11. ELECTROSTATIC FIELD

11.1. FIELD COMPUTATION BY MEANS OF INTEGRAL RELATIONS

11-1 (11.1). Derive the formulae to determine the electric flux in the coordinate function a) from the point charge through the spherical surface of radius R; b) from the linear charge through the cylindrical surface of radius r.

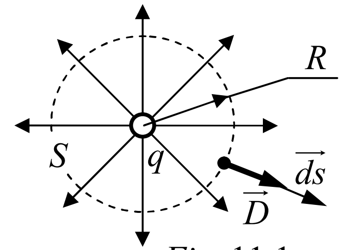


Fig. 11.1

Solution. Case a). Let's construct a spherical surface S of radius R around the point charge q (fig. 11.1) and apply Gauss' law for flux in an integral form: $\oint_S \vec{D}\cdot\vec{ds} = \Sigma q$. From the definition of the electric field intensity, it follows that the force lines are directed radially so they are perpendicular to surface S. Thus, vectors $\vec{D}$ and $\vec{ds}$ have the same direction and their scalar product may be replaced with the product of their length. Furthermore, owing to symmetry the vector $\vec{D}$ has the same value in all the points of surface S which are equally spaced from charge q; that's why it can be put before the sign of integration. Finally, we have:

$$\oint_S \vec{D}\cdot\vec{ds} = \oint_S D\cdot ds = D\cdot\oint_S ds = D\cdot S = D\cdot 4\pi\cdot R^2. \tag{11.1}$$

Consequence: induction and intensity of the point charge field are –

$$D = \frac{q}{4\pi R^2}, \quad E = \frac{D}{\varepsilon_a} = \frac{q}{4\pi\varepsilon_a R^2}, \tag{11.2}$$

so they depend only on coordinate R.

Do the same in case b): construct a cylindrical surface S of radius r and length l around the axis with the linear charge τ and apply Gauss' law for flux in an integral form: $\oint_S \vec{D}\cdot\vec{ds} = \Sigma q$. As illustration, we can again use fig. 11.1, however, we replace q with τ and R with r, axis z being directed perpendicular to the drawing plane. Suppose, the length of object is much bigger than its radius is. Then, it is possible to neglect the distortion of the in-plane field closely to the end faces of cylindrical surface S and consider the force lines being directed radially everywhere. In this case there is no electric flux through the end faces of a cylinder because the angle between $\vec{D}$ and $\vec{ds}$ is 90°. However, in all points of the side face the vectors $\vec{D}$ and $\vec{ds}$ have the same direction, then their scalar product may be replaced with the product of their length. Furthermore, owing to symmetry the value of vector $\vec{D}$ is the same in all the points of the side face of surface S, which are equally spaced from axis z, that's why it can be put before the sign of integration. Finally, we have:

$$\oint_S \vec{D}\cdot\vec{ds} = \int_{S_{side}} D\cdot ds = D\cdot S_{side} = D\cdot 2\pi\cdot r\cdot l. \tag{11.3}$$

Consequence: induction and intensity of the charged axis field are:

$$D = \frac{q}{2\pi rl} = \frac{\tau}{2\pi r}, \quad E = \frac{D}{\varepsilon_a} = \frac{\tau}{2\pi\varepsilon_a r}, \tag{11.4}$$

which means they depend only on coordinate r.

11-2 (11.2). Determine the electric field induced by two charges in the point where the third charge is located as well as the force acting upon the third charge, the system of the charges is presented in fig. 11.2, where

$q_a = 4\cdot 10^{-12}\ C, \quad q_b = 15\cdot 10^{-12}\ C, \quad q_c = 15\cdot 10^{-12}\ C.$

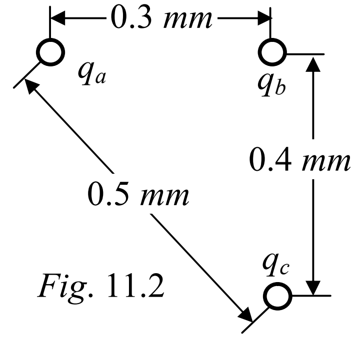


Fig. 11.2

Solution. In accordance with formula (11.3) the formula of intensity of the electrostatic field created by a point charge in vacuum has the view: $E = \dfrac{q}{4\pi\varepsilon_0 R^2}$.

Then the intensities created by charges q_a and q_b in point c are as follows (fig. 11.3):

$$E' = \frac{q_a}{4\pi\varepsilon_0 |ac|^2} = \frac{4\cdot 10^{-12}}{4\pi\cdot 8.85\cdot 10^{-12}(0.5\cdot 10^{-3})^2} = 143.9\cdot 10^3\ V/m = 143.9\ kV/m.$$

$$E'' = \frac{q_b}{4\pi\varepsilon_0 |bc|^2} = \frac{15\cdot 10^{-12}}{4\pi\cdot 8.85\cdot 10^{-12}(0.4\cdot 10^{-3})^2} = 843.0\cdot 10^3\ V/m = 843.0\ kV/m.$$

Let's determine the angle at vertex c of triangle abc:

$cos(\angle acb) = bc/ac = 0.4/0.5 = 0.8; \quad \angle acb = arccos\ 0.8 = 36.9°.$

Then the angle at vertex d of triangle cdf is:

$$\angle cdf = 180° - 36.9° = 143.1°.$$

In accordance with the superposition principle, the resultant vector of the required intensity $\vec{E}$ is equal to the vector sum $\vec{E} = \vec{E}' + \vec{E}''$ (fig. 11.3). Under the cosine rule, we find

$$E = \sqrt{(E')^2 + (E'')^2 - 2\cdot E'\cdot E''\cdot cos(\angle cdf)} =$$

$$= \sqrt{143.9^2 + 843.0^2 - 2\cdot 143.9\cdot 843.0\cdot cos(143{,}1)} = 962\ kV/m.$$

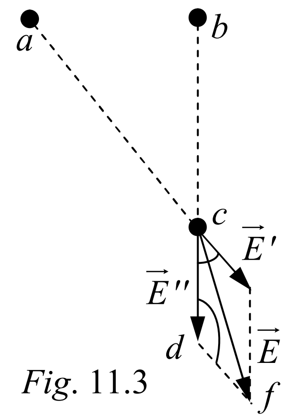


Fig. 11.3

The force acting upon charge q_c is equal to:

$F = q_c\cdot E = 15\cdot 10^{-12}\cdot 962\cdot 10^3 = 14.4\cdot 10^{-6}\ N.$

11-3 (11.3). Two balls of radius $R_1 = 0.2\ cm$ and $R_2 = 0.5\ cm$ are separated by distance $d = 20\ cm$ from each other in dielectric with relative permeability $\varepsilon = 4$ (fig. 11.4). The ball charges are $q = 10^{-10}\ C$ of different signs. Find the field energy and the maximum intensity.

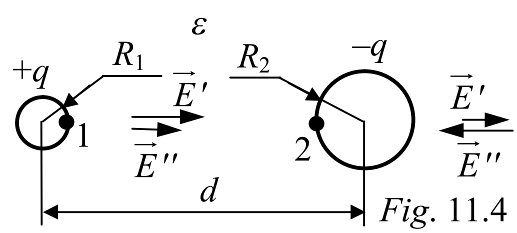


Fig. 11.4

Solution. Distance d greatly exceeds the balls' radius, so it may be supposed that the geometrical and electric centers of the balls coincide with each other.

In area between the balls, the intensities $\vec{E}'$ and $\vec{E}''$ are added, while in the area outside the balls they are subtracted (fig. 11.4). That's why the resultant intensity is bigger in the area between the balls. While moving away from the ball centre, the field intensity decreases, correspondingly, the maximum intensity is expected in the point on the ball's surface. The radius of the first ball is smaller, so the biggest field intensity is observed in point 1. In order to compute the field in this point, the superposition principle is used. On the grounds of formula (11.2) we have

$$E_1 = E_1' + E_1'' = \frac{q}{4\pi\varepsilon_0\varepsilon R_1^2} + \frac{q}{4\pi\varepsilon_0\varepsilon d^2} = \frac{10^{-10}}{4\pi\cdot 8.85\cdot 10^{-12}\cdot 4}\left(\frac{1}{(2\cdot 10^{-3})^2} + \frac{1}{0.2^2}\right) =$$

$= 56141\ V/m.$

The potential of an isolated ball field (similar to problem 11.5, formula 11.8):

$$\varphi = \frac{q}{4\pi\varepsilon_a R} + A.$$

Assume $\varphi = 0$ at $R = \infty$, then $A = 0$. Thus, the ball potentials are

$$\varphi_1 = \varphi_1' + \varphi_1'' = \frac{q}{4\pi\varepsilon_0\varepsilon R_1} + \frac{-q}{4\pi\varepsilon_0\varepsilon d} = \frac{10^{-10}}{4\pi\cdot 8.85\cdot 10^{-12} 4}\left(\frac{1}{2\cdot 10^{-3}} + \frac{-1}{0.2}\right) = 111.1\ V;$$

$$\varphi_2 = \varphi_2' + \varphi_2'' = \frac{q}{4\pi\varepsilon_0\varepsilon d} + \frac{-q}{4\pi\varepsilon_0\varepsilon R_2} = \frac{10^{-10}}{4\pi\cdot 8.85\cdot 10^{-12} 4}\left(\frac{1}{0.2} + \frac{-1}{0.005}\right) = -43.8\ V.$$

Voltage between the balls is $U_{12} = \varphi_1 - \varphi_2 = 111.1 + 43.8 = 154.9\ V.$

Capacity of the installation is $C = q/U_{12} = 10^{-10}/154.9 = 0.645\cdot 10^{-12}\ F.$

The field energy is $W_E = ½CU_{12}^2 = ½\cdot 0.645\cdot 10^{-12}\cdot 154.9^2 = 7.75\cdot 10^{-9}\ J.$

11-4 (11.4). Derive formulae to compute the field and capacity of the plate capacitor possessing the plate square S, the distance between the plates d (fig. 11.5).

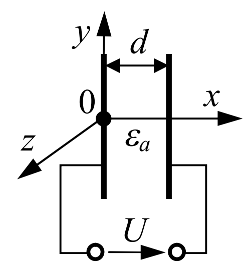


Fig. 11.5

Solution. The plate square S is much bigger then the squared distance d^2 between them. That's why the field may be supposed to be regular (uniform) one. Let the left plate of the capacitor have charge q. Then on the grounds of the boundary condition the electrostatic induction vector is $D = \sigma = q/S$. The electric field intensity is $E = D/\varepsilon_a = q/(S\cdot\varepsilon_a)$. The supplied voltage is $U = E\cdot d = q\cdot d/(S\cdot\varepsilon_a)$. The capacitor capacity is

$$C = q/U = \varepsilon_a\cdot S/d. \qquad (11.5)$$

Suppose the coordinate axes to be situated as in fig. 11.5. Let the origin of the coordinates coincides with the left plate. Derive the formula for potential $\varphi(x)$ on the grounds of the relationship

$$\vec{E} = -\,grad\,\varphi. \qquad (11.6)$$

In Cartesian coordinate system, $grad\,\varphi$ is written as

$$grad\,\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k}$$ (see the table in Appendix).

It is obvious, potential φ depends only on one coordinate x, it means $\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial z} = 0$;

vector $\vec{E}$ being directed along the axis x; that's why we have: $E = -\frac{\partial\varphi}{\partial x}$. From here

$$\varphi = -\int E dx = -E\cdot x + const. \qquad (11.7)$$

If a capacitor has several layers, each layer capacity being – $C_k = \varepsilon_{ak}\cdot S/d_k$, then the capacity of the whole capacitor because of the layer series connection is as follows $C = (\Sigma(C_k)^{-1})^{-1}$; the voltage across each layer is – $U_k = q/C_k$; the field intensity in each layer is – $E_k = U_k/d_k = q/(S\cdot\varepsilon_{ak})$.

11-5 (11.5). Derive the formulae to compute the field and capacity of the spherical capacitor with the insulator layer, the inner and outer radiuses of the layer are r_1 and R_1, respectively, the dielectric permeability is ε_a (fig. 11.6).

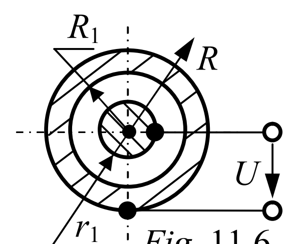


Fig. 11.6

Solution. Let inner ball have charge q. On the grounds of formula (11.2), the electric flux density and the field intensity in dielectric are $D = q/(4\pi\cdot R^2)$; $E = q/(4\pi\varepsilon_a\cdot R^2)$.

We derive a formula for potential $\varphi(R)$ on the grounds of expression (11.6). In a spherical coordinate system $grad\,\varphi$ is expressed as follows

$$grad\,\varphi = \frac{\partial\varphi}{\partial R}\vec{R_0} + \frac{1}{R}\frac{\partial\varphi}{\partial\theta}\vec{\theta_0} + \frac{1}{R\sin\theta}\frac{\partial\varphi}{\partial\alpha}\vec{\alpha_0} \quad \text{(see the table in Appendix).}$$

It is obvious, potential φ depends on but one coordinate R, i.e. $\frac{\partial\varphi}{\partial\theta} = 0$ and $\frac{\partial\varphi}{\partial\alpha} = 0$; vector $\vec{E}$ being directed along the axis R; that's why we have: $E = -\frac{\partial\varphi}{\partial R}$. From here

$$\varphi = -\int E dR = \frac{q}{4\pi\varepsilon_a R} + A. \qquad (11.8)$$

Let potential of the outer ball be equal to zero – $\varphi = 0$ at $R = R_1$. Then $A = -\frac{q}{4\pi\varepsilon_a R_1}$.

The formula for potential is $\varphi = \frac{q}{4\pi\varepsilon_a}\cdot\left(\frac{1}{R} - \frac{1}{R_1}\right)$.

The voltage supplied to the capacitor is $U = \varphi(R{=}r_1) - \varphi(R{=}R_1) = \frac{q}{4\pi\varepsilon_a}\cdot\left(\frac{1}{r_1} - \frac{1}{R_1}\right)$.

Capacity of a single-layer spherical capacitor is $C = q/U = \frac{4\pi\varepsilon_a}{1/r_1 - 1/R_1}$. (11.9)

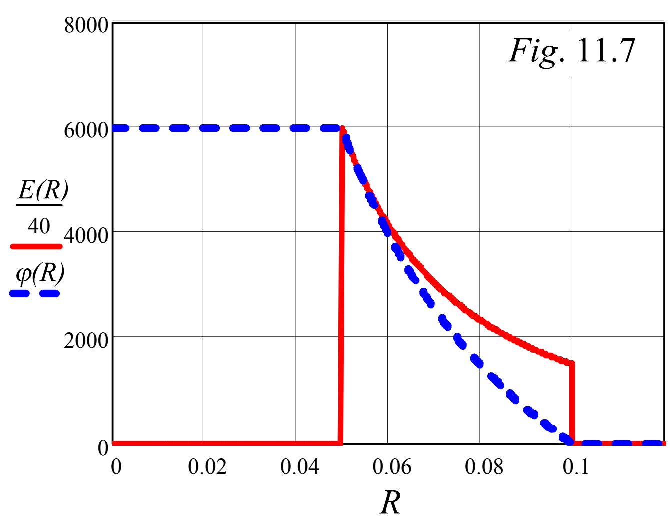
In case of several layers, each layer has the capacity – $C_k = \frac{4\pi\varepsilon_{ak}}{1/r_k - 1/R_k}$, capacity of the capacitor as a whole being $C = (\Sigma(C_k)^{-1})^{-1}$; the voltage across each layer being $U_k = q/C_k$; the field intensity in each layer being $E_k = q/(4\pi\varepsilon_{ak}\cdot R^2)$.

11-6 (11.6). A spherical capacitor (fig. 11.6) is supplied with direct voltage $U = 6000\ V$. Its geometrical sizes are $r_1 = 5\ cm$, $R_1 = 10\ cm$. Relative dielectric permeability of insulation is $\varepsilon = 4$.

Determine the capacitor capacity, plot $E(R)$, $\varphi(R)$. Find the energy stored in dielectric.

Solution. Let's use the formula (11.9). The capacitor capacity is

$$C = \frac{q}{U} = \frac{4\pi\varepsilon\varepsilon_0 r_1 R_1}{R_1 - r_1} = 44.48\ pF.$$



The capacitor charge is $q = CU = 26.69\cdot 10^{-8}\ C$.

The energy of the charged capacitor is $W = ½CU^2 = 0.8\ mJ$.

Intensity and potential in function of coordinate $R\ [m]$ (fig. 11.7) are:

$$E(R) = \frac{q}{4\pi\varepsilon\varepsilon_0 R^2} = \frac{600}{R^2}\ V/m, \qquad \varphi(R) = \frac{q}{4\pi\varepsilon\varepsilon_0 R} + A.$$

Assuming $\varphi = 0$ at $R = R_1$, we get $\qquad \varphi(R) = \frac{600}{R} - 6000\ V.$

11-7 (11.7). A spherical capacitor has two insulation layers (fig. 11.8,a): $R_1 = 5\ cm$, $R_2 = 8\ cm$, $R_3 = 10\ cm$, $R_4 = 10.5\ cm$, $\varepsilon_1 = 6$, $\varepsilon_2 = 2$. The capacitor is connected to a D-C voltage source $U = 36\ kV$. Determine the capacitor capacity, plot $E(R)$. Find the voltage across each layer.

Solution. Capacities of the 1st and the 2nd layers are:

$$C_1 = \frac{4\pi\varepsilon_1\varepsilon_0}{R_1^{-1} - R_2^{-1}} = 88.97\ pF, \qquad C_2 = \frac{4\pi\varepsilon_2\varepsilon_0}{R_2^{-1} - R_3^{-1}} = 88.97\ pF.$$

The capacity of the capacitor formed by series connection of C_1 and C_2 is:

$$C = \frac{C_1 C_2}{C_1 + C_2} = 44.48\ pF.$$

The inner ball charge is: $\qquad q = CU = 1.6\ \mu C.$

The field intensity in the 1st area $R_1 \le R \le R_2$, where $R\ [cm]$

$$E_1(R) = \frac{q}{4\pi\varepsilon_1\varepsilon_0 R^2} = \frac{2.4\cdot 10^5}{R^2}\ V/cm,$$

and that in the 2nd one $R_2 \le R \le R_3$ $\qquad E_2(R) = \frac{q}{4\pi\varepsilon_2\varepsilon_0 R^2} = \frac{7.2\cdot 10^5}{R^2}\ V/cm.$

The graph of the dependence $E(R)$ is presented in fig. 11.8,b.

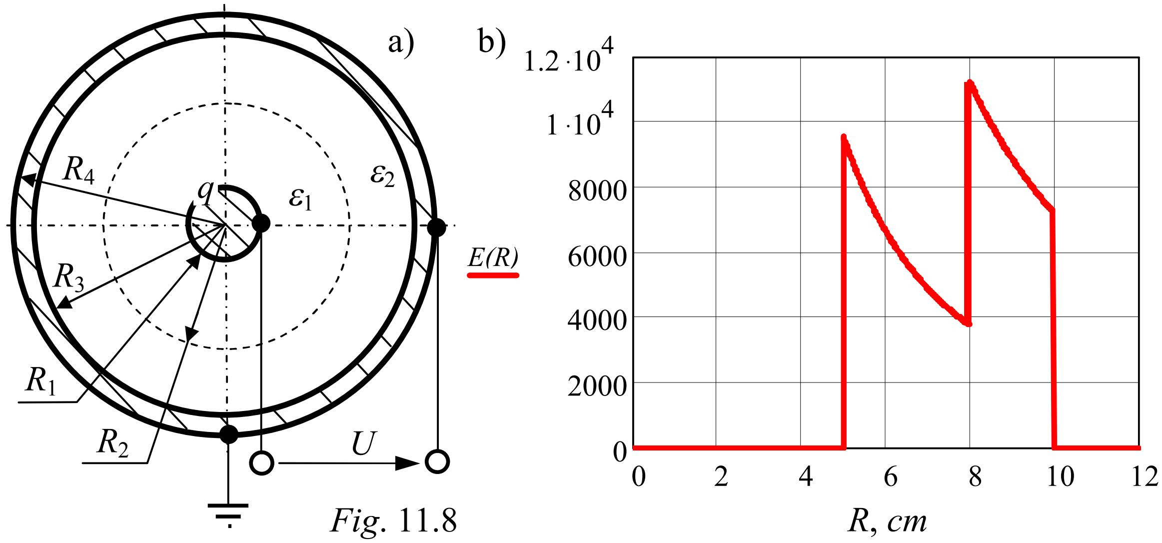


Fig. 11.8

The voltage across the 1st layer of insulation is: $U_1 = \int_{R_1}^{R_2} E_1(R)dR = 18\ kV,$

and that across the 2nd one: $\qquad U_2 = \int_{R_2}^{R_3} E_2(R)dR = 18\ kV.$

Verification: $U_1 + U_2 = 18 + 18 = 36\ kV = U.$

11-8 (11.8). Derive the formulae to compute the field and capacity of the cylindrical capacitor with the insulator layer, its inner and outer radiuses being r_1 and R_1, respectively, furthermore, $l >> R_1$, dielectric permeability is ε_a (fig. 11.9).

Solution. Let inner cylinder have charge q. On the grounds of formula (11.4) taking into account that $q = \tau l$, we determine electric-flux density and the field intensity in dielectric $D = q/(2\pi \cdot r \cdot l)$; $E = q/(2\pi\varepsilon_a \cdot r \cdot l)$.

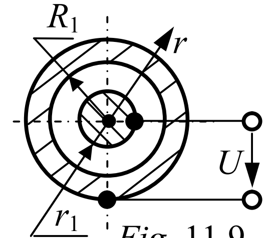


Fig. 11.9

We derive a formula for potential $\varphi(r)$ on the grounds of expression (11.6). In a cylindrical coordinate system $grad\ \varphi$ is expressed as follows

$$grad\ \varphi = \frac{\partial\varphi}{\partial r}\vec{r_0} + \frac{1}{r}\frac{\partial\varphi}{\partial\alpha}\vec{\alpha_0} + \frac{\partial\varphi}{\partial z}\vec{z_0}$$ (see the table in Appendix).

It is obvious, potential φ depends on but one coordinate r, i.e. $\frac{\partial\varphi}{\partial\alpha} = 0$ and $\frac{\partial\varphi}{\partial z} = 0$; vector $\vec{E}$ being directed along axis r; that's why we have:

$$E = -\frac{\partial\varphi}{\partial r}. \qquad \text{From here} \quad \varphi = -\int E dr = -\frac{q}{2\pi\varepsilon_a l} ln\ r + A. \qquad (11.10)$$

Let potential of the outer shell be equal to zero – $\varphi = 0$ at $r = R_1$. Then

$$A = \frac{q}{2\pi\varepsilon_a l} lnR_1. \qquad \text{The formula for potential is} \quad \varphi = \frac{q}{2\pi\varepsilon_a l} ln\frac{R_1}{r}. \quad (11.10,a)$$

The voltage supplied to the capacitor is $U = \varphi(r = r_1) - \varphi(r = R_1) = \frac{q}{2\pi\varepsilon_a l} ln\frac{R_1}{r_1}$.

The capacity of a single-layer cylindrical capacitor is $C = q/U = \frac{2\pi\varepsilon_a l}{ln(R_1/r_1)}$. (11.11)

In case of several layers, each layer has the capacity – $C_k = \frac{2\pi\varepsilon_a l}{ln(R_k/r_k)}$, capacity of the capacitor as a whole being $C = (\Sigma(C_k)^{-1})^{-1}$; voltage across each layer being $U_k = q/C_k$; the field intensity in each layer being $E_k = q/(2\pi\varepsilon_{ak} \cdot l \cdot r)$.

11-9 (11.9). A coaxial cable is supplied with D-C voltage $U = 6\ kV$. The cable length is $l = 20\ m$ (fig. 11.10). A cable conductor radius is $r_1 = 0.5\ cm$, radiuses of the shell are $r_2 = 2\ cm$ and $r_3 = 2.4\ cm$. The insulation relative permeability is $\varepsilon = 4$.

Neglecting the edge effect, determine the cable capacity, plot $E(r)$, $\varphi(r)$, find the energy stored in the cable insulation.

Answers: the cable capacity in accordance with (11.11) and linear charge density are, respectively, as follows: $C = \frac{2\pi\varepsilon\varepsilon_0 l}{ln(r_2/r_1)} = 3.21\ nF$, $\tau = \frac{CU}{l} = \frac{2\pi\varepsilon\varepsilon_0 U}{ln(r_2/r_1)}$;

intensity and potential (formulae (11.4) and (11.10,a)):

$$E(r) = \frac{4328}{r}\ V/m, \quad \varphi(r) = 4328 ln\frac{r_2}{r}\ V, \ \text{where}\ r[m];$$

the graphs $E(r)$ and $\varphi(r)$ are in fig. 11.10.

Energy volume density in dielectric is $w_E = \frac{\varepsilon\varepsilon_0 E^2}{2} = \frac{\tau^2}{8\pi^2\varepsilon\varepsilon_0 r^2}$.

The field energy stored in the cable is

$$W_E = \int_{r_1}^{r_2} \frac{\tau^2 l \cdot 2\pi r}{8\pi^2 \varepsilon\varepsilon_0 r^2} dr = \frac{\tau^2 l}{4\pi\varepsilon\varepsilon_0} ln\frac{r_2}{r_1} = \frac{CU^2}{2} = 57.73\cdot 10^{-9}\ J.$$

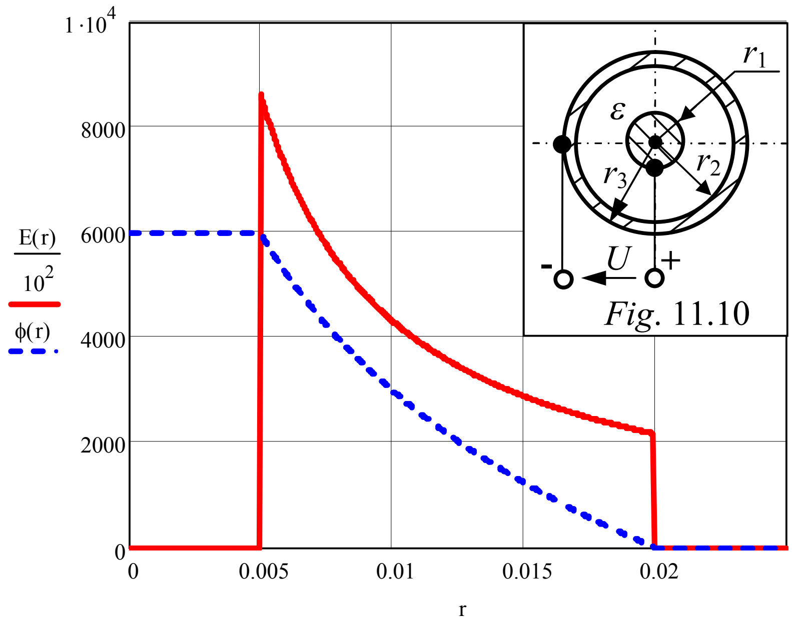


Fig. 11.10

11-10 (11.10). Compute the field of a cylindrical beam of electrons – $\rho = -10^{-10}\ C/cm^3$, $\varepsilon = 2$, $d = 2\ mm$. Solve the problem with the aid of Gauss' theorem in an integral form.

Solution. Gauss' theorem in an integral form is – $\oint_S \vec{D}\cdot d\vec{S} = \Sigma q$.

In cylindrical coordinates

$$\oint_S \vec{D}\cdot d\vec{S} = D\cdot 2\pi rl \text{ (see (11.3))};\quad E = \frac{D}{\varepsilon\varepsilon_0} = -\frac{\partial\varphi}{\partial r},\quad \varphi(r) = -\int E dr \text{ (see problem 11.8)}.$$

In the area $r < d/2$

$$\Sigma q = \rho\cdot\pi r^2 l;\quad D_1 = ½\rho r,\quad E_1(r) = \frac{\rho}{2\varepsilon\varepsilon_0}\cdot r;\quad \varphi_1(r) = -\frac{\rho}{4\varepsilon\varepsilon_0}\cdot r^2 + A_1.$$

Let it be $\varphi_1 = 0$ at $r = 0$. Then $A_1 = 0$. Substituting with the numerical data, we have: $E_1(r) = -2.825\cdot 10^6\cdot r\ V/m$; $\varphi_1(r) = 1.412\cdot 10^6\cdot r^2\ V$.

In the area $r > d/2$ $\Sigma q = \rho\cdot\pi(d/2)^2 l$;

$$D_2 = \frac{\rho\cdot(d/2)^2}{2r};\quad E_2(r) = \frac{\rho(d/2)^2}{2\varepsilon_0 r};\quad \varphi_2(r) = -\frac{\rho(d/2)^2}{2\varepsilon_0}\cdot ln(r) + A_2.$$

At $r = ½d$ $\varphi_2(d/2) = \varphi_1(d/2) = 1.412\cdot 10^6\cdot(10^{-3})^2 = 1.41\ V$, from here

$$A_2 = \varphi_2(d/2) + \frac{\rho(d/2)^2}{2\varepsilon_0} ln(d/2) = 1.41 - 5.65\cdot\ln(0.001) = 40.44.$$

Thus, $E_2(r) = -\dfrac{5.65}{r}\ V/m$; $\varphi_2(r) = 5.65\cdot ln(r) + 40.44\ V$.

11-11 (11.11). The inner and outer radiuses of an earthed hollow metallic ball are $R_3 = 3.2$ *cm* and $R_4 = 3.6$ *cm*, respectively (fig. 11.11); inside this ball there is another metallic ball of radius $R_1 = 0.8$ *cm* with the charge $q = 2.5\cdot10^{-8}$ *C*. In the device under question, there are both the area charged uniformly with volume density $\rho = -10^{-10}$ *C/cm*^{3} and the relative permeability $\varepsilon_1 = 2$, and the area without free charges with the relative permeability $\varepsilon_2 = 1.5$. Size $R_2 = 1.6$ *cm*.

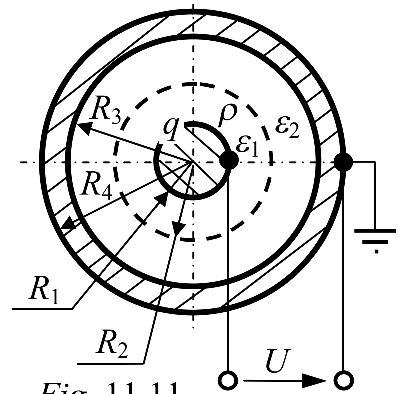


Fig. 11.11

Compute and plot the dependences of the electrostatic field intensity and potentials on the distance to the ball centre.

Solution. Areas $0 < R < R_1$, $R_3 < R < R_4$ are filled in with conducting material, that's why there is no electrostatic field here – $E = 0$, $D = 0$, $\varphi = const$. There is no field beyond the ball bounds either for the ball shell is earthed.

