Functional Analysis and its Applications
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Inversion of the local Pompeiu transform

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Abstract

The construction of inversion of the local Pompeiu transform for some classes of cylinders is obtained.

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1. Introduction

Let $\mathbb{R}^n$ be a real Euclidean space of dimension $n \geq 2$ with the Euclidean norm $|\cdot|$, let $M(n)$ be the group of Euclidean motions in $\mathbb{R}^n$, and let $\mathcal{F} = \{\mu_i\}_{i=1}^k$ be a finite family of distributions with compact supports in $\mathbb{R}^n$. For fixed $g \in M(n)$ we consider the distribution $g\mu_i$ acting on $C^\infty(\mathbb{R}^n)$ by the rule

$$\langle g\mu_i, f\rangle = \langle \mu_i, f \circ g\rangle, \quad f \in C^\infty(\mathbb{R}^n).$$

The (global) Pompeiu transform $\mathcal{P}_\mathcal{F}$ is the map

$$\mathcal{P}_\mathcal{F}: C^\infty(\mathbb{R}^n) \to C^\infty(M(n))^k$$

given by

$$\mathcal{P}_\mathcal{F}(f)(g) = (\langle g\mu_1, f\rangle, \ldots, \langle g\mu_k, f\rangle), \quad g \in M(n). \tag{1}$$

Similarly, for an open set $U \subset \mathbb{R}^n$ the local Pompeiu transform maps $C^\infty(U)$ into the Cartesian product $C^\infty(\Lambda(U,\mu_1)) \times \ldots \times C^\infty(\Lambda(U,\mu_k))$ by the formula (1), where $\Lambda(U,\mu_i) = \{g \in M(n): \mathrm{supp}\, g\mu_i \subset U\}$.

For given $\mathcal{F}$ and U the following problems arise (see [4]).

Problem 1. Find out whether $\mathcal{P}_{\mathcal{F}}$ is an injective map. If it is not, describe the kernel of $\mathcal{P}_{\mathcal{F}}$.

Problem 2. In the case $\mathcal{P}_{\mathcal{F}}$ is injective, find the converse map.

Many authors have studied the injectivity of the Pompeiu transform and related problems for special $\mathcal{F}$ and U (see the survey papers [4,13], which contain an extensive bibliography, and also [2,3,6,8–12]). The most interesting case is the case when $U = B_R(y) = \{x \in \mathbb{R}^n : |x-y| < R\}$, and $\mathcal{F} = \{\chi_E\}$ is the characteristic function (indicator) of a compact set $E \subset B_R(y)$ of positive measure. For this family $\mathcal{F}$ and the set $E \subset \overline{B_r(x_0)}$ having a hyperbolic point $x_1 \in E \cap \partial B_r(x_0)$ and the global Pompeiu property, the Pompeiu transform $\mathcal{P}_{\mathcal{F}}$ is injective with respect to U if $R > 2r$ (see [2,3], and also [6,9–11], where for some E the minimal value of R for which $\mathcal{P}_{\chi_E}$ is injective is found). In [3] for the sets E with the above properties the construction of the inversion of the transform $\mathcal{P}_{\chi_E}$ in the ball $B_R(y)$, $R > 3r$ is obtained. Besides, for the case where E is a square a function $f \in C^\infty(B_R(y))$ is recovered in [3] from its Pompeiu transform $\mathcal{P}_{\chi_E}(f)$ also for $R > 2r$. In connection with this, the inversion of the transform $\mathcal{P}_{\chi_E}$ in the ball $B_R(y)$ of radius $R > 2r$ for other compacts E is of great interest. In the present paper the solution of this problem for some class of cylinders is obtained.

2. Statement of the main result

Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, let ρ and σ be the polar coordinates in $\mathbb{R}^n$ (for each $x \in \mathbb{R}^n$ we set $\rho = |x|$ and if $x \neq 0$, then we set $\sigma = \frac{x}{\rho} \in \mathbb{S}^{n-1}$). Let $\left\{Y_s^{(k)}(\sigma)\right\}$, $1 \le s \le d_k$, be a fixed orthonormal basis in the spase $\mathcal{H}_k$ of spherical harmonics of degree k, regarded as a subspace of $L^2\left(\mathbb{S}^{n-1}\right)$ (see [7, Chapter 4, section 2]). To every function $f \in L_{loc}(B_R)$ we assign its Fourier series

$$f(x) \sim \sum_{k=0}^{\infty} \sum_{s=1}^{d_k} f_{ks}(\rho) Y_s^{(k)}(\sigma), \quad 0 < \rho < R,$$

where

$$f_{ks}(\rho) = \int_{\mathbb{S}^{n-1}} f(\rho\sigma) \overline{Y_s^{(k)}(\sigma)}\, d\sigma.$$

To reconstruct f it is sufficient to find the coefficients f_{ks} of its Fourier series.

