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Γ-CONVERGENCE OF INTEGRAL FUNCTIONALS WITH DEGENERATE INTEGRANDS IN PERIODICALLY PERFORATED DOMAINS

We consider a sequence of integral functionals with degenerate integrands in perforated domains of periodic structure. We establish the Γ-convergence of the sequence under consideration to an integral functional defined on a limit weighted Sobolev space. A representation formula for the integrand of the Γ-limit functional is given.

1. Introduction. In this article we consider a sequence of integral functionals with degenerate integrands in perforated domains of periodic structure. We establish the Γ-convergence of the sequence under consideration to an integral functional defined on a limit weighted Sobolev space. At the same time a representation formula for the integrand of the Γ-limit functional is given.

We note that the Γ-convergence of functionals plays an important part in the study of convergence of solutions to variational problems (see for instance [1], [3], [5], [8], [15] and [16]). In particular, the questions related to the investigation of convergence of minimizers and minimum values of functionals defined on variable weighted Sobolev spaces were studied in [10]–[13].

The Γ-convergence of quadratic integral functionals having periodic quickly oscillating coefficients and defined on a fixed weighted Sobolev space was proved in [2].

In the nonweighted case the Γ-convergence of integral functionals associated with different kinds of periodically perforated domains was established for instance in [6] and [9]. Moreover, in the nonweighted case representation formulae for coefficients of the homogenized problem corresponding to the Neumann variational problems for quadratic integral functionals in periodically perforated domains were given in [4].

Finally, we emphasize that the integral functionals under consideration in the present article combine the following three features: their domains of definition depend on a parameter; their integrands, having a quickly oscillating component, depend on the same parameter; the integrands have a fixed weighted multiplier.

2. Preliminaries. Let Ω be a bounded domain of $\mathbb{R}^n$ $(n \geqslant 2)$, $p \in (1, n)$, and let ν be a nonnegative function on Ω with the properties: $\nu > 0$ almost everywhere in Ω and

$$\nu \in L^1_{\rm loc}(\Omega), \qquad \left(\frac{1}{\nu}\right)^{1/(p-1)} \in L^1_{\rm loc}(\Omega). \tag{2.1}$$

We denote by $L^p(\nu, \Omega)$ the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that the function $\nu|u|^p$ is summable in Ω. $L^p(\nu, \Omega)$ is a Banach space with the norm

$$\|u\|_{L^p(\nu,\Omega)} = \left(\int_\Omega \nu|u|^p\, dx\right)^{1/p}.$$

We note that by virtue of Young's inequality and the second inclusion of (2.1) we have $L^p(\nu,\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$.

We denote by $W^{1,p}(\nu,\Omega)$ the set of all functions $u \in L^p(\nu,\Omega)$ such that for every $i \in \{1,\dots,n\}$ there exists the weak derivative $D_i u$, $D_i u \in L^p(\nu,\Omega)$. $W^{1,p}(\nu,\Omega)$ is a reflexive Banach space with the norm

$$\|u\|_{1,p,\nu} = \biggl(\int_\Omega \nu|u|^p\,dx + \sum_{i=1}^n \int_\Omega \nu|D_i u|^p\,dx\biggr)^{1/p}.$$

Due to the first inclusion of (2.1) we have $C_0^\infty(\Omega) \subset W^{1,p}(\nu,\Omega)$. We denote by $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in $W^{1,p}(\nu,\Omega)$.

Next, let $\{\Omega_s\}$ be a sequence of domains of $\mathbb{R}^n$ which are contained in Ω.

By analogy with the spaces introduced above we define the functional spaces corresponding to the domains Ω_s.

Let $s \in \mathbb{N}$. We denote by $L^p(\nu,\Omega_s)$ the set of all measurable functions $u : \Omega_s \to \mathbb{R}$ such that the function $\nu|u|^p$ is summable in Ω_s. $L^p(\nu,\Omega_s)$ is a Banach space with the norm

$$\|u\|_{L^p(\nu,\Omega_s)} = \biggl(\int_{\Omega_s} \nu|u|^p\,dx\biggr)^{1/p}.$$

