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ON THE CONVERGENCE OF SOLUTIONS OF THE VARIATIONAL PROBLEMS WITH INTEGRAL CONSTRAINTS AND DEGENERATION IN VARIABLE DOMAINS

In this article we deal with a sequence of integral functionals defined on weighted Sobolev spaces associated with a sequence of n-dimensional domains. For the given functionals we consider variational problems with sets of constraints of an integral kind. We establish sufficient conditions of convergence of minimizers and minimum values of the variational problems under consideration.

Keywords: *varying weighted Sobolev spaces, variational problem, integral functional, degeneration, integral constraint,* Γ*-convergence.*

1. Introduction. In this article for a sequence of integral functionals defined on varying weighted Sobolev spaces we consider variational problems with sets of constraints of an integral kind. The strong connectedness of the given weighted Sobolev spaces with a "limit" weighted Sobolev space and the Γ-convergence of the involved integral functionals are the main conditions under which we establish the convergence of the minimizers and minimum values of the given variational problems.

The role of the strong connectedness of the corresponding spaces in the study of the homogenization of boundary value problems and variational problems in variable domains (particularly, in strongly perforated domains) is well known (see for instance [7-12]). The notion of the strong connectedness used in the present work was introduced and studied in [11].

The Γ-convergence is a special kind of the convergence introduced by E. De Giorgi in the seventies of last century to propose a framework for the study of the asymptotic behaviour of families of minimum problems. Its first definition as well as the main properties were presented in [5]. There are many works devoted to the Γ-convergence of functionals including integral functionals with degenerate integrands and variable domains of definitions (see for instance [1-6], [8], [9] and [11-13] and the bibliography in [1], [2]). We note that the effectiveness of the Γ-convergence in the study of the homogenization of variational problems is connected with the possibility of obtaining converging subsequences from sequences of minimizers of minimum problems.

2. Preliminaries. Let $n \in \mathbb{N}$, $n \geqslant 2$, and let Ω be a bounded domain of $\mathbb{R}^n$. Let $p \in (1, n)$, and let ν be a nonnegative function on Ω with the properties: $\nu > 0$ almost everywhere in Ω and

$$\nu \in L^1_{\rm loc}(\Omega), \ \ \left(\frac{1}{\nu}\right)^{1/(p-1)} \in L^1_{\rm loc}(\Omega). \tag{1}$$

We denote by $L^p(\nu, \Omega)$ the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that the function $\nu|u|^p$ is summable in Ω. $L^p(\nu,\Omega)$ is a Banach space with the norm

$$\|u\|_{L^p(\nu,\Omega)} = \left(\int_\Omega \nu|u|^p\,dx\right)^{1/p}.$$

Note that by virtue of Young's inequality and the second inclusion of (1) we have $L^p(\nu,\Omega)\subset L^1_{\text{loc}}(\Omega)$.

We denote by $W^{1,p}(\nu,\Omega)$ the set of all functions $u\in L^p(\nu,\Omega)$ such that for every $i\in\{1,\dots,n\}$ there exists the weak derivative D_iu, $D_iu\in L^p(\nu,\Omega)$. $W^{1,p}(\nu,\Omega)$ is a reflexive Banach space with the norm

$$\|u\|_{1,p,\nu} = \left(\int_\Omega \nu|u|^p\,dx + \sum_{i=1}^n \int_\Omega \nu|D_iu|^p\,dx\right)^{1/p}.$$

Due to the first inclusion of (1) we have $C_0^\infty(\Omega)\subset W^{1,p}(\nu,\Omega)$. We denote by $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in $W^{1,p}(\nu,\Omega)$.

Next, let $\{\Omega_s\}$ be a sequence of domains of $\mathbb{R}^n$ which are contained in Ω.

By analogy with the spaces introduced above we define the functional spaces corresponding to the domains Ω_s.