Let's calculate the field in the areas $R_1 < R < R_2$ and $R_2 < R < R_3$ with the aid of Gauss' theorem in an integral form: $\oint\limits_S \vec{D}\cdot\vec{ds} = \Sigma q$. On the grounds of (11.1)

$$\oint\limits_S \vec{D}\cdot\vec{ds} = D\cdot 4\pi R^2.$$

For the area $R_1 < R < R_2$ $\quad \Sigma q = q + \rho\cdot{}^4\!/\!_3\,\pi(R^3 - R_1^3)$. Then

$$D_1 = \frac{q + \rho\,{}^4\!/\!_3\,\pi(R^3 - R_1^3)}{4\pi R^2};\qquad E_1 = \frac{D_1}{\varepsilon_1\varepsilon_0} = \frac{q - \rho\,{}^4\!/\!_3\,\pi R_1^3}{4\pi\varepsilon_1\varepsilon_0 R^2} + \frac{\rho R}{3\varepsilon_1\varepsilon_0}.$$

On the grounds of (11.8), $\quad \varphi_1 = -\int E_1 dR = \dfrac{q - \rho\,{}^4\!/\!_3\,\pi R_1^3}{4\pi\varepsilon_1\varepsilon_0 R} - \dfrac{\rho R^2}{6\varepsilon_1\varepsilon_0} + A_1.$

For the area $R_2 < R < R_3$ $\quad \Sigma q = q + \rho\cdot{}^4\!/\!_3\,\pi(R_2^3 - R_1^3)$. Then

$$D_2 = \frac{q + \rho\,{}^4\!/\!_3\,\pi(R_2^3 - R_1^3)}{4\pi R^2};\qquad E_2 = \frac{D_2}{\varepsilon_2\varepsilon_0} = \frac{q + \rho\,{}^4\!/\!_3\,\pi(R_2^3 - R_1^3)}{4\pi\varepsilon_2\varepsilon_0 R^2};$$

$$\varphi_2 = -\int E_2 dR = \frac{q + \rho\,{}^4\!/\!_3\,\pi(R_2^3 - R_1^3)}{4\pi\varepsilon_2\varepsilon_0 R} + A_2.$$

We determine integration constants A_1 and A_2 under the condition $\varphi_2(R_3) = 0$, and that the potential is a continuous function.

$$\varphi_2(R_3) = \frac{q + \rho\,{}^4\!/\!_3\,\pi(R_2^3 - R_1^3)}{4\pi\varepsilon_2\varepsilon_0 R_3} + A_2 = 0,\quad \text{from here}$$

$$A_2 = -\frac{q + \rho\,{}^4\!/\!_3\,\pi(R_2^3 - R_1^3)}{4\pi\varepsilon_2\varepsilon_0 R_3} = -\,4402.$$

Thus, after substituting with the numerical data, we have:

$$\varphi_2(R) = \frac{140.86}{R} - 4402\ V,\qquad E_2(R) = \frac{140.86}{R^2}\ V/m, \text{ where } R\ [m].$$

At $\quad R = R_2 \qquad \varphi_1(R_2) = \varphi_2(R_2) = \dfrac{140.86}{0{,}016} - 4402 = 4402\ V.$

I.e. $\quad \dfrac{q - \rho\, {}^4\!/_3\, \pi R_1^3}{4\pi\varepsilon_1\varepsilon_0 R_2} - \dfrac{\rho R_2^2}{6\varepsilon_1\varepsilon_0} + A_1 = 4402 \quad$ and $\quad A_1 = -\,2804.$

Thus, after substituting with the numerical data, we have:

$$\varphi_1(R) = \frac{111.43}{R} + 9.42\cdot 10^5\cdot R^2 - 2804\ V,$$

$$E_1(R) = \frac{111.43}{R^2} - 18.83\cdot 10^5\cdot R\ V/m.$$

The graphs $\varphi(R)$ and $E(R)$ presented in fig. 11.12.

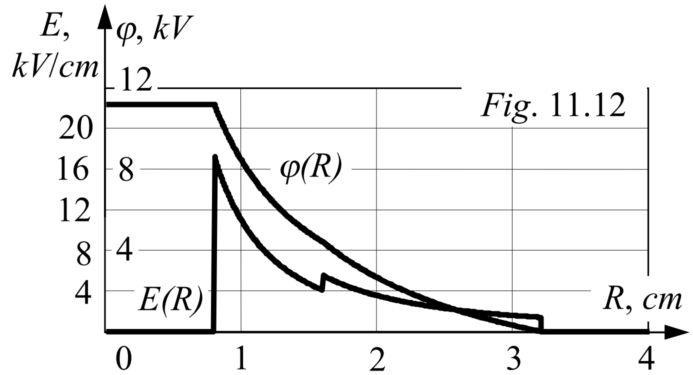


Fig. 11.12

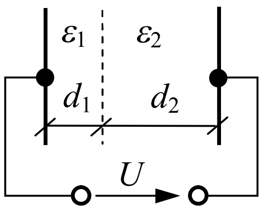


Fig. 11.13

11-12 (11.13). What maximum voltage may be supplied to a two-layer plane capacitor fig. 11.13, if: $\varepsilon_1 = 2$, $\varepsilon_2 = 4$, $d_1 = 2.5\ mm$, $d_2 = 5\ mm$. Disruptive strength of insulation is $E_{dis} = 30\ kV/cm$. Take the stress safety factor equal to $n = 2.5$.

For the voltage found, compute the energy volume density in the 2nd dielectric.

Solution. Taking into account the given stress safety factor, determine the allowable electric intensity in dielectrics:

$$E_{1max} \le \frac{E_{dis}}{n} = \frac{30}{2.5} = 12\ kV/cm, \quad E_{2max} \le \frac{E_{dis}}{n} = 12\ kV/cm.$$

As the edge effect can be ignored, the field intensity of the plane capacitor is constant in each area – E_1 and E_2, respectively. On the dielectric interface, the condition $D_{1n} = D_{2n}$ results in expression $\varepsilon_1 E_1 = \varepsilon_2 E_2$ or $E_1 = 2E_2$.

As $E_1 > E_2$ we assume $E_1 = E_{1max} = 12\ kV/cm$. Then $E_2 = ½E_1 = 6\ kV/cm$.

The maximum permissible voltage across the capacitor is

$$U = E_1 d_1 + E_2 d_2 = 12\cdot 0.25 + 6\cdot 0.5 = 6\ kV.$$

The energy volume density in the 2nd dielectric is

$$w_2 = ½\varepsilon_2\varepsilon_0 E_2^2 = 0.5\cdot 4\cdot 8.85\cdot 10^{-14}\cdot 6000^2 = 6.37\cdot 10^{-6}\ J/cm^3.$$

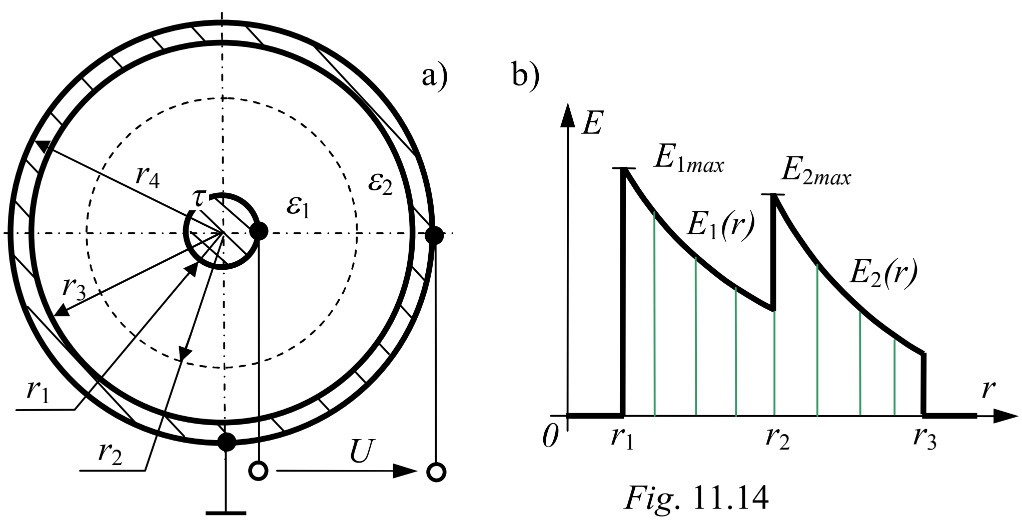


Fig. 11.14

11-13 (11.14). What maximum voltage may be supplied to a two-layer coaxial cable (fig. 11.14,a), if: $r_1 = 2.5\ mm$, $r_2 = 7.5\ mm$, $r_3 = 12\ mm$, $r_4 = 14\ mm$, $\varepsilon_1 = 5$, $\varepsilon_2 = 2$. Disruptive strength of insulation is: $E_{1dis} = E_{2dis} = 30\ kV/cm$. Take the stress safety factor equal to 3.

Compute the cable capacity.

Solution. When supplying the cable from a D-C voltage source U, a cable conductor has the charge τ per unit length. The field intensity is determined under expression (11.4). For the 1st insulation layer: $E_1 = \dfrac{\tau}{2\pi\varepsilon_1\varepsilon_0 r}$.

For the 2nd one: $E_2 = \dfrac{\tau}{2\pi\varepsilon_2\varepsilon_0 r}$.

A sketch of the intensity distribution is presented in fig. 11.14,b.

The maximum field intensity in the insulation layers is

$$E_{1max} = \frac{\tau}{2\pi\varepsilon_1\varepsilon_0 r_1}, \quad E_{2max} = \frac{\tau}{2\pi\varepsilon_2\varepsilon_0 r_2}.$$

In order to prevent the insulation breakdown it is necessary to have:

$$E_{1max} \le \frac{E_{1dis}}{n} = 10\ kV, \quad E_{2max} \le \frac{E_{2dis}}{n} = 10\ kV.$$

Products $\varepsilon_1 r_1 = 5\cdot 2.5 = 12.5$; $\varepsilon_2 r_2 = 2\cdot 7.5 = 15$; $\varepsilon_1 r_1 < \varepsilon_2 r_2$.

That's why $E_{1max} > E_{2max}$. Assume $E_{1max} = 10\ kV$. Then

$$\tau = E_{1max}\cdot 2\pi\varepsilon_1\varepsilon_0 r_1; \qquad E_1 = \frac{E_{1max} r_1}{r}; \qquad E_2 = \frac{E_{1max} r_1 \varepsilon_1}{\varepsilon_2 r}.$$

The voltage across the 1st insulation layer is:

$$U_1 = \int_{r_1}^{r_2} E_1 dr = E_{1max} r_1 ln\frac{r_2}{r_1} = 10^4\cdot 0.25 ln3 = 2747\ V;$$

and that across the 2nd layer is:

$$U_2 = \int_{r_2}^{r_3} E_2 dr = \frac{E_{1max} r_1 \varepsilon_1}{\varepsilon_2} ln\frac{r_3}{r_2} = \frac{10^4\cdot 0.25\cdot 5}{2} ln\frac{12}{7.5} = 2937\ V.$$

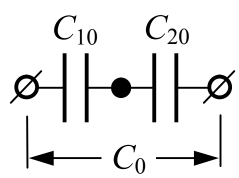


Fig. 11.15

The maximum working voltage for cable is:

$$U = U_1 + U_2 = 2747 + 2937 = 5684\ V.$$

Capacity of the two-layer cable (capacitor) may be calculated representing it as a series connection of two capacitances (fig. 11.15). In accordance with (11.11) the capacities of the layers separately and of the cable as a whole are:

$$C_{10} = \frac{2\pi\varepsilon_1\varepsilon_0}{ln\frac{r_2}{r_1}}, \quad C_{20} = \frac{2\pi\varepsilon_2\varepsilon_0}{ln\frac{r_3}{r_2}}, \quad C_0 = \frac{C_{10}\cdot C_{20}}{C_{10} + C_{20}} = \frac{2\pi\varepsilon_1\varepsilon_2\varepsilon_0}{\varepsilon_2 ln\frac{r_2}{r_1} + \varepsilon_1 ln\frac{r_3}{r_2}}.$$

Or under definition of capacity $C_0 = \dfrac{\tau}{U} = \dfrac{2\pi\varepsilon_1\varepsilon_0}{ln\frac{r_2}{r_1} + \frac{\varepsilon_1}{\varepsilon_2} ln\frac{r_3}{r_2}} = 0.367\ pF/m.$

11-14 (11.15). Maximum electric field intensity in the insulation of a two-layer cylindrical capacitor (fig. 11.16) is $E_{max} = 30\ kV/cm$. It is required to determine the voltage supplied to the capacitor as well as its capacity, if $l = 5\ m$, $\varepsilon_1 = 2$, $\varepsilon_2 = 1$, $r = 1\ cm$, $r_2 = 3\ cm$, $r_3 = 4\ cm$ and $r_4 = 5\ cm$.

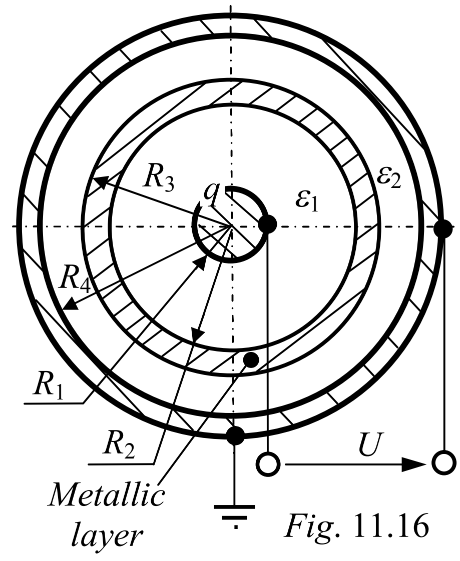


Fig. 11.16

Solution. Let inner conductor possess the charge $-\tau$. Then the intensity and potential of different areas in accordance with (11.4) and (11.8) are:

$$E_1 = \frac{\tau}{2\pi\varepsilon_1\varepsilon_0 r};\quad \varphi_1 = -\frac{\tau}{2\pi\varepsilon_1\varepsilon_0} ln(r) + A_1;$$

$$E_2 = \frac{\tau}{2\pi\varepsilon_2\varepsilon_0 r};\quad \varphi_2 = -\frac{\tau}{2\pi\varepsilon_2\varepsilon_0} ln(r) + A_2.$$

The maximum intensities in the areas are: $E_{1max} = \dfrac{\tau}{2\pi\varepsilon_1\varepsilon_0 r_1}$; $\quad E_{2max} = \dfrac{\tau}{2\pi\varepsilon_2\varepsilon_0 r_3}$.

As $\varepsilon_1 \cdot r_1 = 2 < \varepsilon_2 \cdot r_3 = 4$, then $E_{2max} < E_{1max} = E_{max}$, $\tau = 2\pi\varepsilon_1\varepsilon_0 r_1 E_{max}$,

$$\varphi_1 = -r_1 E_{max} \cdot ln(r) + A_1;\qquad \varphi_2 = -\frac{\varepsilon_1}{\varepsilon_2} r_1 E_{max} \cdot ln(r) + A_2.$$

Let $\varphi_1(r_2) = \varphi_2(r_3) = 0$, then $A_1 = r_1 E_{max} \cdot ln(r_2)$, $A_2 = \dfrac{\varepsilon_1}{\varepsilon_2} r_1 E_{max} \cdot ln(r_3)$,

$$\varphi_1(r) = r_1 E_{max} \cdot ln(r_2/r),\quad \varphi_2(r) = \frac{\varepsilon_1}{\varepsilon_2} r_1 E_{max} \cdot ln(r_3/r).$$

The supplied voltage and the capacitor capacity are:

$$U = \varphi_1(r_1) - \varphi_2(r_4) = r_1 E_{max} \cdot ln\left({r_2}/{r_1} \cdot \left({r_4}/{r_3} \right)^{\varepsilon_1/\varepsilon_2} \right) = 46.37\ kV,$$

$$C = \frac{\tau \cdot l}{U} = \frac{2\pi\varepsilon_1\varepsilon_0}{ln\left({r_2}/{r_1} \cdot \left({r_4}/{r_3} \right)^{\varepsilon_1/\varepsilon_2} \right)} = 71.99\ pF.$$

11.2. METHOD OF ELECTRICAL IMAGES

11-15 (11.25). A conductor with charge $\tau = 10^{-8}\ C/m$ creates the electrostatic field close to a metallic plane. Determine the conductor potential as well as the intensity and potential in points A and B (fig. 11.17,a), if $r_0 = 2\ cm$; $h = 4\ m$; $\varepsilon = 1$.

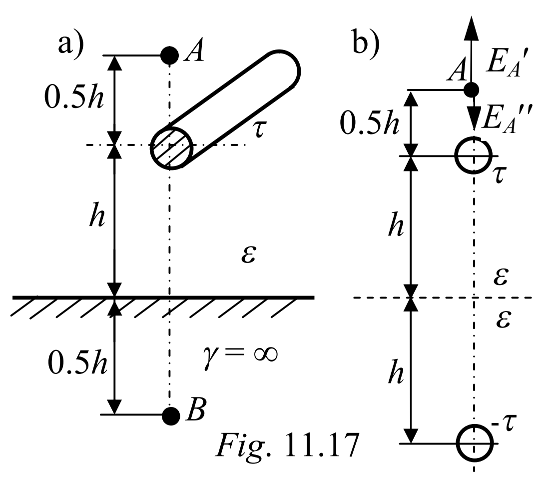


Fig. 11.17

Solution. There is no electrostatic field in point B (inside the metal), that's why $E_B = 0$, $\varphi_B = 0$. In order to calculate the field in the upper half plane, we apply the method of electrical images.

The coefficient of incomplete reflection from ideal conducting medium is equal to –1. The sketch for calculation is in fig. 11.17,b, where two conductors with charges τ and $-\tau$ are placed in homogeneous medium with permeability ε. Further, we apply the superposition principle.

The intensity and potential of a single charged axis in homogeneous medium are set by formulae (11.4) and (11.10,a): $E=\dfrac{\tau}{2\pi\varepsilon_0 r}$, $\varphi=\dfrac{\tau}{2\pi\varepsilon_0}ln\dfrac{h}{r}$.

As it is seen from fig. 11.17,b, resultant intensity in point A is equal to difference of intensities from two conductors. Finally, the required intensity in point A is:

$$E_A=E_A'-E_A''=\frac{\tau}{2\pi\varepsilon_0 0.5h}-\frac{\tau}{2\pi\varepsilon_0 2.5h}=\frac{10^{-8}}{2\pi\cdot 8.85\cdot 10^{-12}}\left(\frac{1}{0.5\cdot 4}-\frac{1}{2.5\cdot 4}\right)=71.93\ V/m.$$

The potentials of point A and the point on the conductor surface (conductor potential) obey the expressions:

$$\varphi_A=\varphi_A'+\varphi_A''=\frac{\tau}{2\pi\varepsilon_0}\left(ln\frac{h}{0.5h}-ln\frac{h}{2.5h}\right)=\frac{10^{-8}}{2\pi\cdot 8.85\cdot 10^{-12}}ln5=290\ V;$$

$$\varphi=\frac{\tau}{2\pi\varepsilon_0}\left(ln\frac{h}{r_0}-ln\frac{h}{2h}\right)=\frac{10^{-8}}{2\pi\cdot 8.85\cdot 10^{-12}}ln\frac{2\cdot 4}{0.02}=1078\ V.$$

11-16 (11.26). The voltage between the conductor of radius $r_0=1\ cm$ placed in air parallel to earth at height $h=1\ m$ and the earth surface is $U=1000\ V$ (fig. 11.18,a).

Determine the capacity C_0, the field energy W_0 and force F_0, acting on the unit length of the conductor.

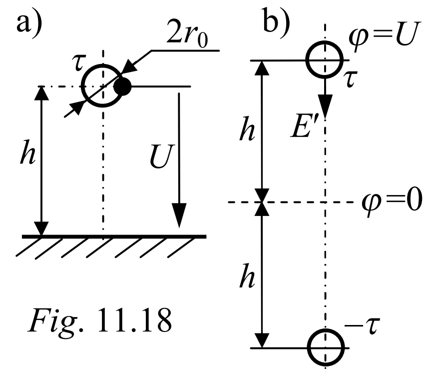


Fig. 11.18

Solution. In accordance with the method of electrical images, we perform a sketch for calculation fig. 11.18,b. Then the conductor potential (see problem 11.15) is

$$\varphi=U=\frac{\tau}{2\pi\varepsilon_0}ln\frac{2h}{r_0}.$$

From here, the wire charge is

$$\tau=\frac{U\cdot 2\pi\varepsilon_0}{ln\dfrac{2h}{r_0}}=\frac{1000\cdot 2\pi\cdot 8.85\cdot 10^{-12}}{ln\dfrac{2}{0.01}}=10.5\cdot 10^{-9}\ C/m.$$

The capacity of the line unit length is $C_0=\dfrac{\tau}{U}=10.5\cdot 10^{-12}\ F/m=10.5\ pF/m.$

The field intensity created by the charge $-\tau$ in the charge τ location area in accordance with (11.4) is

$$E'=\frac{\tau}{2\pi\varepsilon_0\cdot 2h}=\frac{10.5\cdot 10^{-9}}{2\pi\cdot 8.85\cdot 10^{-12}\cdot 2}=94.4\ V/m.$$

The wire is attracted to earth with the following force

$$F_0=\tau E'=10.5\cdot 10^{-9}\cdot 94.4=0.991\cdot 10^{-6}\ N/m.$$

Energy of the line unit length is

$$W_0=½C_0U^2=½10.5\cdot 10^{-12}\cdot 10^6=5.25\cdot 10^{-6}\ J/m.$$

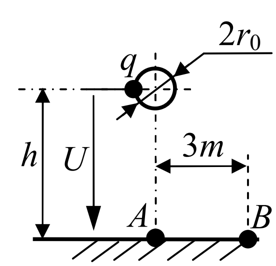


Fig. 11.19

11-17 (11.27). Voltage $U = 500\ kV$ acts between the metallic ball of radius $r_0 = 10\ cm$ and the conducting plane (fig. 11.19). Determine the surface density of the induced charge in points A and B, if $h = 4\ m$.

Answers. $q = \dfrac{4\pi\varepsilon_0 U}{\dfrac{1}{r_0} - \dfrac{1}{2h}}$; $E_A = \dfrac{q}{2\pi\varepsilon_0 h^2}$; $\sigma_A = D_A = \varepsilon_0 E_A = 5.6\cdot 10^{-8}\ C/m^2$;

$$\sigma_B = D_B = \varepsilon_0 E_B = \frac{q}{2\pi(h^2+3^2)}\cdot\frac{h}{\sqrt{h^2+3^2}} = 2.87\cdot 10^{-8}\ C/m^2.$$

11-18 (11.28). There is a ball of radius $R_0 = 1\ cm$ with charge $q = 10^{-10}\ C$ in the right angle formed by two planes of conducting medium (fig. 11.20,a). Determine the ball potential with respect to the conducting plane as well as the force and its direction acting on the ball.

Solution. In accordance with method of mirror images, we get what is given in fig. 11.20,b after the regular reflection with respect to the right interface. After the second regular reflection with respect to the lower interface, we get the field of four balls in homogeneous medium (fig. 11.20,c). We apply the superposition principle (see problems 11.2 and 11.3) and finally obtain

$$\varphi_1 = \frac{q}{4\pi\varepsilon_0}\left(R_0^{-1} - (2h_2)^{-1} + \left(\sqrt{(2h_1)^2+(2h_2)^2}\right)^{-1} - (2h_1)^{-1}\right) = 87.55\ V,$$

$$C = q/\varphi_1 = 1.14\ pF.$$

The force acting on the ball may be determined through the intensity induced by the other three charges in the point of the ball location. However, the superposition principle is to be applied in the vector form (fig. 11.21). For convenience, let's resolve the intensity vectors onto horizontal *(x)* and vertical *(y)* components.

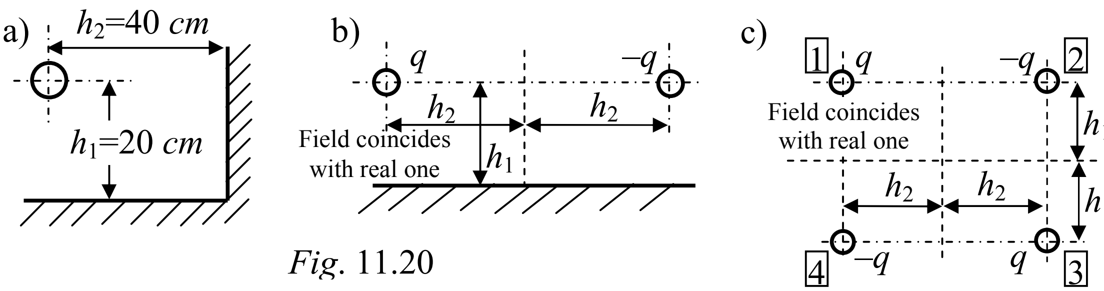


Fig. 11.20

$$E_{2y} = 0,\quad E_{2x} = \frac{q}{4\pi\varepsilon_0(2h_2)^2},$$

$$E_{3x} = \frac{-q\cdot\cos\alpha}{4\pi\varepsilon_0[(2h_1)^2+(2h_2)^2]},\quad E_{3y} = \frac{q\cdot\sin\alpha}{4\pi\varepsilon_0[(2h_1)^2+(2h_2)^2]},$$

where $\cos\alpha = \dfrac{2h_2}{\sqrt{(2h_1)^2+(2h_2)^2}} = 0.8944$; $\sin\alpha = 0.4473$.

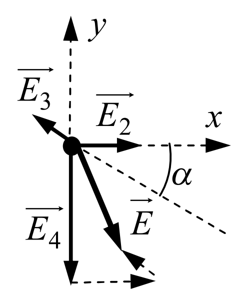


Fig. 11.21

$$E_{4x} = 0,\quad E_{4y} = -\frac{q}{4\pi\varepsilon_0(2h_1)^2},$$

$E_x = E_{2x} + E_{3x} + E_{4x} = 0.40\ V/m$; $E_y = E_{2y} + E_{3y} + E_{4y} = -5.12\ V/m$;

$E = \sqrt{E_x^2 + E_y^2} = 5.13\ V/m; \quad F = q \cdot E = 10^{-10} \cdot 5.13 = 5.13 \cdot 10^{-10}\ N.$

Note. If, instead of the ball, there would be a charged conductor (cylinder) parallel to the angle, it would be enough to apply only the first regular reflection and then to use the groups of Maxwell's formulae for two-wire line.

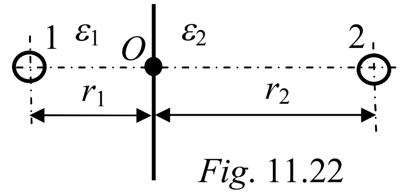


Fig. 11.22

11-19 (11.29). The wires of a two-wire line $U = 500\ V$ are placed in different dielectrics as it is shown in fig. 11.22. Compute the charge on the line wire and the line capacity. Geometrical sizes and properties of mediums are as follows: $r_0 = 0.2\ cm$, $r_1 = 20\ cm$, $r_2 = 40\ cm$, $\varepsilon_1 = 1$, $\varepsilon_2 = 2$.

Solution. Let the charge of the left wire be τ, then the charge of the right one is $-\tau$. As it is necessary to calculate the field in two mediums, we use two figures for computation – fig. 11.23,a and b. Coefficients of incomplete reflection are:

$$k_1' = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} = -0{,}333; \qquad k_2' = \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} = 0{,}667;$$

$$k_1'' = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1 + \varepsilon_2} = 0{,}333; \qquad k_2'' = \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} = 1{,}333.$$

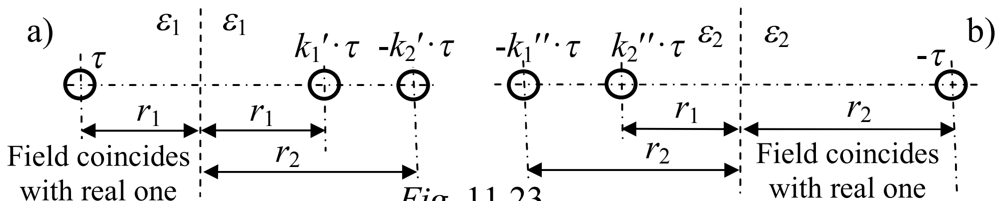


Fig. 11.23

Point O potential (fig. 11.23) is assumed to be equal to zero. The potential of the 1st (left) wire is computed by fig. 11.23,a in accordance with the superposition principle taking into account equation (11.10,a): $\varphi_1 = \frac{\tau}{2\pi\varepsilon_1\varepsilon_0}(ln\frac{r_1}{r_0} + k_1' \cdot ln\frac{r_1}{2r_1} - k_2' \cdot ln\frac{r_2}{r_1 + r_2});$

similarly, the potential of the 2nd (right) wire by fig. 11.23,b is:

$$\varphi_2 = \frac{\tau}{2\pi\varepsilon_2\varepsilon_0}(-ln\frac{r_2}{r_0} - k_1'' \cdot ln\frac{r_2}{2r_2} + k_2'' \cdot ln\frac{r_1}{r_1 + r_2}).$$

The voltage between the wires is

$$U = \varphi_1 - \varphi_2 = \frac{\tau}{2\pi\varepsilon_0}[\varepsilon_1^{-1} \cdot (ln\frac{r_1}{r_0} + k_1' \cdot ln\frac{r_1}{2r_1} - k_2' \cdot ln\frac{r_2}{r_1 + r_2}) -$$

$$- \varepsilon_2^{-1} \cdot (-ln\frac{r_2}{r_0} - k_1'' \cdot ln\frac{r_2}{2r_2} + k_2'' \cdot ln\frac{r_1}{r_1 + r_2})] = \tau \cdot 15.06 \cdot 10^{10}.$$

From here $\tau = U/15.06 \cdot 10^{10} = 0.332 \cdot 10^{-8}\ C/m.$

The line capacity is $C_0 = \tau/U = 6.64\ pF/m.$

12.3. GROUPS OF MAXWELL'S FORMULAE. DIRECT CAPACITANCES.

11-20 (11.30). A transmission line (fig. 11.24) consists of three wires of radius $r_0 = 0.6\ cm$. The wires are installed at a height of $h_1 = h_3 = 6\ m$, $h_2 = 5.2\ m$. The horizontal distances between the wires are $d_{12} = 2\ m$, $d_{23} = 1.6\ m$. The voltages between the wires are $U_{12} = 60\ kV$, $U_{23} = 40\ kV$. Determine the potential and charge of each wire.