By virtue of the second inclusion of (2.1) we have $L^p(\nu,\Omega_s) \subset L^1_{\mathrm{loc}}(\Omega_s)$. We denote by $W^{1,p}(\nu,\Omega_s)$ the set of all functions $u \in L^p(\nu,\Omega_s)$ such that for every $i \in \{1,\dots,n\}$ there exists the weak derivative $D_i u$, $D_i u \in L^p(\nu,\Omega_s)$. $W^{1,p}(\nu,\Omega_s)$ is a Banach space with the norm

$$\|u\|_{1,p,\nu,s} = \biggl(\int_{\Omega_s} \nu|u|^p\,dx + \sum_{i=1}^n \int_{\Omega_s} \nu|D_i u|^p\,dx\biggr)^{1/p}.$$

We denote by $\widetilde{C}_0^\infty(\Omega_s)$ the set of all restrictions on Ω_s of functions from $C_0^\infty(\Omega)$. Due to the first inclusion of (2.1) we have $\widetilde{C}_0^\infty(\Omega_s) \subset W^{1,p}(\nu,\Omega_s)$. We denote by $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ the closure of the set $\widetilde{C}_0^\infty(\Omega_s)$ in $W^{1,p}(\nu,\Omega_s)$.

We observe that if $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and $s \in \mathbb{N}$, then $u|_{\Omega_s} \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$.

Definition 2.1. If $s \in \mathbb{N}$, q_s is the mapping from $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ into $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that for every function $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, $\quad q_s u = u|_{\Omega_s}$.

Definition 2.2. Let for every $s \in \mathbb{N}$, I_s be a functional on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$, and let I be a functional on $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. We say that the sequence $\{I_s\}$ Γ-converges to the functional I if the following conditions are satisfied:

(i) for every function $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ there exists a sequence $w_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that $\lim\limits_{s\to\infty} \|w_s - q_s u\|_{L^p(\nu,\Omega_s)} = 0$ and $\lim\limits_{s\to\infty} I_s(w_s) = I(u)$;

(ii) for every function $u \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and every sequence $u_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that $\lim\limits_{s\to\infty} \|u_s - q_s u\|_{L^p(\nu,\Omega_s)} = 0$ we have $\liminf\limits_{s\to\infty} I_s(u_s) \geqslant I(u)$.

The given definition was introduced in [10], and the corresponding Γ-compactness theorem for integral functionals was established in [10] and [13].

Further, we shall use the following notation: for every $i \in \{1, \dots, n\}$, e^i is the unit vector of the i^{th} axis in $\mathbb{R}^n$; for every $y \in \mathbb{R}^n$ and $\rho > 0$, $B(y, \rho) = \{x \in \mathbb{R}^n : |x - y| < \rho\}$; for every $y \in \mathbb{R}^n$ and $t \in \mathbb{N}$,

$$Q_t(y) = \{x \in \mathbb{R}^n : |x_i - y_i| < 1/(2t),\ i = 1, \dots, n\}.$$

For every $i \in \{1, \dots, n\}$ we set

$$Q^i_- = \{x \in \partial Q_1(0) : x_i = -1/2\}, \qquad Q^i_+ = \{x \in \partial Q_1(0) : x_i = 1/2\}.$$

Clearly, if $i \in \{1, \dots, n\}$ and $x \in Q^i_-$, we have $x + e^i \in Q^i_+$.

For every function $v \in C^1(\overline{Q_1(0)})$ we denote by $\bar{v}$ the unique continuous extension of v on $\overline{Q_1(0)}$.

By $C^1_{\text{per}}(Q_1(0))$ we denote the set of all functions $v \in C^1(\overline{Q_1(0)})$ such that for every $i \in \{1, \dots, n\}$ and $x \in Q^i_-$, $\bar{v}(x + e^i) = \bar{v}(x)$.

Next, we fix $r \in (0, 1/2)$ and set $\Pi = Q_1(0) \setminus \overline{B(0, r)}$. By $C^1_{\text{per}}(\Pi)$ we denote all functions $v \in C^1(\Pi)$ such that $v = w|_\Pi$, where $w \in C^1_{\text{per}}(Q_1(0))$. Finally, by $W^{1,p}_{\text{per}}(\Pi)$ we denote the closure of the set $C^1_{\text{per}}(\Pi)$ in $W^{1,p}(\Pi)$.

