungswert *m* [bedingte mathematische Erwartung *f*]	условное математическое ожидание	умовне математичне сподівання
6. conditional mean, conditional average	moyenne conditionnée [conditionnelle]	bedingtes Mittel *n*, bedingter Mittelwert *m*	условное среднее	умовна середня
7. conditional probability	probabilité *f* conditionée, conditionnelle	bedingte Wahrscheinlicheit *f*	условная вероятность	умовна ймовірність
8. corrécted còrrelátion còefficient	coefficient *m* de corrélation corrigé	korrigíerter Korrelationskoeffizi¦ent *m*	исправленный коэффициент корреляции	виправлений коефіцієнт кореляції
9. corrécted correlátive [còrrelátion] móment	moment *m* corrélatif corrigé	korrigíertes Korrelationsmoment *n*	исправленный корреляционный момент	виправлений кореляційний момент
10.correlated	corrélé	korreliert	коррелированный	корельований
11.correlated random variables	variables *f* aléatoires corrélées	korrelierte Zufallsvariablen *f pl*	коррелированные случайные величины	корельовані випадкові величини

12.correlation	corrélation *f*	Korrelation *f*	корреляция	кореляція
13.correlation [correlátive] depéndence	dépendance *f* corrélative	Korrelations-abhängigkeit *f*	корреляционная зависимость	кореляційна залежність
14.còrrelátion [correlátive] móment	moment *m* corrélatif	Korrelations-moment *n*	корреляционный момент	кореляційний момент
15.correlation còefficient	coefficient *m* de corrélation	Korrelations-koeffizi¦ent *m*	коэффициент корреляции	коефіцієнт кореляції
16.correlation rátio	rapport *m* de corrélation	Korrelations-verhältnis *n*	корреляционное отношение	кореляційне відношення
17.còrrelátion table	table *f* corrélative, tableau *m* corrélatif	Korrelationstabelle *f*	корреляционная таблица	кореляційна таблиця
18.còrrelátion theory	théorie *f* de corrélation *f*	Korrelations-theorie *f*	теория корреляции	теорія кореляції
19.curvilinear [non-linear] còrrelátion	corrélation *f* curviligne	krummlinige [nichtlinere] Korrelation *f*	криволинейная корреляция	криволінійна кореляція
20.deterministic depéndence between two rándom váriables	Dépendence *f* déterminée entre deux variables aléatoires	Determinierte Äbhängigkeit *f* zwischen zwei Zúfallsgrößen	детерминированная зависимость между двумя случайными величинами	детермінована залежність між двома випадковими величинами
21.equation of linear regréssion	équation *f* de régression linéaire	linearer Regressionsgleichung *f*	уравнение линейной регрессии	рівняння лінійної реґресії
22.érror	erreur *f*	Féhler *m* –s, =	ошибка	помилка
23.fúnctional depéndence between two rándom váriables	Dépendence *f* fonctionnelle entre deux variables aléatoires	Funktionalabhängigkeit *f* zwischen zwei Zúfallsgrößen	функцио-нальная зави-симость между двумя случайными ве-личинами	функціональна залежність між двома випадковими величинами
24.least-squares method,method of least squares	méthode *f* des moindres carrés	Methóde *f* der kleinsten Quadrate	метод наименьших квадратов	метод найменших квадратів
25.linear còrrelátion	corrélation *f* linéaire	lineare Korrelation *f*	линейная корреляция	лінійна кореляція
26.méasure of fúnctional de-	Mesure *f* de dépendence *f* fon-	Maß *n* von Funktionalab-	мера функциональной за-	міра функціональної зале-

péndence between two rándom váriables	ctionnelle entre deux variables aléatoires	hängigkeit *f* zwischen zwei Zúfallsgrößen	висимости между двумя случайными величинами	жності між двома випадковими величинами
27.méasure of línear depéndence between two rándom váriables	Mesure *f* de dépendence *f* linéaire entre deux variables aléatoires	Maß *n* von lineárer Abhängigkeit *f* zwischen zwei Zúfallsgrößen	мера линейной зависимости между двумя случайными величинами	міра лінійної залежності між двома випадковими величинами
28.minimize	minimiser	minimieren, minimisieren	Минимизировать	мінімізувати
29.multiple [multivariable] correlation	corrélation *f* multiple	mehrfache [multiple] Korrelation *f*	множественная кореляция	множинна коряція
30.normal system of the least-squares method	système *m* normal de la méthode *f* des moindres carrés	normales System *n* der Methóde der kleinsten Quadrate	нормальная система метода наименьших квадратов	нормальна система методу найменших квадратів
31.regréssion	régression *f*	Regression *f*	регрессия	реґресія
32.regression curve, regréssion line	courbe *f* de régression, ligne *f* de régression	Regressionskurve *f* , Regressionslinie *f*, Beziehungslinie *f*	кривая регрессии, линия регрессии	крива [лінія] реґресії
33.regréssion equation	équation *f* de régression	Regressionsgleichung *f*	уравнение регрессии	рівняння реґресії
34.regréssion function	fonction de régression	Regressionsfunktion *f*	функция регрессии	функція реґресії
35.regréssion line	ligne *f* de régression	Regressionslinie *f* , Beziehungslinie *f*	линия регрессии	лінія реґресії
36.regréssion straight line	ligne droite de régression	Regressionsgerade *f* , Ausgleichende Gerade *f*	прямая регрессии	пряма реґресії
37.sample regréssion equation	équation *f* de régression d'échantillonnage	stichprobenartige Regressionsgleichung *f*	выборочное уравнение регрессии	вибіркове рівняння реґресії
38.sample còrrelátion còefficient	coefficient de corrélation d'échantillonnage	stichprobenartiger Korrelationskoeffizi-	выборочный коэффициент корреляции	вибірковий коефіцієнт кореляції