Further, as usual, $\mathcal{D}(\mathbb{R}^n)$ is the space of infinitely differentiable functions on $\mathbb{R}^n$, with compact supports, and $\mathcal{D}'(\mathbb{R}^n)$ is the space of distributions on $\mathbb{R}^n$. Let $\mu_1 * \mu_2$ be the convolution of two distributions on $\mathbb{R}^n$ one of which is compactly supported. We need a concept of circular symmetrization of a distribution defined as follows: for any $\mu \in \mathcal{D}'(\mathbb{R}^n)$, we define $\mathcal{R}\mu$ setting

$$\langle \mathcal{R}\mu, \varphi \rangle = \langle \mu(x), \int_{\mathbf{SO}(n)} \varphi(kx)\, \mathrm{d}k \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^n), \tag{2}$$

where $\mathbf{SO}(n)$ is the group of rotations of $\mathbb{R}^n$, and $\mathrm{d}k$ is the normalized Haar measure on $\mathbf{SO}(n)$.

Let $\alpha \in (0, 2\pi)$, $\beta_\alpha = \begin{cases} \alpha/2, & \alpha \in (0,\pi], \\ \pi, & \alpha \in (\pi, 2\pi), \end{cases}$ $Sg = \{z \in \mathbb{C}\colon |z| \le 1, |\arg z| \le \beta_\alpha, \operatorname{Re} z \ge \cos\frac{\alpha}{2}\}$ be the circular segment of angular measure α, $H = (Sg - h_\alpha) \times [-b_3, b_3] \times \ldots \times [-b_n, b_n]$, where $h_\alpha = \begin{cases} \cos\dfrac{\alpha}{2}, & \alpha \in (0,\pi], \\ 0, & \alpha \in (\pi, 2\pi), \end{cases}$ $b_k > 0$, $k = 3, \ldots, n$.

Further we need the following differential operators: $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$, $D^\varkappa = \frac{\partial^{|\varkappa|}}{\partial x_1{}^{\varkappa_1} \ldots \partial x_n{}^{\varkappa_n}}$ $(\varkappa = (\varkappa_1, \ldots, \varkappa_n) \in \mathbb{Z}_+^n, |\varkappa| = \varkappa_1 + \cdots + \varkappa_n)$, $D_1 = \frac{\partial}{\partial x_1} + \mathrm{i}\frac{\partial}{\partial x_2}$, $D_2 = \frac{\partial^{n-2}}{\partial x_3 \ldots \partial x_n}$, $D_{ij}(a) = (x_i + a)\frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$, $(a \in \mathbb{R}^1)$. We denote $\mathcal{R}_k = \mathcal{R}(D_2 D_1^k \mu)$, where $\mu = \frac{\partial^2}{\partial x_2^2} D_{12}(h_\alpha) D_2 \chi_H$.

Let r_0 be the radius of the least closed ball containing Sg, $r = \sqrt{r_0^2 + b_3^2 + \cdots + b_n^2}$, $R > 2r$, $f \in C^\infty(B_R)$, where $B_R = B_R(0)$. For $x \in B_{R-r}$ we set $\check{f}(x) = f(-x)$, $f_1(x) = (f * \mathcal{R}\chi_H)(x)$, $f_i(x) = (\check{f} * \nu_i)(x)$, $i = 2, 3$, where $\nu_2 = \mathcal{R}_1$,

$$\nu_3 = \begin{cases} \mathcal{R}_5 + \dfrac{4}{9n^2r^2}(3n-4)(3n+2)\Delta\mathcal{R}_1 - \dfrac{4}{9n^2}\Delta^2\mathcal{R}_1, & 3n\sin^2\dfrac{\alpha}{2} = 2r^2, \\ \mathcal{R}_3 + \dfrac{2}{3n}\Delta\mathcal{R}_1, & 3n\sin^2\dfrac{\alpha}{2} \ne 2r^2, \end{cases}$$

for $\alpha \in (0, \pi]$, and $\nu_3 = \mathcal{R}_2$ for $\alpha \in (\pi, 2\pi)$.

