UDC 517.987.4

Computing the Functional Integral over Nonnegative Frontier of the Null Center Unit Ball

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Abstract

Romanuke V. V. Module for Computing the Functional Integral over Nonnegative Frontier of the Null Center Unit Ball. *There has been stated a problem of finding the functional integral over nonnegative frontier of the null center unit ball, being the set of all the possible probability density distribution functions. The suggested method of computing such functional integral has been based on constructing its integral sum after breaking the space-time discretization grid and differentiating the elements of the functional set by the distance between their sorted norms. The developed in MATLAB module works with the preloaded set of the space-time quantized functions and the parameter of norming them.*

Foresay and problem headline

Functional integration takes its leading positions in the math apparatus of the theoretical nuclear physics, including quantum theory [1 — 7]. And functional (or continual) integrals were adjusted for the control theory, where in the phase space there may be needful to obtain the sum of all the trajectories, which allow for the phase point to get of the initial position and come into the terminal position [1, 2, 4, 7, 8]. However, the functional integration may be applied practically not only within the nuclear quantum physics or control theory. Within antagonistic game theory there [9, 10] had been stated a problem of finding the global aftermath for a player by its selecting the pure or mixed strategy, when the other player makes its selection whatever. This global aftermath lies in finding the specific sum from the double integrals

$$\int\limits_X\int\limits_Y p(x)q(y)K(x,\,y)\,dxdy = V\big(p(x),\,q(y)\big),\quad (1)$$

taken over the whole set of the probability density distribution functions $p(x)$ or $q(y)$, defined on the Borelean subsets X or Y of $\mathbb{R}$, where $x \in X$ and $y \in Y$ are the pure strategies of the first and second players correspondingly, and surface $K(x,\,y)$ is the game kernel, defined on $X \times Y \subset \mathbb{R}^2$. Certainly, for preventing the infinity, each component of such sum must be multiplied by some infinitesimal, that is by the differential of the probability density distribution function, being here $p(x)$ or $q(y)$. So, the problem headline is to find the integral

$$\int\limits_{\mathscr{Y}}\left(\int\limits_X\int\limits_Y p(x)q(y)K(x,\,y)\,dxdy\right)d\big[q(y)\big] =$$

$$= \int\limits_{\mathscr{Y}} V\big(p(x),\,q(y)\big)\,d\big[q(y)\big] \quad (2)$$

for having the first player global aftermath by its selecting the strategy $p(x)$, and (or) the integral

$$\int\limits_{\mathscr{X}}\left(\int\limits_X\int\limits_Y p(x)q(y)K(x,\,y)\,dxdy\right)d\big[p(x)\big] =$$

$$= \int\limits_{\mathscr{X}} V\big(p(x),\,q(y)\big)\,d\big[p(x)\big] \quad (3)$$

for the second player global aftermath by its selecting the strategy $q(y)$, where $\mathscr{X}$ and $\mathscr{Y}$ are the sets of all the possible probability density distribution functions $p(x)$ and $q(y)$ correspondingly. Speaking generally, there is the problem to find the functional integral of the type

$$\int\limits_{f(t)\in\mathscr{F}}\left(\int\limits_T \psi\big[f(t),\,t\big]\,dt\right)d\big[f(t)\big], \quad (4)$$

taken over the set

$$\left\{f(t)\in\mathbb{L}_1(T):\, f(t)\geqslant 0\ \forall\, t\in T\subset\mathbb{R}\right\}\bigcap$$

$$\bigcap\left\{f(t)\in\mathbb{L}_1(T):\, \int\limits_T f(t)\,dt = 1\right\} =$$

$$= \mathscr{F} \subset \mathbb{B}[0,\,1] \subset \mathbb{L}_1(T), \quad (5)$$

having coincided with the nonnegative frontier of the null center $f(t)=0$ unit ball [11, 12].