Let $\hat{c}_1$, $\hat{c}_2 > 0$, $\hat{c} \geqslant 0$, and let $\hat{f} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function such that the following conditions are satisfied:

for every $\xi \in \mathbb{R}^n$ the function $\hat{f}(\cdot, \xi)$ is 1-periodic; (2.2)

for almost every $x \in \mathbb{R}^n$ the function $\hat{f}(x, \cdot)$ is convex in $\mathbb{R}^n$; (2.3)

for almost every $x \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^n$,

$$\hat{c}_1 |\xi|^p - \hat{c} \leqslant \hat{f}(x, \xi) \leqslant \hat{c}_2 |\xi|^p + \hat{c}. \tag{2.4}$$

Let $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ be the function such that for every $\xi \in \mathbb{R}^n$,

$$\tilde{f}(\xi) = \inf_{v \in W^{1,p}_{\text{per}}(\Pi)} \int_\Pi \hat{f}(x, \xi + \nabla v) dx.$$

From (2.4) it follows that for every $\xi \in \mathbb{R}^n$,

$$-\hat{c}\,\text{meas}\,\Pi \leqslant \tilde{f}(\xi) \leqslant (\hat{c}_2 |\xi|^p + \hat{c})\text{meas}\,\Pi. \tag{2.5}$$

We also observe that owing to (2.3) the function $\tilde{f}$ is convex in $\mathbb{R}^n$.

3. Statement of the main result. We shall assume that $\nu \in L^1(\Omega)$. We define $b = \hat{c}\nu$ and for every $s \in \mathbb{N}$ we set $\psi_s = b|_{\Omega_s}$. Moreover, we set

$$\bar{e} = \frac{1}{2} \sum_{i=1}^n e^i.$$

Now let for every $s \in \mathbb{N}$, $f_s : \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ be the function such that for every $(x, \xi) \in \Omega_s \times \mathbb{R}^n$,

$$f_s(x, \xi) = \nu(x)\hat{f}(sx - \bar{e}, \xi).$$

Clearly, for every $s \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$ the function $f_s(\cdot, \xi)$ is measurable in Ω_s. Moreover, owing to conditions (2.3) and (2.4) the following assertions hold true: for every $s \in \mathbb{N}$ and almost every $x \in \Omega_s$ the function $f_s(x, \cdot)$ is convex in $\mathbb{R}^n$; for every $s \in \mathbb{N}$, almost every $x \in \Omega_s$ and every $\xi \in \mathbb{R}^n$ we have

$$\hat{c}_1\nu(x)|\xi|^p - \psi_s(x) \leqslant f_s(x, \xi) \leqslant \hat{c}_2\nu(x)|\xi|^p + \psi_s(x). \tag{3.1}$$

Let for every $s \in \mathbb{N}$, $J_s : \widetilde{W}_0^{1,p}(\nu, \Omega_s) \to \mathbb{R}$ be the functional such that for every $u \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$,

$$J_s(u) = \int_{\Omega_s} f_s(x, \nabla u)dx.$$

We denote by $\mathcal{F}$ the set of all functions $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfying the conditions: for every $\xi \in \mathbb{R}^n$ the function $f(\cdot, \xi)$ is measurable in Ω; for almost every $x \in \Omega$ the function $f(x, \cdot)$ is convex in $\mathbb{R}^n$; for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$ we have

$$-b(x) \leqslant f(x, \xi) \leqslant \hat{c}_2\nu(x)|\xi|^p + b(x).$$

Definition 3.1. If $f \in \mathcal{F}$, $J^f : \overset{\circ}{W}{}^{1,p}(\nu, \Omega) \to \mathbb{R}$ is the functional such that for every $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$,

$$J^f(u) = \int_\Omega f(x, \nabla u)dx.$$

Let $\bar{f} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be the function such that for every $(y, \xi) \in \Omega \times \mathbb{R}^n$,

$$\bar{f}(y, \xi) = \nu(y)\tilde{f}(\xi).$$

Observe that due to (2.5) and the convexity of the function $\tilde{f}$ we have $\bar{f} \in \mathcal{F}$.

