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…rametric Equations of a Plane and the Problem … Description of Plane Curves in Space R^3

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…раметрические уравнения плоскости и задача …исания плоских кривых в пространстве R^3

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…раметричні рівняння площини і завдання опису …оских кривих у просторі R^3

…m of this article is the application of the parametric equations of the plane for the description of plane curves in … R^3. It is found a simple way of parameterization of the plane, which allows you to enter a local coordinate …m on the plane in space R^3. In this local coordinate system of equations of plane curves look easy. In the paper …nsider the parameterization of the plane lines on the example of second-order curves. This view may be useful …alytical geometry and mechanics.

…ywords: plane, curve, parametric, normal vector, plane line, the coordinate system.

…ью статьи является применение параметрических уравнений плоскости для описания плоских кривых в …странстве R^3. Найден простой способ параметризации плоскости, который позволяет ввести …льную систему координат на плоскости. В такой локальной системе координат уравнения плоских …вых выглядят просто. В данной работе рассмотрена параметризация плоских линий на примере …вых второго порядка. Это представление может быть полезным для решения задач аналитической …етрии, механики.

…ючевые слова: плоскость, кривая, параметрический, нормальный вектор, …ская линия, система координат.

…ою статті є застосування параметричних рівнянь площини для опису плоских кривих у просторі R^3. …йдено простий спосіб параметризації площини, який дозволяє ввести локальну систему координат на …щині. У цій локальній системі координат рівняння плоских кривих мають простий вигляд. У робот… …глянута параметризація плоских ліній на прикладі кривих другого порядку. Це подання може бути …рисним для вирішення завдань аналітичної геометрії та механіки.

Ключові слова: площина, крива, параметричний, нормальний вектор, плоска лінія, …стема координат.

Introduction

The parametric equations of the plane are practically not used in the course of analytic geometry, but are one of the important methods of analytical description of the plane. The matter is that in analytical geometry exists, at least, five various representations of a plane, each of which solves the certain problems [1-4].

Using the equation of a plane passing through three given points make it easy to formulate and solve the problem of the parameterization of arbitrary plane. This problem is a particular case of the parameterization of a smooth surface of arbitrary shape in space R^3. In general, this problem is discussed in the section of differential geometry involving apparatus of differential calculus [5-6].

In turn, not involving such complex concepts as the first, second quadratic forms, the description of plane curves in Cartesian coordinates can be solved by means of vector algebra. This approach is made available to a wider audience, not only for specialists in differential geometry.

The aim is a simple way of introducing the parametric equations of the plane and the parametric method of describing plane curves in space R^3. This paper gives examples of second-order curves. Their description in the local coordinate system is very simple. Their equations have the canonical form.

1 Parametric representation of the equation of a plane

The parametric equations of the plane are most easily obtained from the equation of the plane through three given points that do not lie on a straight line

$$M_0(x_0,y_0,z_0),\ M_1(x_1,y_1,z_1),\ M_3(x_3,y_3,z_3).$$

Let the point $M(x,y,z)$ be an arbitrary point in the plane. Then three vectors are coplanar (Fig.1).

$$\begin{aligned}\mathbf{M_0M} &= \{x-x_0,\ y-y_0,\ z-z_0\}\\ \mathbf{M_0M_1} &= \{x_1-x_0,\ y_1-y_0,\ z_1-z_0\}\\ \mathbf{M_0M_2} &= \{x_2-x_0,\ y_2-y_0,\ z_2-z_0\}\end{aligned}$$

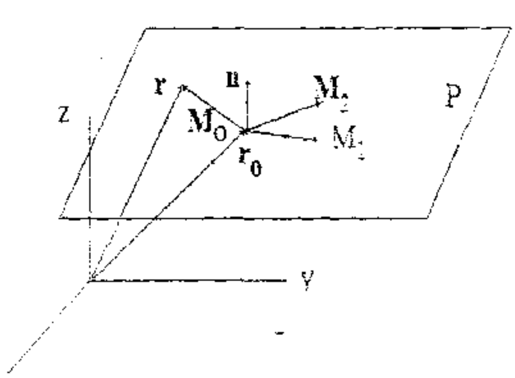

Fig.1 – Derivation of parametric equations of the plane

The mixed product of any coplanar vectors is zero, i.e. $\mathbf{M_0M}\cdot\mathbf{M_0M_1}\times\mathbf{M_0M_2}=0$.

Using the coordinates this condition can be written as the determinant

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \\ x_2-x_0 & y_2-y_0 & z_2-z_0 \end{vmatrix} = 0 .$$

… denote $s_1(l_1, m_1, n_1) = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$, $s_2(l_2, m_2, n_2) = (x_2 - x_0, y_2 - y_0, z_2 - z_0)$ … ding to the notation we write the determinant

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 .$$

… ality to zero of the determinant means that its lines (or columns), and in this … are proportional. This fact means that, for example, the first line of a determinant … linear combination of the other two with some coefficients α and β

$$\begin{cases} x - x_0 = \alpha l_1 + \beta l_2, \\ y - y_0 = \alpha m_1 + \beta m_2, \\ z - z_0 = \alpha n_1 + \beta n_2. \end{cases} \Rightarrow \begin{cases} x = x_0 + \alpha l_1 + \beta l_2, \\ y = y_0 + \alpha m_1 + \beta m_2, \\ z = z_0 + \alpha n_1 + \beta n_2. \end{cases} \quad (2)$$

In the vector form we have the simple equality

$$r = r_0 + \alpha s_1 + \beta s_2 . \quad (3)$$

… α and β are parameters of the plane, and the equations (2) and (3) are called *the parametric equations of the plane*.