Analysis of known origins on functional integration

Some analytical ways of finding the functional integral were discussed in [8, 13 — 15], though there had been investigated a separate class of functional integrals on Wiener or Gaussian measure, and also on conditional Wiener measure with the weight [16, 17]. Various numerical methods from [8, 18] allow to take wider class of functional integrals, but they somehow or other are related to probability measures with predetermined correlation functions. But the common widespread technique in the numerical way is the space-time discretization, approximately substituting the continual integration with the finite number of ordinary Riemannian integrals by applying the Monte-Carlo method [18], though there are huge time resources spendings [18, 19]. Another way for executing the functional integration is constructing the approximate formulas, turning to the exact statements on the determined class of functionals, where [18, 19], particularly, for the continual integrals over Gaussian measures there had been got new approximate formulas, turned out to be exact on the class of functional polynomials of any power. The obtained formulas in the particular case of the conditional Wiener measure [18, 20] had been used for calculating some values in the Euclidean quantum mechanics with approximate computation of the Feynman integrals without time discretization, where quadrature formulas brought the time saving and desired precision. Furthermore, there in [18] had been obtained the approximate formulas for multiple continual integrals over conditional Wiener measure with the weight, where also investigated the measure of the continual integration within the two-dimensional Euclidean quantum field theory with polynomial interactions of boson fields [4, 6, 7, 19]. Nevertheless, there have not been exposed the origins, where the proposed ways of numerical functional integration computing would have been implemented within some math program environment [18, 19]. Moreover, the integral (4), even not regarding its simplicity, has not been programmatically implemented until now.

Paper line destination and main points

In the initial approximation, while integrating (4), the Borelean subset $T \subset \mathbb{R}$ may be predefined as the unit segment $[0;\,1]$. Then this paper line destination is to develop the program module for computing the functional integral (4) by $T = [0;\,1]$. For this the following points should be accomplished successively. Firstly, there should be stated properly and digitized the differential $d\left[f(t)\right]$ within $\mathbb{B}[0,\,1]$. Secondly, there should be stated and digitized the integral sum for the functional integral (4). And thirdly, that actually must be accomplished primarily, the number of the integral sum components is defined by the space-time discretization grid and its structure along the space dimension. At last, those digitally stated categories will be typed accordingly within the powerful math environment MATLAB for running the module for digital functional integration from the MATLAB Command Window line.

Breaking the space-time into grid

Before digitizing the set (5) from the infinite number of functions of the half-ball frontier towards the finite number of the time-sampled staircase functions [18], each function $f(t) \in \mathscr{F}$ is going to be essentially sampled. For this the unit segment $[0;\,1]$ is divided into N equal segments

$$\left\{[t_{i-1};\, t_i]\right\}_{i=1}^{N} = \left\{\left[\frac{i-1}{N};\, \frac{i}{N}\right]\right\}_{i=1}^{N} \qquad (6)$$

by as great N as needful or possible (that depends on the computation resources, duration resources or on all taken together). Then the time differential in the functional-integrand of (4) is

$$dt \approx t_i - t_{i-1} = \frac{i}{N} - \frac{i-1}{N} = \frac{1}{N} \qquad (7)$$

approximately by

$$\max_{\substack{i=\overline{1,\,N}\\ N\to\infty}} \left(t_i - t_{i-1}\right) \to 0\,. \qquad (8)$$

So, the integrand of the functional in (4) on the segment $[t_{i-1};\, t_i]$ is

$$\psi\left[f(t),\, t\right] \approx \psi\left[f(t_i),\, t_i\right] \quad \forall\, i = \overline{1,\, N}\,. \qquad (9)$$

Then the functional-integrand of (4) is

$$\int_0^1 \psi\left[f(t),\, t\right] dt \approx \sum_{i=1}^{N} \psi\left[f(t_i),\, t_i\right]\left(t_i - t_{i-1}\right) =$$

$$= \sum_{i=1}^{N} \psi\left[f(t_i),\, t_i\right]\frac{1}{N} = \frac{1}{N}\sum_{i=1}^{N} \psi\left[f(t_i),\, t_i\right] =$$

$$= \frac{1}{N}\sum_{i=1}^{N} \psi\left[f\left(\frac{i}{N}\right),\, \frac{i}{N}\right]. \qquad (10)$$

Going further, suppose, that the time-sampled function $f(t) \in \mathscr{F}$ has M quantization levels. First of them is that $f(t) = 0$. About the rest of the $M-1$ quantization levels there is the lemma.