In what follows we shall suppose that

$$\Omega = \{x \in \mathbb{R}^n : |x_i| < 1,\ i = 1, \ldots, n\}.$$

For every $s \in \mathbb{N}$ we set $\tilde{Z}_s = \{z \in \Omega : sz_i - 1/2 \in \mathbb{Z},\ i = 1, \ldots, n\}$. We have

$$\forall\, s \in \mathbb{N}, \qquad \bigcup_{z \in \tilde{Z}_s} \overline{Q_s(z)} = \overline{\Omega}, \tag{3.2}$$

$$\forall\, s \in \mathbb{N}, \forall\, z,\ z' \in \tilde{Z}_s,\ z \neq z', \qquad Q_s(z) \cap Q_s(z') = \emptyset. \tag{3.3}$$

We shall assume that the domains Ω_s have the following structure: for every $s \in \mathbb{N}$,

$$\Omega_s = \Omega \setminus \bigcup_{z \in \tilde{Z}_s} \overline{B(z, r/s)}.$$

Theorem 3.2. *Suppose that the function ν is positive and continuous in $\Omega \setminus \{0\}$. Then the sequence $\{J_s\}$ Γ-converges to the functional $J^{\bar f}$.*

4. Scheme of the proof of Theorem 3.2. *Step* 1. For every $k \in \mathbb{N}$ we set

$$\Omega^{(k)} = \{x \in \mathbb{R}^n : |x_i| < 1 - 1/(2k),\ i = 1, \dots, n\} \setminus \overline{Q_{2k}(0)}.$$

Evidently, $\{\Omega^{(k)}\}$ is a sequence of nonempty open sets of $\mathbb{R}^n$, and the following assertions hold true: for every $k \in \mathbb{N}$, $\overline{\Omega^{(k)}} \subset \Omega^{(k+1)} \subset \Omega$; $\operatorname{meas}(\Omega \setminus \Omega^{(k)}) \to 0$; for every $k \in \mathbb{N}$ the functions ν and b are bounded in $\Omega^{(k)}$.

These assertions along with the properties of the functions b, ψ_s and f_s provide the fulfilment of all the conditions under which in [13] Theorem 2 on the Γ-compactness of a sequence of integral functionals was proved. Thus, some necessary constructions given in the proof of this theorem may be utilized. These ones are as follows.

A. For every $t \in \mathbb{N}$ we set $Y_t = \{y \in \mathbb{R}^n : ty_i \in \mathbb{Z},\ i = 1, \dots, n\}$. Observe that

$$\forall\, t \in \mathbb{N}, \qquad \bigcup_{y \in Y_t} \overline{Q_t(y)} = \mathbb{R}^n;$$

$$\forall\, t \in \mathbb{N},\ \forall\, y, y' \in Y_t,\ y \neq y', \qquad Q_t(y) \cap Q_t(y') = \emptyset.$$

For every $t \in \mathbb{N}$ we define $Y'_t = \{y \in Y_t : \overline{Q_t(y)} \subset \Omega\}$. Obviously, there exists $t_0 \in \mathbb{N}$ such that for every $t \in \mathbb{N}$, $t \geqslant t_0$, the set Y'_t is nonempty.

Let for every $t \in \mathbb{N}$, $t \geqslant t_0$, $s \in \mathbb{N}$ and $y \in Y'_t$,

$$V_{t,s}(y) = \Big\{ u \in \widetilde{W}^{1,p}_0(\nu, \Omega_s) : \int_{Q_t(y) \cap \Omega_s} \nu |u|^p \, dx \leqslant t^{-n-3p} \Big\}.$$

Now for every $t \in \mathbb{N}$, $t \geqslant t_0$, $s \in \mathbb{N}$, $y \in Y'_t$ and $\xi \in \mathbb{R}^n$ we set

$$F_{t,s}(y, \xi) = t^n \inf_{u \in V_{t,s}(y)} \int_{Q_t(y) \cap \Omega_s} f_s(x, \xi + \nabla u) dx.$$

B. Let $\{\bar s_k\} \subset \mathbb{N}$ be an arbitrary increasing sequence. From (3.1) and the convexity of the functions $f_s(x, \cdot)$ for almost every $x \in \Omega_s$ it follows that there exist an increasing sequence $\{s_j\} \subset \{\bar s_k\}$ and a sequence of functions $\Phi_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that for every $t \in \mathbb{N}$, $t \geqslant t_0$, $y \in Y'_t$ and $\xi \in \mathbb{R}^n$ we have

$$\lim_{j \to \infty} F_{t,s_j}(y, \xi) = \Phi_t(y, \xi). \tag{3.4}$$

C. Let for every $t \in \mathbb{N}$ and $y \in \Omega$ such that $\overline{Q_t(y)} \subset \Omega$, $\chi_{t,y} : \Omega \to \mathbb{R}$ be the characteristic function of the set $Q_t(y)$.