The main result of this paper is the following

Theorem 1. Let $R > 2r$. Then for any $k \in \mathbb{Z}_+$, $1 \le s \le d_k$, $\rho \in (0, R)$ there exist distributions $\mathcal{U}_{l,i}$, $(l \in \mathbb{N}, i = 1,2,3,4)$ with the following properties:
(1) $\operatorname{supp}\mathcal{U}_{l,i} \subset B_{R-r}$ $(l \in \mathbb{N}, i = 1,2,3)$, $\operatorname{supp}\mathcal{U}_{l,4} \subset B_R$ $(l \in \mathbb{N})$;
(2) for any $f \in C^\infty(B_R)$

$$\left(\Delta^n \check{f}\right)_{ks}(\rho) = \lim_{l\to\infty} \left(\langle \mathcal{U}_{l,1}, f_2 \rangle + \langle \mathcal{U}_{l,2}, f_3 \rangle\right), \tag{3}$$

$$\check{f}_{ks}(\rho) = \lim_{l\to\infty} \left(\langle \mathcal{U}_{l,3}, f_1 \rangle + \langle \mathcal{U}_{l,4}, \Delta^n \check{f} \rangle\right). \tag{4}$$

Let us make some remarks. Lemmas 1 - 3 (see section 3 below) show how to reconstruct the functions f_i, $i = 1,2,3$ in terms of $\mathcal{P}_{\chi_H}(f)$. Therefore, applying the equalities (3), (4) we can compute the Fourier coefficients of $\check{f}$ in terms of the Pompeiu transform of f. Thus, in Theorem 1 the procedure for the local inversion of the Pompeiu transform $\mathcal{P}_{\chi_H}$ is obtained. This method enabled us to obtain similar results for other compact sets E. In particular, analogues of Theorem 1 can be proved for cylinders of the form $K \times [-b_3, b_3] \times \ldots \times [-b_n, b_n]$, where K belongs to some class of polygons or circular domains.

3. Auxiliary assertions

Let $1 \le i < j \le n$, $h, \theta \in \mathbb{R}^1$, let $\tau_{i,h}$ be a shift on h along x_i axis, and let $k_{i,j,\theta}$ be the rotation in the plane (x_i, x_j) by an angle θ.

Lemma 1. Let $E \subset \overline{B}_{r_1}$, $R > r_1$, $f \in C^\infty(B_R)$. Then

$$\mathcal{P}_{\chi_E}\left(\frac{\partial f}{\partial x_i}\right)(g) = \frac{\mathrm{d}}{\mathrm{d}h}\left(\mathcal{P}_{\chi_E}(f)(\tau_{i,h} \circ g)\right)\Big|_{h=0}, \tag{5}$$

$$\mathcal{P}_{\chi_E}\left(D_{ij}(0)f\right)(g) = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\mathcal{P}_{\chi_E}(f)(k_{i,j,\theta} \circ g)\right)\Big|_{\theta=0}, \tag{6}$$

where $gE \subset B_R$.

Proof. For small h we have

$$\mathcal{P}_{\chi_E}(f)(\tau_{i,h} \circ g) = \mathcal{P}_{\chi_E}(f \circ \tau_{i,h})(g). \tag{7}$$

Differentiating (7) with respect to h and setting $h = 0$ we obtain (5). Equality (6) is proved analogously. □

Lemma 2. Let $R > r_1$, $E \subset \overline{B}_{r_1}$, $\nu = D^\varkappa D_{ij}(a)\chi_E$. Then for any $f \in C^\infty(B_R)$ and $x \in B_{R-r_1}$ we have

$$(\check{f} * \mathcal{R}\nu)(x) = (-1)^{|\varkappa|+1} \int\limits_{\mathbf{SO}(n)} \left(\mathcal{P}_{\chi_E}\left(D_{ij}(a)D^\varkappa\right)(y)\left(f(ky - x)\right)\right)(\mathbf{e})\,\mathrm{d}k, \tag{8}$$

where $\mathbf{e}$ is the unit element of $\mathbf{SO}(n)$.