Let $s\in\mathbb{N}$. We denote by $L^p(\nu,\Omega_s)$ the set of all measurable functions $u:\Omega_s\to\mathbb{R}$ such that the function $\nu|u|^p$ is summable in Ω_s. $L^p(\nu,\Omega_s)$ is a Banach space with the norm

$$\|u\|_{L^p(\nu,\Omega_s)} = \left(\int_{\Omega_s} \nu|u|^p\,dx\right)^{1/p}.$$

By virtue of Young's inequality and the second inclusion of (1) we have $L^p(\nu,\Omega_s)\subset L^1_{\text{loc}}(\Omega_s)$.

We denote by $W^{1,p}(\nu,\Omega_s)$ the set of all functions $u\in L^p(\nu,\Omega_s)$ such that for every $i\in\{1,\dots,n\}$ there exists the weak derivative D_iu, $D_iu\in L^p(\nu,\Omega_s)$. $W^{1,p}(\nu,\Omega_s)$ is a Banach space with the norm

$$\|u\|_{1,p,\nu,s} = \left(\int_{\Omega_s} \nu|u|^p\,dx + \sum_{i=1}^n \int_{\Omega_s} \nu|D_iu|^p\,dx\right)^{1/p}.$$

We denote by $\widetilde{C}_0^\infty(\Omega_s)$ the set of all restrictions on Ω_s of functions from $C_0^\infty(\Omega)$. Due to the first inclusion of (1) we have $\widetilde{C}_0^\infty(\Omega_s)\subset W^{1,p}(\nu,\Omega_s)$. We denote by $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ the closure of the set $\widetilde{C}_0^\infty(\Omega_s)$ in $W^{1,p}(\nu,\Omega_s)$.

We observe that if $u\in\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and $s\in\mathbb{N}$, then $u|_{\Omega_s}\in\widetilde{W}_0^{1,p}(\nu,\Omega_s)$.

Definition 1. If $s\in\mathbb{N}$, then $q_s:\overset{\circ}{W}{}^{1,p}(\nu,\Omega)\to\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ is the mapping such that for every function $u\in\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, $q_su=u|_{\Omega_s}$.

Definition 2. We say that the sequence of the spaces $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ is strongly connected with the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ if there exists a sequence of linear continuous operators $l_s:\widetilde{W}_0^{1,p}(\nu,\Omega_s)\to\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that:

(i) the sequence of the norms $\|l_s\|$ is bounded;

(ii) for every $s \in \mathbb{N}$ and for every $u \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$ we have $q_s(l_s u) = u$ a. e. in Ω_s.

Proposition 1. Suppose that the embedding of $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ into $L^p(\nu, \Omega)$ is compact, and the sequence of the spaces $\widetilde{W}_0^{1,p}(\nu, \Omega_s)$ is strongly connected with the space $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$. Let for every $s \in \mathbb{N}$, $u_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$, and let the sequence of the norms $\|u_s\|_{1,p,\nu,s}$ be bounded. Then there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ such that $\lim\limits_{j \to \infty} \|u_{s_j} - q_{s_j} u\|_{L^p(\nu, \Omega_{s_j})} = 0$.

The proof of the proposition is simple (see [11]).

Definition 3. Let for every $s \in \mathbb{N}$, I_s be a functional on $\widetilde{W}_0^{1,p}(\nu, \Omega_s)$, and let I be a functional on $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$. We say that the sequence $\{I_s\}$ Γ-converges to the functional I if the following conditions are satisfied:

(i) for every function $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ there exists a sequence $w_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$ such that $\lim\limits_{s \to \infty} \|w_s - q_s u\|_{L^p(\nu, \Omega_s)} = 0$ and $\lim\limits_{s \to \infty} I_s(w_s) = I(u)$;

(ii) for every function $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ and for every sequence $u_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$ such that $\lim\limits_{s \to \infty} \|u_s - q_s u\|_{L^p(\nu, \Omega_s)} = 0$ we have $\liminf\limits_{s \to \infty} I_s(u_s) \geqslant I(u)$.