Solution. Determine the distances between wires as well as the distances between the wires and their mirror images (fig. 11.24):

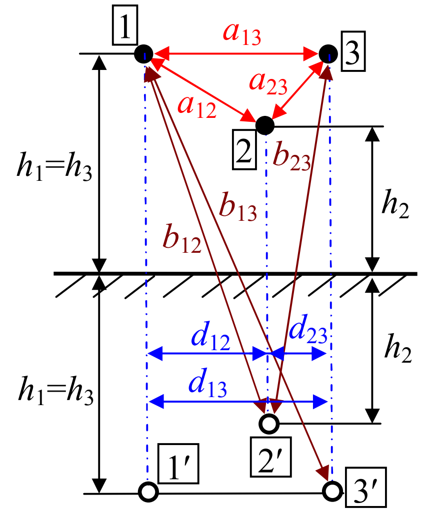


Fig. 11.24

$$a_{12}=\sqrt{(h_1-h_2)^2+d_{12}^2}=\sqrt{(6-5.2)^2+2^2}=2.13\ m,$$
$$a_{13}=\sqrt{(h_1-h_3)^2+d_{13}^2}=\sqrt{(6-6)^2+3.6^2}=3.6\ m,$$
$$a_{23}=\sqrt{(h_2-h_3)^2+d_{23}^2}=\sqrt{(5.2-6)^2+1.6^2}=1.79\ m,$$
$$b_{12}=\sqrt{(h_1+h_2)^2+d_{12}^2}=\sqrt{(6+5.2)^2+2^2}=11.4\ m,$$
$$b_{13}=\sqrt{(h_1+h_3)^2+d_{13}^2}=\sqrt{(6+6)^2+3.6^2}=12.5\ m,$$
$$b_{23}=\sqrt{(h_2+h_3)^2+d_{23}^2}=\sqrt{(5.2+6)^2+1.6^2}=11.3\ m.$$

Calculate the potential coefficients (per a unit length):

$$\alpha_{11}=\alpha_{33}=\frac{1}{2\pi\varepsilon_0}ln\frac{2h_1}{r_0}=\frac{1}{2\pi\cdot 8.85\cdot 10^{-12}}ln\frac{2\cdot 6}{0.006}=13.66\cdot 10^{10}\ m/F,$$
$$\alpha_{22}=\frac{1}{2\pi\varepsilon_0}ln\frac{2h_2}{r_0}=\frac{1}{2\pi\cdot 8.85\cdot 10^{-12}}ln\frac{2\cdot 5.2}{0.006}=13.41\cdot 10^{10}\ m/F,$$
$$\alpha_{12}=\alpha_{21}=\frac{1}{2\pi\varepsilon_0}ln\frac{b_{12}}{a_{12}}=\frac{1}{2\pi\cdot 8.85\cdot 10^{-12}}ln\frac{11.4}{2.13}=3.02\cdot 10^{10}\ m/F,$$
$$\alpha_{13}=\alpha_{31}=\frac{1}{2\pi\varepsilon_0}ln\frac{b_{13}}{a_{13}}=\frac{1}{2\pi\cdot 8.85\cdot 10^{-12}}ln\frac{12.5}{3.6}=2.24\cdot 10^{10}\ m/F,$$
$$\alpha_{23}=\alpha_{32}=\frac{1}{2\pi\varepsilon_0}ln\frac{b_{23}}{a_{23}}=\frac{1}{2\pi\cdot 8.85\cdot 10^{-12}}ln\frac{11.3}{1.79}=3.32\cdot 10^{10}\ m/F.$$

To determine the wire charges, we use the 1st group of Maxwell's formulae:

$$\begin{cases}\varphi_1=\tau_1\alpha_{11}+\tau_2\alpha_{12}+\tau_3\alpha_{13},\\ \varphi_2=\tau_1\alpha_{21}+\tau_2\alpha_{22}+\tau_3\alpha_{23},\\ \varphi_3=\tau_1\alpha_{31}+\tau_2\alpha_{32}+\tau_3\alpha_{33}.\end{cases}$$

The missing equations to determine six required quantities are generated based on the additional problem conditions: $\varphi_1-\varphi_2=U_{12}$, $\varphi_2-\varphi_3=U_{23}$, furthermore, as the wires form the isolated system, not connected with earth, then

$$\tau_1+\tau_2+\tau_3=0.$$

Solving the system of six above-generated equations, we find the necessary:

$\tau_1=0.466\cdot 10^{-6}\ C/m$, $\tau_2=-0.058\cdot 10^{-6}\ C/m$, $\tau_3=-0.408\cdot 10^{-6}\ C/m$,
$\varphi_1=52.8\ kV$, $\varphi_2=-7.2\ kV$, $\varphi_3=-47.2\ kV$.

11-21 (11.31). Solve the problem 11.16, using the groups of Maxwell's formulae.

Answers. $\alpha_{11}=\frac{1}{2\pi\varepsilon_0}ln\frac{2h}{r_0}=9.522\cdot 10^{10}\ m/F$; $\varphi_1=U=1000\ V$;
$\tau_1=\varphi_1/\alpha_{11}=1.05\cdot 10^{-8}\ C/m$; $C_0=C_{11}=10.5\ pF/m$.

11-22 (11.32). Determine the direct and mutual (C_0) capacitances of 1 *m* of a two-wire overhead transmission line (fig. 11.25), if $h=d=2\ m$, $r_0=1\ cm$.

Answers. $a_{12} = d = 2\ m$,

$b_{12} = \sqrt{d^2 + (2h)^2} = 2\sqrt{5}\ m$;

$\alpha_{11} = \alpha_{22} = 10.77 \cdot 10^{10}\ m/F$,

$\alpha_{12} = \alpha_{21} = 1.45 \cdot 10^{10}\ m/F$;

$\beta_{11} = \beta_{22} = 9.46 \cdot 10^{-12}\ F/m$,

$\beta_{12} = \beta_{21} = -1.27 \cdot 10^{-12}\ F/m$;

$C_{11} = C_{22} = 8.19\ pF/m$,

$C_{12} = C_{21} = 1.27\ pF/m$; $C_0 = 5.37\ pF/m$.

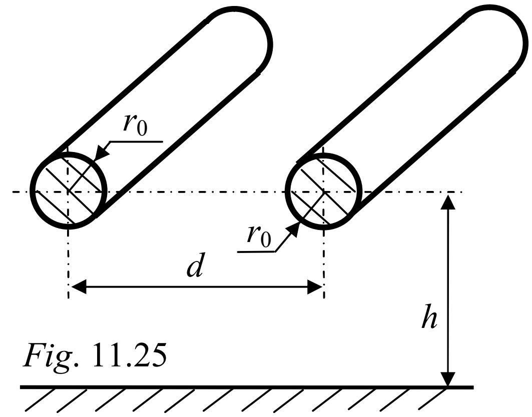


Fig. 11.25

11-23 (11.33). Determine the direct capacitances of a three-wire overhead transmission line (fig. 11.26), if $r_0 = 1\ cm$, $h = 6\ m$, $d = 2\ m$.

Answers. $a_{12} = a_{13} = a_{23} = d = 2\ m$,

$b_{12} = b_{23} = \sqrt{(0.5d)^2 + (2h + 0.5d\sqrt{3})^2} = 13.77\ m$,

$b_{13} = \sqrt{d^2 + (2h)^2} = 12.17\ m$,

$\alpha_{11} = \alpha_{33} = 12.76 \cdot 10^{10}\ m/F$, $\alpha_{22} = 13.21 \cdot 10^{10}\ m/F$,

$\alpha_{12} = \alpha_{23} = 3.47 \cdot 10^{10}\ m/F$, $\alpha_{13} = 3.25 \cdot 10^{10}\ m/F$,

$\beta_{11} = \beta_{33} = 8.783 \cdot 10^{-12}\ F/m$, $\beta_{22} = 8.54 \cdot 10^{-12}\ F/m$,

$\beta_{12} = \beta_{23} = -1.852 \cdot 10^{-12}\ F/m$, $\beta_{13} = -1.732 \cdot 10^{-12}\ F/m$, $C_{11} = C_{33} = 5.2\ pF/m$,

$C_{22} = 4.836\ pF/m$, $C_{12} = C_{23} = 1.852\ pF/m$, $C_{13} = 1.732\ pF/m$.

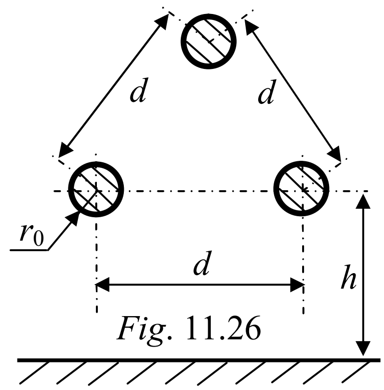


Fig. 11.26

11-24 (11.34). Self and mutual direct capacitances of a three-conductor cable are equal to, respectively: $C_{11} = C_{22} = C_{33} = 0.064\ \mu F/km$, $C_{12} = C_{23} = C_{13} = 0.076\ \mu F/km$. When testing the cable, one of the cable conductors was earthed, while the 2nd one possesses the potential $\varphi_2 = 2\ kV$, the 3rd one – $\varphi_3 = -3\ kV$. The cable shell was not earthed.

Find the shell potential as well as the conductor charges.

Solution. In accordance with the problem task we construct a sketch fig. 11.27, for which on the grounds of the 3rd group of Maxwell's formulae we have:

$$\begin{cases} \tau_1 = \varphi_1' C_{11} + U_{12} C_{12} + U_{13} C_{13}, \\ \tau_2 = U_{21} C_{21} + \varphi_2' C_{22} + U_{23} C_{23}, \\ \tau_3 = U_{31} C_{31} + U_{32} C_{32} + \varphi_3' C_{33}, \end{cases}$$

here the conductor potentials are determined with respect to the shell.

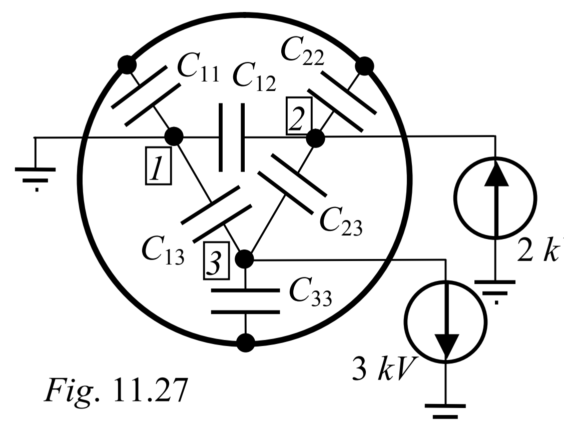


Fig. 11.27

If the shell is considered as the 4th conductor, then it is possible to generate an equation for it taking into account that it is not connected with earth and therefore it does not have any charge: $\tau_4 = (\varphi_4 - \varphi_1)C_{11} + (\varphi_4 - \varphi_2)C_{22} + (\varphi_4 - \varphi_3)C_{33} = 0$.

Here potentials are determined with respect to earth.

Since $C_{11} = C_{22} = C_{33}$ and $\varphi_1 = 0$, $\varphi_2 = 2\ kV$, $\varphi_3 = -3\ kV$, then

$$\varphi_4 = \frac{\varphi_1 + \varphi_2 + \varphi_3}{3} = \frac{2000 - 3000}{3} = -333\ V.$$

We return to the equation system and knowing that

$\varphi_1' = \varphi_1 - \varphi_4 = 333\ V,\ U_{21} = -U_{12} = 2000\ V,\ U_{13} = -U_{31} = 3000\ V,$

$U_{23} = -U_{32} = 5000\ V,\ \varphi_2' = \varphi_2 - \varphi_4 = 2333\ V,\ \varphi_3' = \varphi_3 - \varphi_4 = -2667\ V,$

we obtain $\tau_1 = (333\cdot0.064 - 2000\cdot0.076 + 3000\cdot0.076)\cdot10^{-6} = 97\cdot10^{-6}\ C/km,$

$\tau_2 = (2000\cdot0.076 + 2333\cdot0.064 + 5000\cdot0.076)\cdot10^{-6} = 681\cdot10^{-6}\ C/km,$

$\tau_3 = (-3000\cdot0.076 - 5000\cdot0.076 - 2667\cdot0.064)\cdot10^{-6} = -779\cdot10^{-6}\ C/km.$

11-25 (11.35). Compute the direct and mutual capacitances of a two-wire shielded line (fig. 11.28), if $r_0 = 2\ mm,\ a = 4\ cm,\ r_1 = 10\ cm,\ \varepsilon = 4.$

The shell (shield) is supposed to be earthed.

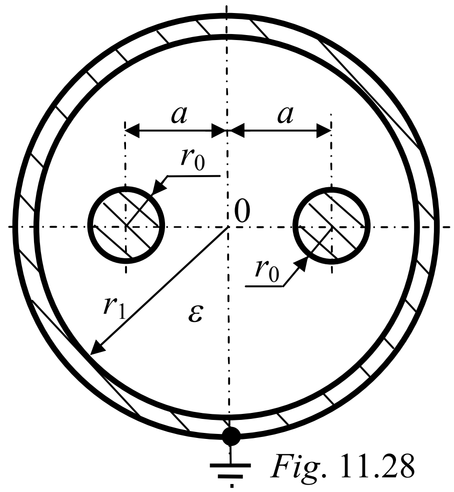


Fig. 11.28

Solution. Assume there is the charge separation between the 1st (left) and the 2nd (right) wires owing to the action of the voltage source U connected to the wires. Then the 1st wire has the charge $\tau_1 = \tau$ per a unit length, while the 2nd one – $\tau_2 = -\tau$.

For further calculations, the method of electrical images for axis located inside the cylinder is used. A sketch for the field calculation inside the cylinder has a view fig. 11.29, furthermore $a\cdot b = r_1^2$.

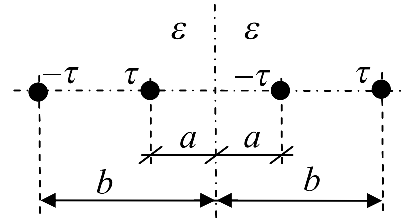


Fig. 11.29

In our case $b = \dfrac{r_1^2}{a} = \dfrac{10^2}{4} = 25\ cm.$

The 1st group of Maxwell's formulae for a system of the charged bodies located close to the cylindrical conducting surface is as follows:

$$\begin{cases}\varphi_1 = \tau_1\alpha_{11} + \tau_2\alpha_{12},\\ \varphi_2 = \tau_1\alpha_{21} + \tau_2\alpha_{22},\end{cases}$$

where $\alpha_{11} = \alpha_{22} = \dfrac{1}{2\pi\varepsilon\varepsilon_0} ln\dfrac{b-a}{r_0} = \dfrac{10^{12}}{2\pi\cdot4\cdot8.85} ln\dfrac{25-4}{0.2} = 2.091\cdot10^{10}\ m/F,$

$$\alpha_{12} = \alpha_{21} = \frac{1}{2\pi\varepsilon\varepsilon_0} ln\frac{b+a}{2a} = \frac{10^{12}}{2\pi\cdot4\cdot8.85} ln\frac{25+4}{2\cdot4} = 0.579\cdot10^{10}\ m/F.$$

The 2nd group of Maxwell's formulae is:

$$\begin{cases}\tau_1 = \beta_{11}\varphi_1 + \beta_{12}\varphi_2,\\ \tau_2 = \beta_{21}\varphi_1 + \beta_{22}\varphi_2,\end{cases}$$

where $\beta_{11} = \beta_{22} = \dfrac{\alpha_{22}}{\begin{vmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{vmatrix}} = \dfrac{2.091\cdot10^{-10}}{2.091^2 - 0.579^2} = 0.518\cdot10^{-10}\ F/m,$

$$\beta_{12} = \beta_{21} = \frac{-\alpha_{21}}{\begin{vmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{vmatrix}} = \frac{-0{,}579\cdot10^{-10}}{2.091^2 - 0.579^2} = -0.143\cdot10^{-10}\ F/m.$$

The 3rd group of Maxwell's formulae is:

$$\begin{cases}\tau_1 = U_{10}C_{11} + U_{12}C_{12},\\ \tau_2 = U_{21}C_{21} + U_{20}C_{22},\end{cases}$$

where $C_{11} = C_{22} = \beta_{11} + \beta_{12} = 0.375\cdot10^{-10}\ F/m,\ C_{12} = C_{21} = -\beta_{12} = 0.143\cdot10^{-10}\ F/m.$

Direct capacitancies of a line are shown in fig. 11.30.

Mutual capacitance of a two-wire shielded line per a unit length is:

$$C_0 = C_{12} + \frac{C_{11} \cdot C_{22}}{C_{11} + C_{22}} = \left(0.143 + \frac{0.375}{2}\right) \cdot 10^{-10} = = 0.331 \cdot 10^{-10}\ F/m.$$

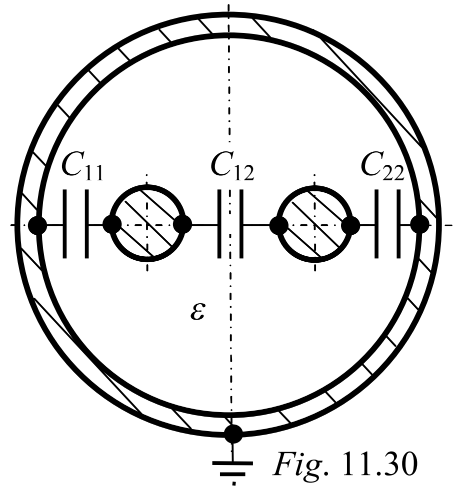


Fig. 11.30

11-26 (11.36). In a two wire system (fig. 11.31) located in air close to the conducting surface, there is an emf source $E = 127\ V$, the second wire is connected with earth. Geometrical sizes are: $r_0 = 6\ mm$, $d_{12} = 1\ m$, $h_1 = 3\ m$, $h_2 = 4\ m$.

Determined the wire charges.

Answers. $a_{12} = \sqrt{2}\ m$, $b_{12} = 5\sqrt{2}\ m$,
$\alpha_{11} = 12.4 \cdot 10^{10}\ m/F$, $\alpha_{22} = 12.9 \cdot 10^{10}\ m/F$,
$\alpha_{12} = 2.9 \cdot 10^{10}\ m/F$, $\varphi_1 = 127\ V$, $\varphi_2 = 0$,
$\tau_1 = 0.852 \cdot 10^{-11}\ C/m$, $\tau_2 = -0.191 \cdot 10^{-11}\ C/m$.

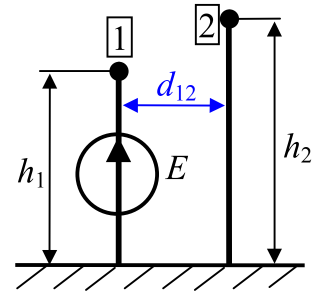


Fig. 11.31

11-27 (11.37). Determine the potentials and charges of a wire system (fig. 11.32), where switches 1 and 2 are closed, while 3 is opened, $U = 10\ kV$, the wire radius is $r_0 = 5\ mm$.

How will the solution change if at first wire 2 is disconnected from earth, then wire 1 is disconnected from the source and finally wire 3 is connected to earth?

Answers. Before commutations $\varphi_1 = 10\ kV$, $\varphi_2 = 0$, $\varphi_3 = 0.605\ kV$, $\tau_1 = 80\ \mu C/km$, $\tau_2 = -12\ \mu C/km$, $\tau_3 = 0$, $\alpha_{11} = \alpha_{33} = 11.5 \cdot 10^{10}\ m/F$, $\alpha_{22} = 12.3 \cdot 10^{10}\ m/F$, $\alpha_{12} = \alpha_{23} = 0.2 \cdot 10^{10}\ m/F$, $\alpha_{13} = 0.106 \cdot 10^{10}\ m/F$.

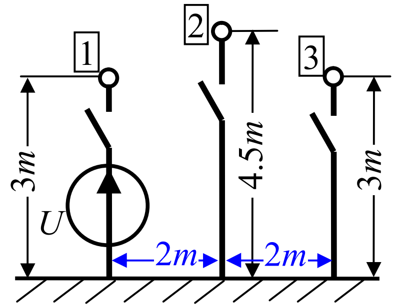


Fig. 11.32

After commutations $\tau_1' = \tau_1$, $\tau_2' = \tau_2$, $\varphi_3' = 0$. Then we determine the rest of the potentials and charge using the 1st group of Maxwell's formulae with the same potential coefficients:

$\varphi_1' = 9950\ V$, $\varphi_2' = -88\ V$, $\tau_3' = -4.76\ \mu C/km$.

12. ELECTRIC FIELD IN THE CONDUCTING MEDIUMS

12.1. THE FIELD COMPUTATION WITH THE AID OF THE INTEGRAL CORRELATIONS. APPLICATION OF THE ANALOGY BETWEEN THE ELECTROSTATIC FIELD AND THAT IN CONDUCTING MEDIUM.

12-1 (12.1). Derive the formulae to compute the field of a spherical electrode in homogeneous medium (fig. 12.1). Electrode radius is R_0, the medium conductivity is γ, current drainage is I.

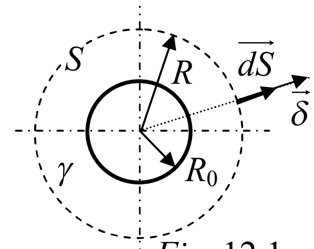


Fig. 12.1

Solution. Let's draw a spherical surface S of radius R around the electrode and determine the current through it: $I = \oint_S \vec{\delta} \cdot \overrightarrow{dS}$.

In all the points of surface S directions of vectors $\vec{\delta}$ and $\overrightarrow{dS}$ coincide, and the meaning of δ is the same. That's why

$$\oint_S \vec{\delta} \cdot \overrightarrow{dS} = \delta \cdot S = \delta \cdot 4\pi R^2 = I, \quad \text{from here} \quad \delta = I/(4\pi R^2).$$

Taking into account Ohm's law in a differential form $\delta = \gamma \cdot E$, we get the formula for electric intensity –
$$E = \frac{I}{4\pi\gamma R^2}.$$

Field point potential is $\varphi = -\int E dR = \dfrac{I}{4\pi\gamma R} + A.$

Assume the potential of infinite point equal to zero $\varphi_{(R=\infty)} = 0$, then

$$A = 0 \quad \text{and} \quad \varphi = \frac{I}{4\pi\gamma R}.$$

So, for the field of the ball electrode we have

$$\delta = \frac{I}{4\pi R^2}; \quad E = \frac{I}{4\pi\gamma R^2}; \quad \varphi = -\int E dR = \frac{I}{4\pi\gamma R} + A. \qquad (12.1)$$

12-2 (12.2). Derive the formulae to compute the field of a cylindrical electrode in a homogeneous medium (fig. 12.1). Radius and length of electrode are r_0 and l, the medium conductivity is γ, current drainage per unit length is I_0.

Solution. Fig. 12.1 can be regarded as a cross-section of a long cylindrical electrode. Let's draw a cylindrical surface S of radius r, length l and determine the current through its side surface: $I_0 \cdot l = \int_{S_{side}} \vec{\delta} \cdot \overrightarrow{dS}$. In all the points of the side cylindrical surface S the vectors $\vec{\delta}$ and $\overrightarrow{dS}$ have the same directions, and the meaning of δ is the same. That's why $\int_{S_{side}} \vec{\delta} \cdot \overrightarrow{dS} = \delta \cdot S_{side} = \delta \cdot 2\pi r l = I_0 \cdot l$, from here $\delta = I_0/(2\pi r)$.

Taking into account Ohm's law in a differential form $\delta = \gamma \cdot E$, we get the formula for electric intensity – $E = \dfrac{I_0}{2\pi\gamma r}$.

The field point potential is $\varphi = -\int E dr = \dfrac{-I_0}{2\pi\gamma} ln(r) + A$. Assume the potential of the collecting electrode situated at distance H equal to zero $(\varphi_{(r=H)} = 0)$, then

$$A=\frac{I_0}{2\pi\gamma}lnH \quad \text{and} \quad \varphi=\frac{I_0}{2\pi\gamma}ln\frac{H}{r}.$$

So, for the field of the cylindrical electrode we have

$$\delta=\frac{I_0}{2\pi r};\quad E=\frac{I_0}{2\pi\gamma r};\quad \varphi=-\int Edr=\frac{-I_0}{2\pi\gamma}ln(r)+A=\frac{I_0}{2\pi\gamma}ln\frac{H}{r}. \qquad (12.2)$$

12-3 (12.3). Two metallic balls of radiuses $r_1 = 2\ cm$ and $r_2 = 4\ cm$ are deeply drown into sea-water. The distance between the balls greatly exceeds their radiuses: $d = 2\ m$.

Determine the water resistance between balls, if sea-water resistivity is $\rho = 100\ Ohm{\cdot}m$.

Solution. Let's obtain the formula of conductance g in a general view. Although the balls are of different size, the current drainage is the same: $\pm I$, and sea-water conductivity is $\gamma = 1/\rho = 0.01\ S/m$.

In accordance with (12.1) and the superposition principle, we write down the voltage between the balls as potential difference

$$U=\varphi_1-\varphi_2=\left[\frac{I}{4\pi\gamma}\cdot\frac{1}{r_1}+\frac{-I}{4\pi\gamma}\cdot\frac{1}{d}\right]-\left[\frac{-I}{4\pi\gamma}\cdot\frac{1}{r_2}+\frac{I}{4\pi\gamma}\cdot\frac{1}{d}\right]=\frac{I}{4\pi\gamma}\left[r_1^{-1}+r_2^{-1}-2\cdot d^{-1}\right].$$

From here, the conductance and resistance of water between the balls are

$$g=\frac{I}{U}=\frac{4\pi\gamma}{r_1^{-1}+r_2^{-1}-2\cdot d^{-1}}=0.001698\ S,\quad R=1/g=589\ Ohm.$$

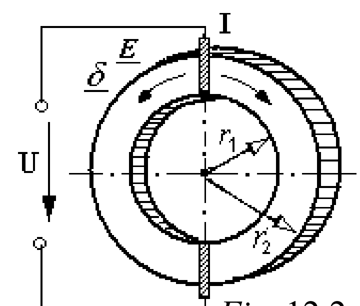


Fig. 12.2

12-4 (12.4). A D-C voltage source is connected to a conducting hollow disk with the aid of the copper plates fitted into the disk perpendicularly (fig. 12.2). The disk material resistivity is $\rho=0.5{\cdot}10^{-6}\ Ohm{\cdot}m$, the disk sizes are: thickness $a = 1\ mm$, inner radius $r_1 = 5\ cm$, outer radius $r_2 = 8\ cm$.

Determine the maximum and minimum values of the current density in the disk as well as the current through the voltage source $U = 1.57\ V$.

Solution. Two halves of the disk are two halves of a cylinder $a = 1\ mm$ long. The lines of the current density δ and the field intensity E coincide with disk semi-circles, their values depending on but one coordinate – radius r. That's why:

$$E(r) = U/(\pi{\cdot}r);\qquad \delta(r) = \gamma{\cdot}E = \gamma{\cdot}U/(\pi{\cdot}r);\qquad \gamma = 1/\rho = 2{\cdot}10^6\ S/m.$$

Then the minimum and maximum values of the current density are

$$\delta_{min} = \gamma{\cdot}U/(\pi{\cdot}r_2) = 2{\cdot}10^6{\cdot}1.57/(\pi{\cdot}0.08) = 12.5{\cdot}10^6\ A/m^2,$$
$$\delta_{max} = \gamma{\cdot}U/(\pi{\cdot}r_1) = 2{\cdot}10^6{\cdot}1.57/(\pi{\cdot}0.05) = 20{\cdot}10^6\ A/m^2.$$

The total source current is found as an integral quantity:

$$I=2\int\delta\cdot dS=2\int_{r1}^{r2}\gamma E\cdot(a{\cdot}dr)=2\cdot\gamma\frac{U\cdot a}{\pi}\cdot\int_{r1}^{r2}\frac{1}{r}\cdot dr=$$
$$=\frac{2\gamma U\cdot a}{\pi}(ln\ r_2-ln\ r_1)=$$
$$=(2{\cdot}2{\cdot}10^6{\cdot}1.57{\cdot}0{,}001/\pi){\cdot}(ln0.08-ln0.05)=940\ A.$$

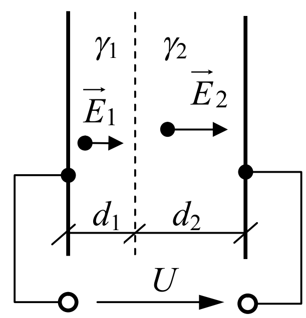


Fig. 12.3

12-5 (12.5). Compute the leakage current of the plane two-layer capacitor as well as the heat loss in the unit volume of the second

dielectric (fig. 12.3). $d_1 = 1\ cm$, $\gamma_1 = 5\cdot10^{-8}\ S/m$, $U = 1.8\ kV$,
$d_2 = 2\ cm$, $\gamma_2 = 2\cdot10^{-8}\ S/m$, $S = 0.01\ m^2$.

Solution. On the grounds of Kirchhoff's voltage law we have:

$$U = U_1 + U_2 = E_1\cdot d_1 + E_2\cdot d_2.$$

The force lines are perpendicular to the interface of dielectrics, that's why the boundary condition is as follows $\delta_1 = \delta_2$ or $\gamma_1\cdot E_1 = \gamma_2\cdot E_2$.

Then $E_1 = (\gamma_2/\gamma_1)\cdot E_2 = 0.4E_2$; $U = E_2\cdot(0.4\cdot d_1 + d_2)$; from here
$E_2 = U/(0.4\cdot d_1 + d_2) = 75\ kV/m$; $E_1 = 30\ kV/m$;
$\delta = \delta_1 = \delta_2 = \gamma_1\cdot E_1 = 1.5\cdot10^{-3}\ A/m^2$; $I = \delta\cdot S = 15\ \mu A$;

$$p_2 = \delta^2/\gamma_2 = (1.5\cdot10^{-3})^2/(2\cdot10^{-8}) = 112.5\ W/m^3.$$

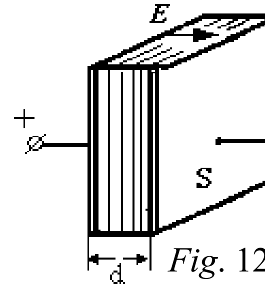


Fig. 12.4

12-6 (12.6). Determine the heat loss P in a plane capacitor (fig. 12.4) with imperfect dielectric (mica).

$U = 1000\ V$, $S = 100\ cm^2$, $d = 1\ cm$, $\gamma = 1\cdot10^{-12}\ S/m$.

Answers: $E = U/d = 100\ kV/m$, $P = \gamma E^2\cdot V = 1\ \mu W$.

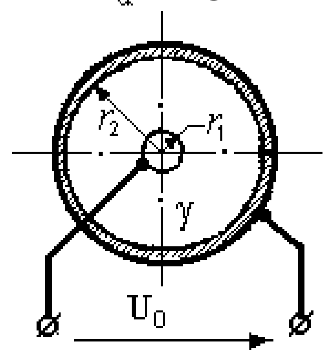


Fig. 12.5

12-7 (12.7). Coaxial cable is under the voltage $U_0 = 10\ kV$. The insulation between a current-carrying conductor and shell is imperfect and its conductivity is $\gamma = 1\cdot10^{-8}\ S/m$. Compute the leakage current and the heat loss in the cable insulation of length $l = 1\ km$ (fig. 12.5). $r_1 = 1.2\ mm$, $r_2 = 3.26\ mm$.