For every k, $t \in \mathbb{N}$ we set $Y_{k,t} = \{y \in Y_t : Q_t(y) \subset \Omega^{(k)}\}$.

Let us give the following definition: if k, $t \in \mathbb{N}$ and $Y_{k,t} \neq \emptyset$, $H^{(k)}_t$ is the function on $\Omega \times \mathbb{R}^n$ such that for every pair $(x, \xi) \in \Omega \times \mathbb{R}^n$,

$$H^{(k)}_t(x, \xi) = \sum_{y \in Y_{k,t}} \chi_{t,y}(x) \Phi_t(y, \xi);$$

if $k,\ t \in \mathbb{N}$ and $Y_{k,t} = \emptyset$, $H_t^{(k)}$ is the function on $\Omega \times \mathbb{R}^n$ such that for every pair $(x,\xi) \in \Omega \times \mathbb{R}^n$, $H_t^{(k)}(x,\xi) = 0$.

D. In accordance with the considerations given within steps 4–11 of the proof of Theorem 2 in [13] there exist an increasing sequence $\{t_i\} \subset \mathbb{N}$ and a Carathéodory function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ such that $f \in \mathcal{F}$ and the following assertions hold:

$$k \in \mathbb{N},\ \xi \in \mathbb{R}^n,\ \varphi \in L^\infty(\Omega) \Rightarrow \lim_{i\to\infty} \int_{\Omega^{(k)}} H_{t_i}^{(k)}(\cdot,\xi)\varphi\, dx = \int_{\Omega^{(k)}} f(\cdot,\xi)\varphi\, dx; \tag{3.5}$$

$$\text{the sequence } \{J_{s_j}\}\ \Gamma\text{-converges to the functional } J^f. \tag{3.6}$$

Now the aim is to prove that for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, $f(x,\xi) = \bar{f}(x,\xi)$.

Step 2. Taking into account the inclusions $\nu \in L^1(\Omega)$ and $f \in \mathcal{F}$, we establish that there exists a set $E \subset \Omega$ with measure zero such that

$$\text{for every } z \in \Omega \setminus E \text{ and } \xi \in \mathbb{R}^n, \quad \tau^n \int_{Q_\tau(z)} f(\cdot,\xi)dx \to f(z,\xi). \tag{3.7}$$

Step 3. We fix $z_0 \in \Omega \setminus (E \cup \{0\})$ and $\xi \in \mathbb{R}^n$. Clearly, there exists $\tau_0 \in \mathbb{N}$ such that $0 \notin \overline{Q_{\tau_0}(z_0)}$ and $\overline{Q_{\tau_0}(z_0)} \subset \Omega$. Then there exists $k \in \mathbb{N}$ such that $Q_{\tau_0}(z_0) \subset \Omega^{(k)}$. Since the function ν is positive and continuous in $\Omega \setminus \{0\}$, there exists $M_k > 0$ such that

$$\forall\, x \in \Omega^{(k)}, \quad 1/M_k \leqslant \nu(x) \leqslant M_k. \tag{3.8}$$

We fix $\varepsilon > 0$. Due to the continuity of ν in $\Omega \setminus \{0\}$ there exists $\delta > 0$ such that

$$\text{for every } x',\, x'' \in \Omega^{(k)},\ |x' - x''| \leqslant \delta, \text{ we have } |\nu(x') - \nu(x'')| \leqslant \varepsilon. \tag{3.9}$$

Let $\tau \in \mathbb{N}$, $\tau > \tau_0 + 1 + n/\delta$. We fix $t \in \mathbb{N}$ such that $t > \max\{t_0,\, 2\tau(\tau - 1)\}$ and define $X_t = \{y \in Y_t : Q_t(y) \cap Q_\tau(z_0) \neq \emptyset\}$. It is easy to see that $X_t \neq \emptyset$.

Step 4. We fix $y \in X_t$ and take a function $w \in W^{1,p}_{\mathrm{per}}(\Pi)$ such that

$$\tilde{f}(\xi) = \int_\Pi \hat{f}(x, \xi + \nabla w)dx. \tag{3.10}$$

The existence of such a function follows from (2.3), (2.4) and the known results on the existence of minimizers of functionals (see for instance [14]).