The geometric meaning of the parametric equations of the plane

Explain the geometric meaning of the parametric equations of the plane. For this we … vectors s_1 and s_2 in the plane P and take into account that the point M_0 corresponds … value of parameters $\alpha = \beta = 0$. If $\alpha = \beta = 1$ the point M_1 is at the top of the parallelogram … ed by the vectors s_1 and s_2 (Fig. 2). The further scenario consists in that, at the any fixed … es of parameters α *and* β vectors s_1 and s_2 are "extended" accordingly in α *and* β … (if $\alpha > 1$, $\beta > 1$) or "shortened" in α *and* β times (if $\alpha < 1$, $\beta < 1$), and directed opposite … negative values of parameters $\alpha < 0$ и $\beta < 0$. Since the values of the parameters … $-\infty < \beta < \infty$ are independent from each other, then the vector r runs through all … plane P.

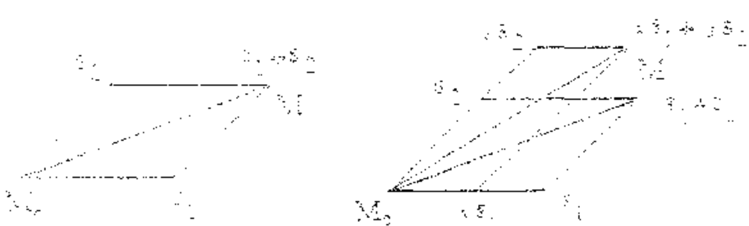

Fig. 2 – Geometrical meaning of the parametrical equations of a plane

From the above arguments follows, if parameters α *and* β will be connected among themselves by functional dependence the vector **r** will describe some flat line L on the plane P.

Let's consider the simplest case when parameters are equal $\alpha = \beta = t$. The equation (3) will become

$$\mathbf{r} = \mathbf{r}_0 - \mathbf{s}t, \quad \mathbf{s} = \mathbf{s}_1 - \mathbf{s}_2. \qquad (4)$$

These are the parametrical equations of the straight line which is passing through the point M_0 with the directed vectors $\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2$. Let's pay attention to the vectors $\mathbf{s}_1$ and $\mathbf{s}_2$, which allow you to orient any straight line in a plane in any desired direction.

3. Parametric representation of plane curves in space R^3

A choice of the vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ is simple. They should not be collinear.

It is convenient to choose vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ perpendicular each others $\mathbf{s}_1 \perp \mathbf{s}_2$. In that case, for example, the ellipse equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ will be $\frac{x^2}{|\mathbf{s}_1|^2} + \frac{y^2}{|\mathbf{s}_2|^2} = 1$. Let's consider a parametric form $x = |\mathbf{s}_1| \cos t$, $y = |\mathbf{s}_2| \sin t$, $0 \le t < 2\pi$. Using (3), we find representation of an ellipse centered at the point M_0 and semi axes $|\mathbf{s}_1| = a$, $|\mathbf{s}_2| = b$ and the normal vector $\mathbf{n} = \mathbf{s}_1 \times \mathbf{s}_2$ in the space R^3.

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{e}_1 a \cos t + \mathbf{e}_2 b \sin t, \qquad (5)$$

where $\mathbf{e}_1 = \mathbf{s}_1 / |\mathbf{s}_1|$, $\mathbf{e}_2 = \mathbf{s}_2 / |\mathbf{s}_2|$ are mutual perpendicular unit vectors.

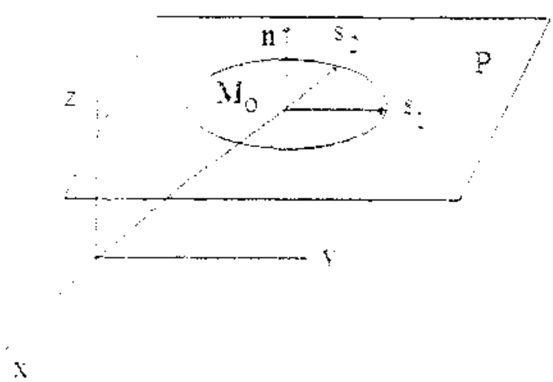

Fig. 3 – Presentation of an ellipse in space

Here are more examples of other second order curves. Parameterize the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ believing $a = |\mathbf{s}_1|$, $b = |\mathbf{s}_2|$ and $x = |\mathbf{s}_1| ch t$, $y = |\mathbf{s}_2| sh t$, $-\infty < t < \infty$. As a result we have

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{e}_1 a ch t + \mathbf{e}_2 b sh t.$$

In the case of a parabola $y^2 = 2px$ suppose $x = \frac{p}{2} t^2$, $y = pt$.