Lemma 1. If the time-sampled function $f(t) \in \mathscr{F}$ has M equidistant quantization levels, starting with the zeroth, then on the m-th quantization level, if only $f(t) \neq 0$ there,

$$f(t) = (m-1)\frac{N}{M-1},\; m = \overline{1,\, M}\,. \qquad (11)$$

Proof. Let $f(t)$ correspond to the unit probability in the point $t = \tau$. Then it is the Dirac function: $f(t) = \delta(t-\tau)$ and $\int_0^1 \delta(t-\tau)\,dt = 1$. But with (10) there should be

$$\int_0^1 \delta(t-\tau)\,dt \approx \sum_{i=1}^{N} \delta(t_i - \tau)\left(t_i - t_{i-1}\right) =$$

$$=\frac{1}{N}\sum_{i=1}^{N}\delta(t_i-\tau)=1 \text{ by } \tau\in\{t_i\}_{i=1}^{N}, \qquad (12)$$

what means that $\delta(t_i-\tau)\approx N$ by $\tau=t_i$ or even $\forall\,\tau\in[t_{i-1};\,t_i]$ and $\delta(t_i-\tau)=0$ $\forall\,\tau\notin[t_{i-1};\,t_i]$. So, on the last M-th quantization level $f(t)=0$ or $f(t)=N$ and this is the maximal value of the quantized function $f(t)\in\mathscr{F}$. If those quantization levels are equidistant, then the difference between two neighboring levels is $\frac{N-0}{M-1}=\frac{N}{M-1}$. And on the m-th quantization level, $m=\overline{1,M}$, if only $f(t)\neq 0$ there, $f(t)=(m-1)\frac{N}{M-1}$. The lemma has been proved.

Note, that here, as a corollary, the space-time quantized function $f(t)\in\mathscr{F}$ may be better represented in the form of the discrete probability distribution.

Theorem 1. For the time-sampled function $f(t)\in\mathscr{F}$ with M equidistant quantization levels, starting with zeroth, the probability $P_i=P([t_{i-1};\,t_i])$ on the segment $[t_{i-1};\,t_i]$ is found as $\frac{f(t_i)}{N}$ and $P_i\in\left\{0,\,\frac{m-1}{M-1}\right\}$.

Proof. It is just

$$P_i=P([t_{i-1};\,t_i])=\int_{t_{i-1}}^{t_i}f(t)\,dt\approx\int_{t_{i-1}}^{t_i}f(t_i)\,dt=$$

$$=f(t_i)\int_{t_{i-1}}^{t_i}dt=f(t_i)[t_i-t_{i-1}]=\frac{f(t_i)}{N}. \qquad (13)$$

And as on the m-th quantization level the value $f(t_i)\in\left\{0,\,(m-1)\frac{N}{M-1}\right\}$ then $P_i\in\left\{0,\,\frac{m-1}{M-1}\right\}$ on this level. The theorem has been proved.

The set of the space-time quantized functions $f(t)\in\mathscr{F}$ may be accumulated with the MATLAB script “general_ri_N10” (figure 1) through the corresponding discrete probability distributions, presented for $N=10$ and $M=17$. For the greater N and M this code is obviously added with proper nested loops.

However, the number of the space-time quantized probability density distribution functions $f(t)\in\mathscr{F}$ is not strictly determined by the integers N and M. It is determined by some operation, allowing to discriminate each couple of probability density distribution functions. Such operation runs with differentiating the function $f(t)\in\mathscr{F}$.

Digitized differential $d[f(t)]$

One of the simplest ways to define the differential over the functional set is to line up the sampled elements of this set in the ascending order. After such conversion from the functional set into some compact closed subset of $\mathbb{R}$ there is the single way to define the differential as the difference between the current and foregoing element, where module is not yet needed due to the ascending order. The conversion of the function $f(t)\in\mathscr{F}$ to some nonnegative value here cannot be ruled over the norm in the space $\mathbb{L}_1[0;\,1]$ as then there would be the same norm value for any function $f(t)\in\mathscr{F}$. Then it should be accepted the norm of the space $\mathbb{L}_\beta[0;\,1]$ by $\beta>1$. And then

$$\|f(t)\|=\left(\int_0^1|f(t)|^\beta\,dt\right)^{\frac{1}{\beta}}=\left(\int_0^1[f(t)]^\beta\,dt\right)^{\frac{1}{\beta}}\approx$$

$$\approx\left(\sum_{i=1}^{N}[f(t_i)]^\beta(t_i-t_{i-1})\right)^{\frac{1}{\beta}}=$$

$$=\left(\frac{1}{N}\sum_{i=1}^{N}[f(t_i)]^\beta\right)^{\frac{1}{\beta}}. \qquad (14)$$