Proof. Using (2), we have

$$(\check{f} * \mathcal{R}\nu)(x) = \langle \nu(y), \int\limits_{\mathbf{SO}(n)} f(ky - x)\,\mathrm{d}k\rangle. \tag{9}$$

By the relation (9) and the definition of $D_{ij}(a)$, we obtain

$$(\check{f} * \mathcal{R}\nu)(x) = (-1)^{|\varkappa|+1} \int\limits_E \int\limits_{\mathbf{SO}(n)} \left(D_{ij}(a)D^\varkappa\right)(y)\left(f(ky - x)\right)\mathrm{d}k\,\mathrm{d}y.$$

Hence, from (1) we obtain the required assertion. □

Remark. Let $x \in B_{R-r_1}$, $k \in \mathbf{SO}(n)$ be fixed, and let g_1 be the element of $\mathbf{M}(n)$ acting by the formula $g_1 y = ky - x$. Then $\mathcal{P}_{\chi_E}(f(ky-x))(g) = \mathcal{P}_{\chi_E}(f)(g_1 g)$, where $gE \subset B_{R-|x|}$. Due to this relation and Lemma 1 the values $\check{f} * \mathcal{R}\nu$ can be recovered from Pompeiu transform $\mathcal{P}_{\chi_E}(f)$ via formula (8).

Let δ be the Dirac delta measure at the origin of $\mathbb{R}^n$.

Lemma 3 ([3]). Let $R > 2r_1$, $E \subset \overline{B}_{r_1}$, $\mu(\varkappa) = \mathcal{R}(D^{\varkappa}\chi_E)$. Then for any $f \in C^{\infty}(B_R)$, and every $x \in B_{R-r_1}$ we have

$$(f * \mu(\varkappa))(x) = \int\limits_{\mathbf{SO}(n)} \langle D^{\varkappa}\delta(y), (\mathcal{P}_{\chi_E} f)\left(\begin{Vmatrix} -k^{-1} & x-ky \\ 0 & 1 \end{Vmatrix}\right)\rangle \, \mathrm{d}k,$$

where $\mathbf{M}(n)$ is considered as the group of $(n+1)\times(n+1)$ matrices of the form

$$\begin{Vmatrix} k & x \\ 0 & 1 \end{Vmatrix}, \quad k \in \mathbf{SO}(n), \quad x \in \mathbb{R}^n,$$

and $\mathbb{R}^n$ is identified with the affine subspace $\{x_{n+1} = 1\}$ of $\mathbb{R}^{n+1}$.

For each $m \in \{1, \dots, n\}$ let η_m be a map $\mathbb{R}^n \to \mathbb{R}^n$ acting as follows: if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then $\eta_m x = ((\eta_m x)_1, \dots, (\eta_m x)_n)$, where $(\eta_m x)_k = x_k$ for $k \neq m$ and $(\eta_m x)_m = -x_m$. Let G_+^n (respectively, G_-^n) be the set of maps $\mathbb{R}^n \to \mathbb{R}^n$ representable as a composition of an even (respectively, odd) number of maps η_m, $1 \le m \le n$.

Lemma 4. For any $f \in C^{n+1}(H)$ we have

$$\int\limits_H D_2 D_{12}(h_\alpha) \frac{\partial^2}{\partial x_2^2} f(x_1, \dots, x_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_n =$$

$$= \left(\sum_{\eta \in G_+^{n-2}} - \sum_{\eta \in G_-^{n-2}} \right) \Bigg[f(z_1, \eta b) - f(z_2, \eta b) + \sin\frac{\alpha}{2} \frac{\partial f}{\partial x_2}(z_1, \eta b) +$$

$$+ \sin\frac{\alpha}{2} \frac{\partial f}{\partial x_2}(z_2, \eta b) \Bigg],$$

where

$$z_1 = \begin{cases} -\mathrm{i}\sin\dfrac{\alpha}{2} &, \alpha \in (0, \pi], \\ \mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}} &, \alpha \in (\pi, 2\pi), \end{cases} \quad z_2 = \overline{z}_1, \quad b = (b_3, \dots, b_n).$$