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{e}_1 \frac{p}{2} t^2 + \mathbf{e}_2 pt.$$

In the polar coordinate system $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, therefore

$$\mathbf{r} = \mathbf{r}_o + \rho(\varphi)(\mathbf{e}_1\cos\varphi + \mathbf{e}_2\sin\varphi).$$

…he point M_o in the right focus curve of the second order (ellipse, hyperbole or parab…) …en the equations of curves of the second order in the polar coordinate system will … the form

$$\mathbf{r} = \mathbf{r}_o + \frac{p}{1-\varepsilon\cos\varphi}(\mathbf{e}_1\cos\varphi + \mathbf{e}_2\sin\varphi).$$

…re ε is the eccentricity of a second order curve. At $\varepsilon < 1$ we have an ellipse, at $\varepsilon > 1$ … right branch a hyperbole (for the left branch of a hyperbole in the general equation (8) …uld be put $\rho = \frac{-p}{1+\varepsilon\cos\varphi}$), at $\varepsilon = 1$ we have a parabola, p is the parameter of the curve …the second order (9). In particular, the equation of a circle of radius R is obtained from … general equation (8) if we set $\rho = R$.

$$\mathbf{r} = \mathbf{r}_o + R(\mathbf{e}_1\cos\varphi + \mathbf{e}_2\sin\varphi).$$

… Other representations of parametric equations of the plane

Let's build a local basis of mutually perpendicular unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ on a given plane. …en equation (3) takes the form

$$\mathbf{r} = \mathbf{r}_o + \alpha\mathbf{e}_1 + \beta\mathbf{e}_2. \quad (10)$$

The scalar multiplication of the vectors $\mathbf{e}_1, \mathbf{e}_2$ is satisfied to the equalities $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$ …d $\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1$. Then the scalar multiplication of the vectors $\mathbf{e}_1, \mathbf{e}_2$ by the equation (10) …ves the system of the equations

$$\begin{cases} \mathbf{e}_1 \cdot (\mathbf{r} - \mathbf{r}_o) = \alpha \\ \mathbf{e}_2 \cdot (\mathbf{r} - \mathbf{r}_o) = \beta. \end{cases}$$

We summarize both equalities and denote $\alpha + \beta = t$. Then we get a one-parameter …ctor equation of a plane $(\mathbf{e}_1 + \mathbf{e}_2)\cdot(\mathbf{r} - \mathbf{r}_o) = t$, which corresponds to a family of the …rametrical equations of straight lines in the perpendicular direction to the vector $\mathbf{e}_1 + \mathbf{e}_2$.

Vector multiplication of the vectors $\mathbf{e}_1, \mathbf{e}_2$ is satisfied to the equalities $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 = \mathbf{n}$ …d $\mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_2 = 0$. Then the vector multiplication of the vectors $\mathbf{e}_1, \mathbf{e}_2$ by the equation … gives the next vector system of the equations

$$\begin{cases} \mathbf{e}_1 \times (\mathbf{r} - \mathbf{r}_o) = -\alpha\mathbf{n} \\ \mathbf{e}_2 \times (\mathbf{r} - \mathbf{r}_o) = \beta\mathbf{n} \end{cases}$$

We summarize both the last equalities and denote $-\alpha + \beta = t$. Then we get another …e-parameter vector equation of a plane $(\mathbf{e}_1 + \mathbf{e}_2)\times(\mathbf{r} - \mathbf{r}_o) = t\mathbf{n}$.

We can exclude the parameter t using the triple vector multiplication of the equation $\mathbf{n}\times(\mathbf{e}_1-\mathbf{e}_2)\times(\mathbf{r}-\mathbf{r}_o)$. According to the formula $\mathbf{a}\times\mathbf{b}\times\mathbf{c}=\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})$ we obtain $\mathbf{n}\times(\mathbf{e}_1+\mathbf{e}_2)\times(\mathbf{r}-\mathbf{r}_o)=t\mathbf{n}\times\mathbf{n}=0\Rightarrow(\mathbf{e}_1+\mathbf{e}_2)(\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_o))-(\mathbf{r}-\mathbf{r}_o)(\mathbf{n}\cdot(\mathbf{e}_1+\mathbf{e}_2))=0$.

Now $\mathbf{n}\perp(\mathbf{e}_1+\mathbf{e}_2)$, therefore $\mathbf{n}\cdot(\mathbf{e}_1+\mathbf{e}_2)=0$ and $\mathbf{e}_1+\mathbf{e}_2\neq 0$. As a result we have the equation $\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_o)=0$. This is well-known equation of a plane. In the coordinate representation the equality looks like

$$A(x-x_o)+B(y-y_o)+C(z-z_o)=0.$$

Findings

1. The simple way of introduction of the parametrical equations of a plane is offered.
2. The effective way of the description of plane curves in space is offered.
3. Results of the work are approved on curves of the second order.

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