Solution. We make use of formal analogy between the expressions for capacity and conductance of single-layer coaxial cable per unit length:

$$C_0 = \frac{2\pi\cdot\varepsilon_0\varepsilon}{ln\ r_2/r_1},\quad g_0 = \frac{2\pi\cdot\gamma}{ln\ r_2/r_1} = \frac{2\pi\cdot1\cdot10^{-8}}{ln\ 3.26/1.2} = 0.0628\cdot10^{-6}\ S/m. \qquad (12.3)$$

Then the leakage current through the cable insulation at length 1 km is

$$I = g_0 l U = 0.0628\cdot10^{-6}\cdot1000\cdot10\cdot10^3 = 0.628\ A.$$

The heat loss in insulation is $P = U\cdot I = 10000\cdot0.628 = 6280\ W$.

12-8 (12.8). Single-layer coaxial cable is under the voltage 600 V and its sizes are: $r_1 = 4\ mm$, $r_2 = 8\ mm$, $l = 10\ km$, the insulation conductivity is $\gamma = 1\cdot10^{-9}\ S/m$. Determine the leakage current and its density in insulation on the conductor surface and on the inner side of the cable shell as well as the cable heat loss.

Answers: $I = 54.4\ mA$, $\delta_1 = 216\ \mu A/m^2$,
$\delta_2 = 108\ \mu A/m^2$, $P = 32.6\ W$.

12-9 (12.9). Compute the leakage current between two conductors of the coaxial cable (fig. 12.6). Insulation is made of imperfect dielectric and has two layers (conductivities $\gamma_1 = 5\cdot10^{-8}\ S/m$ and $\gamma_2 = 2\cdot10^{-8}\ S/m$, relative dielectric permeability $\varepsilon_1 = 2$ and $\varepsilon_2 = 5$). Supplying voltage is $U = 1\ kV$. Geometrical sizes are – $r_1 = 1\ mm$, $r_2 = 2\ mm$, $r_3 = 3\ mm$.

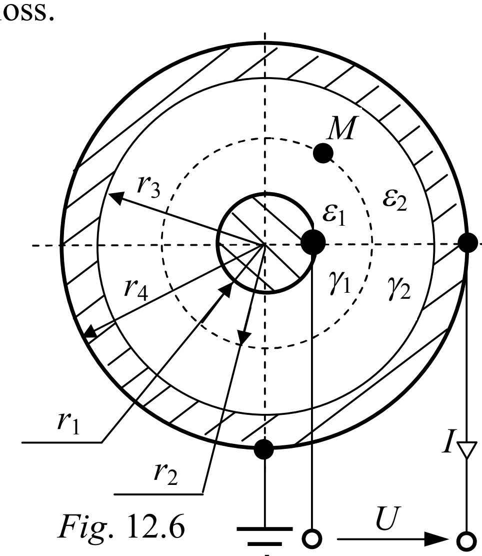


Fig. 12.6

Find specific heat loss in the surroundings of the point M, conductance and capacitance between conductors, construct the equivalent scheme of the system. The cable is considered to be rather long; calculations are to be performed per unit length.

Determine additionally the maximum permissible length of a cable used as a transmission line.

Solution. 1. Let the leakage current be I. Then in accordance with formulae (12.2), the current density, intensity and potential in layers of dielectric obey to formulae

$$\delta = \frac{I}{2\pi l r};\quad E_1 = \frac{I}{2\pi\gamma_1 l r},\quad E_2 = \frac{I}{2\pi\gamma_2 l r};\quad \varphi_1 = \frac{-I}{2\pi\gamma_1 l} ln r + A_1,\quad \varphi_2 = \frac{-I}{2\pi\gamma_2 l} ln r + A_2.$$

2. The voltage supplied to an installation is

$$U = \varphi_1(r_1) - \varphi_1(r_2) + \varphi_2(r_2) - \varphi_2(r_3) =$$

$$= \frac{I}{2\pi\gamma_1 l} ln\frac{r_2}{r_1} + \frac{I}{2\pi\gamma_2 l} ln\frac{r_3}{r_2} = \frac{I}{2\pi\gamma_2 l} ln\left[\left(\frac{r_2}{r_1}\right)^{\gamma_2/\gamma_1}\cdot\left(\frac{r_3}{r_2}\right)\right].$$

3. From here, the leakage current per unit length is

$$I_0 = \frac{I}{l} = \frac{U\cdot 2\pi\gamma_2}{ln\left[\left(r_2/r_1\right)^{\gamma_2/\gamma_1}\cdot\left(r_3/r_2\right)\right]} = \frac{1000\cdot 2\pi\cdot 2\cdot 10^{-8}}{ln\left[\left(2/1\right)^{2/5}\cdot\left(3/2\right)\right]} = 1.841\cdot 10^{-4}\ A/m = 0.1841\ mA/m.$$

4. The cable conductance is found under Ohm's law:

$$g_0 = I_0/U = 1.841\cdot 10^{-4}/10^3 = 0.1841\cdot 10^{-6}\ S/m = 0.1841\ \mu S/m.$$

5. The same answer may be obtained with the aid of the analogy between an electric field in the conducting medium and electrostatic one. Formula (12.3) for capacity of single-layer coaxial cable is: $C_0 = \dfrac{2\pi\varepsilon_a}{ln\left(r_2/r_1\right)}$.

The capacity of layers and that of the whole cable is:

$$C_{10} = \frac{2\pi\varepsilon_1\varepsilon_0}{ln\left(r_2/r_1\right)} = \frac{2\pi\cdot 2\cdot 8.85\cdot 10^{-12}}{ln\left(2/1\right)} = 160.4\ pF/m;$$

$$C_{20} = \frac{2\pi\varepsilon_2\varepsilon_0}{ln\left(r_3/r_2\right)} = \frac{2\pi\cdot 5\cdot 8.85\cdot 10^{-12}}{ln\left(3/2\right)} = 873.5\ pF/m;$$

since capacitors are connected in series then

$$C_0 = \frac{C_{10}C_{20}}{C_{10}+C_{20}} = \frac{160.4\cdot 873.5}{160.4+873.5} = 135.5\ pF/m.$$

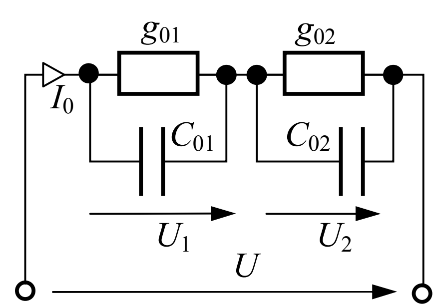


Fig. 12.7

The conductance of layers and that of the whole cable per unit length is:

$$g_{10} = \frac{2\pi\gamma_1}{ln\left(r_2/r_1\right)} = \frac{2\pi\cdot 5\cdot 10^{-8}}{ln\left(2/1\right)} = 0.454\cdot 10^{-6}\ S/m;$$

$$g_{20} = \frac{2\pi\gamma_2}{ln\left(r_2/r_1\right)} = \frac{2\pi \cdot 2 \cdot 10^{-8}}{ln\left(3/2\right)} = 0.310 \cdot 10^{-6}\ S/m;$$

$$g_0 = \frac{g_{10} g_{20}}{g_{10} + g_{20}} = \frac{0.454 \cdot 0.310}{0.454 + 0.310} = 0.1842\ \mu S/m.$$

6. The electric equivalent scheme of an installation is shown in fig. 12.7.

7. The current density in the surroundings of point M is as follows:

$$\delta = I_0/(2\pi r_2) = 1.841 \cdot 10^{-4}/(2\pi \cdot 2 \cdot 10^{-3}) = 0.0147\ A/m^2;$$

heat loss in the surroundings of point M is found by Joule's effect:

$$p_1 = \delta^2/\gamma_1 = 0.0147^2/(5 \cdot 10^{-8}) = 0.432 \cdot 10^4\ W/m^3 = 4.32\ kW/m^3;$$

$$p_2 = \delta^2/\gamma_2 = 0.0147^2/(2 \cdot 10^{-8}) = 1.08 \cdot 10^4\ W/m^3 = 10.8\ kW/m^3.$$

8. The cross-section of the inner conductor of cable is less than that of the outer one and is equal to $S = \pi r_1^2 = 3.14\ mm^2$. If to assume the conductors are made of aluminum and permissible current density is 1 A/mm^2, then permissible current through the cable is 3.14 A. Then permissible cable length is $l_p = 3.14/I_0 = 3.14 \cdot 1000/0.1841 = 17050\ m$.

However, in order to have more or less reasonable efficiency, this length is to be even less.

12-10 (12.10). A cylindrical capacitor with two-layer insulation (fig. 12.6) operates at voltage $U = 1\ kV$. Known: the capacitor sizes ($r_1 = 0.08\ cm$, $r_2 = 0.2\ cm$, $r_3 = 0.6\ cm$, $l = 5\ cm$) and the insulation features ($\gamma_1 = 1 \cdot 10^{-8}\ S/cm$, $\varepsilon_1 = 2$, $\gamma_2 = 4 \cdot 10^{-8}\ S/cm$, $\varepsilon_2 = 6$). Compute the leakage current through the capacitor insulation, find expressions of conductance g_0 and capacity C_0 per unit length.

Answers: $I = 0.2634\ mA$,

$$g_0 = \frac{2\pi\gamma_1}{ln\frac{r_2}{r_1} + \frac{\gamma_1}{\gamma_2} ln\frac{r_3}{r_2}} = 5.273 \cdot 10^{-8}\ S/m, \quad C_0 = \frac{2\pi\varepsilon_0\varepsilon_1}{ln\frac{r_2}{r_1} + \frac{\varepsilon_1}{\varepsilon_2} ln\frac{r_3}{r_2}} = 43.34\ pF/m.$$

12-11 (12.11). A plane capacitor of capacity 10 μF operates at voltage 1 kV. Dielectric permeability of insulation is $\varepsilon = 4$, its conductivity being $\gamma = 1 \cdot 10^{-12}\ S/m$. Determine the leakage current between the capacitor plates.

Methodical instructions: formula for capacity $C = q/U = \varepsilon_0\varepsilon\ S/d$ gives the ratio $S/d = C/(\varepsilon_0\varepsilon)$, then it may be used in identical formula for the capacitor conductance: $g = \gamma \cdot C/(\varepsilon_0\varepsilon)$.

Answer: $I = 0.283\ mA$.

12-12 (12.12). Two plane capacitors $C_1 = 0.2\ \mu F$, $C_2 = 0.5\ \mu F$ with imperfect dielectrics are connected in series, they being supplied with voltage $U = 1200\ V$. Dielectric permeability of dielectrics is: $\varepsilon_1 = 2.4$, $\varepsilon_2 = 4$; conductivity is: $\gamma_1 = 0.2 \cdot 10^{-9}\ S/m$, $\gamma_2 = 5 \cdot 10^{-9}\ S/m$. Determine the voltage across each capacitor at the moment of commutation as well as in the steady-state condition.

Solution. Let's use the equivalent scheme of the system presented in fig. 12.7. At the commutation moment, the total current flows through the discharged capacitors in accordance with the 2nd commutation law. It means one and the same charge passes through the capacitors $q = C_1U_1 = C_2U_2$. However, in accordance with Kirchhoff's voltage law for circuits $U_1 + U_2 = U$. From here, the voltages across the capacitors at the initial moment after commutation are

$$U_1(0) = \frac{U}{1+\dfrac{C_1}{C_2}} = \frac{1200}{1+\dfrac{0.2}{0.5}} = 857\ V, \quad U_2(0) = \frac{U}{1+\dfrac{C_2}{C_1}} = \frac{1200}{1+\dfrac{0.5}{0.2}} = 343\ V.$$

In steady-state condition, the total current flows through the conductance g_1 and g_2, which can be found from the formula for capacity of the plane capacitor $C = \varepsilon_0\varepsilon S/d$:

$$g_1 = \gamma_1 \cdot \frac{S_1}{d_1} = \gamma_1 \cdot C_1/\varepsilon_0\varepsilon_1 = 0.2 \cdot 10^{-9} \cdot \frac{0.2 \cdot 10^{-6}}{8.85 \cdot 10^{-12} \cdot 2.4} = 1.883 \cdot 10^{-6}\ S;$$

$$g_2 = \gamma_2 \cdot C_2/\varepsilon_0\varepsilon_2 = 5 \cdot 10^{-9} \cdot \frac{0.5 \cdot 10^{-6}}{8.85 \cdot 10^{-12} \cdot 4} = 70.62 \cdot 10^{-6}\ S;$$

$$g = \frac{g_1 \cdot g_2}{g_1 + g_2} = 10^{-6} \cdot \frac{1.883 \cdot 70.62}{1.883 + 70.62} = 1.834 \cdot 10^{-6}\ S.$$

The leakage current and voltages across the capacitor are found under Ohm's law:

$I = g \cdot U = 1.834 \cdot 10^{-6} \cdot 1200 = 2.201 \cdot 10^{-3}\ A = 2.201\ mA$,

$U_1 = I/g_1 = 2.201 \cdot 10^{-3}/1.883 \cdot 10^{-6} = 1169\ V$;

$U_2 = I/g_2 = 2.201 \cdot 10^{-3}/70.62 \cdot 10^{-6} = 31\ V$.

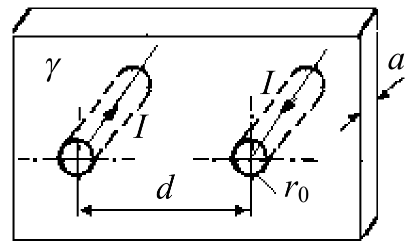


Fig. 12.8

12-13 (12.13). Two cylindrical wires pass through a marble board (fig. 12.8) of thickness $a = 3\ cm$. The distance between the wire axes is $d = 20\ cm$, the wire radius is $r_0 = 4\ mm$. Supposing the board area is unrestrictedly large, determine the leakage current between the wires, if: $\gamma = 1 \cdot 10^{-10}\ S/m$, $U = 240\ V$.

Methodical instructions. It is possible to use the traditional way of calculation with the help of formula (12.2). In this case it is necessary to write down the wire potentials φ_1, φ_2 taking any value of leakage current I as it is still unknown and then the leakage current is found from the expression $U = \varphi_1 - \varphi_2$.

However, much simpler way is to make use of analogy between the formulae of capacity and conductance of a two-wire line in unlimited medium (formulae 12.3):

$$C_0 = \frac{\pi \cdot \varepsilon_0\varepsilon}{ln(d - r_0)/r_0}; \quad g_0 = \frac{\pi \cdot \gamma}{ln(d - r_0)/r_0}.$$

Answers: $g_0 = 0.6837\ S/m$, $I = g_0 \cdot a \cdot U = 4.92 \cdot 10^{-10}\ A$.

12.2. THE FIELD COMPUTATION BY METHOD OF ELECTRICAL IMAGES

12-14 (12.19). A tram contact-wire line is suspended at height $h = 5\ m$, the wire radius is $r_0 = 6\ mm$, its potential with respect to earth is $U = 600\ V$ (fig. 12.9). Compute the wire capacity with respect to earth; find the leakage current at the length $l = 500\ m$ in usual and foggy weather. $\gamma_a = 0.2 \cdot 10^{-6}\ S/m$, $\gamma_f = 1 \cdot 10^{-4}\ S/m$.

Additionally: using analogy with an electrostatic field, find the electric field intensity under the wire at the height of a man $h_1 = 1.7\ m$.

Solution. The principal field characteristics in infinite homogeneous medium are set by expressions (12.2):

$$\delta(r) = \frac{I_0}{2\pi r}, \quad E(r) = \frac{I_0}{2\pi\gamma r}, \quad \varphi(r) = -\int E dr = \frac{-I_0}{2\pi\gamma} \cdot ln\, r + A.$$

A contact wire create a field around the plane conducting surface, and in such cases the field is computed using the method of electrical images.

Introduce a dummy wire with current $I_f = -I_0$. Work out the voltage at the contact wire received from the action of both current elements ($h >> r_0$): $U = \dfrac{-I_0}{2\pi\gamma_a}\cdot(\ln r_0 - \ln 2h) = \dfrac{I_0}{2\pi\gamma_a}\cdot \ln\dfrac{2h}{r_0}$.

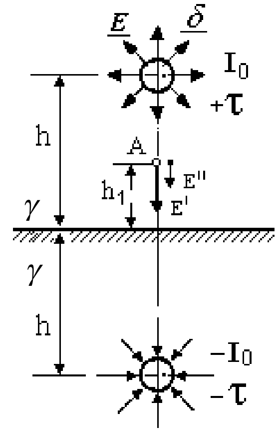


Fig. 12.9

From here, we can compute the wire conductance for two above-mentioned cases

$$g_{0a} = \frac{I_0}{U} = \frac{2\pi\gamma_a}{\ln(2h/r_0)} = \frac{2\pi\cdot 0.2\cdot 10^{-6}}{\ln(2\cdot 5/0.006)} = 0.17\cdot 10^{-6}\ S/m;$$

$$g_{0f} = \frac{2\pi\gamma_f}{\ln(2h/r_0)} = \frac{2\pi\cdot 10^{-4}}{\ln(2\cdot 5/0.006)} = 0.0847\cdot 10^{-3}\ S/m.$$

By analogy, the capacity of the contact wire close to a conducting surface can be determined by the expression:

$$C_0 = \frac{\tau}{U} = \frac{2\pi\varepsilon_0}{\ln(2h/r_0)} = \frac{2\pi\cdot 8.85\cdot 10^{-12}}{\ln(2\cdot 5/0.006)} = 7.5\cdot 10^{-12}\ F/m = 7.5\ pF/m.$$

The leakage current of the contact wire at the length $l = 500\ m$ for two cases is, respectively: $I_a = g_{0a}\cdot U\cdot l = 0.17\cdot 10^{-6}\cdot 600\cdot 500 = 0.051\ A$, $I_f = 25.4\ A$.

Concerning the additional task of problem.

Using the found value of the wire capacity, it is possible to write down the formula for the linear charge τ on the contact wire: $\tau = C_0\cdot U = 7.5\cdot 10^{-12}\cdot 600 = 4.49\cdot 10^{-9}\ C/m$.

The field intensity in point A under the wire is computed as a sum of components induced by both charged axis (fig. 12.9) τ and $-\tau$:

$$E_A = E' + E'' = \frac{\tau}{2\pi\varepsilon_0(h-h_1)} + \frac{\tau}{2\pi\varepsilon_0(h+h_1)} = \frac{4.49\cdot 10^{-9}}{2\pi\cdot 8.85\cdot 10^{-12}}(0.303 + 0.149) = 36.6\ V/m.$$

12-15 (12.20). The contact wires of a trolleybus are suspended at height $h = 6\ m$, the wire radius is $r_0 = 6\ mm$, the distance between wires is $d = 0.5\ m$. The line is isolated from earth and operates at voltage $U = 600\ V$.

Determine the line conductance and the leakage current at length $l = 500\ m$ for foggy weather $\gamma_f = 1\cdot 10^{-4}\ S/m$.

Find additionally: the mutual capacity C_m of the section of line $l = 500\ m$ as well as the electric field intensity under the wires at a man height $h_1 = 1.7\ m$.

Answers: the sketch for calculation is given in fig. 12.10, $E_A = 1.26\ V/m$,

$$g_0 = \frac{\pi\gamma_f}{\ln(2h/r_0) + \ln(d/\sqrt{(2h)^2 + d^2})} = 0.0711\cdot 10^{-3}\ S/m,$$

$I = 21.3\ A$,

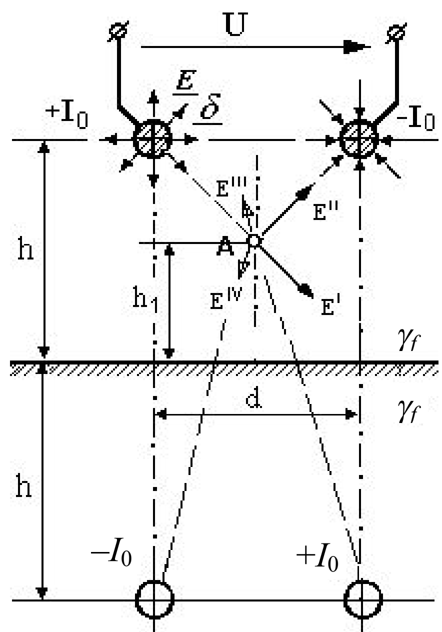


Fig. 12.10

$$C_0 = \frac{\pi\varepsilon_0}{ln(2h/r_0) + ln(d/\sqrt{(2h)^2 + d^2})} = 6{,}29\cdot 10^{-12}\ F/m, \qquad C_m = 3.14\ nF.$$

12-16 (12.21). A two-wire line with bare wires is used to heat the soil in a greenhouse (fig. 12.11,a). The numerical data are: $h = 50\ cm$, $d = 80\ cm$, $r_0 = 3\ mm$, $l = 20\ m$, $U = 36\ V$, $\gamma = 0.2\ S/m$. Determine the line current and power.

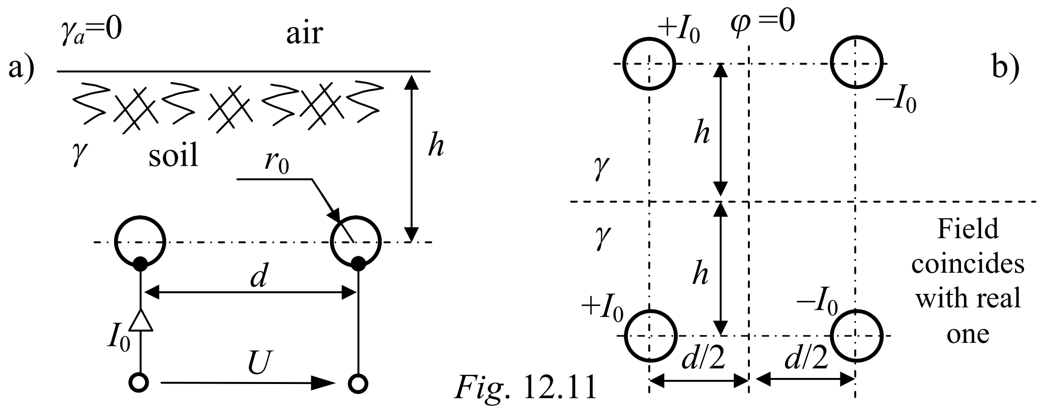


Fig. 12.11

Solution. To solve the problem the method of electrical images is used. The scheme for calculation of the field in the lower half plane is in fig. 12.11,b. Here it is taken into account that the incomplete reflection coefficient is $k_1 = \dfrac{\gamma - \gamma_a}{\gamma + \gamma_a} = 1.$

Further we apply the superposition principle. The field pattern is symmetrical with respect to the vertical plane between the wires, so, its potential may be assumed to be equal to zero. The distance from any wire to this plane (to reference point) is equal to $d/2$. A single wire potential (12.2) is $\varphi = \dfrac{I_0}{2\pi\gamma} ln \dfrac{d/2}{r}$.

The potential of the left real wire is:

$U/2 = \varphi' + \varphi'' + \varphi''' + \varphi'''' =$

$$= \frac{I_0}{2\pi\gamma}\left[ln\frac{d/2}{r_0} - ln\frac{d/2}{d} + ln\frac{d/2}{2h} - ln\frac{d/2}{\sqrt{d^2 + (2h)^2}}\right] = \frac{I_0}{2\pi\gamma} ln \frac{d\cdot\sqrt{d^2 + (2h)^2}}{r_0 \cdot 2h},$$

from here the line current and power per unit length of line is:

$$I_0 = \frac{U\cdot\pi\gamma}{ln\dfrac{d\cdot\sqrt{d^2 + (2h)^2}}{r_0\cdot 2h}} = \frac{36\cdot\pi\cdot 0.2}{ln\dfrac{80\cdot\sqrt{80^2 + 100^2}}{0.3\cdot 100}} = 3.878\ A/m,$$

$P_0 = U\cdot I_0 = 36\cdot 3.878 = 139.6\ W/m.$

12-17 (12.22). There is a voltage $U = 220\ V$ between a ball of radius $R_0 = 10\ cm$ placed in sea-water with conductivity $\gamma = 0.2\ S/m$ at depth $h = 2\ m$ and a metallic plate (fig. 12.12). Determine the current flowing between the ball and unrestrictedly large metallic plate.

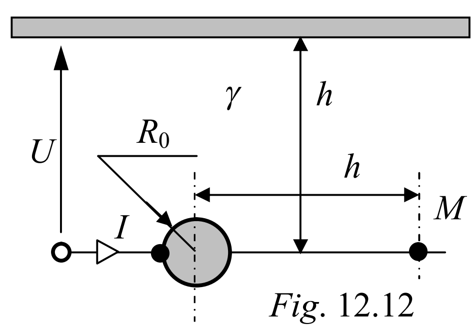


Fig. 12.12

Find the specific heat loss in point M.

Answers: the calculation sketch is in fig. 12.13.

$$U=\frac{I}{4\pi\gamma}(R_0^{-1}-(2h)^{-1}),\ \ I=\frac{U\cdot 4\pi\gamma}{R_0^{-1}-(2h)^{-1}}=56.71\ A.$$

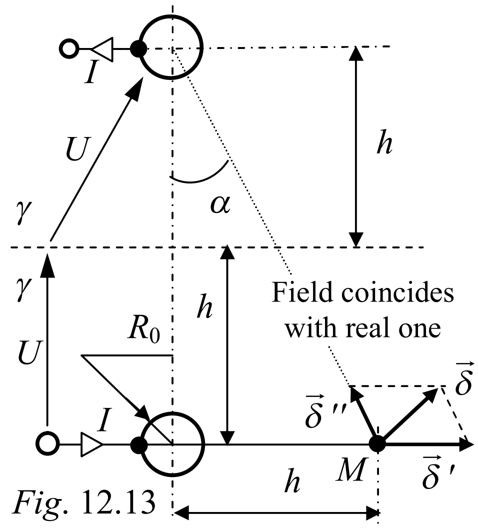


Fig. 12.13

The results of the auxiliary calculations when resolving the current density vectors into the projections are:

$$cos\alpha = 2h/\sqrt{h^2+(2h)^2}=2/\sqrt{5}=0.8944;$$

$$sin\alpha = 1/\sqrt{5}=0.4472,$$

$$\delta'=\frac{I}{4\pi h^2};\qquad \delta'_x=\frac{I}{4\pi h^2};\qquad \delta'_y=0;$$

$$\delta''=\frac{I}{4\pi(h^2+(2h)^2)}=\frac{I}{20\pi h^2};$$

$$\delta''_x=-\frac{I\cdot sin\alpha}{20\pi h^2};\qquad \delta''_y=\frac{I\cdot cos\alpha}{20\pi h^2};$$

$$\delta=\sqrt{(\delta'_x+\delta''_x)^2+(\delta'_y+\delta''_y)^2}=\frac{I}{4\pi h^2}\sqrt{(1-0.2\,sin\alpha)^2+(0.2\,cos\alpha)^2}=1.047\ A/m^2.$$

Specific heat loss in point M is: $p_M=\delta^2/\gamma=5.48\ W/m^3$.

Note. If there is a wire close to the conducting surface, a two-wire line arises after the mirror image, its capacity being – $C=\frac{\pi\varepsilon_a l}{ln\left(d/r_0\right)}$. Then its conductance is $g=\frac{\pi\gamma l}{ln\left(d/r_0\right)}$, while the required current is – $I=g\cdot U$.

12-18 (12.23). Wires of a two-wire line of voltage $U=500\ V$ are placed in different dielectrics as it is shown in fig. 12.14. Compute the leakage current of the line. Geometrical sizes and the medium properties are as follows:

$r_0=0.2\ cm,\ \ r_1=20\ cm,\ \ r_2=40\ cm,$

$\gamma_1=2\cdot10^{-8}\ S/m,\ \ \gamma_2=10^{-8}\ S/m.$

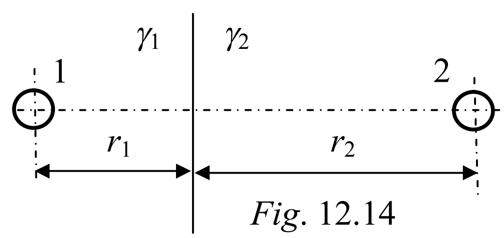


Fig. 12.14

Methodical instructions and answers. The problem is solved in the same way as problem 11.19 after the replacement of τ by I_0 and ε by γ. The calculation sketches are identical to fig. 11.23,a and b.

$$k_1'=0.333;\ \ k_2'=1.333;\ \ k_1''=-0.333;\ \ k_2''=0.667.$$

Interface point potential is assumed to be equal to zero. The wire potentials are:

$$\varphi_1=\frac{I_0}{2\pi\gamma_1}(ln\frac{r_1}{r_0}+k_1'\cdot ln\frac{r_1}{2r_1}-k_2'\cdot ln\frac{r_2}{r_1+r_2});$$

$$\varphi_2=\frac{I_0}{2\pi\gamma_2}(-ln\frac{r_2}{r_0}-k_1''\cdot ln\frac{r_2}{2r_2}+k_2''\cdot ln\frac{r_1}{r_1+r_2}).$$

The voltage between the wires is

$$U = \varphi_1 - \varphi_2 = \frac{I_0}{2\pi}\left[\gamma_1^{-1}\cdot\left(\ln\frac{r_1}{r_0} + k_1'\cdot\ln\frac{r_1}{2r_1} - k_2'\cdot\ln\frac{r_2}{r_1+r_2}\right) - \right.$$

$$\left. - \gamma_2^{-1}\cdot\left(-\ln\frac{r_2}{r_0} - k_1''\cdot\ln\frac{r_2}{2r_2} + k_2''\cdot\ln\frac{r_1}{r_1+r_2}\right)\right] = I_0\cdot 1.388\cdot 10^8.$$

From here $I_0 = U/(1.388\cdot10^8) = 3.603\cdot10^{-6}\ A/m = 3.603\ \mu A/m$.

12.3. COMPUTAION OF THE GROUNDING CONDUCTORS

12-19 (12.24). A transmission tower is installed on a hemispherical reinforced concrete foundation not far from a steep (fig. 12.15); $R_0 = 2\ m$, $h = 25\ m$. Reinforced concrete conductivity is much higher than the earth conductivity $\gamma = 10^{-4}\ S/cm$. The short-circuit current $I = 200\ A$ is supposed to drain through the foundation.