Proof. Let $S_\alpha = Sg - h_\alpha$, $u \in C^3(S_\alpha)$. Then

$$\int\limits_{S_\alpha} \left(D_{12}(h_\alpha) \frac{\partial^2}{\partial x_2^2} u \right) (x_1, x_2)\, dx_1\, dx_2 \tag{10}$$

$$= \int\limits_{\cos\frac{\alpha}{2} - h_\alpha}^{1-h_\alpha} \Big[(\mathcal{A}u) \left(x_1, \sqrt{1-(x_1+h_\alpha)^2} \right)$$

$$- (\mathcal{A}u) \left(x_1, -\sqrt{1-(x_1+h_\alpha)^2} \right) \Big]\, dx_1,$$

where $\mathcal{A} = \frac{\partial}{\partial x_1} - x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + (x_1 + h_\alpha) \frac{\partial^2}{\partial x_2^2}$. Bearing in mind that

$$(\mathcal{A}u) \left(x_1, \sqrt{1-(x_1+h_\alpha)^2} \right)$$

$$= \frac{\partial}{\partial x_1} \Big[u \left(x_1, \sqrt{1-(x_1+h_\alpha)^2} \right)$$

$$- \sqrt{1-(x_1+h_\alpha)^2}\, \frac{\partial u}{\partial x_2} \left(x_1, \sqrt{1-(x_1+h_\alpha)^2} \right) \Big],$$

$$(\mathcal{A}u) \left(x_1, -\sqrt{1-(x_1+h_\alpha)^2} \right)$$

$$= \frac{\partial}{\partial x_1} \Big[u \left(x_1, -\sqrt{1-(x_1+h_\alpha)^2} \right)$$

$$+ \sqrt{1-(x_1+h_\alpha)^2}\, \frac{\partial u}{\partial x_2} \left(x_1, -\sqrt{1-(x_1+h_\alpha)^2} \right) \Big]$$

from (10) we find

$$\int\limits_{S_\alpha} \left(D_{12}(h_\alpha) \frac{\partial^2 u}{\partial x_2^2} \right) (x_1, x_2)\, dx_1\, dx_2 \tag{11}$$

$$= u(z_1) - u(z_2) + \sin\frac{\alpha}{2} \frac{\partial u}{\partial x_2}(z_1) + \sin\frac{\alpha}{2} \frac{\partial u}{\partial x_2}(z_2).$$

Since for every $v \in C^{n-1}([-b_3, b_3] \times \ldots \times [-b_n, b_n])$

$$\int\limits_{-b_3}^{b_3} \ldots \int\limits_{-b_n}^{b_n} (D_2 v)(x_3, \ldots, x_n)\, dx_3 \ldots dx_n = \sum_{\eta \in G_+^{n-2}} v(\eta b) - \sum_{\eta \in G_-^{n-2}} v(\eta b),$$

from (11) we obtain the assertion of Lemma 4. □

Let J_q be the Bessel function of the first kind of order $q \geq 0$, and let $j_q(z) = J_q(z)/z^q$. The spherical transform of the radial distribution μ with the compact support in $\mathbb{R}^n$ is defined by

$$\widetilde{\mu}(\lambda) = \langle \mu(x), j_{\frac{n-2}{2}}(\lambda|x|) \rangle, \quad \lambda \in \mathbb{C}. \tag{12}$$

Lemma 5. Let $k \in \mathbb{Z}_+$. Then

$$\widetilde{\mathcal{R}}_k(\lambda) = (-1)^{n-1}2^{n-2}b_3 \dots b_n \lambda^{2(k+n-2)} \left\{ C_{1k} j_{\frac{3n+2k-6}{2}}(\lambda r) + \lambda^2 C_{2k} j_{\frac{3n+2k-4}{2}}(\lambda r) \right\},$$

where $C_{1k} = z_1^k - z_2^k + \mathrm{i}k \sin\frac{\alpha}{2}\left(z_1^{k-1} + z_2^{k-1}\right), C_{2k} = \sin^2\frac{\alpha}{2}\left(z_1^k - z_2^k\right)$.