Finally, we fix $s_0 \in \mathbb{N}$ such that

$$2^n s_0^{-p} M_k \int_\Pi |w|^p\, dx \leqslant t^{-n-3p},$$

and after that fix $s \in \mathbb{N}$, $s \geqslant \max\{s_0, 2t\}$.

Let $w_s : \Omega_s \to \mathbb{R}$ be a function such that

$$w_s(x) = s^{-1} w(s(x - z)) \text{ if } z \in \tilde{Z}_s \text{ and } x \in Q_s(z) \setminus \overline{B(z, r/s)}.$$

Taking into account assertions (3.2) and (3.3) and the inclusion $w \in W^{1,p}_{\rm per}(\Pi)$, we establish that $w_s \in W^{1,p}(\Omega_s)$. Then involving into consideration the function $w_s\varphi_t$, where φ_t is a function in $C_0^\infty(\Omega)$ such that $\varphi_t = 1$ in $Q_t(y)$, we obtain the inequality

$$F_{t,s}(y,\xi) \leqslant t^n \int_{Q_t(y)\cap\Omega_s} f_s(x, \xi + \nabla w_s)dx. \tag{3.11}$$

Using the definitions of the functions w_s, f_s and $\bar f$ along with (2.2), (2.4) and (3.8)–(3.11), we get

$$F_{t,s}(y,\xi) - \bar f(y,\xi) \leqslant 2^n n(\hat c_2/\hat c_1 + 1)(|\tilde f(\xi)| + \hat c)M_k(\varepsilon + ts^{-1}). \tag{3.12}$$

Step 5. With the use of the definition of $F_{t,s}(y,\xi)$, properties of the function f_s and (3.8) we establish that there exists a function $v_s \in \widetilde{C}_0^\infty(\Omega_s)$ such that $v_s = 0$ in $\Omega_s \setminus Q_t(y)$ and

$$\int_{Q_t(y)\cap\Omega_s} f_s(x, \xi + \nabla v_s)dx \leqslant t^{-n} F_{t,s}(y,\xi) + \hat c_3(1 + |\xi|^p)M_k t^{-n-1}, \tag{3.13}$$

where $\hat c_3$ is a positive constant depending only on n, p, $\hat c_1$, $\hat c_2$ and $\hat c$.

Step 6. By means of the function v_s we construct a function belonging to $C^1_{\rm per}(\Pi)$. We need this in order to obtain a suitable estimate from below for the left-hand side of inequality (3.13).

First, we observe that due to the inclusion $v_s \in \widetilde{C}_0^\infty(\Omega_s)$ there exists a function $v \in C^1(\mathbb{R}^n)$ such that $\operatorname{supp} v \subset \Omega$ and $v_s = v|_{\Omega_s}$.

We define $\tilde Z_s' = \{z \in \tilde Z_s : Q_s(z) \cap Q_t(y) \neq \emptyset\}$. Owing to (3.2) the set $\tilde Z_s'$ is nonempty. We denote by n_s' the number of elements of the set $\tilde Z_s'$.

For every $z \in \tilde Z_s'$ we define the function $g_{s,z} : \Pi \to \mathbb{R}$ by $g_{s,z}(x) = sv(s^{-1}x + z)$, $x \in \Pi$, and after that we set

$$g_s = \frac{1}{n_s'} \sum_{z\in\tilde Z_s'} g_{s,z}.$$

With the use of considerations analogous to those given in the proof of Lemma 2.2.1 of [7] we establish that $g_s \in C^1_{\rm per}(\Pi)$. Therefore, $g_s \in W^{1,p}_{\rm per}(\Pi)$. Then, taking into account the definitions of the functions $\tilde f$ and g_s and (2.3), we get

$$\tilde f(\xi) \leqslant \int_\Pi \hat f(x, \xi + \nabla g_s)dx \leqslant \frac{1}{n_s'} \sum_{z\in\tilde Z_s'} \int_\Pi \hat f(x, \xi + \nabla g_{s,z})dx. \tag{3.14}$$