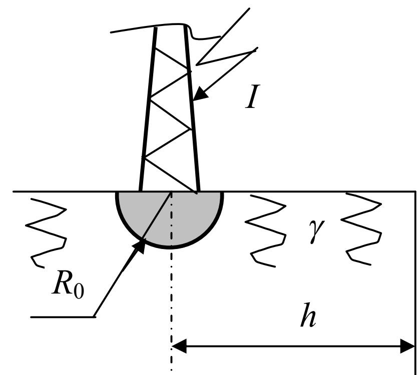


Fig. 12.15

Compute the resistance of the grounding conductor, plot the potential variations on the earth surface around the grounding conductor, find the maximum step voltage. A pace is assumed to be equal to $l_p = 0.8\ m$.

Solution. The problem is solved by the electrical image method. A sketch to calculate the field in the lower half plane is in fig. 12.16. The incomplete reflection coefficient is taken into account – $k_1 = \dfrac{\gamma - \gamma_a}{\gamma + \gamma_a} = 1.$

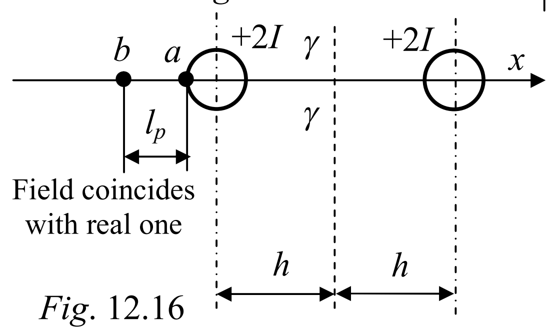


Fig. 12.16

Further we apply the superposition principle. For a single ball with current I in a homogeneous medium, potential is $\varphi = \dfrac{I}{4\pi\gamma R}$ (12.1). In our case, current $2I$ is drained through the ball, that's why $\varphi = \dfrac{I}{2\pi\gamma R}$.

Grounding conductor potential is $\varphi = \varphi' + \varphi'' =$

$$= \frac{I}{2\pi\gamma}\left(R_0^{-1} + (2h)^{-1}\right) = 1656\ V.$$

Grounding conductor resistance is

$$R_g = \frac{\varphi_g}{I} = 8.28\ Ohm.$$

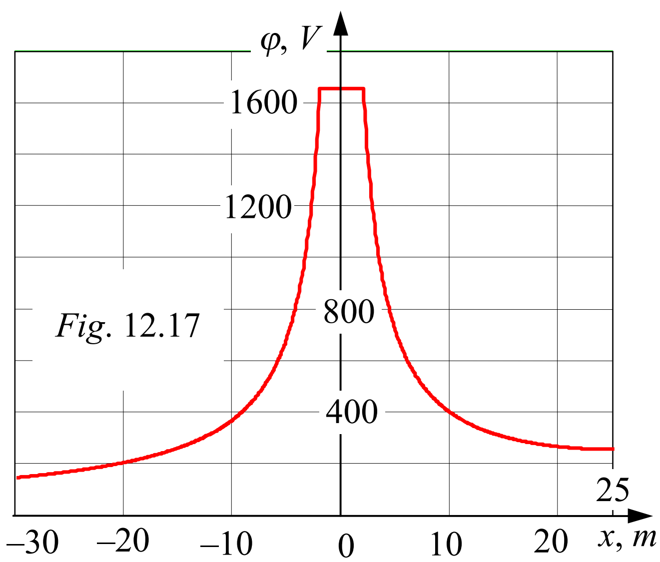


Fig. 12.17

Let's take a coordinate axis Ox on the earth surface from left to right (fig. 12.17). The coordinate origin coincides with the centre of the left ball. Potential of any point on the earth surface (with the exception of points $(-R_0) < x < (+R_0)$, the potential of which is equal to 1656 V) for $x < h$ is $\varphi(x) = \dfrac{I}{2\pi\gamma}\left(|x|^{-1} + (2h - x)^{-1}\right).$

As it is seen from graph fig. 12.17, the potential decreases sharply in the area of negative "x". That's why the maximum step voltage is observed between points a and b shown in fig. 12.16.

$$U_{p\,max} = \varphi_a - \varphi_b = \frac{I}{2\pi\gamma}(R_0^{-1} + (2h + R_0)^{-1} - (R_0 + l_p)^{-1} - (2h + R_0 + l_p)^{-1}) = 1458\ V.$$

12-20 (12.25). A grounding conductor consists of two cylinders (fig. 12.18), the following data is known $h = 1\ m$, $d = 1.5\ m$, $r_0 = 0.1\ m$, $\gamma = 0.1\ S/m$. The step voltage between points a and b ($l_p = 0.8\ m$) must not exceed $50\ V$. Compute the maximum permissible current drained by the grounding conductor.

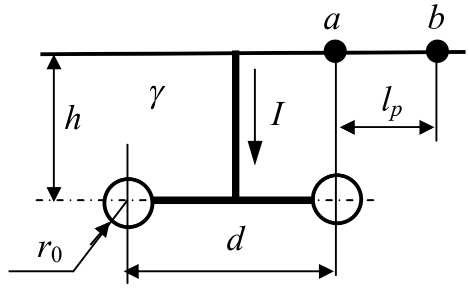


Fig. 12.18

Methodical instructions. Suppose I_0 is the current drained per unit length. In accordance with the electrical image method, the field of two cylinders (each with current $I_0/2$) in a heterogeneous medium (soil-air) is presented by the field of four identical cylinders in a homogeneous medium. The potential of a single cylindrical electrode (12.2) is: $\varphi = \frac{I_0/2}{2\pi\gamma} ln\frac{H}{r}$,

where H – distance to the collecting electrode. The given step voltage is:

$$U_{ab} = \varphi_a - \varphi_b = \frac{2\cdot I_0}{4\pi\gamma}\left(ln\frac{H^2}{\sqrt{d^2+h^2}\cdot h} - ln\frac{H^2}{\sqrt{(d+l_p)^2+h^2}\cdot\sqrt{l_p^{\,2}+h^2}}\right) =$$

$$= \frac{I_0}{2\pi\gamma} ln\frac{\sqrt{(d+l_p)^2+h^2}\cdot\sqrt{l_p^{\,2}+h^2}}{\sqrt{d^2+h^2}\cdot h}.$$

From here, the required current is

$$I_0 = \frac{U_{ab}\cdot 2\pi\gamma}{ln\left(\sqrt{(d+l_p)^2+h^2}\sqrt{l_p^{\,2}+h^2} \Big/ \sqrt{d^2+h^2}\cdot h\right)} = 54.4\ A/m.$$

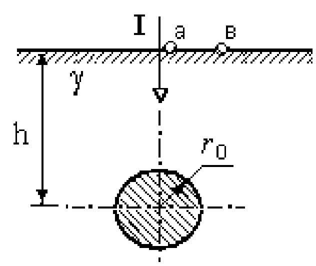


Fig. 12.19

12-21 (12.26). A ball-shaped grounding conductor of radius $r_0 = 28\ cm$ is buried into homogeneous soil at depth $h = 2\ m$. Current drainage is $I = 150\ A$ (fig. 12.19). A collecting electrode is situated sufficiently far. $\gamma = 1.2\cdot 10^{-2}\ S/m$. Determine the resistance R_g, pointing out what factors it depends on; find the maximum step voltage U_p on the earth surface. $l_p = 0.8\ m$.

Answers: the grounding conductor potential is $\varphi_g = \frac{I}{4\pi\gamma}\left(\frac{1}{r_0} + \frac{1}{2h}\right) = 3820\ V$,

its resistance is $R_g = \frac{1}{4\pi\gamma}\left(\frac{1}{r_0} + \frac{1}{2h}\right) = 25.5\ Ohm$,

the grounding conductor resistance depends on its size, burial depth and the soil conductivity; if $h >> r_0$, then resistance depends mainly on the conductor radius and the soil conductivity.

Step voltage is $U_p = \varphi_a - \varphi_b = \dfrac{I}{2\pi\gamma}\left(\dfrac{1}{h} - \dfrac{1}{\sqrt{h^2 + l_p^2}}\right) = 63.4\ V.$

12-22 (12.27). For the grounding conductor described in problem 12-21, determine its resistance and the step voltage for the following cases:

- current I draining from the conductor is increased by 1.5 times;
- depth h is 1.5 times deeper;
- radius r_0 of a conductor is increased by 1.5 times;
- the soil conductivity γ is increased by 1.5 times because of wetting.

Answers are presented in table 12.1.

Table 12.1

Initial data	$\varphi_g = 3820\ V$	$R_g = 25.5\ Ohm$	$\varphi_a = 3553\ V$	$\varphi_b = 3490\ V$	$U_p = 63\ V$
$I' = 1.5I$	5730	25.5	5329	5234	95
$h' = 1.5h$	3727	24.8	663	643	20
$r_0' = 1.5r_0$	2646	17.6	995	931	64
$\gamma' = 1.5\gamma$	2547	17.0	663	621	42

12-23 (12.28). Current of the grounding conductor described in problem 12.21 is increased by 1.5 times. Determine:

- radius r_0' of a conductor to obtain the same step voltage U_p;
- burial depth h' of a conductor to obtain the same step voltage U_p.

Answers:

- U_p does not depend on the conductor radius, that's why it is not possible to maintain the same step voltage by changing the radius r_0;
- in order to determine the depth h' necessary to maintain the same step voltage $U_p = 63.4\ V$ when the current increases by 1.5 times, let's rewrite the formula for U_p (see problem 12.21) in the following view

$$U_p = \frac{1.5 \cdot 150}{2\pi \cdot 0.012}\left(\frac{1}{h'} - \frac{1}{\sqrt{(h')^2 + 0.8^2}}\right) = 2984 \cdot \left(\frac{1}{h'} - \frac{1}{\sqrt{(h')^2 + 0.8^2}}\right) = 63.4\ V$$

$$\text{or} \quad \frac{1}{h'} - \frac{1}{\sqrt{(h')^2 + 0.8^2}} = 0.0212.$$

Finally, we obtain the equation of the 4th order:

$$(h')^4 - 94.34(h')^3 + 0.64(h')^2 - 60.38h' + 1424 = 0, \qquad h' = 2.41\ m.$$

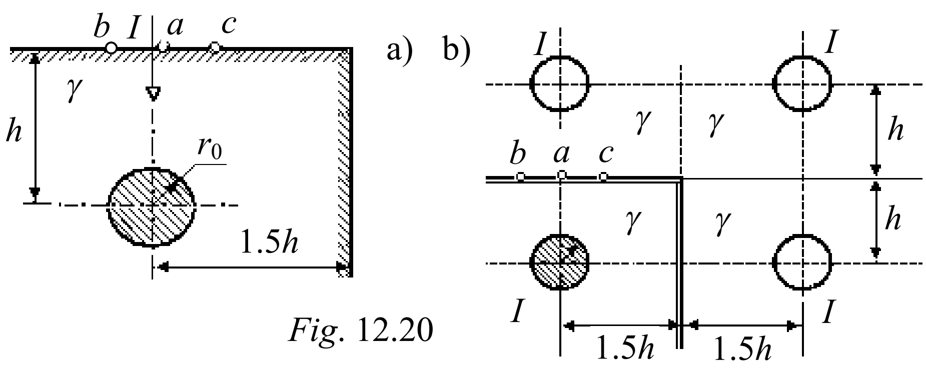


Fig. 12.20

12-24 (12.29). A ball-shaped grounding conductor (fig. 12.20,a) of radius $r_0 = 28$ *cm* is buried at the depth $h = 2$ *m* in soil with conductivity $\gamma = 1.2 \cdot 10^{-2}$ *S*/*m* at distance 1.5*h* from the steep. $I = 150$ *A*.

Determine the conductor resistance R_g, find the step voltage on the soil surface U_p.

Answers: the calculation sketch is in fig. 12.20,b; conductor potential is

$$\varphi_g = \frac{I}{4\pi\gamma r_0} + \frac{I}{4\pi\gamma \cdot 2h} + \frac{I}{4\pi\gamma \cdot 2 \cdot 1.5h} + \frac{I}{4\pi\gamma\sqrt{(2h)^2 + (3h)^2}} = 4140\ V;$$

resistance is $R_g = 27.6$ *Ohm*,

step voltage is $U_{ab} = 95.4\ V$, $U_{ac} = 24.2\ V$.

12-25 (12.30). A grounding conductor in the form of a lengthy pipe is buried into soil vertically (fig. 12.21). The collecting electrode is situated at the distance $H = 100$ *m*.

$I_0 = 10$ *A*/*m*, $d = 10$ *cm*, $\gamma = 0.1$ *S*/*m*.

Determine the conductor resistance R_g as well as the step voltage U_p.

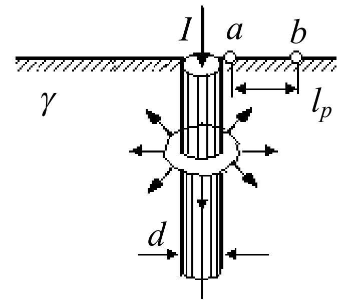


Fig. 12.21

Answers: a conductor in form of a lengthy pipe creates an in-plane field. It is not distorted when interfacing with air. Then, current flows away from a conductor along the radial lines.

The conductor potential and its resistance per unit length are:

$$\varphi_g = \frac{I_0}{2\pi\gamma} \cdot ln\frac{H}{d/2} = 121\ V, \qquad R_g = 12.1\ Ohm \cdot m.$$

The maximum step voltage is $U_p = \varphi_a - \varphi_b = \frac{I_0}{2\pi\gamma} \cdot (ln\frac{H}{d/2} - ln\frac{H}{l_p + d/2}) = 45.1\ V.$

12-26 (12.31). A grounding conductor in the form of a lengthy pipe (fig. 12.22) is buried into soil vertically, not far from the steep. The collecting electrode is situated at the distance $H = 100$ *m*. $I_0 = 10$ *A*/*m*, $d = 10$ *cm*, $\gamma = 0.1$ *S*/*m*.

Determine the conductor resistance R_g as well as the step voltage on the soil surface U_p.

Answers: the conductor potential and its resistance per unit length are: $\varphi_g = \frac{I_0}{2\pi\gamma} \cdot ln\frac{H^2}{hd} = 172\ V$, $R_g = 17.2\ Ohm \cdot m$.

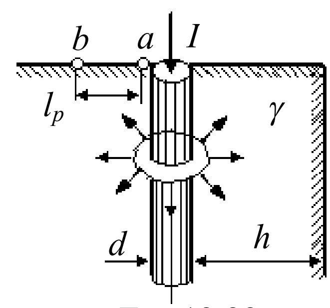


Fig. 12.22

The maximum step voltage is

$$U_p = \varphi_a - \varphi_b = \frac{I_0}{2\pi\gamma} \cdot (ln\frac{H^2}{hd} - ln\frac{H^2}{(l_p + d/2)(2h + l_p + d/2)}) = 48.4\ V.$$

12-27 (12.32). A grounding conductor is made in the form of a cemicylinder and it is half buried into soil (fig. 12.23). The current draining from the conductor is $I_0 = 50$ *A*/*m*. The collecting electrode is situated at distance $H = 100$ *m*.

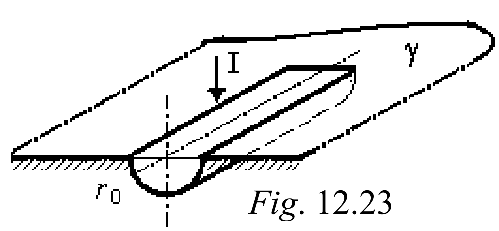


Fig. 12.23

$r_0 = 0.2\ m, \quad \gamma = 0.1\ S/m.$

Determine the conductor resistance R_g, find the step voltage on the soil surface U_p.

Answers: the conductor potential and its resistance per unit length are:

$$\varphi_g = \frac{I_0}{\pi\gamma} \cdot ln\frac{H}{r_0} = 989\ V, \quad R_g = 19.8\ Ohm{\cdot}m.$$

The step voltage is $U_p = \dfrac{I_0}{\pi\gamma} \cdot ln\dfrac{r_0 + l_p}{r_0} = 256\ V.$

12-28 (12.33). A grounding conductor in the form of a pipe is buried into soil horizontally at depth h (fig. 12.24,a). The collecting electrode is situated at distance $H = 100\ m$.

$I_0 = 10\ A/m$, $r_0 = 5\ cm$, $h = 1\ m$, $\gamma = 0.1\ S/m$.

Determine the conductor resistance R_g as well as the step voltage on the soil surface U_p.

Answers: the calculation sketch is in fig. 12.24,b, the conductor potential and its resistance per unit length are: $\varphi_g = \dfrac{I_0}{2\pi\gamma} \cdot [ln\dfrac{H}{r_0} + ln\dfrac{H}{2h}] = 184\ V$, $R_g = 18.4\ Ohm{\cdot}m$;

the step voltage is $U_p = \dfrac{I_0}{\pi\gamma} \cdot ln\dfrac{\sqrt{h^2 + l_p^2}}{h} = 7.88\ V.$

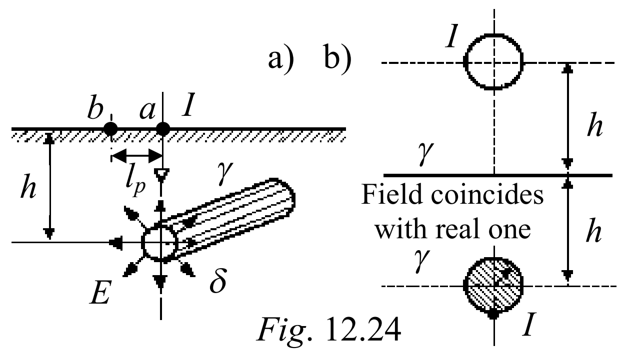


Fig. 12.24

12-29 (12.34). A grounding conductor in the form of a pipe is buried into soil horizontally, not far from the steep (fig. 12.25,a). The collecting electrode is situated at distance $H = 100\ m$. $I_0 = 10\ A/m$, $r_0 = 5\ cm$, $h_1 = 1\ m$, $h_2 = 2\ m$, $\gamma = 0.1\ S/m$.

Determine the conductor resistance R_g as well as the step voltage on the soil surface U_p.

Answers: the sketch for calculation is in fig. 12.25,b,
the conductor potential and its resistance per unit length are:

$$\varphi_g = \frac{I_0}{2\pi\gamma} \cdot [ln\frac{H}{r_0} + ln\frac{H}{2h_1} + ln\frac{H}{2h_2} + ln\frac{H}{\sqrt{(2h_1)^2 + (2h_2)^2}}] = 284\ V, \quad R_g = 28.4\ Ohm{\cdot}m;$$

the step voltage is $U_{p\,ab} = 13.4\ V$, $U_{p\,ac} = 1.3\ V$.

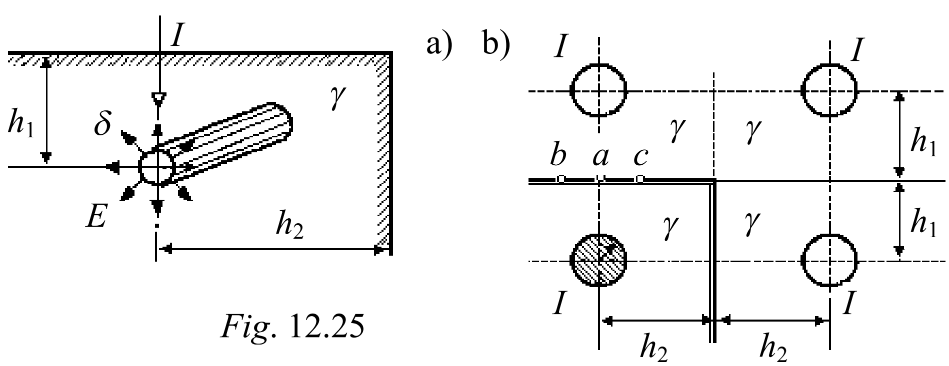


Fig. 12.25

13. MAGNETIC FIELD

13.1. THE FIELD COMPUTATION WITH THE AID OF THE INTEGRAL CORRELATIONS. APPLICATION OF THE SCALAR MAGNETIC POTENTIAL.

13-1 (13.5). Calculate the magnetic field intensity created by a single cylindrical current-carrying conductor I in a homogeneous medium in point A at distance r from the conductor axis (fig. 13.1).

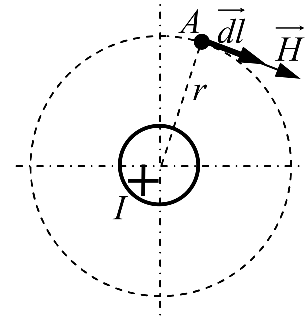


Fig. 13.1

Solution. Let's draw a circle of radius r through point A and apply Ampere's law in an integral form $\oint_L \vec{H}\cdot\vec{dl} = I.$

As it is seen from fig. 13.1, because of symmetry, vectors $\vec{H}$ and $\vec{dl}$ have the same direction in all the points of the circle, the intensity H having one and the same value. So we replace the scalar product of the vectors $\vec{H}$ and $\vec{dl}$ with the product of their absolute values, at the same time we take H out of the integral sign as a constant. We have:

$$\oint_L \vec{H}\cdot\vec{dl} = \oint_L H\cdot dl = H\oint_L dl = H\cdot L = H\cdot 2\pi r = I,$$

from here $$H = \frac{I}{2\pi r}. \qquad (13.1)$$

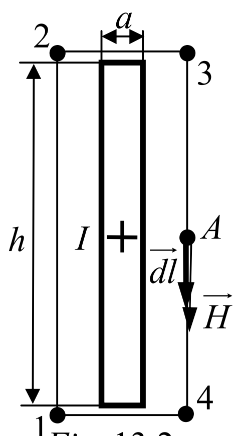


Fig. 13.2

13-2 (13.6). Calculate the intensity of a magnetic field created by a flat bus-bar carrying the current I and having the sizes $a\times h$, $h >> a$ in a homogeneous medium in point A not far from the bus-bar (fig. 13.2).

Solution. Let's draw a rectangular loop 1-2-3-4-1 through point A around the bus-bar and apply Ampere's law in an integral form:

$$\oint_L \vec{H}\cdot\vec{dl} = I.$$

The regular magnetic field is created in points of segments 1-2 and 3-4 situated closely to the bus-bar surface. As the bus-bar height is much bigger than its thickness, the result of the integration along sides 1-4 and 2-3 can be neglected. In the points of sides 1-2 and 3-4 the directions of vectors $\vec{H}$ and $\vec{dl}$ coincide, the intensity H having one and the same value. Then we replace the scalar product of vectors $\vec{H}$ and $\vec{dl}$ with the product of their absolute values, at the same time we take H out of the integral sign as a constant. We have:

$$\oint_L \vec{H}\cdot\vec{dl} = H\cdot l_{12} + H\cdot l_{34} = H\cdot 2h = I; \quad H = \frac{I}{2h}. \qquad (13.2)$$

13-3 (13.7). Calculate the magnetic flux induced by a single conductor in a homogeneous medium through the rectangular frame AB of length l, long sides of which are parallel to the conductor (fig. 13.3).

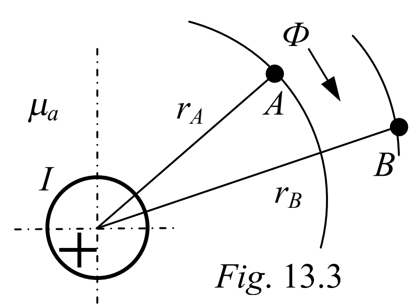


Fig. 13.3

Solution. Magnetic flux Φ through the frame AB is determined by the flux through the field tube restricted by circles of radiuses r_A and r_B: $\Phi = \int_{r_A}^{r_B} B(r)\cdot l\cdot dr$.

However, in accordance with (13.1) $H=\frac{I}{2\pi r}$; $B(r)=\mu_a \cdot H=\mu_a \cdot \frac{I}{2\pi r}$. Thus,

$$\Phi=\frac{\mu_a I \cdot l}{2\pi} ln\frac{r_B}{r_A}. \qquad (13.3)$$

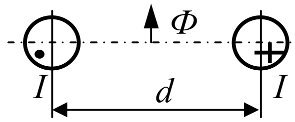


Fig. 13.4

Note. If start turning the frame around its axis, the maximum flux flows through the frame when it is situated perpendicular to the force line (along the radial lines).

13-4 (13.8). Calculate the external inductance of a two-wire line (fig. 13.4). The wire radius and the distance between the wires are $r_0 = 1\ cm$, $d = 1\ m$.

Solution. The external inductance is caused by the magnetic flux between the wires. Since one and the same current flows through wires, the magnetic fluxes induced by wires are identical. The flux of one conductor is in accordance with (13.3)

$$\Phi=\frac{\mu_0 I \cdot l}{2\pi} ln\frac{d-r_0}{r_0}.$$

The required external inductance per 1 *m* of the line length is:

$$L=\frac{2\Phi}{I}=\frac{\mu_0 l}{\pi} ln\frac{d-r_0}{r_0}=\frac{4\pi\cdot 10^{-7}\cdot 1}{\pi} ln\frac{100-1}{1}=1.84\cdot 10^{-6}\ H=1.84\ \mu H.$$

13-5 (13.9). Calculate the field and inductance of a coaxial cable (fig. 13.5). Current *I* flows through the cable.

Solution. There is no magnetic field outside of cable as $\Sigma I=0$. Thus, it is possible to set three different areas (denoted with Roman numerals in fig. 13.5,a) with magnetic permeability μ_{a1}, μ_{a2}, μ_{a3}.

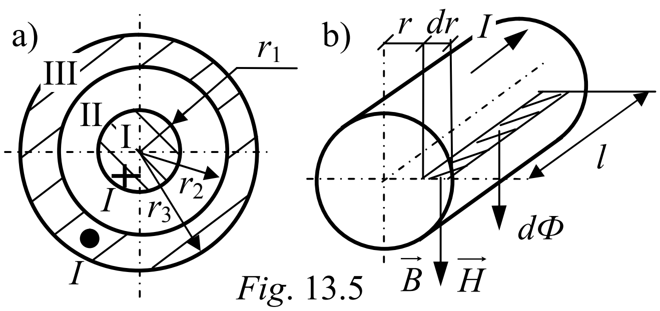


Fig. 13.5

In order to calculate the field we use Ampere's law $\oint_L \vec{H}\cdot\vec{dl}=\Sigma I.$

1. Area I – $0 < r < r_1$ (fig. 13.5,b). $\oint_L \vec{H}\cdot\vec{dl}=H\cdot 2\pi r;$

$$\Sigma I=I\frac{\pi r^2}{\pi r_1^2};\quad H=\frac{I}{2\pi r_1^2}\cdot r;\quad B=\mu_{a1}\cdot H=\frac{\mu_{a1}I}{2\pi r_1^2}\cdot r;\quad d\Phi=B\cdot dS=\frac{\mu_{a1}I}{2\pi r_1^2}\cdot r\cdot l\cdot dr.$$

Magnetic flux $d\Phi$ inside the inner conductor links only with the part of current *I*, which is proportional to ratio $\frac{r^2}{r_1^2}$; that's why the magnetic flux linkage is $d\Psi=d\Phi\cdot\frac{r^2}{r_1^2}$.

The internal inductance of the 1st area is calculated by the formula

$$L_1=\frac{\Psi}{I}=\frac{\int_0^{r_1} d\Psi(r)}{I}=\frac{\mu_{a1}l}{2\pi r_1^2}\cdot\frac{1}{r_1^2}\int_0^{r_1} r^3 dr=\frac{\mu_{a1}l}{2\pi r_1^4}\cdot\frac{r^4}{4}\Big|_0^{r_1}=\frac{\mu_{a1}l}{8\pi}$$

and it does not depend on the conductor radius.

2. Area II – $r_1 < r < r_2$.

$$\Sigma I = I; \quad H = \frac{I}{2\pi r}; \quad B = \frac{\mu_{a2} I}{2\pi r}; \quad d\Psi = d\Phi = B \cdot dS = \frac{\mu_{a2} I}{2\pi r} \cdot l \cdot \mathrm{dr}.$$

The external inductance is $L_2 = \frac{\int_{r_1}^{r_2} d\Psi(r)}{I} = \frac{\mu_{a2} l}{2\pi} ln \frac{r_2}{r_1}$.

3. Area III – $r_2 < r < r_3$.

$$\Sigma I = I - I\frac{\pi(r^2 - r_2^2)}{\pi(r_3^2 - r_2^2)}; \quad H = \frac{I}{2\pi r}\frac{r_3^2 - r^2}{r_3^2 - r_2^2}; \quad B = \frac{\mu_{a3} I}{2\pi r}\frac{r_3^2 - r^2}{r_3^2 - r_2^2}; \quad d\Phi = \frac{\mu_{a3} I}{2\pi r}\frac{r_3^2 - r^2}{r_3^2 - r_2^2} \cdot l \cdot dr.$$

This flux links with current I and with part of the reverse current which is equal to $I\frac{r^2 - r_2^2}{r_3^2 - r_2^2}$.

That's why the elementary flux linkage is $d\Psi = d\Phi \cdot \left(1 - \frac{r^2 - r_2^2}{r_3^2 - r_2^2}\right) = d\Phi \cdot \frac{r_3^2 - r^2}{r_3^2 - r_2^2}$.

The internal inductance of the 3rd area is:

$$L_3 = \frac{\int_{r_2}^{r_3} d\Psi(r)}{I} = \frac{\mu_{a3} l}{2\pi(r_3^2 - r_2^2)^2}\int_{r_2}^{r_3}\frac{(r_3^2 - r^2)^2}{r} dr =$$

$$= \frac{\mu_{a3} l}{2\pi(r_3^2 - r_2^2)^2} \cdot [\int_{r_2}^{r_3}\frac{r_3^4}{r} dr - 2r_3^2 \int_{r_2}^{r_3} r \cdot dr + \int_{r_2}^{r_3} r^3 \cdot dr] =$$

$$= \frac{\mu_{a3} l}{2\pi(r_3^2 - r_2^2)^2} \cdot [r_3^4 ln\frac{r_3}{r_2} - r_3^2(r_3^2 - r_2^2) + \frac{1}{4}(r_3^4 - r_2^4)] =$$

$$= \frac{\mu_{a3} l}{2\pi}[\frac{r_3^4}{(r_3^2 - r_2^2)^2} ln\frac{r_3}{r_2} - \frac{3r_3^2 - r_2^2}{4(r_3^2 - r_2^2)}].$$

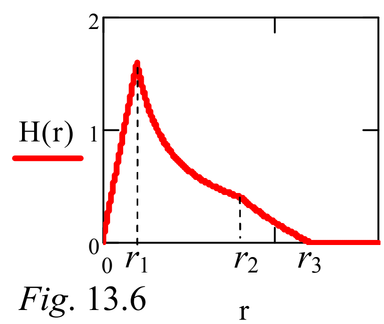


Fig. 13.6

The external cable inductance is – $L_e = L_2$; its internal inductance is – $L_i = L_1 + L_3$; the total inductance is – $L = L_1 + L_2 + L_3$.

The dependence diagram $H(r)$ is presented in fig. 13.6.

The second way of coaxial cable inductance calculation.