3. Main result. Let c_1, $c_2 > 0$, and let for every $s \in \mathbb{N}$, $\psi_s \in L^1(\Omega_s)$ and $\psi_s \geqslant 0$ in Ω_s. We shall assume that

$$\text{the sequence of the norms } \|\psi_s\|_{L^1(\Omega_s)} \text{ is bounded.} \tag{2}$$

Let $f_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$, $s \in \mathbb{N}$, be a sequence of functions such that:

$$\text{for every } s \in \mathbb{N} \text{ and for every } \xi \in \mathbb{R}^n \text{ the function } f_s(\cdot, \xi) \text{ is measurable in } \Omega_s; \tag{3}$$

$$\text{for every } s \in \mathbb{N} \text{ and for almost every } x \in \Omega_s \text{ the function } f_s(x, \cdot) \text{ is convex in } \mathbb{R}^n; \tag{4}$$

$$\begin{cases} \text{for every } s \in \mathbb{N}, \text{ for almost every } x \in \Omega_s \text{ and for every } \xi \in \mathbb{R}^n, \\ c_1 \nu(x) |\xi|^p - \psi_s(x) \leqslant f_s(x, \xi) \leqslant c_2 \nu(x) |\xi|^p + \psi_s(x). \end{cases} \tag{5}$$

From (3)-(5) it follows that for every $s \in \mathbb{N}$, f_s is a Carathéodory function and if $s \in \mathbb{N}$ and $u \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$, the function $f_s(x, \nabla u)$ is summable in Ω_s.

For every $s \in \mathbb{N}$ we define the functional $J_s : \widetilde{W}_0^{1,p}(\nu, \Omega_s) \to \mathbb{R}$ by

$$J_s(u) = \int_{\Omega_s} f_s(x, \nabla u)\, dx, \quad u \in \widetilde{W}_0^{1,p}(\nu, \Omega_s).$$

Next, let c_3, $c_4 > 0$, and let ψ be a function in $L^1(\Omega)$ such that $\psi \geqslant 0$ in Ω.

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that:

$$\text{for every } \eta \in \mathbb{R} \text{ the function } g(\cdot, \eta) \text{ is measurable in } \Omega; \tag{6}$$

$$\text{for almost every } x \in \Omega \text{ the function } g(x, \cdot) \text{ is strictly convex in } \mathbb{R}; \tag{7}$$

$$\begin{cases} \text{for almost every } x \in \Omega \text{ and for every } \eta \in \mathbb{R}, \\ c_3\nu(x)|\eta|^p - \psi(x) \leqslant g(x,\eta) \leqslant c_4\nu(x)|\eta|^p + \psi(x). \end{cases} \tag{8}$$

From (6)-(8) it follows that g is a Carathéodory function and if $s \in \mathbb{N}$ and $u \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$, the function $g(x,u)$ is summable in Ω_s. Moreover, for every $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$, the function $g(x,u)$ is summable in Ω.

For every $s \in \mathbb{N}$ we define the functional $G_s : \widetilde{W}_0^{1,p}(\nu, \Omega_s) \to \mathbb{R}$ by

$$G_s(u) = \int_{\Omega_s} g(x,u)\,dx, \quad u \in \widetilde{W}_0^{1,p}(\nu, \Omega_s).$$

We observe that by virtue of (2), (4), (5) and (7), (8) for every $s \in \mathbb{N}$ the functional $J_s + G_s$ is weakly lower semicontinuous on $\widetilde{W}_0^{1,p}(\nu, \Omega_s)$, strictly convex and there exist c', c'', $c''' > 0$ such that for every $s \in \mathbb{N}$ and $u \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$,

$$c'\|u\|^p_{1,p,\nu,s} - c''' \leqslant (J_s + G_s)(u) \leqslant c''\|u\|^p_{1,p,\nu,s} + c'''. \tag{9}$$

In view of known results on the existence of the minimizers of functionals (see for instance [14]), these properties of the functionals $J_s + G_s$ imply that the next assertion holds true:

$$\begin{cases} \text{if } s \in \mathbb{N} \text{ and } U \text{ is a nonempty convex and closed set in } \widetilde{W}_0^{1,p}(\nu, \Omega_s), \\ \text{there exists a unique function } u \in U \text{ minimizing the functional } J_s + G_s \text{ on } U. \end{cases} \tag{10}$$