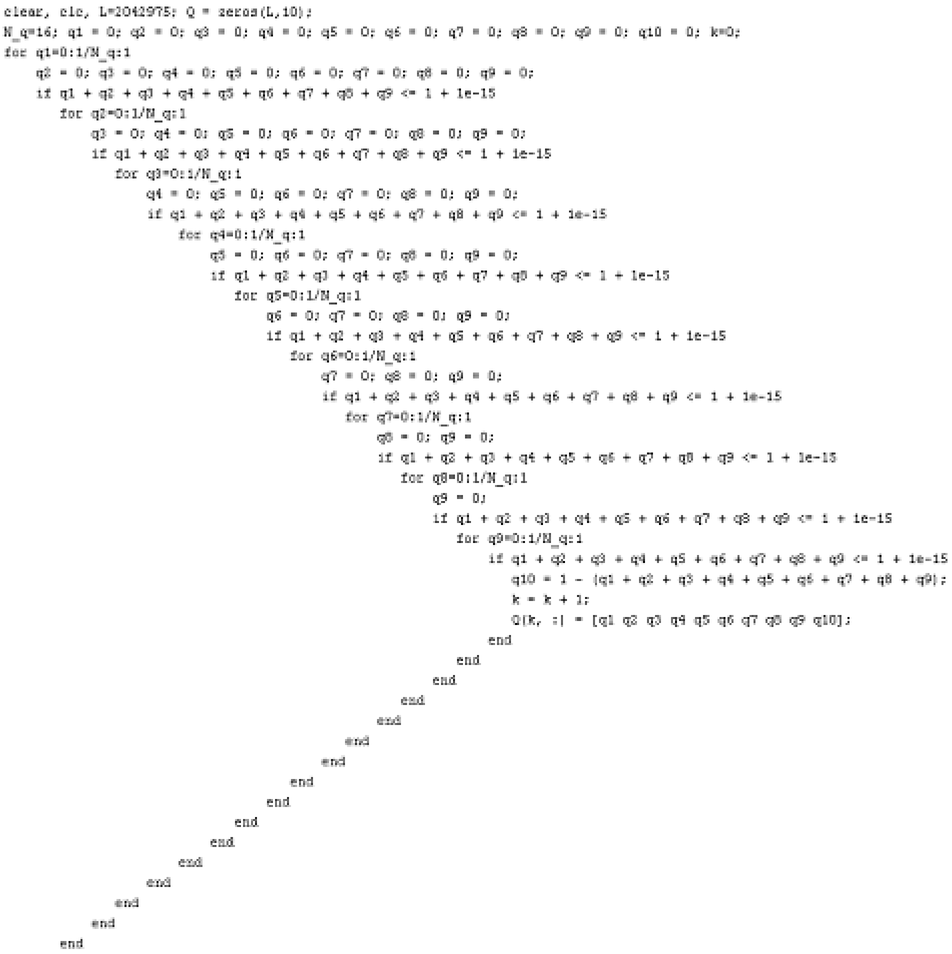
However, it is easy to demonstrate that by $\beta>1$ the integral (14) turns to infinity for $f(t)=\delta(t-\tau)$. Then it will be natural to redefine the norm (14) as

$$\|f(t)\|=\frac{\left(\int_0^1[f(t)]^\beta\,dt\right)^{\frac{1}{\beta}}}{\max\limits_{f(t)\in\mathscr{F}\subset\mathbb{B}[0,\,1]\subset\mathbb{L}_\beta[0;\,1]}\left(\int_0^1[f(t)]^\beta\,dt\right)^{\frac{1}{\beta}}}=$$