1. Area I – $0 < r < r_1$. $\quad H = \frac{I}{2\pi r_1^2} \cdot r$.

The energy of the elementary layer dr (fig. 13.5,b) at the distance r from axis is:

$$dW = \frac{B \cdot H}{2} dV = \mu_{a1}\frac{H^2}{2} \cdot 2\pi r l \cdot dr = \mu_{a1}\frac{I^2 l}{4\pi^2 r_1^4 \cdot 2} r^2 \cdot 2\pi r dr = \mu_{a1}\frac{I^2 l}{4\pi r_1^4} r^3 \cdot dr.$$

$$L_1 = \frac{2W_1}{I^2} = \frac{2}{I^2}\int_0^{W_1} dW = \frac{2}{I^2}\mu_{a1}\frac{I^2 l}{4\pi r_1^4}\int_0^{r_1} r^3 dr = \frac{\mu_{a1} l}{2\pi r_1^4}\frac{r_1^4 - 0^4}{4} = \frac{\mu_{a1} l}{8\pi}.$$

2. Area II – $r_1 < r < r_2$.

$$\Sigma I = I; \qquad H = \frac{I}{2\pi r}; \qquad dW = \mu_{a2}\frac{I^2 l}{4\pi^2 r^2 \cdot 2}2\pi r\cdot dr = \mu_{a2}\frac{I^2 l}{4\pi}\frac{dr}{r}.$$

$$L_2 = \frac{2}{I^2}\mu_{a2}\frac{I^2 l}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r} = \frac{\mu_{a2} l}{2\pi}\ln\frac{r_2}{r_1}.$$

3. Area III – $r_2 < r < r_3$. $H = \frac{I}{2\pi r}\frac{r_3^2 - r^2}{r_3^2 - r_2^2}$;

$$dW = \mu_{a3}\frac{I^2 l}{4\pi^2 r^2 \cdot 2}\frac{(r_3^2 - r^2)^2}{(r_3^2 - r_2^2)^2}2\pi r\cdot dr = \mu_{a3}\frac{I^2 l}{4\pi(r_3^2 - r_2^2)^2}\frac{(r_3^2 - r^2)^2}{r}dr.$$

$$L_3 = \frac{2}{I^2}\mu_{a3}\frac{I^2 l}{4\pi(r_3^2 - r_2^2)^2}[\int_{r_2}^{r_3}\frac{r_3^4}{r}dr - 2r_3^2\int_{r_2}^{r_3} r\cdot dr + \int_{r_2}^{r_3} r^3\cdot dr] =$$

$$= \frac{\mu_{a3} l}{2\pi(r_3^2 - r_2^2)^2}\cdot[r_3^4 \ln\frac{r_3}{r_2} - r_3^2(r_3^2 - r_2^2) + \frac{1}{4}(r_3^4 - r_2^4)].$$

13-6 (13.10). Compute the internal inductance of a rectangular bus-bar (fig. 13.7).

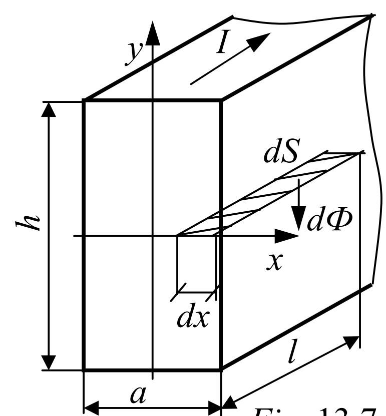


Fig. 13.7

Solution. The current density in a bus-bar is – $\delta = I/(a\cdot h)$. In accordance with Ampere's law $H\cdot 2h = \delta\cdot 2xh$, from here $H = \delta x$, it means the intensity H inside the bus-bar depends on x by a linear law. On the bus-bar surface

$$H = \delta\cdot ½a = \frac{I}{ah}\frac{a}{2} = \frac{I}{2h}.$$

The induction inside the bus-bar is $B = \mu_a H = \mu_0\mu\delta x$. The magnetic flux through the cross-section $dS = l\cdot dx$ is $d\Phi = B\cdot dS = \mu_0\mu\delta x l\cdot dx$. This magnetic flux links only with the part of current I, which is proportional to $\frac{hx}{ha/2} = \frac{2x}{a}$, that's why the elementary flux linkage is $d\Psi = d\Phi\frac{2x}{a}$. The internal inductance is

$$L_i = \frac{\Psi}{I} = \frac{\int_0^{a/2} d\Psi}{I} = \frac{\frac{2}{a}\mu_0\mu\delta l\int_0^{a/2} x^2 dx}{\delta a h} = \frac{2\mu_0\mu l}{3ha^2}x^3\Big|_0^{a/2} = \frac{\mu_0\mu l a}{12h}.$$

The internal inductance does not depend on the bus-bar size if the proportion h/a is kept.

If there are two bus-bars with oppositely directed currents (fig. 13.8,a), the bus-bars happen to be connected in series and their internal inductance is

$$L_{i\Sigma} = 2\cdot L_i = \frac{\mu_0\mu l a}{6h}.$$

But if currents directions coincide (fig. 13.8,b), the bus-bars are connected in parallel and then

$$L_{i\Sigma} = ½\cdot L_i = \frac{\mu_0\mu l a}{24h}.$$

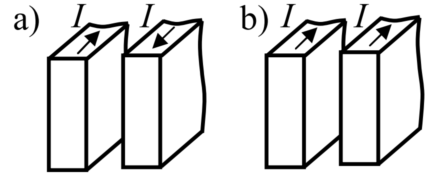


Fig. 13.8

13-7 (13.11). Current I flows through a metallic hollow pipe placed in air; the magnetic permeability of the pipe material is μ_r (see fig. 13.9). Calculate the magnetic field inside, outside the pipe and in the pipe body.

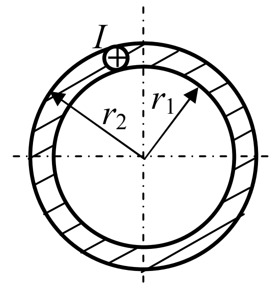


Fig. 13.9

Solution. Ampere's law in an integral form is used for calculation. A circle of radius r is assumed as an integration loop, the circle centre coincides with the pipe axis. For the area inside the pipe ($0 \le r \le r_1$), total current is zero, that's why here both the intensity H and the induction B are equal to zero.

In the pipes body ($r_1 \le r \le r_2$) $\oint_L \vec{H} \cdot \vec{dl} = \oint_L H \cdot dl = H \oint_L dl = H \cdot 2\pi r$,

while the total current is equal to $\delta \cdot S$, where δ – current density in the pipe body, S – the ring area with radiuses r_1 and r.

Since $\delta = \dfrac{I}{\pi(r_2^2 - r_1^2)}$; $S = \pi(r^2 - r_1^2)$, then

$$H = \frac{\delta \cdot S}{2\pi r} = \frac{I(r^2 - r_1^2)}{2\pi r(r_2^2 - r_1^2)} \quad \text{and} \quad B = \mu_0\mu_r H = \frac{\mu_0\mu_r I(r^2 - r_1^2)}{2\pi r(r_2^2 - r_1^2)}.$$

Outside the pipe ($r_2 \le r \le \infty$) $\oint_L \vec{H} \cdot \vec{dl} = H \cdot 2\pi r$, while the total current is equal to I,

accordingly, $H = \dfrac{I}{2\pi r}$ and $B = \dfrac{\mu_0\mu_r I}{2\pi r}$.

13-8 (13.12). A plane triangular frame (fig. 13.10) the number of turns in which is $w = 400$ is placed in a medium with permeability $\mu = 10$ in the same plane with a lengthy circular conductor of radius r_0. Sizes: $a = 10$ *cm*, $b = 20$ *cm*, $c = 15$ *cm*. Determine the mutual inductance of the conductor and the frame.

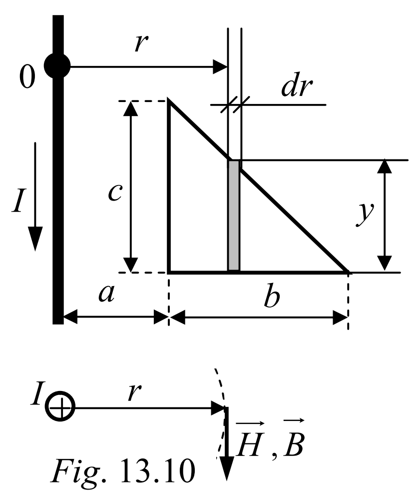


Fig. 13.10

Solution. By definition $M_{12} = \dfrac{\Psi_{12}}{I_1} = M_{21} = \dfrac{\Psi_{21}}{I_2} = M$. In this case it is more convenient to set a current I through the conductor and calculate the magnetic flux through frame.

Let's choose an elementary area in the triangle $dS = y \cdot dr$ (hatched strip in the diagram) at distance r from the conductor and obtain the formula for magnetic flux through it:
$$y = k \cdot r + d; \quad k = -\frac{c}{b}; \quad d = \frac{c}{b}(a + b);$$

$$H = \frac{I}{2\pi r}; \quad B = \frac{\mu\mu_0 I}{2\pi r}; \quad d\Phi = B \cdot dS = B \cdot y \cdot dr = \frac{\mu\mu_0 I}{2\pi r}\left(-\frac{c}{b}r + \frac{c}{b}(a + b)\right)dr.$$

$$\Phi = \int_{r=a}^{r=a+b} d\Phi = \frac{\mu\mu_0 Ic}{2\pi b}\int_a^{a+b}(-r + a + b)\frac{dr}{r} =$$

$$= \frac{\mu\mu_0 Ic}{2\pi b}[-r + (a + b)ln(r)]\Bigg|_a^{a+b} = \frac{\mu\mu_0 Ic}{2\pi b}[-b + (a + b)ln\frac{a + b}{a}];$$

$$\Psi = w\cdot\Phi; \quad M = \frac{\Psi}{I} = \frac{\mu\mu_0 wc}{2\pi b}[-b + (a+b)ln\frac{a+b}{a}] = 77.75\ \mu H.$$

13-9 (13.13). The energy of the magnetic field between wires of a two-wire overhead line per unit length is $W = 6\cdot 10^{-3}\ J/m$ (fig. 13.11,a); furthermore, $r_0 = 3\ mm$; $d = 40\ cm$.

It is required to: a) determine the current through the line wires, plot the variation of the magnetic induction between wires; b) compute the mutual inductance of the line and a rectangular frame with the number of turns $w = 200$, which is in the same plane with a line, furthermore, $a = 10\ cm$; $b = 20\ cm$; $c = 30\ cm$.

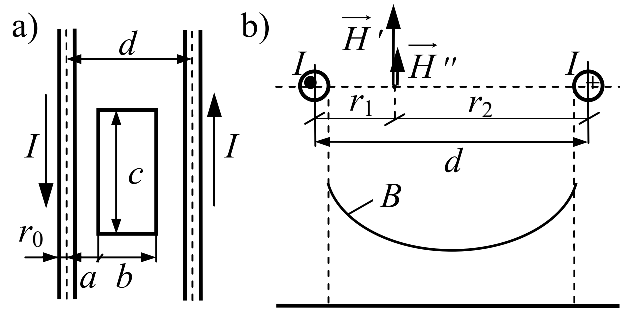


Fig. 13.11

Solution. The energy of the magnetic field between the line wires is $W = \frac{L_0 I^2}{2}$, where L_0 – external inductance of the unit length.

Suppose, the current I flows through the line wires as it is shown in fig. 13.11,b. Under Ampere's law with application of the superposition principle, we determine the components of the magnetic field intensity H' and H'', their direction being determined under Ampere's rule (right-hand screw rule), they being perpendicular to the plane of two-wire line. Then we determine the magnetic induction and magnetic flux between the wires which is external with respect to wires. For that, we mark the distance from the axis of the left wire to the point under consideration as r_1, and that from the axis of the right wire – as r_2. Then in accordance with (13.1) and (13.3), we have

$$H' = \frac{I}{2\pi r_1}; \quad H'' = \frac{I}{2\pi r_2}; \quad B' = \frac{\mu_0 I}{2\pi r_1}; \quad B'' = \frac{\mu_0 I}{2\pi r_2} = \frac{\mu_0 I}{2\pi(d - r_1)};$$

$$\Phi' = \Phi'' = \frac{\mu_0 I\cdot l}{2\pi}ln\frac{d - r_0}{r_0}; \quad \Phi = \Phi' + \Phi'' = \frac{\mu_0 I\cdot l}{\pi}ln\frac{d - r_0}{r_0}.$$

The graph of the magnetic induction variation between wires $B = B' + B''$ is presented in its approximate view in fig. 13.11,b.

In this case $d >> r_0$, that's why it may be adequately assumed that $\Phi = \frac{\mu_0 I\cdot l}{\pi}ln\frac{d}{r_0}$, while the external inductance of the unit length line is

$$L_0 = \frac{\Phi}{I\cdot l} = \frac{\mu_0}{\pi}ln\frac{d}{r_0} = \frac{4\pi\cdot 10^{-7}}{\pi}ln\frac{0.4}{0.003} = 19.57\cdot 10^{-7}\ H/m.$$

Then the line current is $I = \sqrt{\frac{2W}{L_0}} = 78.31\ A.$

The mutual inductance between the line and frame is $M = \frac{w\cdot\Phi_{12}}{I}$, where Φ_{12} is a magnetic flux linked with the frame turns. We determine it under the formula (13.3):

$$\Phi_{12}' = \frac{\mu_0 I \cdot c}{2\pi} ln\frac{a+b}{a}; \qquad \Phi_{12}'' = \frac{\mu_0 I \cdot c}{2\pi} ln\frac{d-a}{d-a-b};$$

$$\Phi_{12} = \Phi_{12}' + \Phi_{12}'' = \frac{\mu_0 I \cdot c}{2\pi}\left(ln\frac{a+b}{a} + ln\frac{d-a}{d-a-b}\right) = \frac{\mu_0 I \cdot c}{2\pi} ln\frac{(a+b)(d-a)}{a \cdot (d-a-b)}.$$

Then mutual inductance is

$$M = \frac{\mu_0 \cdot w \cdot c}{2\pi} ln\frac{(a+b)(d-a)}{a \cdot (d-a-b)} = 2.637 \cdot 10^{-5}\ H.$$

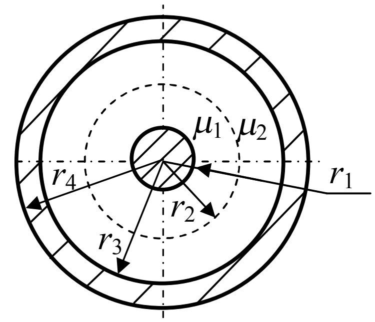


Fig. 13.12

13-10 (13.16). Direct current $I = 80\ A$ flows through a conductor of a two-layer coaxial cable (fig. 13.12). $\mu_1 = 3$, $\mu_2 = 8$, $r_1 = 3\ mm$, $r_2 = 8\ mm$, $r_3 = 15\ mm$, $r_4 = 18\ mm$. Both the conductor and sheath are made of non-magnetic material.

Plot the magnetic induction variation inside the cable. Compute the external inductance of the cable $l = 25\ m$ long, find the energy stored in the magnetic field.

Answer: $$B(r) = \begin{cases} \dfrac{\mu_0 I r}{2\pi r_1^2} = 1.778 \cdot r\ T\ if\ 0 \le r \le r_1, \\ \dfrac{\mu_1 \mu_0 I}{2\pi r} = \dfrac{4.8 \cdot 10^{-5}}{r}\ T\ if\ r_1 \le r \le r_2, \\ \dfrac{\mu_2 \mu_0 I}{2\pi r} = \dfrac{1.28 \cdot 10^{-4}}{r}\ T\ if\ r_2 \le r \le r_3, \\ \dfrac{\mu_0 I (r_4^2 - r^2)}{2\pi r (r_4^2 - r_3^2)} = \dfrac{0.162 \cdot (3.24 \cdot 10^{-4} - r^2)}{r}\ T\ if\ r_3 \le r \le r_4, \\ 0\ if\ r > r_4. \end{cases} \qquad r\ [m];$$

graph $B(r)$ is presented in fig. 13.13;
$L_e = 4 \cdot 10^{-5}\ H$; $W = 0.128\ J$.

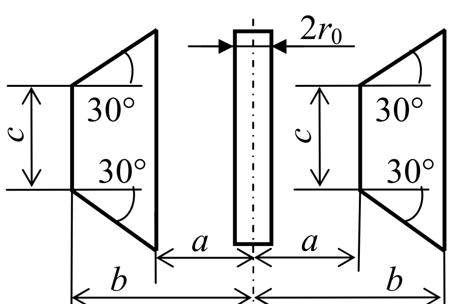


Fig. 13.14

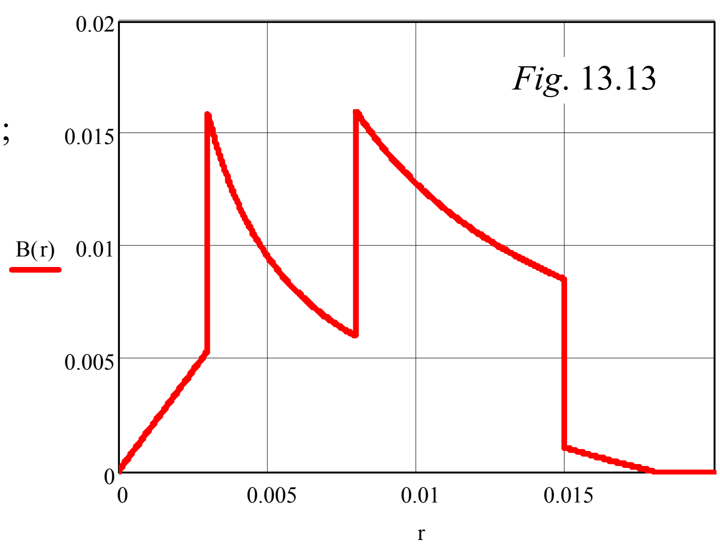


Fig. 13.13

13-11 (13.15). A straight conductor and two frames with the number of turns $w = 300$ are in air in one plane (fig. 13.14). Sizes: $r_0 = 3\ mm$, $a = 5\ cm$, $b = 15\ cm$, $c = 30\ cm$.

1) Compute the mutual inductance between the conductor and the right frame.
2) Compute the mutual inductance between the conductor and the left frame.

Answers: 1) $h(r) = 2 \cdot (0.577r + 0.1211)\ m$,

$dS = dr \cdot h(r)$, $B = \mu_0 I/(2\pi r)$, $d\Phi = B \cdot dS = I \cdot \dfrac{\mu_0}{\pi} \cdot (0.577r + 0.1211) \cdot \dfrac{dr}{r}$,

$\Phi = \int_{r=a}^{r=b} d\Phi = I \cdot 7.63 \cdot 10^{-8}\ Wb$, $M = \dfrac{w \cdot \Phi}{I} = 22.89\ \mu H.$

2) $h(r) = 2 \cdot (-\,0.577r + 0.2366)\ m$, $d\Phi = B \cdot dS = I \cdot \dfrac{\mu_0}{\pi} \cdot (-\,0.577r + 0.2366) \cdot \dfrac{dr}{r}$,

$\Phi = \int_{r=a}^{r=b} d\Phi = I \cdot 8089 \cdot 10^{-8}\ Wb$, $M = \dfrac{w \cdot \Phi}{I} = 24.27\ \mu H.$

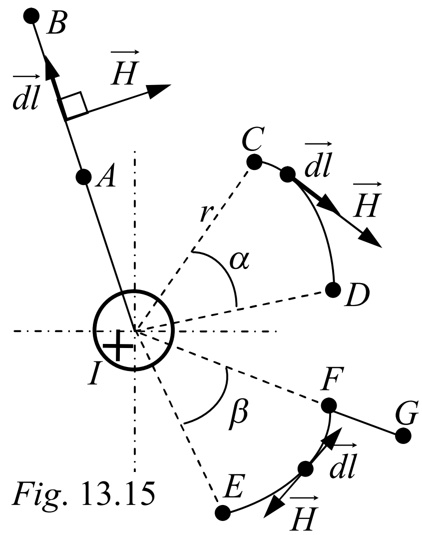


Fig. 13.15

13-12 (13.17). Compute the magnetic voltages U_{mAB}, U_{mCD}, U_{mEG} of the field induced by a single current-carrying conductor I in a homogeneous medium with magnetic permeability μ_a (fig. 13.15).

Solution. The magnetic voltage between points A and B which are on the radial line is $U_{MAB} = \int_A^B \vec{H} \cdot \vec{dl}$.

In all the points of segment AB, the angle between the vectors $\vec{H}$ and $\vec{dl}$ is equal to 90°. That's why their scalar product is equal to zero, the voltage $U_{mAB} = \varphi_A - \varphi_B = 0$ is zero too. Accordingly, scalar magnetic potentials φ_A and φ_B are equal to each other, hence, radial line is an *equipotential line*.

The magnetic voltage between points C and D which are on the arc is $U_{mCD} = \int_C^D \vec{H} \cdot \vec{dl}$.

In all the points of the arc CD the angle between vectors $\vec{H}$ and $\vec{dl}$ is 0°, their scalar product may be replaced with the product of their absolute values. The value of intensity H is the same – $H = \dfrac{I}{2\pi r}$, so, it may be taken out of the integral sign, while the integral of dl gives the arc length CD:

$$U_{mCD} = H \cdot \int_C^D dl = H \cdot l_{CD} = \frac{I}{2\pi r} \cdot r \cdot \alpha = \frac{I}{2\pi} \cdot \alpha, \quad \alpha\ [rad].$$

The magnetic voltage between the arbitrary points E and G is $U_{mEG} = \int_E^G \vec{H} \cdot \vec{dl}$. In the areas free from conductors, the magnetic field is a potential one, so, the magnetic voltage does not depend on the integration paths if they do not form the loops coupled with currents. That's why it is reasonable to choose such an integration path EG so that the integral evaluation is simple – along the radial lines and a circumference. Thus,

$$U_{mEG} = U_{mEF} + U_{mFG} = -\frac{I}{2\pi} \cdot \beta + 0 = -\frac{I}{2\pi} \cdot \beta.$$

Conclusions. 1. The points lying on the line radial to a conductor have one and the same potential, i.e. a radial line is equipotential one.

2. Magnetic voltage depends on the angle between points and does not depend on the distance from the points to a conductor. The voltage for the arbitrary curve LN is calculated under formula

$$U_{mLN} = \pm\frac{I}{2\pi}\cdot\gamma; \qquad (13.4)$$

here γ – the angle in radians, at which the arc LN can be seen from the conductor centre.

3. If the integration is along the forced lines, magnetic voltage is positive, otherwise it is negative.

13-13 (13.18). The current $I = 360\,A$ flows through a conductor of a coaxial cable (fig. 13.16). Determine the magnetic voltage between points A and B, if $\alpha = 30°$.

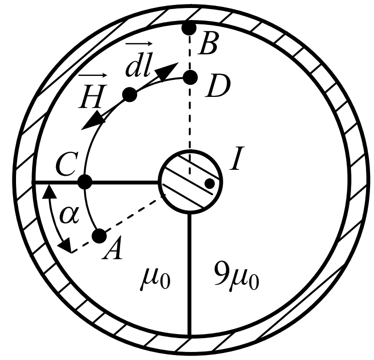


Fig. 13.16

Solution. The magnetic voltage between points A and B is determined under the formula $U_{mAB} = \int_A^B \vec{H}\vec{dl}$. Let a path of integration be A-C-D-B (fig. 13.16), taking into account that the intensities in the different magnetic layers are different. Then

$$U_{mAB} = \int_A^B \vec{H}\vec{dl} = \int_A^C \vec{H}\vec{dl} + \int_C^D \vec{H}\vec{dl} + \int_D^B \vec{H}\vec{dl} = -H_1\cdot l_{AC} - H_2\cdot l_{CD},$$

where $\int_D^B \vec{H}\vec{dl} = 0$ because of the perpendicular vectors $\vec{H_2}$ and $\vec{dl}$;

$\vec{H_1}$ and $\vec{H_2}$ – intensities in the areas with μ_0 and $9\mu_0$, respectively;

voltages $H\cdot l$ are taken with minus, because the directions of vectors $\vec{H}$ and $\vec{dl}$ are opposite.

We determine the quantities H_1 and H_2 applying Ampere's law

$$\oint_L \vec{H}\cdot\vec{dl} = H_1\cdot\frac{\pi\cdot r_A}{2} + H_2\cdot\frac{3\pi\cdot r_A}{2} = \frac{\pi\cdot r_A}{2}(H_1 + 3H_2) = I.$$

As the normal component of vector $\vec{B}$ is continuous on the interface of two mediums, we have $\mu_0 H_1 = 9\mu_0 H_2$.

Solving two latter equations simultaneously, we get $H_1 = \dfrac{3I}{2\pi r_A}$; $H_2 = \dfrac{I}{6\pi r_A}$.

As $l_{AC} = r_A\cdot\alpha$, where α is expressed in *radians* and $l_{CD} = r_A\cdot\dfrac{\pi}{2}$, we get

$$U_{mAB} = -H_1\cdot l_{AC} - H_2\cdot l_{CD} = -\frac{3I}{2\pi r_A}\cdot r_A\cdot\frac{\pi}{6} - \frac{I}{6\pi r_A}\cdot r_A\cdot\frac{\pi}{2} = \frac{-I}{3} = -120\,A.$$

13-14 (13.19). A single current-carrying conductor $I = 10\,A$ is placed on the interface of two mediums (fig. 13.17): $\mu_1 = 2$, $\mu_2 = 4$, $\mu_3 = 6$. The point coordinates are $x_A = y_A = 10\,cm$, $x_B = -5\,cm$, $y_B = -15\,cm$. It is required:

1. Compute the magnetic field intensity in points A and B as well as the magnetic voltage between them.

2. Supposing A and B are the intersection points of sides of the rectangular frame $l = 1$ *m* long and the number of turns $w = 100$, find the magnetic flux through the frame and the mutual inductance between the conductor and the frame.

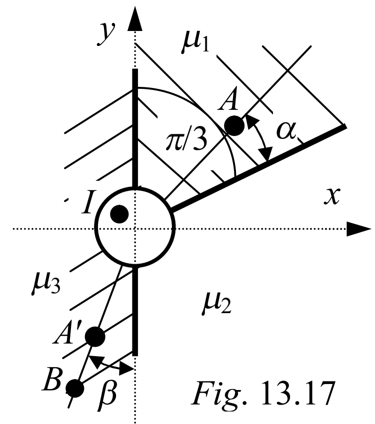


Fig. 13.17

Solution. Let's calculate the distances from the conductor centre to the points A and B:

$$r_A = 10\sqrt{2} = 14.14\ cm,\quad r_B = \sqrt{5^2 + 15^2} = 15.81\ cm.$$

In order to compute the magnetic field intensity in point A, we draw a circle through it, the centre of which coincides with the conductor axis. On the grounds of Ampere's law, one may write down: $H_1 \cdot \frac{\pi}{3} r_A + H_2 \cdot \frac{2\pi}{3} r_A + H_3 \cdot \pi \cdot r_A = I,$

where H_1, H_2, H_3 – the field intensity in the medium with relative permeability μ_1, μ_2, μ_3, respectively.

The interface is radially situated, that's why on the interface the vectors $\vec{B}$ and $\vec{H}$ have but a normal component. Boundary condition is $B_{1n} = B_{2n} = B_{3n}$, from here $\mu_1 \cdot H_1 = \mu_2 \cdot H_2 = \mu_3 \cdot H_3$.

Thus, $H_1 \cdot (\frac{\pi}{3} r_A + \frac{\mu_1}{\mu_2} \cdot \frac{2\pi}{3} r_A + \frac{\mu_1}{\mu_3} \cdot \pi r_A) = I$, from here

$$H_A = H_1 = I/(\frac{\pi}{3} r_A + \frac{\mu_1}{\mu_2} \cdot \frac{2\pi}{3} r_A + \frac{\mu_1}{\mu_3} \cdot \pi r_A) = 10 \cdot 3/((1 + 0.5 \cdot 2 + ⅓ \cdot 3) \cdot \pi \cdot 0.1414) = 22.51\ A/m.$$

Similarly, the intensity in point B is:

$$H_B = H_3 = \frac{\mu_1}{\mu_3} H_1 = \frac{\mu_1}{\mu_3} I/(\frac{\pi}{3} r_B + \frac{\mu_1}{\mu_2} \cdot \frac{2\pi}{3} r_B + \frac{\mu_1}{\mu_3} \cdot \pi r_B) =$$

$$= 10/((1 + 0.5 \cdot 2 + ⅓ \cdot 3) \cdot \pi \cdot 0.1581) = 6.71\ A/m.$$

The magnetic voltage between points A and B in clockwise direction (opposite to the force lines) may be calculated under the formula (see problem 13.13):

$$U_{mAB} = -H_1 \cdot r_A \cdot \alpha - H_2 \cdot r_A \cdot \frac{2\pi}{3} - H_3 \cdot r_A \cdot \beta;$$

$$H_1 \cdot r_A = 30/(3\pi),\quad H_2 \cdot r_A = 15/(3\pi),\quad H_3 \cdot r_A = 10/(3\pi),$$

$$\alpha = \frac{\pi}{3} - arctg\frac{x_A}{y_A} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}\ rad,\quad \beta = arctg\frac{x_B}{y_B} = arctg\frac{5}{15} = 0.3218\ rad.$$

Thus, $U_{mAB} = -\frac{30}{3\pi}\frac{\pi}{12} - \frac{15}{3\pi}\frac{2\pi}{3} - \frac{10}{3\pi} \cdot 0.3218 = -5.24\ A.$

The flux tubes have the form of a ring around the conductor, that's why the magnetic flux through cross-section AB is equal to the magnetic flux through the cross-section $A'B$, where A' is the point at a distance r_A from the conductor centre but within the medium with μ_3. The magnetic flux value in accordance with (13.3) is $\Phi = \frac{\mu_a I_3 \cdot l}{2\pi} ln\frac{r_B}{r_A}$, where $I_3 = H_3 \cdot r_A \cdot 2\pi$, $\mu_a = \mu_3 \cdot \mu_0$.

Thus, $\Phi = \mu_3 \cdot \mu_0 \cdot H_3 \cdot r_A \cdot l \cdot ln\frac{r_B}{r_A} = 6 \cdot 4\pi \cdot 10^{-7} \cdot \frac{10}{3\pi} \cdot 1 \cdot ln\frac{15.81}{14.14} = 0.893 \cdot 10^{-6}\ Wb.$

The mutual inductance is $M = \frac{w \cdot \Phi}{I} = \frac{100 \cdot 0.893 \cdot 10^{-6}}{10} = 8.93 \cdot 10^{-6}\ Гн = 8.93\ \mu H.$

13-15 (13.20). Direct current $I = 80\ A$ flows along a two-wire line (fig. 14.22): $r_0 = 2\ mm$, $d = 30\ cm$.