Moreover, using (2.2), (2.4), (3.8) and (3.9), we obtain that for every $z \in \tilde Z_s'$,

$$\nu(y)\int_\Pi \hat f(x, \xi + \nabla g_{s,z})dx \leqslant s^n \int_{Q_s(z)\setminus\overline{B(z,r/s)}} f_s(x, \xi + \nabla v_s)dx$$
$$+ \varepsilon \hat c_2 M_k s^n \int_{Q_s(z)\setminus\overline{B(z,r/s)}} \nu|\xi + \nabla v_s|^p\, dx + \varepsilon \hat c M_k^2. \tag{3.15}$$

From (3.13)–(3.15), taking into account (3.1), (3.8) and the definitions of $\bar f$ and $F_{t,s}(y,\xi)$, we get the inequality

$$\bar f(y,\xi) - F_{t,s}(y,\xi) \leqslant \hat c_4(1+|\xi|^p)M_k^2(\varepsilon + t^{-1} + ts^{-1}), \tag{3.16}$$

where $\hat c_4$ is a positive constant depending only on n, p, $\hat c_1$, $\hat c_2$ and $\hat c$.

Step 7. From (2.5), (3.12) and (3.16) we deduce that

$$|F_{t,s}(y,\xi) - \bar f(y,\xi)| \leqslant \hat c_5(1+|\xi|^p)M_k^2(\varepsilon + t^{-1} + ts^{-1}),$$

where $\hat c_5$ is a positive constant depending only on n, p, $\hat c_1$, $\hat c_2$ and $\hat c$.

Hence, taking into account that s is an arbitrary natural number greater than or equal to $\max\{s_0, 2t\}$ and using (3.4), we infer that for every $y \in X_t$,

$$|\Phi_t(y,\xi) - \bar f(y,\xi)| \leqslant \hat c_5(1+|\xi|^p)M_k^2(\varepsilon + t^{-1}). \tag{3.17}$$

Step 8. Taking into account the definition of the function $H_t^{(k)}$ and the equality

$$\sum_{y\in X_t} \mathrm{meas}[Q_\tau(z_0)\cap Q_t(y)] = \tau^{-n}, \tag{3.18}$$

we obtain that

$$\Big|\int_{Q_\tau(z_0)} H_t^{(k)}(\cdot,\xi)dx - \bar f(z_0,\xi)\tau^{-n}\Big| \leqslant \sum_{y\in X_t} |\Phi_t(y,\xi) - \bar f(y,\xi)|\mathrm{meas}[Q_\tau(z_0)\cap Q_t(y)] + \sum_{y\in X_t} |\bar f(y,\xi) - \bar f(z_0,\xi)|\mathrm{meas}[Q_\tau(z_0)\cap Q_t(y)].$$

Hence, taking into account (3.17), (3.18), (3.9), the inequality $\tau > 1 + n/\delta$ and the definition of the function $\bar f$, we derive that for every $t \in \mathbb{N}$, $t > \max\{t_0, 2\tau(\tau-1)\}$,

$$\Big|\int_{Q_\tau(z_0)} H_t^{(k)}(\cdot,\xi)dx - \bar f(z_0,\xi)\tau^{-n}\Big| \leqslant (\hat c_5+1)(1+|\xi|^p + |\tilde f(\xi)|)M_k^2(\varepsilon + t^{-1})\tau^{-n}.$$

This and (3.5) imply that for every $\tau \in \mathbb{N}$, $\tau > \tau_0 + 1 + n/\delta$,

$$\Big|\tau^n \int_{Q_\tau(z_0)} f(\cdot,\xi)dx - \bar f(z_0,\xi)\Big| \leqslant (\hat c_5+1)(1+|\xi|^p + |\tilde f(\xi)|)M_k^2\varepsilon.$$

Hence, using (3.7) and after that taking into account the arbitrariness of $\varepsilon > 0$, we obtain $f(z_0,\xi) = \bar f(z_0,\xi)$.

Thus, for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$ we have $f(x,\xi) = \bar f(x,\xi)$. Therefore, $J^f = J^{\bar f}$. From this and (3.6) it follows that the sequence $\{J_{s_j}\}$ Γ-converges to the functional $J^{\bar f}$.

Step 9. The result obtained allows us to affirm that the following assertion holds true: for every increasing sequence $\{\bar{s}_k\} \subset \mathbb{N}$ there exists an increasing sequence $\{s_j\} \subset \{\bar{s}_k\}$ such that the sequence $\{J_{s_j}\}$ Γ-converges to the functional $J^{\bar{f}}$. Hence we deduce the conclusion of the theorem.

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