Proof. Since $j_q'(t) = -t j_{q+1}(t)$ (see [1, Chapter 7, § 7.2, formula (51)]), we have

$$\frac{\partial}{\partial x_1}\left((x_1 + \mathrm{i}x_2)^k j_q(\lambda|x|)\right) \tag{13}$$

$$= k(x_1 + \mathrm{i}x_2)^{k-1} j_q(\lambda|x|) - \lambda^2 x_1 (x_1 + \mathrm{i}x_2)^k j_{q+1}(\lambda|x|),$$

$$\frac{\partial}{\partial x_2}\left((x_1 + \mathrm{i}x_2)^k j_q(\lambda|x|)\right) \tag{14}$$

$$= \mathrm{i}k(x_1 + \mathrm{i}x_2)^{k-1} j_q(\lambda|x|) - \lambda^2 x_2 (x_1 + \mathrm{i}x_2)^k j_{q+1}(\lambda|x|).$$

Using induction on k from (13), (14) we find

$$(-1)^k D_1^k \left(j_q(\lambda|x|)\right) = \lambda^{2k}(x_1 + \mathrm{i}x_2)^k j_{q+k}(\lambda|x|).$$

Hence (see (12))

$$\widetilde{\mathcal{R}}_k(\lambda) = \lambda^{2(k+n-2)}\langle \mu, x_3 \dots x_n (x_1 + \mathrm{i}x_2)^k j_{\frac{3n+2k-6}{2}}(\lambda|x|)\rangle,$$

where $\mu = \frac{\partial^2}{\partial x_2^2} D_{12}(h_\alpha) D_2 \chi_H$. Using Lemma 4, we obtain the required assertion. □

By the Paley – Wiener theorem [5, Theorem 7.3.1] there exist radial distributions μ_1 and μ_2 with supports in B_r, for which $\widetilde{\mu}_1(\lambda) = (-1)^n \widetilde{\nu}_2(\lambda)/\lambda^{2n}$, $\widetilde{\mu}_2(\lambda) = (-1)^n \widetilde{\nu}_3(\lambda)/\lambda^{2n}$. ¿From Lemma 5 and properties of Bessel functions (see [1, Chapter 7, § 7.9]) it follows that $\widetilde{\mu}_1(\lambda)$ and $\widetilde{\mu}_2(\lambda)$ have no common zeros. Further, we shall need a lower estimate for the function $\widetilde{\mu}_1(\lambda)\widetilde{\mu}_2(\lambda) j_{\frac{n}{2}+k-1}(\varepsilon\lambda)$, where $\varepsilon > 0$.

Lemma 6. Let $a_1, a_2, a_3 > 0$, $k \in \mathbb{Z}_+$,

$$\theta(\lambda) = j_{\frac{3n-2}{2}}(a_1\lambda) j_{\frac{3n}{2}}(a_2\lambda) j_{\frac{n}{2}+k-1}(a_3\lambda).$$

Then there are constants L_{1k}, L_{2k}, such that for any integer $l \geq L_{1k}$ there exists $\rho_l \in (l, l+1)$ such that if either $|z| = \rho_l$ or $|\operatorname{Im}\lambda| \geq 1$ and $|\lambda| \geq L_{1k}$, then

$$|\theta(\lambda)| \geq \frac{L_{2k}}{|\lambda|^{\frac{7n+2k-1}{2}}} e^{(a_1+a_2+a_3)|\operatorname{Im}\lambda|}.$$

Proof. The function $\theta(\lambda)$ is an even entire function, therefore to prove the assertion of Lemma 6 we can assume that $\operatorname{Re}\lambda \geq 0$. From the asymptotic development of the Bessel function (see [1, Chapter 7, § 7.13, formula (3)]) we find

$$\theta(\lambda) = \frac{2}{\pi}\sqrt{\frac{2}{\pi}} \frac{(a_1)^{-\frac{3n-1}{2}}(a_2)^{-\frac{3n+1}{2}}(a_3)^{-\frac{n+2k-1}{2}}}{\lambda^{\frac{7n+2k-1}{2}}} \cos\left(a_1\lambda - (3n-1)\frac{\pi}{4}\right) \times$$

$$\times \cos\left(a_2\lambda - (3n+1)\frac{\pi}{4}\right)\cos\left(a_3\lambda - (n+2k-1)\frac{\pi}{4}\right) + \mathrm{O}\left(\frac{\mathrm{e}^{(a_1+a_2+a_3)|\operatorname{Im}\lambda|}}{|\lambda|^{\frac{7n+2k+1}{2}}}\right).$$