$$=\frac{\left(\int_0^1[f(t)]^\beta\,dt\right)^{\frac{1}{\beta}}}{\left(\int_0^1[\delta(t-\tau)]^\beta\,dt\right)^{\frac{1}{\beta}}}\approx\frac{\left(\frac{1}{N}\sum_{i=1}^{N}[f(t_i)]^\beta\right)^{\frac{1}{\beta}}}{\left(\frac{1}{N}N^\beta\right)^{\frac{1}{\beta}}}=$$

$$=\frac{\left(\sum_{i=1}^{N}[f(t_i)]^\beta\right)^{\frac{1}{\beta}}}{\left(N^\beta\right)^{\frac{1}{\beta}}}=\frac{1}{N}\left(\sum_{i=1}^{N}[f(t_i)]^\beta\right)^{\frac{1}{\beta}}. \qquad (15)$$

Suppose, that after getting the norm (15), there are the K space-time quantized functions $f_k(t)\in\mathscr{F}$, arranged as $\{f_k(t)\}_{k=1}^{K}$ in ascending order of their norms (15). Then the differential



```
clear, clc, L=2042975; Q = zeros(L,10);
N_q=16; q1 = 0; q2 = 0; q3 = 0; q4 = 0; q5 = 0; q6 = 0; q7 = 0; q8 = 0; q9 = 0; q10 = 0; k=0;
for q1=0:1/N_q:1
    q2 = 0; q3 = 0; q4 = 0; q5 = 0; q6 = 0; q7 = 0; q8 = 0; q9 = 0;
    if q1 + q2 + q3 + q4 + q5 + q6 + q7 + q8 + q9 <= 1 + 1e-15
        for q2=0:1/N_q:1
            q3 = 0; q4 = 0; q5 = 0; q6 = 0; q7 = 0; q8 = 0; q9 = 0;
            if q1 + q2 + q3 + q4 + q5 + q6 + q7 + q8 + q9 <= 1 + 1e-15
               for q3=0:1/N_q:1
                   q4 = 0; q5 = 0; q6 = 0; q7 = 0; q8 = 0; q9 = 0;
                   if q1 + q2 + q3 + q4 + q5 + q6 + q7 + q8 + q9 <= 1 + 1e-15
                       for q4=0:1/N_q:1
                           q5 = 0; q6 = 0; q7 = 0; q8 = 0; q9 = 0;
                           if q1 + q2 + q3 + q4 + q5 + q6 + q7 + q8 + q9 <= 1 + 1e-15
                              for q5=0:1/N_q:1
                                  q6 = 0; q7 = 0; q8 = 0; q9 = 0;
                                  if q1 + q2 + q3 + q4 + q5 + q6 + q7 + q8 + q9 <= 1 + 1e-15
                                     for q6=0:1/N_q:1
                                         q7 = 0; q8 = 0; q9 = 0;
                                         if q1 + q2 + q3 + q4 + q5 + q6 + q7 + q8 + q9 <= 1 + 1e-15
                                            for q7=0:1/N_q:1
                                                q8 = 0; q9 = 0;
                                                if q1 + q2 + q3 + q4 + q5 + q6 + q7 + q8 + q9 <= 1 + 1e-15
                                                   for q8=0:1/N_q:1
                                                       q9 = 0;
                                                       if q1 + q2 + q3 + q4 + q5 + q6 + q7 + q8 + q9 <= 1 + 1e-15
                                                          for q9=0:1/N_q:1
                                                              if q1 + q2 + q3 + q4 + q5 + q6 + q7 + q8 + q9 <= 1 + 1e-15
                                                                 q10 = 1 - (q1 + q2 + q3 + q4 + q5 + q6 + q7 + q8 + q9);
                                                                 k = k + 1;
                                                                 Q(k, :) = [q1 q2 q3 q4 q5 q6 q7 q8 q9 q10];
                                                              end
                                                          end
                                                       end
                                                   end
                                                end
                                            end
                                         end
                                     end
                                  end
                              end
                           end
                       end
                   end
               end
            end
        end
```

Figure 1 — The script for accumulating all the discrete probability distributions with $N=10$ and $M=17$ into the set

$$d\left[f(t)\right] \approx \left\|f_k(t)\right\| - \left\|f_{k-1}(t)\right\| =$$

$$= \frac{1}{N}\left(\sum_{i=1}^{N}\left[f_k(t_i)\right]^{\beta}\right)^{\frac{1}{\beta}} - \frac{1}{N}\left(\sum_{i=1}^{N}\left[f_{k-1}(t_i)\right]^{\beta}\right)^{\frac{1}{\beta}} \qquad (16)$$

in the k-th function, conversed into the point $\left\|f_k(t)\right\|$ by $k=\overline{2, K}$. Surely, that the difference in (16) is not the same with changing k.