		ent *m*		
39.sample còr-relátion rátio	rapport *m* de corrélation d'é-chantillonnage	stichprobenar-tiges Korrela-tionsverhältnis *n*	выборочное корреляцион-ное отноше-ние	вибіркове ко-реляційне від-ношення
40.sample correlátive [còrrelátion] móment	moment *m* cor-rélatif d'échan-tillonnage	stichprobenar-tiges Korrela-tionsmoment *n*	выборочный корреляцион-ный момент	вибірковий кореляційний момент
41.sum of squares	somme *f* des carrés	Quadrátsumme *f*	сумма ква-дратов	сума квадра-тів
42.sum of squares of the érrors	somme des carrés des er-reurs	Quadrátsumme von Féhler	сумма ква-дратов оши-бок	сума квадра-тів помилок
43.uncorrela-ted	non-corrélé	nichtkorreliert, unkorreliert	некоррелиро-ванный	некорельова-ний
44.uncorrela-ted random va-riables	variables *f* alé-atoires non-corrélées	nichtkorrelierte [unkorrelierte] Zufallsgrößen [Zufallsvariab-len]	некоррелиро-ванные слу-чайные вели-чины	некорельова-ні випадкові величини

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Tasks for individual work

Linear and curvilinear correlations

PROBLEM. The correlation table, which is the result of trials on a two-dimensional random variable (ξ, η), is given.

1. Form variation series for every component ξ, η, estimate their mathematical expectations, dispersions, root-mean-square deviations.

2. Find all values of conditional means for every component, plot approximate graphs of the regression functions.

3. With the help of the least-squares method estimate parameters of the linear correlation, write the sample regression equations of η on ξ and ξ on η.

4. Using obtained regression equations find the values of the sample regression functions of η on ξ and ξ on η for observed values of ξ and η respectively and differences between these values and corresponding values of the conditional means. Are these differences large? What does this fact indicate about?

5. Estimate the correlation coefficient of the random variables ξ, η. What a conclusion can one draw from the obtained result?

6. Test the significance of the sample correlation coefficient of the random variables ξ, η for a significance level $\alpha = 0.05$.

7. With the help of the least-squares method estimate parameters of the parabolic correlation, write the sample regression equations of η on ξ and ξ on η.

8. Find the values of the sample regression functions of η on ξ and ξ on η for observed values of ξ and η respectively and differences between these values and corresponding values of the conditional means (compare the point 4). Are these differences large? What does this fact indicate about?

9. Estimate the correlation ratio of the random variable η on ξ and ξ on η. Draw a conclusion from the obtained result.

Multiple correlation

PROBLEM. The system of tables, that is the result of trials on three random variables ξ, η, ζ, is given.

1. Form variation series for every random variable, estimate their mathematical expectations, dispersions, root-mean-square deviations.
2. Compile the correlation tables for every pair of the random variables, estimate the correlation coefficients of these pairs.
3. Write the sample equation of the linear regression of ζ on ξ, η.
4. Estimate the total correlation coefficient of the random variables ξ, η, ζ. Do a linear dependence between ξ, η and ζ expressed essentially?
5. Estimate the partial correlation coefficients. Which of two random variables ξ, η has the most influence on ζ?

Variations of tasks

Linear and curvilinear correlations

Linear and curvilinear correlations Variant 1

η \ ξ	10	30	50	70	n_y	$m_\xi^*(y_j)$
10	9	1				
12	4	10	2			
14	1	9	6	1		
16		3	14	10		
18			6	18		
20				6		
n_x					$n =$	
$m_\eta^*(x_i)$						

Linear and curvilinear correlations Variant 2

η \ ξ	10	30	50	70	n_y	$m_\xi^*(y_j)$
10	7	3				
12	2	11	4			
14	3	8	7			
16		1	10	9		
18			5	8		
20				2		
n_x					$n =$	
$m_\eta^*(x_i)$						

Linear and curvilinear correlations Variant 3

$\eta \backslash \xi$	10	30	50	70	n_y	$m_\xi^*(y_j)$
10	2					
12	3	13	3			
14		12	7			
16		4	1	5		
18			6	8		
20				9		
n_x					$n =$	
$m_\eta^*(x_i)$						

Linear and curvilinear correlations Variant 4

$\eta \backslash \xi$	10	30	50	70	n_y	$m_\xi^*(y_j)$
10				2		
12			1	7		
14		8	5			
16	5	7				
18	4	2				
20	9					
n_x					$n =$	
$m_\eta^*(x_i)$						