Compute the line field, find the energy stored in the line $l = 1.5\ m$ long. Find the magnetic voltage between the points

A[40 cm; –10 cm] and B[10 cm; 10 cm].

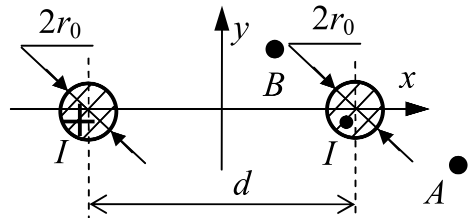


Fig. 13.18

Solution. The line field is calculated by the superposition method. Then, the magnetic field intensity in any point is determined as $\vec{H} = \vec{H}' + \vec{H}''$, where $\vec{H}'$ is a component induced by the left conductor, while $\vec{H}''$ is induced by the right conductor.

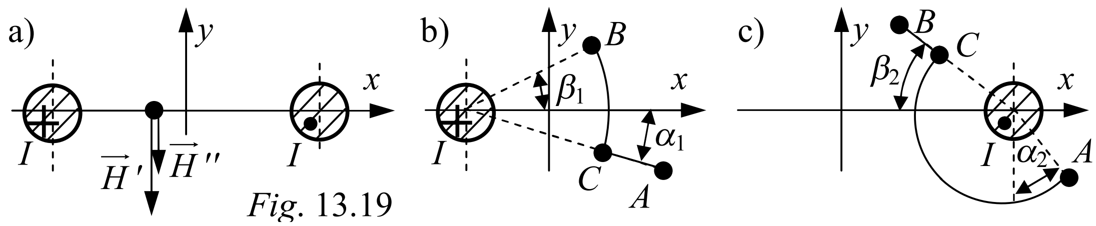


Fig. 13.19

In the points lying on axis x, the field intensity is determined easily as the components $\vec{H}'$ and $\vec{H}''$ have the same direction (fig. 13.19,a), and every component can be calculated with the aid of Ampere's law. Thus, in the space between the wires

$$H' = \frac{I}{2\pi(0{,}5d + x)}; \quad H'' = \frac{I}{2\pi(0{,}5d - x)};$$

$$H = H' + H'' = \frac{I}{2\pi}\left(\frac{1}{0{,}5d + x} + \frac{1}{0{,}5d - x}\right) = \frac{I}{\pi} \cdot \frac{2d}{d^2 - 4x^2}.$$

The energy of the magnetic field between the wires is $W = ½LI^2$, where L is the external inductance of the line of the required length l. The latter one is (see problem 13.4):

$$L = \frac{\mu_0 \cdot l}{\pi} ln\frac{d - r_0}{r_0};$$

$$W = \frac{\mu_0 I^2 \cdot l}{2\pi} ln\frac{d - r_0}{r_0} = \frac{4\pi \cdot 10^{-7} \cdot 80^2 \cdot 1.5}{2\pi} ln\frac{30 - 0.2}{0.2} = 9.608 \cdot 10^{-3}\ J.$$

To determine the magnetic voltage between points A and B we again apply the superposition method $U_{mAB} = U_{mAB}' + U_{mAB}''$, where U_{mAB}' is a component induced by the left conductor, while U_{mAB}'' is induced by the right conductor. In accordance with (13.4) and fig. 13.19,b

$$U_{mAB}' = \frac{-I}{2\pi}(\alpha_1 + \beta_1),$$

$$\alpha_1 = arctg\frac{|y_A|}{0.5d + x_A} = 0.180\ rad; \qquad \beta_1 = arctg\frac{|y_B|}{0.5d + x_B} = 0.381\ rad;$$

$$U_{mAB}' = \frac{-80}{2\pi}(0.18 + 0.381) = -7.14\ A.$$

Similarly, the component U_{mAB}'' induced by the current of the right conductor (fig. 13.19,c) is
$$U_{mAB}'' = \frac{-I}{2\pi}(\alpha_2 + \tfrac{1}{2}\pi + \beta_2),$$

$$\alpha_2 = arctg\frac{|0.5d - x_A|}{|y_A|} = 1.190\ rad; \quad \beta_2 = arctg\frac{y_B}{0.5d - x_B} = 1.107\ rad;$$

$$U_{mAB}'' = \frac{-80}{2\pi}(1.190 + 1.571 + 1.107) = -49.25\ A.$$

Then $\quad U_{mAB} = U_{mAB}' + U_{mAB}'' = -7.14 - 49.25 = -56.39\ A.$

13.2. COMPUTATION OF THE MAGNETIC VECTOR POTENTIAL AND ITS APPLICATION

13-16 (13.21). Direct current $I = 100\ A$ flows through a bimetallic bus-bar (fig. 13.20,a). It is made of the material with relative permeability $\mu = 6$ and is located in air. The conductivity of the bus-bar layers is 3γ and γ. Size is $a = 2\ mm$. Plot the vector potential and the magnetic field intensity versus coordinates. Applying the magnetic vector potential, calculate the magnetic flux through the left part of the bus-bar per unit length.

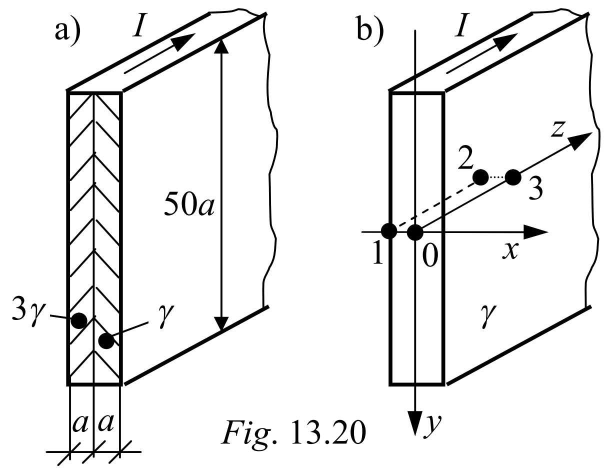


Fig. 13.20

Solution. 1. Determine the bus-bar current density (δ_1 – in the left part, δ_2 – in the right one). On the one hand, the bus-bar current is equal to $I = \delta_1 \cdot 50a^2 + \delta_2 \cdot 50a^2$.

On the other hand, as the current density vector has only a tangent component at the medium interface, on the ground of the boundary condition $E_{1t} = E_{2t}$ and Ohm's law in a differential form $\delta = \gamma \cdot E$, we obtain additional equation $\delta_1/(3\gamma) = \delta_2/\gamma$ or $\delta_1 = 3\delta_2$. Thus,
$$4\delta_2 \cdot 50a^2 = I;$$
$$\delta_2 = I/(200a^2) = 100/(200 \cdot 4 \cdot 10^{-6}) = 125000\ A/m^2; \quad \delta_1 = 375000\ A/m^2.$$

2. The field is calculated applying Poisson's equation:
$$\nabla^2 \vec{A} = -\mu_a \cdot \vec{\delta}\ .$$

Cartesian coordinate system axes are directed as shown in fig. 13.20,b. The current density vector is directed along axis z, that's why the magnetic vector potential possesses but a single component also directed along axis z. Furthermore, as the bus-bar height is a lot bigger than its width, all the values depend on but a single coordinate x. Thus, $A_x = A_y = 0$, $\frac{\partial A}{\partial y} = \frac{\partial A}{\partial z} = 0$. With account of this: $\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2} = \frac{\partial^2 A}{\partial x^2}$.

Perform the double integration of the latter equation for four field areas:

1) $x < -a$, $\qquad \frac{\partial^2 A_1}{\partial x^2} = 0, \qquad A_1 = C_1 \cdot x + C_2,$

2) $-a \le x \le 0,\qquad \dfrac{\partial^2 A_2}{\partial x^2} = -\mu_a\cdot\delta_1,\quad A_2 = -\dfrac{\mu_a}{2}\,\delta_1\cdot x^2 + C_3\cdot x + C_4,$

3) $0 \le x \le a,\qquad \dfrac{\partial^2 A_3}{\partial x^2} = -\mu_a\cdot\delta_2,\quad A_3 = -\dfrac{\mu_a}{2}\,\delta_2\cdot x^2 + C_5\cdot x + C_6,$

4) $x > a,\qquad \dfrac{\partial^2 A_4}{\partial x^2} = 0,\qquad A_4 = C_7\cdot x + C_8.$

3. The equations obtained are not sufficient to determine the integration constants $C_1 \div C_8$, that's why we use formulae for the magnetic field intensity. Vector potential is connected with intensity in the following way: $rot\,\vec{A} = \vec{B} = \mu_a\cdot\vec{H}$.

However, $rot\,\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & 0 & 0 \\ 0 & 0 & A_z \end{vmatrix} = -\vec{j}\,\dfrac{\partial A}{\partial x}.$

Then $\vec{H} = -\vec{j}\,\dfrac{1}{\mu_a}\dfrac{\partial A}{\partial x}$ and $H = -\dfrac{1}{\mu_a}\dfrac{\partial A}{\partial x}$.

Thus, $H_1 = -\dfrac{C_1}{\mu_0};\quad H_2 = \delta_1\cdot x - \dfrac{C_3}{\mu\cdot\mu_0};\quad H_3 = \delta_2\cdot x - \dfrac{C_5}{\mu\cdot\mu_0};\quad H_4 = -\dfrac{C_7}{\mu_0}.$

4. Let's write down and solve the equations to determine the integration constants.

At $x = -a$ in accordance with Ampere's law ($H_{-a}\cdot 2\cdot 50a = -I$ – a minus sign appears because on the left side of the bus-bar the intensity is directed oppositely to axis y):

$$H_1(x=-a) = -I/(100a) = -C_1/\mu_0,$$

$$\text{from here}\quad C_1 = \frac{\mu_0 I}{100a} = \frac{4\pi\cdot 10^{-7}\cdot 100}{100\cdot 0.002} = 6.283\cdot 10^{-4}.$$

At $x = -a$ in accordance with the boundary condition $H_{1t} = H_{2t}$

$H_1(x=-a) = H_2(x=-a)$ or $-C_1/\mu_0 = \delta_1\cdot(-a) - C_3/(\mu\mu_0)$; from here

$C_3 = \delta_1\cdot(-a)\cdot\mu\mu_0 + C_1\cdot\mu = -375000\cdot 0.002\cdot 6\cdot 4\pi\cdot 10^{-7} + 6.283\cdot 10^{-4}\cdot 6 = -1.885\cdot 10^{-3}.$

At $x = 0$ $H_2(x=0) = H_3(x=0)$ or $C_3 = C_5$.

At $x = a$ $H_3(x=a) = H_4(x=a)$ or $\delta_2\cdot a - C_5/(\mu\mu_0) = -C_7/\mu_0$; from here

$C_7 = -\delta_2\cdot a\cdot\mu_0 + C_5/\mu = -125000\cdot 0.002\cdot 4\pi\cdot 10^{-7} + (-1.885\cdot 10^{-3})/6 = -6.283\cdot 10^{-4}.$

Assume that at $x = 0$ $A_2(x=0) = A_3(x=0) = 0$, then $C_4 = C_6 = 0$.

At $x = -a$ $A_1(x=-a) = A_2(x=-a)$ or

$C_1\cdot(-a) + C_2 = -½\mu_a\delta_1\cdot(-a)^2 + C_3\cdot(-a) + C_4$, from here

$C_2 = -½\mu_a\delta_1\cdot(-a)^2 + C_3\cdot(-a) + C_4 - C_1\cdot(-a) =$

$= -½\cdot 6\cdot 4\pi\cdot 10^{-7}\cdot 375000\cdot 0.002^2 + 1.885\cdot 10^{-3}\cdot 0.002 + 0 + 6.283\cdot 10^{-4}\cdot 0.002 = -6.283\cdot 10^{-7}.$

At $x = a$ $A_3(x=a) = A_4(x=a)$ or $-½\mu_a\delta_2\cdot a^2 + C_5\cdot a + C_6 = C_7\cdot a + C_8$,

from here $C_8 = -½\mu_a\delta_2\cdot a^2 + C_5\cdot a + C_6 - C_7\cdot a = -½\cdot 6\cdot 4\pi\cdot 10^{-7}\cdot 125000\cdot 0.002^2 -$

$- 1.885\cdot 10^{-3}\cdot 0.002 + 0 + 6.283\cdot 10^{-4}\cdot 0.002 = -4.398\cdot 10^{-6}.$

5. Finally, we have:

$$A(x)=\begin{cases} 6.283\cdot 10^{-4}\cdot x-6.283\cdot 10^{-7}\ Wb/m\ if\ x<-a \\ -1.414\cdot x^2-1.885\cdot 10^{-3}\cdot x\ Wb/m\ if\ -a<x<0 \\ 0.471\cdot x^2-1.885\cdot 10^{-3}\cdot x\ \ Wb/m\ if\ 0<x<a \\ -6.283\cdot 10^{-4}\cdot x-4.398\cdot 10^{-6}\ Wb/m\ if\ x>a \end{cases},$$

$$H(x)=\begin{cases} -500\ A/m\quad if\quad x<-a \\ 375000\cdot x+250\quad A/m\quad if\ -a<x<0 \\ 125000\cdot x+250\quad A/m\quad if\ 0<x<a \\ 500\quad A/m\quad if\ x>a \end{cases}.$$

The graphs *A(x)* and *H(x)* plotted according to these formulae are given in fig. 13.21.

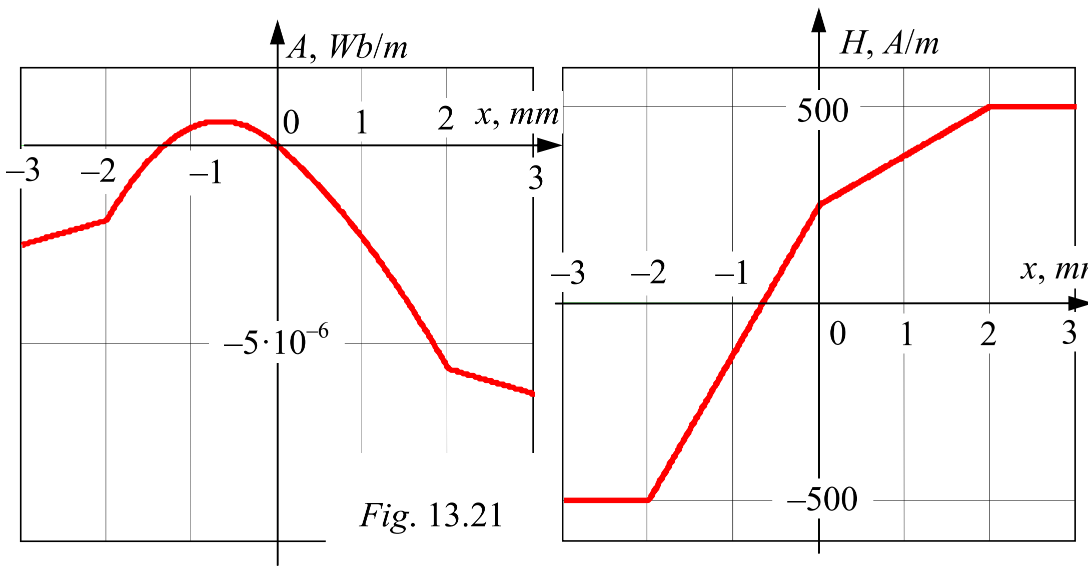


Fig. 13.21

6. In order to determine the magnetic flux flowing through the left part of the bus-bar per unit length, we apply formula $\Phi=\oint_L \vec{A}\cdot\vec{dl}$. A rectangular 0-1-2-3-0 1 *m* long is taken as the integration path (fig. 13.20,b). Furthermore, the integration along sides 0-1 and 2-3 results in zero as the angle between $\vec{A}$ and $\vec{dl}$ is 90° in all the points of the sides (vector $\vec{A}$ is directed along axis *z*, while vector $\vec{dl}$ is parallel to axis *x*). The value of *A* along the side 3-0 is equal to zero. Thus, $\Phi=\int_1^2 \vec{A}\,\vec{dl}=A(x=-a)\cdot 1=-1.885\cdot 10^{-6}\ Wb$.

The minus sign in answer for the magnetic flux means that it is directed oppositely to the positive normal to the integration loop[1], that is upwards.

13-17 (13.22). Direct current $I=350\ A$ flows through a metallic cylindrical bus-bar of radius $a=20\ cm$ placed in air (fig. 13.22). The bus-bar is made of the material with relative permeability $\mu=4$. Plot the magnetic vector potential and the magnetic field intensity versus coordinates.

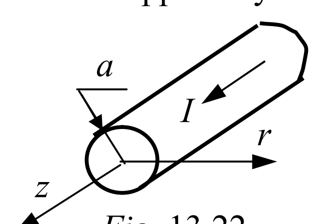


Fig. 13.22

[1] Positive normal direction is connected with the loop direction through the right-hand screw rule: if you watch from the normal vector end, then loop is traversed in the counter-clockwise direction.

Solution. The procedure is very the same as in problem 13.16.

1. The bus-bar current density is $\delta = I/(\pi a^2) = 2785\ A/m^2$.

2. The magnetic vector potential has a single component directed parallel to axis z and depends only on a single coordinate r. Under these circumstances, Poisson's equation in cylindrical coordinate system takes a view:

$$\nabla^2 A = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) = \begin{cases} -\mu\mu_0\delta & if \quad r \le a, \\ 0 \quad if & r > a. \end{cases}$$

Its solution is $A_1 = -\frac{1}{4}\mu\mu_0\delta r^2 + C_1 ln(r) + C_2$, $\qquad A_2 = C_3 ln(r) + C_4$.

In order that A_1 is not infinite at $r = 0$, there should not be any item $C_1 ln(r)$, so $C_1 = 0$. Furthermore, assume $A_1 = 0$ at $r = 0$, then $C_2 = 0$ and $A_1 = -\frac{1}{4}\mu\mu_0\delta r^2$.

3. The magnetic field intensity is $\vec{H} = rot\,\vec{A}/\mu_a = -\dfrac{1}{\mu_a}\dfrac{dA}{dr}$.

Thus, $\quad H_1 = \frac{1}{2}\delta r - \dfrac{C_1}{\mu\mu_0 r} = \frac{1}{2}\delta r; \qquad H_2 = -\dfrac{C_3}{\mu_0 r}$.

4. At $r = a$ $\quad H_1 = H_2$, $\quad$ from here $\quad C_3 = -\frac{1}{2}\mu_0\delta a^2 = -7\cdot 10^{-5}$,

$A_1 = A_2$, $\quad$ from here $\quad C_4 = -\frac{1}{4}\mu\mu_0\delta a^2 - C_3 ln(a) = -25.26\cdot 10^{-5}$.

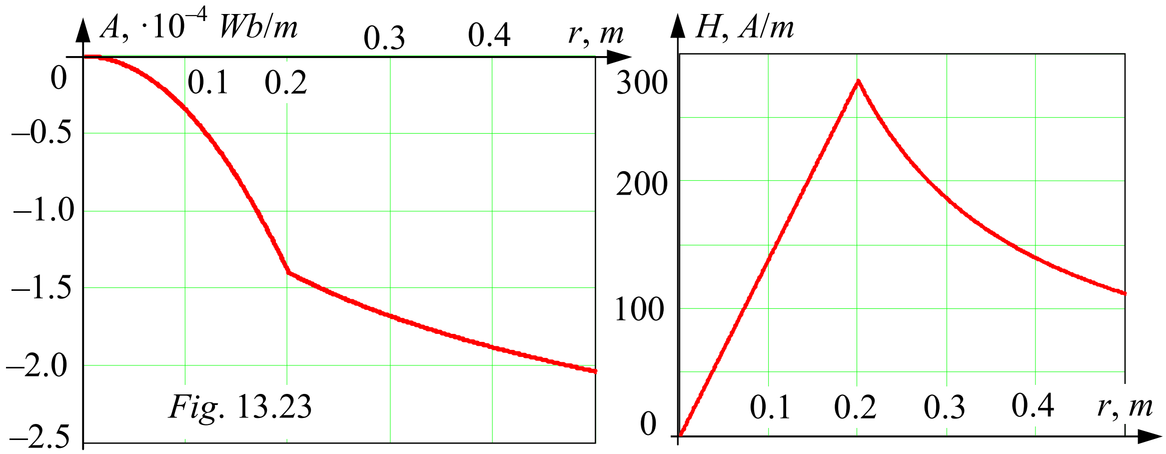


Fig. 13.23

5. Finally, we have

$$A(r) = \begin{cases} -3.498\cdot 10^{-3} r^2\ Wb/m & if \quad r \le a, \\ -7\cdot 10^{-5}\ ln(r) - 25.26\cdot 10^{-5}\ Wb/m & if \quad r > a, \end{cases}$$

$$H(r) = \begin{cases} 1393r\ A/m & if \quad r \le a, \\ \dfrac{55.70}{r}\ A/m & if \quad r > a. \end{cases}$$

The gaphs $A(r)$ and $H(r)$ plotted according to these formulae are given in fig. 13.23.

13-18 (13.23). Direct current $I = 35\ A$ flows through a two-wire line. The wire radius is $a = 2\ mm$, the distance between wires is $d = 50\ cm$ (fig. 13.24). The wires are made of the material with relative magnetic permeability $\mu = 4$ and are placed in air. Plot the dependences of the magnetic vector potential and the magnetic field intensity versus coordinate x.

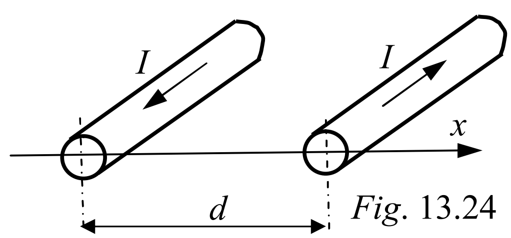


Fig. 13.24

Solution. Let's apply the superposition principle. First, we calculate the magnetic vector potential and the magnetic field intensity produced by each wire separately, then, we add them algebraically.

When calculating the field produced by one, say a left wire, we use the results obtained in problem 13.17. The current density in wires is $\delta = I/(\pi a^2) = 2785000\ A/m^2$. Assume the coordinate origin is on the axis of the left wire. Then

$$A_1 = -\tfrac{1}{4}\mu\mu_0\delta x^2 = -3.498\cdot x^2\ Wb/m, \quad H_1 = \tfrac{1}{2}\delta x = 1.393\cdot 10^6\cdot x\ A/m;$$

$$C_3 = -\tfrac{1}{2}\mu_0\delta a^2 = -7\cdot 10^{-6}, \qquad C_4 = -\tfrac{1}{4}\mu\mu_0\delta a^2 - C_3 ln(a) = -57.50\cdot 10^{-6}.$$

Thus,
$$A'(x) = \begin{cases} -3.498\cdot x^2\ Wb/m & if \quad x \le a, \\ -7\cdot 10^{-6}\ ln(x) - 57.50\cdot 10^{-6}\ Wb/m & if \quad x > a, \end{cases}$$

$$H'(x) = \begin{cases} 1.393\cdot 10^6 x\ \ A/m & if \quad x \le a, \\ \dfrac{5.57}{x}\ \ A/m & if \quad x > a. \end{cases}$$

Furthermore, one should remember that as the current-carrying wire is a structure symmetrical with respect to the plane $y0z$, then function $A'(x)$ is an even one, while $H'(x)$ is odd one[1)].

Assume that the coordinate origin is in the centre between the wires. In this case, the formulae $A'(x)$ and $H'(x)$ transform in the following way (the coordinate origin shift by $d/2$ is taken into account):

$$A'(x) = \begin{cases} -7\cdot 10^{-6}\ ln(-x-d/2) - 57.5\cdot 10^{-6}\ Wb/m\ if\ x < -a-d/2, \\ -3.498\cdot(x+d/2)^2 Wb/m \quad if \quad -a-d/2 \le x \le a-d/2, \\ -7\cdot 10^{-6}\ ln(x+d/2) - 57.5\cdot 10^{-6}\ Wb/m\ if\ x > a-d/2. \end{cases}$$

$$H'(x) = \begin{cases} \dfrac{5.57}{x+d/2} \quad A/m\ \ if\ \ x < -a-d/2, \\ 1.393\cdot 10^6\cdot(x+d/2)\ A/m\ if\ -a-d/2 \le x \le a-d/2, \\ \dfrac{5.57}{x+d/2} \quad A/m\ \ if\ x > a-d/2. \end{cases}$$

The current in the right wire is oppositely directed, that's why the functions change their sign; and in this case the coordinate origin shift is $-d/2$. That's why

$$A''(x) = \begin{cases} 7\cdot 10^{-6}\ ln(-x+d/2) + 57.5\cdot 10^{-6}\ Wb/m\ if\ x < -a+d/2, \\ 3.498\cdot(x-d/2)^2 Wb/m \quad if \quad -a+d/2 \le x \le a+d/2, \\ 7\cdot 10^{-6}\ ln(x-d/2) + 57.5\cdot 10^{-6}\ Wb/m\ if\ x > a+d/2. \end{cases}$$

[1)] When plotting $A(x)$ and $H(x)$, one should pay attention if the installation in question is symmetrical or not. If it is symmetrical with respect to the axis plane (e.g. a flat bus-bar) or has point symmetry (e.g. two-wire line) the graph $A(x)$ is also symmetrical with respect to either the ordinate axis or the coordinate origin, respectively. The functions $H(x)$ and $A(x)$ are differentially constrained, that's why they possess different symmetries: if the graph $A(x)$ is symmetrical with respect to the ordinate axis, then graph $H(x)$ is symmetrical with respect to the coordinate origin and vice versa.

$$H''(x)=\begin{cases}\dfrac{-5.57}{x-d/2}\ A/m\ \ if\ \ x<-a+d/2,\\ -1.393\cdot 10^{6}\cdot (x-d/2)\ A/m\ if\ -a+d/2\le x\le a+d/2\,,\\ \dfrac{-5.57}{x-d/2}\ A/m\ \ if\ x>a+d/2.\end{cases}$$

To determine the resultant field we superimpose the fields of different wires. As a rule, we can add up the potential functions providing they have a zero value at some common point. Let the zero value of the magnetic vector potential be in the coordinate origin. Then $A'(x)$ is to be changed everywhere to the constant quantity

$$N=-A'(0)=7\cdot 10^{-6}\, ln(d/2)+57.5\cdot 10^{-6}=const,$$

and the function $A''(x)$ – to the quantity $A''(0)=-N$, it means the sum $A(x)=A'(x)+A''(x)$ remains the same as the constants N and $-N$ give zero at addition. Finally, we have:

$$A(x)=\begin{cases}7\cdot 10^{-6}\cdot ln\dfrac{-x+d/2}{-x-d/2}\ Wb/m\ \ \ if\ \ \ x<-a-d/2,\\ 7\cdot 10^{-6}\, ln(-x+d/2)+57.5\cdot 10^{-6}-3.498\cdot (x+d/2)^{2}\ Wb/m\\ \qquad\qquad if\ -a-d/2\le x\le a-d/2,\\ 7\cdot 10^{-6}\cdot ln\dfrac{-x+d/2}{x+d/2}\ Wb/m\ if\ \ a+d/2<x\le -a+d/2,\\ -7\cdot 10^{-6}\, ln(x+d/2)-57.5\cdot 10^{-6}+3.98\cdot (x-d/2)^{2}\ Wb/m\\ \qquad\qquad if\ \ \ -a+d/2<x\le a+d/2,\\ 7\cdot 10^{-6}\cdot ln\dfrac{x-d/2}{x+d/2}\ Wb/m\ \ \ if\ x>a+d/2.\end{cases}$$

$$H(x)=H'(x)+H''(x)=\begin{cases}\dfrac{-5.57}{x+d/2}+\dfrac{-5.57}{x-d/2}\ \ A/m\ \ \ if\ \ \ x<-a-d/2,\\ 1.393\cdot 10^{-6}(x+d/2)-\dfrac{5.57}{x+d/2}\ A/m\ if\ -a-\dfrac{d}{2}\le x\le a-\dfrac{d}{2},\\ \dfrac{5.57}{x+d/2}+\dfrac{-5.57}{x-d/2}\ \ A/m\ \ \ if\ \ a-d/2<x\le -a+d/2,\\ \dfrac{5.57}{x+d/2}-1.393\cdot 10^{-6}(x-d/2)\ A/m\ \ if\ -a+\dfrac{d}{2}<x\le a+\dfrac{d}{2},\\ \dfrac{5.57}{x+d/2}+\dfrac{-5.57}{x-d/2}\ \ A/мm\ \ \ if\ \ x>a+d/2.\end{cases}$$

The graphs $A(x)$ and $H(x)$ plotted according to these formulae are presented in fig. 13.25 and 13.26.

13-19 (13.24). Direct current $I=1500\,A$ flows through a bimetallic rectangular bus-bar (fig. 13.27). The relative magnetic permeability of the bus-bar materials is $\mu_1=6\mu_0$, $\mu_2=\mu_0$, conductivities are $\gamma_1=2\cdot 10^7\ S/m$, $\gamma_2=4\cdot 10^7\ S/m$. $a=1\ cm$, $h=50\ cm$.

Compute and plot the magnetic vector potential versus the coordinate.