Integral sum

With the differential (16) the integral sum for the functional integral (4) over nonnegative frontier of the null center unit ball by $T=[0;\,1]$ is

$$\int\limits_{f(t)\in\mathscr{F}}\left(\int\limits_{0}^{1}\psi\left[f(t),\, t\right]dt\right)d\left[f(t)\right] \approx$$

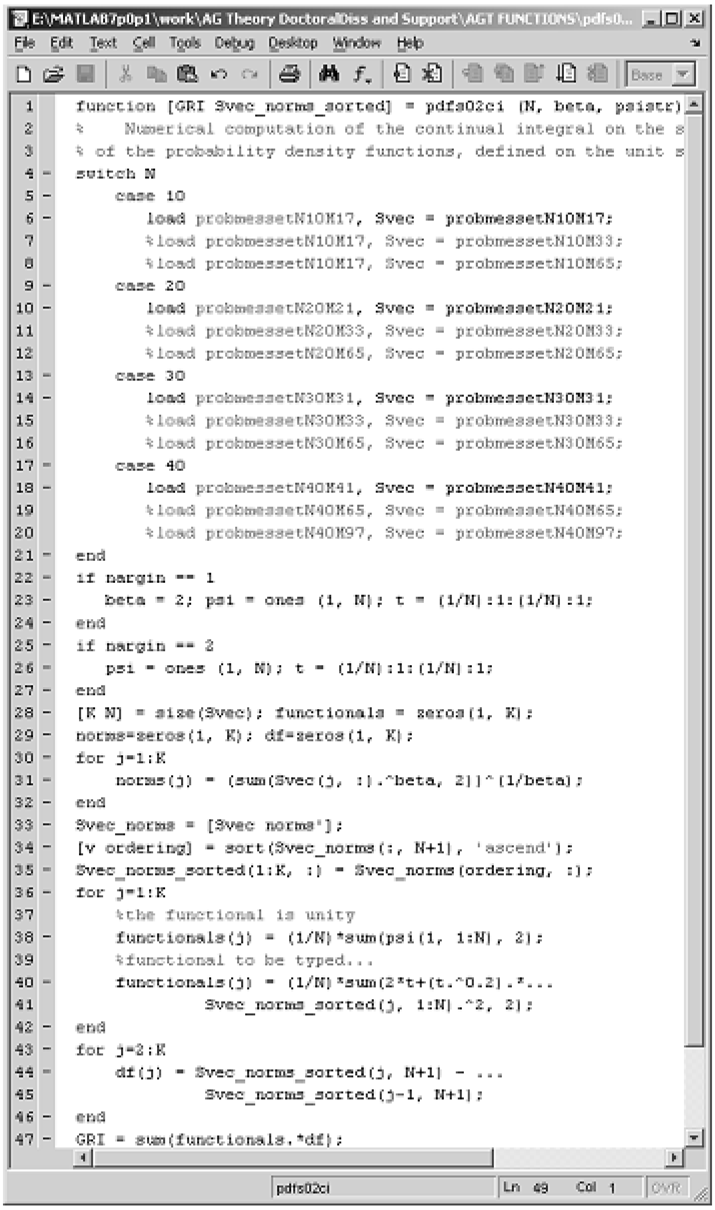
$$\approx \sum_{k=2}^{K}\left(\frac{1}{N}\sum_{i=1}^{N}\psi\left[f_k\left(\frac{i}{N}\right), \frac{i}{N}\right]\right)\left(\left\|f_k(t)\right\| - \left\|f_{k-1}(t)\right\|\right) =$$

$$= \sum_{k=2}^{K}\left(\frac{1}{N}\sum_{i=1}^{N}\psi\left[f_k\left(\frac{i}{N}\right), \frac{i}{N}\right]\right)\times$$

$$\times\left[\frac{1}{N}\left(\sum_{i=1}^{N}\left[f_k(t_i)\right]^{\beta}\right)^{\frac{1}{\beta}} -\right.$$