By the Łojasiewicz inequality (see [3]) we have

$$|\cos z| \geq \frac{1}{\pi \mathrm{e}} d(z,V)\mathrm{e}^{|\operatorname{Im} z|}, \tag{15}$$

where $V = \{(2l+1)\pi/2, l \in \mathbb{Z}\}$, $d(z,V) = \min(1, dist(z,V))$. Using (15) and repeating arguments from the proof of Lemma 7 in [3], we obtain the assertion of Lemma 6. □

In what follows let us assume that $R > 2r$. Choose a strictly increasing sequence of positive numbers $\{\varepsilon_m\}_{m=1}^{\infty}$ with limit $\frac{R}{2r} - 1$, and a corresponding increasing sequence of radii $R_m = 2r(1+\varepsilon_m)$, $m \geq 1$, $R_0 = 0$.

Lemma 7. Let $R > 2r$.Then for any $k \in \mathbb{Z}_+$, $m \in \mathbb{N}$, $t \in [R_{m-1}, R_m)$ there are two sequences of radial distributions satisfying the following conditions:

(1) $\operatorname{supp}\mu_{l,i} \subset B_{R_m - r}$, $i = 1,2$, $l \in \mathbb{N}$,

(2) there exist constants $L = L(k,R,r,\varepsilon_1,n), C = C(R,r,\varepsilon_1,n) > 0$ such that for $l \geq L$ the equality

$$\left|j_{\frac{n}{2}+k-1}(t\lambda) - (\widetilde{\mu}_1(\lambda)\widetilde{\mu}_{l,1}(\lambda) + \widetilde{\mu}_2(\lambda)\widetilde{\mu}_{l,2}(\lambda))\right| \leq$$

$$\frac{C(R,r,\varepsilon_1,n)}{l}\frac{\|\lambda\|^{-\frac{n}{2}-k+\frac{13}{2}}}{t^{\frac{n}{2}+k-1}}\mathrm{e}^{R_m|\operatorname{Im}\lambda|}, \quad \|\lambda\| = \max(1,|\lambda|),$$

holds.

To prove Lemma 7 it is sufficient to use Lemma 6 and repeat the arguments from the proof of Proposition 8 in [3].

4. The proof of main result

By Lemma 7 it follows (see [3, proof of Theorem 9]) that for any $m \in \mathbb{N}$, $\rho \in [R_{m-1}, R_m)$ there are distributions $\mathcal{U}_{l,i}$ ($l \in \mathbb{N}, i = 1,2$) with supports in R_{R-r} such that for $l \geq L$ and any $f \in C^{\infty}(B_R)$ the following estimate is valid

$$|f_{ks}(\rho) - \langle \mathcal{U}_{l,1}, f * \mu_1\rangle - \langle \mathcal{U}_{l,2}, f * \mu_2\rangle| \tag{16}$$

$$\leq \frac{c_3}{l}\frac{\rho^{-\frac{n}{2}+1}}{(R-R_m)^M}\sup_{\substack{x \in B_{R'_m} \\ |\varkappa| \leq M}}\left|\frac{\partial^{|\varkappa|}}{\partial x^{\varkappa}}f(x)\right|,$$

where $R'_m = \frac{2}{3}R + \frac{1}{3}R_m$, $M = \left[\frac{n+13}{2}\right] + 1$, and the constant c_3 does not depend on R, r, ε_1, n. Substituting in (16) f for $\Delta^n \check{f}$ and having in mind that $\Delta^n \mu_i = \nu_{i+1}$,

$i = 1, 2$, we obtain the equality (3). Let now $T_1 = \mathcal{R}\chi_H$, $T_2 = \Delta^n \delta$. Then $\widetilde{T_1}(0) \neq 0$, $\widetilde{T_2}(\lambda) = (-1)^n \lambda^{2n}$. Thus, $\widetilde{T_1}$ and $\widetilde{T_2}$ have no common zeros. Moreover, $\widetilde{T_1}$ has an asymptotic behavior of the same type as that of the Bessel function (see [2,3]). Hence we can apply the same procedure as above to the two radial distributions T_1 and T_2. By the same reasons there exist distributions $\mathcal{U}_{l,i}$ ($l \in \mathbb{N}, i = 3, 4$) such that (4) holds. Theorem is proved.

References

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