Answers: current density $\delta_1=10^5\ A/m^2$, $\delta_2=2\cdot 10^5\ A/m^2$,
the graph $A(y)$ is in fig. 13.28:

$$A(y)=\begin{cases}1.885\cdot 10^{-3}y-5.655\cdot 10^{-5}\ Wb/m \text{ if } y\le -a,\\ -0.377\cdot y^{2}+3.77\cdot 10^{-3}y\ \ Wb/m \text{ if } -a\le y\le 0,\\ -0.1257\cdot y^{2}+6.283\cdot 10^{-4}y\ \ Wb/m \text{ if } 0\le y\le a,\\ -1.885\cdot 10^{-3}y+1.257\cdot 10^{-4}\ Wb/m \text{ if } y>a.\end{cases}$$

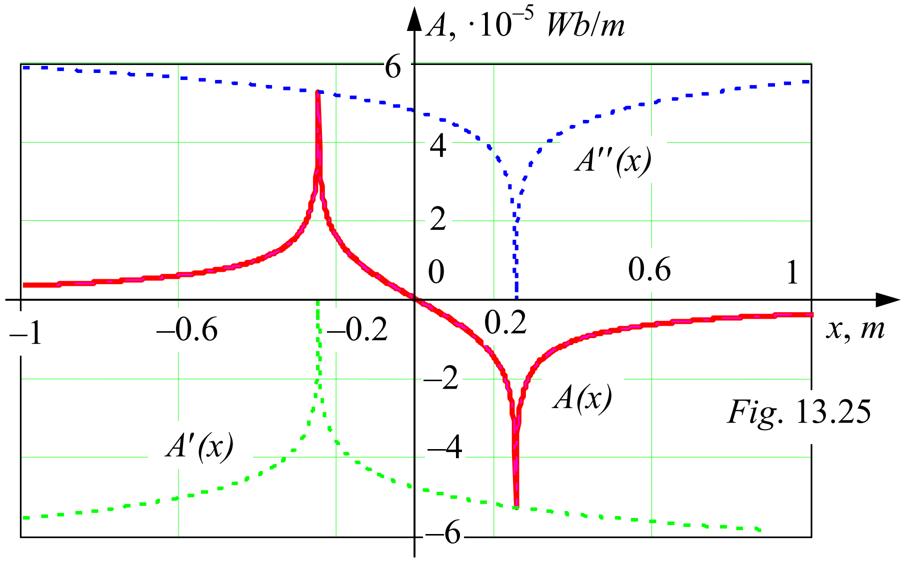


Fig. 13.25

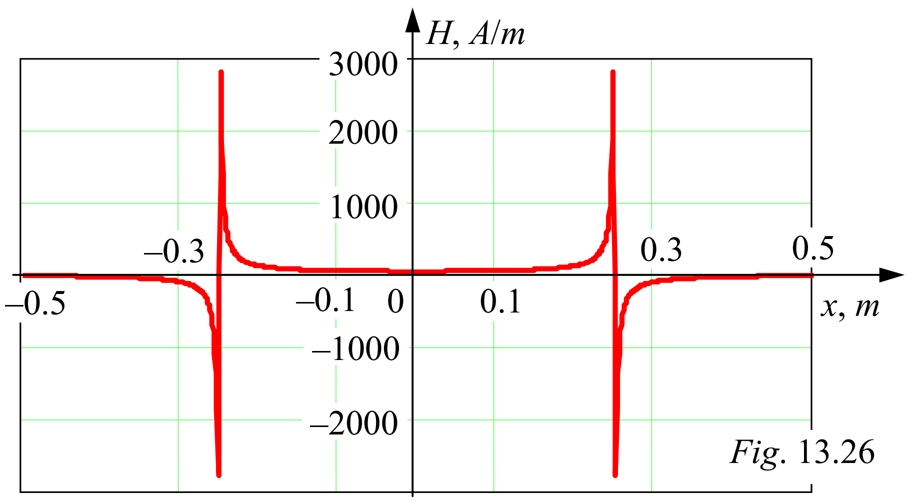


Fig. 13.26

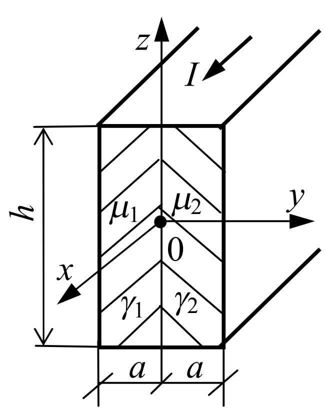


Fig. 13.27

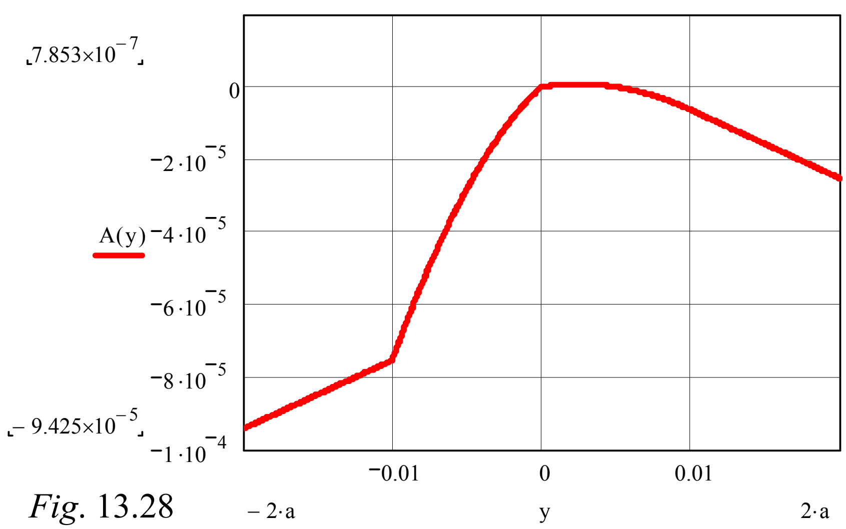


Fig. 13.28

13-20 (13.25). The same current $I = 200\,A$ flows through two parallel bus-bars (fig. 13.29). The relative magnetic permeability of bus-bars is $\mu = 5$, ambient medium is air.

Compute and plot the magnetic vector potential for positive meanings of the coordinate x. Compute the magnetic flux through a rectangular frame of length $l = 2\ m$, if $A[3\ cm;\ 0]$; $B[5\ cm;\ 0]$ and $a = 1\ cm$, $b = 2\ cm$, $h = 40\ cm$.

Answers: the current density in a bus-bar is $\delta = 5 \cdot 10^4\ A/m^2$;

$$A(x) = \begin{cases} 0 \quad if\ 0 \le x \le 0.5b, \\ -0.157x^2 + 3.142 \cdot 10^{-3} x - 1.571 \cdot 10^{-5}\ Wb/m\ if\ 0.5b \le x \le 0.5b + a, \\ -6.283x - 3.142 \cdot 10^{-6}\ Wb/m\ if\ x > 0.5b + a; \end{cases}$$

the graph $A(x)$ is in fig. 13.30; $A(x_B) = -3.456 \cdot 10^{-5}$, $A(x_A) = -2.199 \cdot 10^{-5}$,
the magnetic flux through the frame is $\Phi = -l \cdot (A(x_B) - A(x_A)) = 25.13\ \mu Wb$.

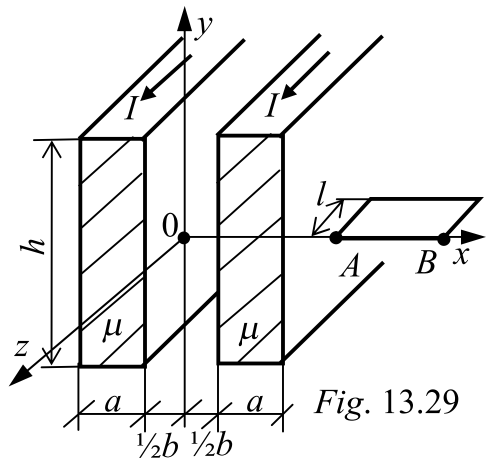


Fig. 13.29

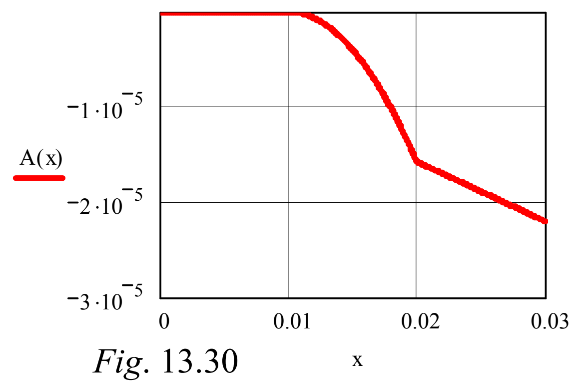


Fig. 13.30

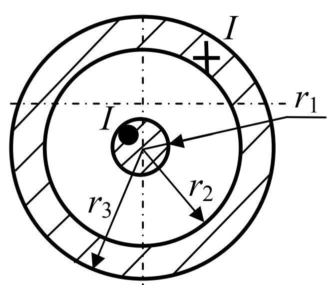


Fig. 13.31

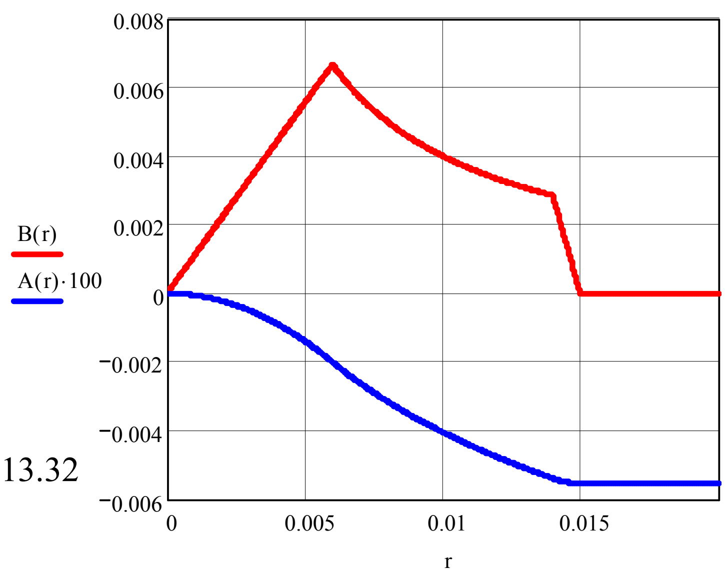


Fig. 13.32

13-21 (13.26). Compute and plot the magnetic vector potential and magnetic induction for the coaxial cable, the insulation, conductor and sheath of which are made of non-magnetic material (fig. 13.31). $r_1 = 6\ mm$, $r_2 = 14\ mm$, $r_3 = 15\ mm$, $I = 200\ A$.

Answers: current density $\delta_1 = 1.768 \cdot 10^6\ A/m^2$, $\delta_2 = 2.195 \cdot 10^6\ A/m^2$,

$$A(r)=\begin{cases}-0.556r^2 Wb/m \; if \; 0 \le r \le r_1, \\ -4\cdot10^{-5}\ln r - 2.246\cdot10^{-4}\; Wb/m \; if \; r_1 \le r \le r_2, \\ 0.6897r^2 - 3.103\cdot10^{-4}\ln r - 0.01514\; Wb/m \; if \; r_2 \le r \le r_3, \\ -5.530\cdot10^{-4}\; Wb/m \; if \; r > r_3;\end{cases}$$

$$B(r)=\begin{cases}1.111r\; T \; if \; 0 \le r \le r_1, \\ 4\cdot10^{-5}/r\; T\; if \; r_1 \le r \le r_2, \\ -1.379r - 3.103\cdot10^{-4}/r\; T\; if \; r_2 \le r \le r_3, \\ 0 \quad if \; r > r_3.\end{cases}$$

The graphs $A(r)$ and $B(r)$ are presented in fig. 13.36.

13.3. MAGNETIC IMAGE METHOD APPLICATION

13-22 (13.27). Direct current $I = 130\ A$ flows through a single wire located close to the medium interface (fig. 13.33,a). The distance is $d = 70\ cm$, the wire radius is negligible in comparison with d.

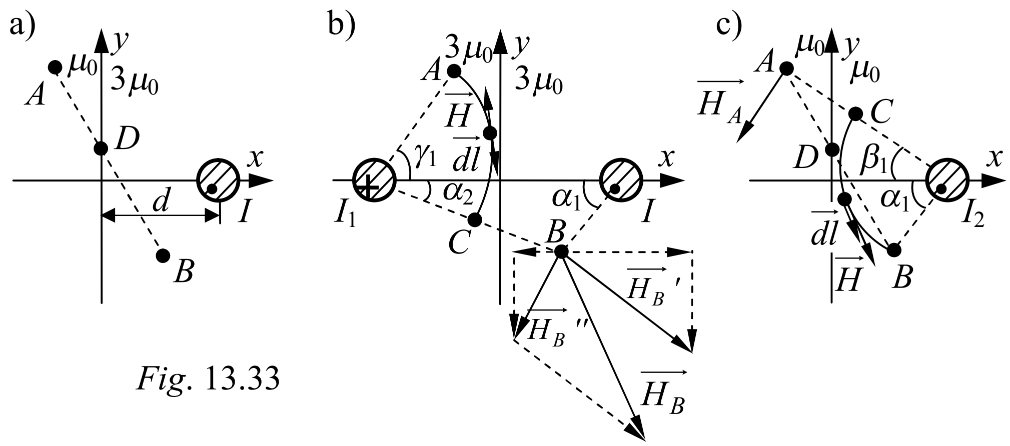


Fig. 13.33

It is required to: 1) compute the magnetic field intensity in points $A(x_A = -15\ cm;\ y_A = 30\ cm)$ and $B(x_B = 20\ cm;\ y_B = -35\ cm)$ as well as the magnetic voltage between them; 2) supposing A and B are the intersection points of the long sides of a rectangular frame $l = 5\ m$ long with number of turns $w = 200$, find the magnetic flux through the frame and the mutual inductance between the wire and the frame.

Solution. Since the wire is located close to the medium interface, we apply the method of magnetic images in accordance with which the field in the right half-space may be computed by fig. 13.33,b, while the field in the left half-space – by fig. 13.33,c.

Determine the fictitious currents coefficients:

$$k_1 = \frac{\mu_0 - 3\mu_0}{\mu_0 + 3\mu_0} = -0.5; \qquad k_2 = \frac{2\cdot3\mu_0}{\mu_0 + 3\mu_0} = 1.5.$$

Then $I_1 = k_1 I = -65\ A$; $I_2 = k_2 I = 195\ A$. Further, the current value I_1 is considered to be positive but its direction is reversed in comparison with the real current I, which is taken into account in fig. 13.33,b.

1. Computation of the intensities.

In point A, the intensity is produced by current I_2 (fig. 13.33,c) and in accordance with Ampere's law it is equal to

$$H_A = \frac{I_2}{2\pi\sqrt{y_A^2 + (d - x_A)^2}} = \frac{195}{2\pi\sqrt{0.3^2 + (0.7 + 0.15)^2}} = 34.4 \ \frac{A}{m}.$$

In point B, the intensity is produced by currents I and I_1 (fig. 13.33,b). Let's calculate the projections of the components H'_B and H''_B upon axes x and y:

$$H'_{Bx} = H'_B \cdot cos\alpha_1 =$$

$$= \frac{I}{2\pi\sqrt{y_B^2 + (d - x_B)^2}} \cdot \frac{-y_B}{\sqrt{y_B^2 + (d - x_B)^2}} = \frac{-I \cdot y_B}{2\pi \cdot (y_B^2 + (d - x_B)^2)} = 19.4 \ \frac{A}{m};$$

$$H'_{By} = H'_B \cdot sin\alpha_1 = \frac{I \cdot (d - x_B)}{2\pi \cdot (y_B^2 + (d - x_B)^2)} = 27.77 \ \frac{A}{m};$$

$$H''_{Bx} = H''_B \cdot cos\alpha_2 =$$

$$= \frac{I_1}{2\pi\sqrt{y_B^2 + (d + x_B)^2}} \cdot \frac{-y_B}{\sqrt{y_B^2 + (d + x_B)^2}} = \frac{-I_1 \cdot y_B}{2\pi \cdot (y_B^2 + (d + x_B)^2)} = 3.88 \ \frac{A}{m};$$

$$H''_{By} = H''_B \cdot sin\alpha_2 = \frac{I_1 \cdot (d + x_B)}{2\pi \cdot (y_B^2 + (d + x_B)^2)} = 9.99 \ \frac{A}{m}.$$

Then $H_{Bx} = H'_{Bx} - H''_{Bx} = 15.52\ A/m$; $H_{By} = H'_{By} + H''_{By} = 37.76\ A/m$;

$$H_B = \sqrt{H_{Bx}^2 + H_{By}^2} = 40.84\ A/m.$$

2. Computation of the magnetic voltage.

Let's apply the superposition method. The voltage component created by currents I and I_2 (see fig. 13.33) in accordance with (13.4) is as follows

$$U_{mAB}' = \frac{I + I_2}{2\pi}(\alpha_1 + \beta_1),$$

where α_1 and β_1 – angles in radians shown in fig. 13.33,c.

$$\alpha_1 = arctg\frac{|y_B|}{d - x_B} = 0.61; \quad \beta_1 = arctg\frac{y_A}{d - x_A} = 0.34;$$

$$U_{mAB}' = \frac{130 + 195}{2\pi}(0.61 + 0.34) = 49.14\ A.$$

The component U_{mAB}'' created by current I_1 (fig. 13.33,b) is

$$U_{mAB}'' = \frac{-I_1}{2\pi}(\alpha_2 + \gamma_1), \quad \alpha_2 = arctg\frac{|y_B|}{d + x_B} = 0.371; \quad \gamma_1 = arctg\frac{y_A}{d + x_A} = 0.5;$$

$$U_{mAB}'' = \frac{-65}{2\pi}(0.371 + 0.5) = -9\ A.$$

Then $U_{mAB} = U_{mAB}' + U_{mAB}'' = 40.14\ A.$

3. Computation of the magnetic flux.

Since the frame parts are located in the different mediums, we use the auxiliary point D with the coordinate (fig. 14.33,c) $\quad y_D = y_A - x_A\dfrac{y_A - y_B}{x_A - x_B} = 0.021\ m.$

Then the magnetic flux component through the frame part located in the left half-space is determined in accordance with (13.3)

$$\Phi' = \frac{\mu_0 I_2 \cdot l}{2\pi} ln \frac{r_A}{r_D} = \frac{\mu_0 I_2 \cdot l}{2\pi} ln \frac{\sqrt{y_A^2 + (d - x_A)^2}}{\sqrt{d^2 + y_D^2}} = 4.92 \cdot 10^{-5}\ Wb.$$

The magnetic flux component through the frame part located in the right half-space is

$$\Phi'' = \frac{3\mu_0 \cdot l}{2\pi} \left(I \cdot ln \frac{\sqrt{d^2 + y_D^2}}{\sqrt{y_B^2 + (d - x_B)^2}} + I_1 \cdot ln \frac{\sqrt{d^2 + y_D^2}}{\sqrt{y_B^2 + (d + x_B)^2}} \right) = 1.16 \cdot 10^{-4}\ Wb.$$

The magnetic flux through the frame is $\Phi = \Phi' + \Phi'' = 1.65 \cdot 10^{-4}\ Wb.$

The mutual inductance between the wire and frame is $M = \frac{w \cdot \Phi}{I} = 2.55 \cdot 10^{-4}\ H.$

13-23 (13.28). A current-carrying $I = 500\ A$ flat bus-bar is placed close to the medium interface (fig. 13.34,a) at distance $a = 5\ mm$ from the interface. It is required to determine the magnetic induction in point A as well as the mutual inductance between the bus-bar and a rectangular frame possessing $w = 200$ *turns*.

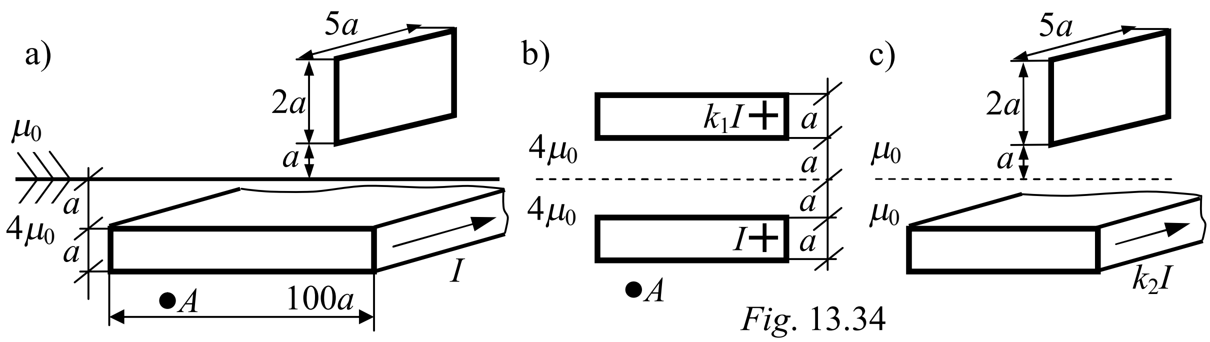


Fig. 13.34

Solution. In this problem, the method of magnetic images is applicable. Let's determine the fictitious currents coefficients:

$$k_1 = \frac{\mu_0 - 4\mu_0}{\mu_0 + 4\mu_0} = -0.6; \quad k_2 = \frac{2 \cdot 4\mu_0}{\mu_0 + 4\mu_0} = 1.6.$$

Then the magnetic field in the lower half-space is computed in accordance with fig. 13.34,b. Since the bus-bar width is much bigger than its thickness, it is possible to neglect the edge effect and to determine the magnetic intensity and magnetic induction in point A with the aid of Ampere's law and the superposition method:

$$H_A = \frac{I \cdot (1 + k_1)}{2 \cdot 100a} = \frac{500 \cdot (1 - 0.6)}{200 \cdot 0.005} = 200\ A/m; \quad B_A = 4\mu_0 H_A = 1.0 \cdot 10^{-3}\ T.$$

The mutual inductance between the bus-bar and frame is $M = \frac{w \cdot \Phi}{I}$, where Φ – the magnetic flux linked with the frame turns. We calculate it by fig. 13.34,c:

$$\Phi = S \cdot \mu_0 H = 2a \cdot 5a \cdot \mu_0 \cdot k_2 \cdot I / (2 \cdot 100a).$$

Then $M = 10a \cdot \mu_0 \cdot k_2 \cdot w / 200 = 0.10 \cdot 10^{-6}\ H.$

13-24 (13.29). A single current-carrying $I = 10\,A$ wire is located close to the medium interface (fig. 13.35,a); furthermore, $\mu_1 = 2$, $\mu_2 = 6$, $r_1 = 10\ cm$. The points coordinates are $x_A = y_A = -10\ cm$, $x_B = 5\ cm$, $y_B = 15\ cm$. It is required to:

1. Compute the magnetic field intensity in points A and B as well as the magnetic voltage between them.

2. Supposing that A and B are the intersection points of long sides of the rectangular frame $l = 1$ m long with the number of turns $w = 100$, find the magnetic flux through the frame and the mutual inductance between the wire and the frame.

Solution. 1. Calculate the field in the left half-space according to fig. 13.35,b, which is drawn in accordance with the method of magnetic images. To pass from one medium into another, we take any point at the interface, for example, the coordinate origin 0 as a reference point.

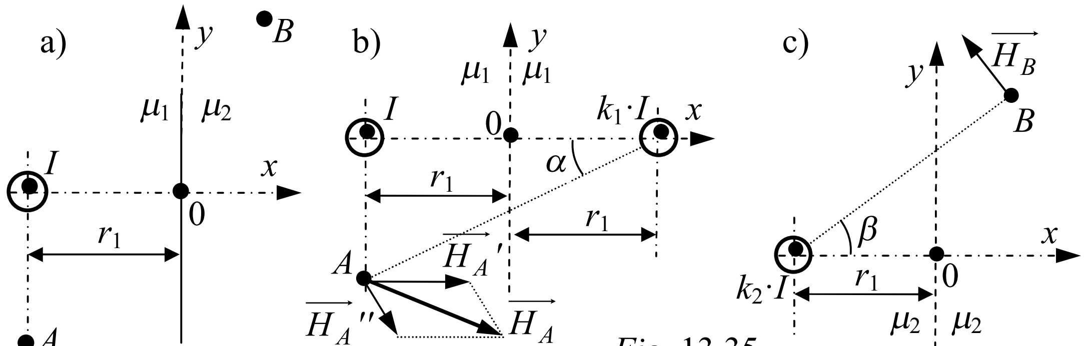


Fig. 13.35

The incomplete reflection coefficient is $k_1 = \dfrac{\mu_2 - \mu_1}{\mu_2 + \mu_1} = \dfrac{6-2}{6+2} = 0.5.$

The magnetic field intensity in point A is:

$$H_A' = \frac{I}{2\pi(-y_a)} = \frac{10}{2\pi(-10)} = 0.159\ A/cm;$$

$$H_{Ax}' = H_A' = 0.159\ A/cm; \qquad H_{Ay}' = 0;$$

$$H_A'' = \frac{k_1 \cdot I}{2\pi\sqrt{(2r_1)^2 + y_A^2}} = \frac{5}{2\pi\sqrt{20^2 + 10^2}} = 0.036\ A/cm;$$

$$tg\alpha = \frac{-y_A}{2r_1} = 10/20 = 0.5, \quad \alpha = 0.4636\ rad;$$

$$H_{Ax}'' = H_A''\cdot sin\alpha = 0.036\cdot 0.4472 = 0.016\ A/cm;$$

$$H_{Ay}'' = -\,H_A''\cdot cos\alpha = -\,0.036\cdot 0.8944 = -\,0.032\ A/cm;$$

$$H_A = \sqrt{(H_{Ax}'+H_{Ax}'')^2 + (H_{Ay}'+H_{Ay}'')^2} = \sqrt{(0.159+0.016)^2 + (0-0.032)^2} = 0.178\ A/cm.$$

The magnetic voltage in accordance with (13.4) is:

$$U_{mA0} = U_{mA0}' + U_{mA0}'' = \frac{I}{2\pi}\frac{\pi}{2} - \frac{k_1 \cdot I}{2\pi}\alpha = \frac{10}{4} - \frac{5}{2\pi}0.4636 = 2.131\ A.$$

The magnetic flux through the section $A0$ in accordance with (13.3) is:

$$\Phi_{A0} = \Phi' + \Phi'' = \frac{\mu_1\mu_0 I \cdot l}{2\pi} ln\frac{-y_A}{r_1} + \frac{\mu_1\mu_0 k_1 I \cdot l}{2\pi} ln\frac{\sqrt{(2r_1)^2 + y_A^2}}{r_1} =$$

$$= 0 + \frac{2\cdot 4\pi\cdot 10^{-7}\cdot 5\cdot 1}{2\pi} ln\frac{\sqrt{500}}{10} = 1.609\cdot 10^{-6}\ Wb \text{ (direction downwards).}$$

2. Calculate the field in the right half-space according to fig. 13.35,c.

$$k_2 = \frac{2\mu_1}{\mu_2 + \mu_1} = \frac{4}{6+2} = 0.5;$$

$$H_B = \frac{k_2 I}{2\pi\sqrt{(r_1 + x_B)^2 + y_B^2}} = \frac{5}{2\pi\sqrt{15^2 + 15^2}} = 0.038\ A/cm;$$

$$tg\beta = \frac{y_B}{r_1 + x_B} = 15/15 = 1,\ \ \alpha = \pi/4\ rad;$$

$$U_{m0B} = \frac{k_2 \cdot I}{2\pi}\beta = \frac{5}{2\pi}\frac{\pi}{4} = 0.625\ A;$$

$$\Phi_{0B} = \frac{\mu_2\mu_0 k_2 I \cdot l}{2\pi} ln\frac{\sqrt{(r_1 + x_B)^2 + y_B^2}}{r_1} = \frac{6 \cdot 4\pi \cdot 10^{-7} \cdot 5 \cdot 1}{2\pi} ln\frac{\sqrt{450}}{10} =$$

$= 4.512 \cdot 10^{-6}\ Wb$ (direction upwards).

3. Finally, we have:

- the field intensities – $H_A = 0.178\ A/cm$; $H_B = 0.038\ A/cm$;
- the magnetic voltage $U_{mAB} = U_{mA0} + U_{m0B} = 2.131 + 0.625 = 2.756\ A$;
- the magnetic flux $\Phi_{AB} = -\Phi_{A0} + \Phi_{0B} = (-1.609 + 4.512)\cdot 10^{-6} = 2.903 \cdot 10^{-6}\ Wb$;
- the mutual inductance $M = \frac{w \cdot \Phi_{AB}}{I} = \frac{100 \cdot 2.903}{10} = 29.03\ \mu H.$

13-25 (13.30). Direct current $I = 130\ A$ flows through a two-wire line, its wires being situated in different magnetic mediums (fig. 13.36,a). The wire radius is $r_0 = 1\ cm$, $d = 50\ cm$. Compute the magnetic voltage between points $A(-15\ cm; 30\ cm)$ and $B(20\ cm; 10\ cm)$.

Answers: the incomplete reflection coefficients are:

$$k_1' = \frac{6-16}{6+16} = -0.455,\ \ k_2' = \frac{2\cdot 6}{6+16} = 0.545,\ \ k_1'' = -k_1' = 0.455,\ \ k_2'' = \frac{2 \cdot 16}{6+16} = 1.455;$$

angles in radians shown in fig. 13.36,b and c:

$$\alpha_1 = 1.249,\ \ \alpha_2 = 0.644,\ \ \beta_1 = 0.219,\ \ \beta_2 = 1.107;$$

calculation sketches for voltages U_{mAO} and U_{mOB} are presented in fig. 13.36,b and c, respectively;

$$U_{mAO} = \frac{I}{2\pi}(\alpha_1 + \alpha_2) = 39.2\ A,\ \ U_{mOB} = \frac{-I}{2\pi}(\beta_1 + \beta_2) = -27.4\ A,\ \ U_{mAB} = 11.8\ A.$$

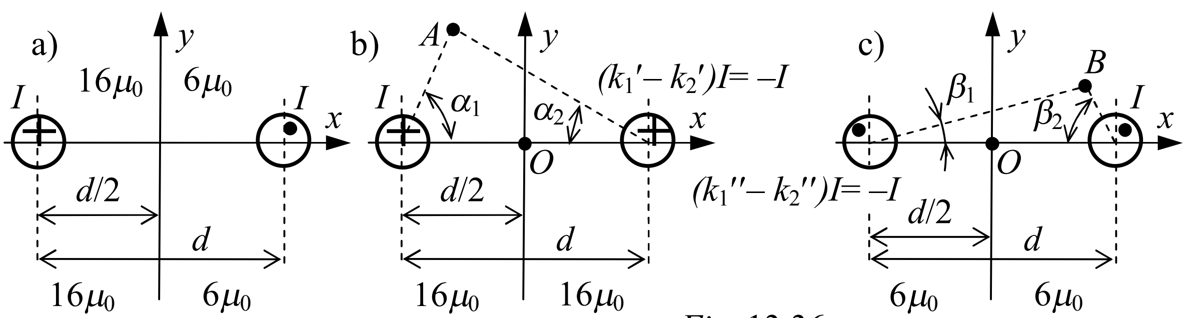


Fig. 13.36

13-26 (13.31). A cylindrical wire of radius $r_0 = 6\ mm$ is located in air at distance $h = 40\ cm$ (fig. 13.37) from the reinforced concrete wall with relative magnetic permeability $\mu_r = 4$.

It is required to: 1. Determine necessary value and direction of current through the wire to obtain at point $N(b = 30\ cm)$ the magnetic field with a tangent component of the magnetic intensity $H_t = 50\ A/m$.

2. Determine the force acting upon each $1\ m$ of wire under these circumstances.

3. Determine the direction and value of the magnetic induction in point K.

4. Find the volume energy density of the magnetic field in points K and K', latter one being the mirror image of point K.

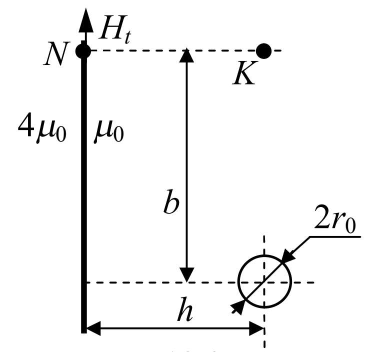


Fig. 13.37

Answers: $I = 490.6\ A$, the current is directed away from us;

$F = 0.0361\ N/m$, the wire is attracted to the wall;

$B_K = 3.572 \cdot 10^{-4}\ T$;

$w_K = 50.8 \cdot 10^{-3}\ J/m^3$, $w_{K'} = 3.36 \cdot 10^{-3}\ J/m^3$.