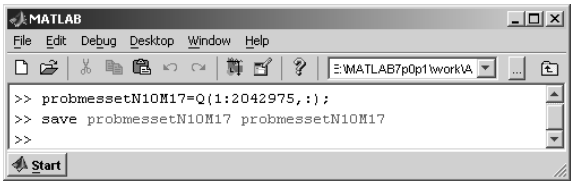
$$\left. -\frac{1}{N}\left(\sum_{i=1}^{N}\left[f_{k-1}(t_i)\right]^{\beta}\right)^{\frac{1}{\beta}}\right]. \qquad (17)$$

Now on the figure 2 there is the code of the module “pdfs02ci” for computing the integral (4) as its sum (17). This module uses the set, saved with the script “general_ri_N10” (figure 3), when the user enters by it typing just the number N. Actually, there is possibility to change the being preloaded set manually within this module by typing the comment at the undesirable preload.


E:\MATLAB7p0p1\work\AG Theory DoctoralDiss and Support\AGT FUNCTIONS\pdfs0...
File Edit Text Cell Tools Debug Desktop Window Help
Base
pdfs02ci
Ln 49 Col 1
OVR

```
function [GRI Svec_norms_sorted] = pdfs02ci (N, beta, psistr)
%    Numerical computation of the continual integral on the s
% of the probability density functions, defined on the unit s
switch N
    case 10
       load probmessetN10M17, Svec = probmessetN10M17;
       %load probmessetN10M17, Svec = probmessetN10M33;
       %load probmessetN10M17, Svec = probmessetN10M65;
    case 20
       load probmessetN20M21, Svec = probmessetN20M21;
       %load probmessetN20M33, Svec = probmessetN20M33;
       %load probmessetN20M65, Svec = probmessetN20M65;
    case 30
       load probmessetN30M31, Svec = probmessetN30M31;
       %load probmessetN30M33, Svec = probmessetN30M33;
       %load probmessetN30M65, Svec = probmessetN30M65;
    case 40
       load probmessetN40M41, Svec = probmessetN40M41;
       %load probmessetN40M65, Svec = probmessetN40M65;
       %load probmessetN40M97, Svec = probmessetN40M97;
end
if nargin == 1
   beta = 2; psi = ones (1, N); t = (1/N):1:(1/N):1;
end
if nargin == 2
   psi = ones (1, N); t = (1/N):1:(1/N):1;
end
[K N] = size(Svec); functionals = zeros(1, K);
norms=zeros(1, K); df=zeros(1, K);
for j=1:K
    norms(j) = (sum(Svec(j, :).^beta, 2))^(1/beta);
end
Svec_norms = [Svec norms'];
[v ordering] = sort(Svec_norms(:, N+1), 'ascend');
Svec_norms_sorted(1:K, :) = Svec_norms(ordering, :);
for j=1:K
    %the functional is unity
    functionals(j) = (1/N)*sum(psi(1, 1:N), 2);
    %functional to be typed...
    functionals(j) = (1/N)*sum(2*t+(t.^0.2).*...
             Svec_norms_sorted(j, 1:N).^2, 2);
end
for j=2:K
    df(j) = Svec_norms_sorted(j, N+1) - ...
             Svec_norms_sorted(j-1, N+1);
end
GRI = sum(functionals.*df);
```

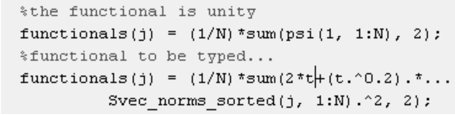
Figure 2 — The module “pdfs02ci” code for computing the integral (4) approximately by the fixed N and M on some $\beta > 1$



```
>> probmessetN10M17=Q(1:2042975,:);
>> save probmessetN10M17 probmessetN10M17
>>
```

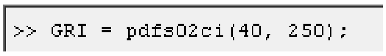
Figure 3 — Saving the set of the space-time quantized functions $f(t) \in \mathscr{F}$ in the form of discrete probability distributions by the fixed integers N and M after having run the script “general_ri_N10”