Further, let $c > 0$, $b \in L^1(\Omega)$, $b \geqslant 0$ in Ω, and let $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that:

$$\text{for almost every } x \in \Omega \text{ the function } \varphi(x, \cdot) \text{ is convex in } \mathbb{R}; \tag{11}$$

$$\text{for almost every } x \in \Omega \quad \varphi(x, 0) = 0; \tag{12}$$

$$\begin{cases} \text{for almost every } x \in \Omega \text{ and for every } \eta \in \mathbb{R}, \\ |\varphi(x,\eta)| \leqslant c\nu(x)|\eta|^p + b(x). \end{cases} \tag{13}$$

For every $s \in \mathbb{N}$ we define the functional $\Phi_s : \widetilde{W}_0^{1,p}(\nu, \Omega_s) \to \mathbb{R}$ by

$$\Phi_s(u) = \int_{\Omega_s} \varphi(x,u)\,dx, \quad u \in \widetilde{W}_0^{1,p}(\nu, \Omega_s).$$

Using (8) and (13) along with Egoroff's theorem, we establish the following fact:

$$\begin{cases} \text{for every } v \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega) \text{ and for every sequence } v_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s) \text{ such that} \\ \|v_s - q_s v\|_{L^p(\nu,\Omega_s)} \to 0 \text{ we have } G_s(v_s) - G_s(q_s v) \to 0 \text{ and } \Phi_s(v_s) - \Phi_s(q_s v) \to 0. \end{cases} \tag{14}$$

For every $s \in \mathbb{N}$ we define

$$V_s = \{u \in \widetilde{W}_0^{1,p}(\nu, \Omega_s) : \ \Phi_s(u) \leqslant 1\}.$$

Due to (12) for every $s \in \mathbb{N}$ the set V_s is nonempty.

For every function $\sigma \in L^\infty(\Omega)$ we define the functionals G^σ, $\Phi^\sigma : \overset{\circ}{W}{}^{1,p}(\nu, \Omega) \to \mathbb{R}$ by

$$G^\sigma(u) = \int_\Omega \sigma g(x, u)\, dx, \quad \Phi^\sigma(u) = \int_\Omega \sigma \varphi(x, u)\, dx, \quad u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega),$$

and we set

$$V^\sigma = \{u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega) : \ \Phi^\sigma(u) \leqslant 1\}.$$

In view of (11) for every $s \in \mathbb{N}$ the set V_s is convex. This fact and (10) imply that for every $s \in \mathbb{N}$ there exists a unique function $u_s \in V_s$ minimizing the functional $J_s + G_s$ on V_s.

Theorem 1. *Suppose that the following conditions are satisfied:*

($*_1$) *the embedding of* $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ *into* $L^p(\nu, \Omega)$ *is compact*;

($*_2$) *the sequence of the spaces* $\widetilde{W}_0^{1,p}(\nu, \Omega_s)$ *is strongly connected with* $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$;

($*_3$) *there exists a positive bounded measurable function* σ *on* Ω *such that for every open cube* $Q \subset \Omega$,

$$\lim_{s\to\infty} \operatorname{meas}(Q \cap \Omega_s) = \int_Q \sigma\, dx;$$

($*_4$) *the sequence* $\{J_s\}$ Γ*-converges to a functional* $J : \overset{\circ}{W}{}^{1,p}(\nu, \Omega) \to \mathbb{R}$.

Assume that for every $s \in \mathbb{N}$, u_s *is the function in* V_s *minimizing the functional* $J_s + G_s$ *on* V_s.

Then there exists a function $u \in V^\sigma$ *such that the following assertions hold true:*

$$\textit{the function } u \textit{ is a unique minimizer of the functional } J + G^\sigma \textit{ on } V^\sigma; \tag{15}$$

$$\lim_{s\to\infty} \|u_s - q_s u\|_{L^p(\nu, \Omega_s)} = 0; \tag{16}$$

$$\lim_{s\to\infty} (J_s + G_s)(u_s) = (J + G^\sigma)(u). \tag{17}$$

Proof. We observe that in view of condition ($*_3$) for every function $v \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ $G_s(q_s v) \to G^\sigma(v)$ and $\Phi_s(q_s v) \to \Phi^\sigma(v)$. This and (14) imply that

$$\begin{cases} \text{for every } \ v \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega) \ \text{ and for every sequence } \ v_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s) \text{ such that} \\ \|v_s - q_s v\|_{L^p(\nu, \Omega_s)} \to 0 \ \text{ we have } \ G_s(v_s) \to G^\sigma(v) \ \text{ and } \ \Phi_s(v_s) \to \Phi^\sigma(v). \end{cases} \tag{18}$$