Linear and curvilinear correlations Variant 5

η \ ξ	10	30	50	70	n_y	$m_\xi^*(y_j)$
10				3		
12			1	5		
14		2	7			
16	1	8	9			
18	5	6				
20	4					
n_x					$n =$	
$m_\eta^*(x_i)$						

Linear and curvilinear correlations Variant 6

η \ ξ	10	30	50	70	n_y	$m_\xi^*(y_j)$
10	4					
12	1	5				
14		2	5			
16		8	2			
18			7	8		
20				1		
n_x					$n =$	
$m_\eta^*(x_i)$						

Linear and curvilinear correlations Variant 7

η \ ξ	10	30	50	70	n_y	$m_\xi^*(y_j)$
10				2		
12				1		
14		2	4	7		
16		4	6			
18	5	7				
20	4					
n_x					$n =$	
$m_\eta^*(x_i)$						

Linear and curvilinear correlations Variant 8

η \ ξ	10	30	50	70	n_y	$m_\xi^*(y_j)$
10	5					
12	4	2				
14		7	8			
16			1	7		
18			6	1		
20				4		
n_x					$n =$	
$m_\eta^*(x_i)$						

Linear and curvilinear correlations Variant 9

η \ ξ	10	30	50	70	n_y	$m_\xi^*(y_j)$
10				1		
12				2		
14			2	4		
16		1	4			
18	3	4	5			
20	5	3				
n_x					$n =$	
$m_\eta^*(x_i)$						

Linear and curvilinear correlations Variant 10

η \ ξ	10	30	50	70	n_y	$m_\xi^*(y_j)$
10	5	4				
12		6	1			
14			8			
16			5			
18				5		
20				3		
n_x					$n =$	
$m_\eta^*(x_i)$						

Multiple correlation

Multiple correlation Variant 1

ζ $\zeta = z_1$ $\zeta = z_2$ $\zeta = z_3$

ξ / η	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y
y_1	3	4			3	1			1	3		
y_2	5	1	4		5	2	1		4	3		
y_3			3				4			2	5	

Multiple correlation Variant 2

ζ $\zeta = z_1$ $\zeta = z_2$ $\zeta = z_3$

ξ / η	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y
y_1			1				3			3	3	
y_2	5	1	4		5	2	1		4	3		
y_3	2				2	1			5			

Multiple correlation Variant 3

ζ $\zeta = z_1$ $\zeta = z_2$ $\zeta = z_3$

ξ / η	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y
y_1	3				3				1			
y_2	2	3	5		5	2	5		4	3	4	
y_3			6				4			1	5	

Multiple correlation Variant 4

ζ $\zeta = z_1$ $\zeta = z_2$ $\zeta = z_3$

ξ / η	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y
y_1	2				3				1	4		
y_2		5	7		1	7				2	2	
y_3		1	4			5	6			1	9	

Multiple correlation Variant 5

ζ $\zeta = z_1$ $\zeta = z_2$ $\zeta = z_3$

ξ / η	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y
y_1	1				3				2			
y_2	6	4	3			2	8		5	3		
y_3		1	10			7	4			5	7	

Multiple correlation Variant 6

ζ $\zeta = z_1$ $\zeta = z_2$ $\zeta = z_3$

ξ / η	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y
y_1			8				2			1	6	
y_2		6	1		1	5	9		1	4		
y_3	2	4			1	9			3			

Multiple correlation Variant 7

ζ | $\zeta = z_1$ | $\zeta = z_2$ | $\zeta = z_3$

ξ / η	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y
y_1			1			5	5			8	7	
y_2		5	2		1	4	2		7	1	2	
y_3	4	1			8				8			

Multiple correlation Variant 8

ζ | $\zeta = z_1$ | $\zeta = z_2$ | $\zeta = z_3$

ξ / η	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y
y_1	3				5	2			6			
y_2	4	2	4			3	2		5	5		
y_3		1	6				7			3	8	

Multiple correlation Variant 9

ζ | $\zeta = z_1$ | $\zeta = z_2$ | $\zeta = z_3$

ξ / η	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y
y_1		3	2			1	8			1	6	
y_2	1	7	6		4	2	9		2	5	2	
y_3	5	1			5				2	3		

Multiple correlation Variant 10

ζ $\zeta = z_1$ $\zeta = z_2$ $\zeta = z_3$

ξ η	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y
y_1	3	2			2				1	3		
y_2	6	8			7	9			4	2	1	
y_3		5	9			5	7				7	

Multiple correlation Variant 11

ζ $\zeta = z_1$ $\zeta = z_2$ $\zeta = z_3$

ξ η	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y
y_1		2	7				2				9	
y_2	1	8	9			7	3			8	4	
y_3	2				5	2			5	2		

Multiple correlation Variant 12

ζ $\zeta = z_1$ $\zeta = z_2$ $\zeta = z_3$

ξ η	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y	x_1	x_2	x_3	n_y
y_1	3	5			4	1			2	2		
y_2	1	4				5	8		1	5		
y_3		2	9				6			4	8	

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