If the parameter β is omitted, then $\beta = 2$ by default. The integrand of the functional (10) may be typed as a insertion string within the module (figure 4), though by default it is put to unity. Then the module is saved and rerun from the MATLAB Command Window line (figure 5).



```
%the functional is unity
functionals(j) = (1/N)*sum(psi(1, 1:N), 2);
%functional to be typed...
functionals(j) = (1/N)*sum(2*t+(t.^0.2).*...
         Svec_norms_sorted(j, 1:N).^2, 2);
```

Figure 4 — Typing the string of the integrand $\psi\left[f(t), t\right] = 2t + t^{0.2}\left(\frac{f(t)}{N}\right)^2$ of the functional (10) within the module “pdfs02ci” code



```
>> GRI = pdfs02ci(40, 250);
```

Figure 5 — Running the module “pdfs02ci” from the MATLAB Command Window line

Some examples on applying the developed module, reflecting the functional integration results with

$$\psi\left[f(t),t\right]=2t+1, \tag{18}$$

$$\psi\left[f(t),t\right]=f(t), \tag{19}$$

$$\psi\left[f(t),t\right]=t^2+f(t), \tag{20}$$

$$\psi\left[f(t),t\right]=e^t, \tag{21}$$

$$\psi\left[f(t),t\right]=\frac{1}{t^2+f(t)}, \tag{22}$$

$$\psi\left[f(t),t\right]=\frac{e^{-t}}{2+f(t)}, \tag{23}$$

$$\psi\left[f(t),t\right]=\frac{1-e^{-t}}{2-f(t)}, \tag{24}$$

$$\psi\left[f(t),t\right]=\frac{4t}{2f(t)-3t^2-7}, \tag{25}$$

have been screen-shot off the MATLAB Command Window alongside the elapsed times of their accomplishment, though it had been taken only $N=10$ and $M=17$. Those elapsed times are grouped within the table 1.

It remained only to note, that the precision of such computation strongly depends on the space-time discretization grid frequency, and may be increased as high as needed by working with greater integers N and M. But if the time resource is restricted, and the integral (4) needs to be found faster, then there is another problem of the precision and computation time reconciliation.

Table 1
Elapsed times for computing the functional integrals with integrands (18) — (25) of the functional in (4)

Formula for $\psi\left[f(t),t\right]$	Elapsed time, seconds
(18)	186.531
(19)	190.797
(20)	242.562
(21)	377.187
(22)	278.094
(23)	276.485
(24)	307.453
(25)	228.953

Conclusion

The stated numerical method for taking the functional integral (4) and the developed within MATLAB program module “pdfs02ci” for its accomplishment are the fundamental basis for further working on integration over functional spaces or infinite-dimensional spaces, that here has not been strung with some Gaussian or Wiener measures. Undoubtedly, that would have been splendiferous to construct the exact analytic formulas for taking the functional integral over nonnegative frontier of the null center unit ball, but this cannot be visible of how to do it as there is put the general type of the functional, being the inner integral (10). Thus for now only numerically there is possibility to evaluate the specific sum of all, say, mathematical expectations (1) of the first player payoff as (2) or (3), and then speak about the global aftermath for a player. In further investigations and programming it should be thought of how to compute the differential $d\left[f(t)\right]$ without sorting the norms (15) or other. That would give a way to compute functional integrals without accumulating the set of space-time quantized function $f(t)\in\mathscr{F}$, because this accumulation may last for very long period, and then the total elapsed time of the functional integral computation might be shortened. Besides, the computer memory (RAM) would be almost free of processing the large arrays, as in this way functional integral computation might be organized as the consecutive process of accumulating the integral sum (17), that is computation on the j-th iteration (not a loop, at all) of the sum

$$I_{j+1}=I_j+ \\ +\left(\frac{1}{N}\sum_{i=1}^{N}\psi\left[f_j\left(\frac{i}{N}\right),\frac{i}{N}\right]\right)\times \\ \times\left(\left\|f_j(t)\right\|-\left\|f_{j-1}(t)\right\|\right)\quad\forall\ j=\overline{1,K-1} \tag{26}$$

by $I_1=0$. And all that will be realizable with another redefining the differential (16), involving, maybe, some other norms.

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Received 30.03.2010