For every $s \in \mathbb{N}$ we set $I_s = J_s + G_s$ and $I = J + G^\sigma$. From condition ($*_4$) and assertion of (18) it follows that

$$\text{the sequence } \{I_s\} \ \Gamma\text{-converges to the functional } I. \tag{19}$$

Moreover, taking into account that for every $s \in \mathbb{N}$ the function u_s minimizes the functional $J_s + G_s$ on V_s and using (9) we obtain that

$$\text{the sequence of the norms } \|u_s\|_{1,p,\nu,s} \text{ is bounded.} \tag{20}$$

This fact along with conditions $(*_1)$, $(*_2)$ and Proposition 1 implies that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that

$$\lim_{j\to\infty} \|u_{s_j} - q_{s_j} u\|_{L^p(\nu,\Omega_{s_j})} = 0. \tag{21}$$

Now we define the sequence $\{\bar{u}_s\}$ by

$$\bar{u}_s = \begin{cases} u_s & \text{if } s = s_j \text{ for some } j \in \mathbb{N}, \\ q_s u & \text{if } s \neq s_j \text{ for every } j \in \mathbb{N}. \end{cases}$$

It is evident that for every $s \in \mathbb{N}$, $\bar{u}_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$. Due to (21) we have

$$\lim_{s\to\infty} \|\bar{u}_s - q_s u\|_{L^p(\nu,\Omega_s)} = 0.$$

Then by virtue of assertions (18) and (19),

$$\lim_{s\to\infty} \Phi_s(\bar{u}_s) = \Phi^\sigma(u), \tag{22}$$

$$\liminf_{s\to\infty} I_s(\bar{u}_s) \geqslant I(u). \tag{23}$$

Due to (22)

$$\lim_{j\to\infty} \Phi_{s_j}(u_{s_j}) = \Phi^\sigma(u).$$

Taking into account that for every $s \in \mathbb{N}$, $\Phi_s(u_s) \leqslant 1$, from the latter equality we derive that $\Phi^\sigma(u) \leqslant 1$. Consequently, $u \in V^\sigma$. Moreover, from (23) we obtain

$$\liminf_{j\to\infty} I_{s_j}(u_{s_j}) \geqslant I(u). \tag{24}$$

Further, we fix $v \in V^\sigma$. Let us show that

$$\limsup_{s\to\infty} I_s(u_s) \leqslant I(v). \tag{25}$$

From (19) it follows that there exists a sequence $v_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that

$$\lim_{s\to\infty} \|v_s - q_s v\|_{L^p(\nu,\Omega_s)} = 0, \tag{26}$$

$$\lim_{s\to\infty} I_s(v_s) = I(v). \tag{27}$$

For every $s \in \mathbb{N}$ we set $\tau_s = (1 + |\Phi_s(v_s) - \Phi^\sigma(v)|)^{-1}$. From (18) and (26) it follows that

$$\lim_{s\to\infty} \tau_s = 1. \tag{28}$$

For every $s \in \mathbb{N}$ we define $w_s = \tau_s v_s$. Clearly, $w_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$, $s \in \mathbb{N}$. Moreover, using (11) and (12), and the inclusion $v \in V^\sigma$, we establish that for every $s \in \mathbb{N}$,

$$\Phi_s(w_s) \leqslant \tau_s \Phi_s(v_s) \leqslant \tau_s(1 + |\Phi_s(v_s) - \Phi^\sigma(v)|) \leqslant 1.$$

This implies that for every $s \in \mathbb{N}$, $w_s \in V_s$. Then, taking into account that for every $s \in \mathbb{N}$ the function u_s minimizes the functional I_s on V_s, we get

$$\forall\, s \in \mathbb{N}, \quad I_s(u_s) \leqslant I_s(w_s). \tag{29}$$

Using (4), (5), (7) and (8), we obtain that for every $s \in \mathbb{N}$,

$$I_s(w_s) \leqslant \tau_s I_s(v_s) + (1 - \tau_s)(\|\psi_s\|_{L^1(\Omega_s)} + \|\psi\|_{L^1(\Omega)}).$$

This along with (2) and (27)-(29) implies that inequality (25) is valid. From this inequality and inequality (24) we infer that the function u minimizes the functional I on V^σ.

We observe that, due to (4), (7), condition $(*_4)$ and the fact that the function σ is positive, the functional I is strictly convex. Therefore, since the set V^σ is convex, the function u is the unique minimizer of the functional I on V^σ. Thus, assertion (15) holds true.

Next, let us show that assertion (16) holds true. For every $s \in \mathbb{N}$ we define $\alpha_s = \|u_s - q_s u\|_{L^p(\nu, \Omega_s)}$. Suppose that the sequence $\{\alpha_s\}$ does not converge to zero. Then there exist $\varepsilon > 0$ and an increasing sequence $\{\bar{s}_k\} \subset \mathbb{N}$ such that

$$\forall\, k \in \mathbb{N}, \quad \alpha_{\bar{s}_k} \geqslant \varepsilon. \tag{30}$$

Taking into account (20) and conditions $(*_1)$ and $(*_2)$, we establish that there exist an increasing sequence $\{\tilde{s}_j\} \subset \{\bar{s}_k\}$ and a function $w \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ such that

$$\lim_{j \to \infty} \|u_{\tilde{s}_j} - q_{\tilde{s}_j} w\|_{L^p(\nu, \Omega_{\tilde{s}_j})} = 0. \tag{31}$$

Thus, by analogy with the above result for the function u, we obtain that $w \in V^\sigma$ and w minimizes the functional I on V^σ. This fact along with the uniqueness of the minimizer of the functional I on V^σ allows us to deduce that $w = u$ a. e. in Ω. Hence, by (31), $\alpha_{\tilde{s}_j} \to 0$. However, this contradicts (30). The contradiction obtained proves that $\alpha_s \to 0$. Thus, assertion (16) holds true.

Now, from (19) and (16) we get

$$\liminf_{s \to \infty} I_s(u_s) \geqslant I(u).$$

This and (25), with $v = u$, imply that assertion (17) holds true. The theorem is proved.

We note that the convergence of minimizers and minimum values of variational problems with certain pointwise constraints for a sequence of functionals like $J_s + G_s$ was studied in [12]. Moreover, in [12] a rather extensive review of the works related to the topic is contained.

In the nondegenerate case, the questions concerning convergence of minimizers of variational problems with general sets of constraints, and in particular sets of constraints of an integral kind, for integral functionals defined on varying Sobolev spaces were studied in [8]. In this connection see also Subsections 1.4 and 2.5 of [10] where convergence of solutions of variational inequalities with different kinds of sets of constraints was investigated.

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О. А. Рудакова

О сходимости решений вариационных задач с интегральными ограничениями и вырождением в переменных областях.

В настоящей статье для последовательности интегральных функционалов, определенных на весовых пространствах Соболева, связанных с последовательностью n-мерных областей, рассмотрены вариационные задачи с множествами ограничений интегрального вида. Установлены достаточные условия сходимости минимизантов и минимальных значений рассматриваемых вариационных задач.

Ключевые слова: *переменные весовые пространства Соболева, вариационная задача, инте-*

гральный функционал, вырождение, интегральное ограничение, Γ*-сходимость.*

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Про збіжність розв'язків варіаційних задач з інтегральними обмеженнями і виродженням в змінних областях.

Для послідовності інтегральних функціоналів, визначених на вагових просторах Соболєва, пов'язаних з послідовністю n-вимірних областей, розглянуто варіаційні задачі з множинами обмежень інтегрального вигляду. Встановлено достатні умови збіжності мінімізантів і мінімальних значень розглянутих варіаційних задач.

Ключові слова: *змінні вагові простори Соболева, варіаційна задача, інтегральний функціонал, виродження, інтегральне обмеження,* Γ*-збіжність.*

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