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A Compact System of Inequalities for the Standard Limits in the Theory of Limits

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Компактная система неравенств для стандартных пределов в теории пределов

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Компактна система нерівностей для стандартних границь у теорії границь

The purpose of the paper is an alternative way of obtaining of the standard limits in the elementary theory of limits. The approach uses two double inequalities $xe^{-x} \le \sin x \le xe^{x}$ and $xe^{-x} \le shx \le xe^{x}$ and the limit transition. The method is applied to both standard limits simultaneously, that makes the theory more universal. Apart from that our theory gives many new representations of the standard limits (total number is 37).
Keywords: theory of limits, standard limits, function, sine, hyperbolic sine, method, inequalities, standard limits, limit transition.

В работе предложен единый подход к стандартным пределам в теории пределов. Подход основан на использовании двойных неравенств $xe^{-x} \le \sin x \le xe^{x}$ и $xe^{-x} \le shx \le xe^{x}$ и предельном переходе в них. Метод применим к обоим стандартным пределам одновременно, что значительно упрощает общепринятые подходы. Кроме того, получены новые следствия из стандартных пределов.
Ключевые слова: теория пределов, стандартные пределы, неравенство, функция, синус, гиперболический синус, двойное неравенство.

У роботі запропоновано підхід, який забезпечує єдиний засіб отримання стандартних границь у теорії границь. Підхід заснований на використанні нерівностей $xe^{-x} \le \sin x \le xe^{x}$ та $xe^{-x} \le shx \le xe^{x}$ і граничному переході в них. Підхід достатньо простий у використанні. Єдина система нерівностей забезпечує одночасне отримання формул першого, другого стандартних границь і, практично, всіх наслідків з них. Більш того, теорія приводить до великої кількості нових наслідків із стандартних границь.
Ключові слова: теорія границь, стандартні границі, функція, синус, гіперболічний синус, нерівність, граничний перехід.

Introduction

The standards limits in mathematical analysis are known as the first and second fundamental (standard) limits. They are used mainly for the determination of derivatives of the elementary functions $\sin x, \cos x, e^x, a^x$. Thus it is built the standard table of derivatives in differential calculus [1], [2]. The standard limits are used practically in all branches of mathematics, also for calculation of an entire class of limits.

Each of the fundamental limits is proved by two different ways. The first standard limit is based on a limit procedure in the trigonometric circle. For the second limit it is used Newton's binomial. Each of the approaches is effective and attractive. Nevertheless they are not universal [1], [2].

Meanwhile there are universal methods for an obtaining of the limits. For example, the first and second limits can be connected one with another by Euler's formula [3].

The next universal method of proof is based on the undetermined coefficient method which allows find limits by the standard expansion of the functions $\sin x$ and e^x [4].

At last there are the approaches that like to the proposal method in the paper. They are based also on inequalities and the limiting in the inequalities. Such methods have some disadvantages that are connected with difficulties of the proof of the inequalities. This is one of the serious disadvantages of the theory.

According to the logic of our consideration the inequalities must be proved by elementary mathematical methods. Other words, the proof must be performed without using of a derivative concept.

We found a simple method to proof the standard limits. This method is based on a specified inequality system and a limiting in the system. The approach may be applied to both limits simultaneously. The proof of each inequality is simple and clear. Besides from the inequalities we have got new important corollaries from the second standard limit.

1 The basic theorem of the method

The main theorem that we use in our theory is well-known theorem about a function $f(x)$ which is enclosed between two given functions $f_1(x)$ and $f_2(x)$ [5], [6].

Theorem. If in any vicinity of the point x_o a function $f(x)$ is satisfied to the inequalities

$$f_1(x) \le f(x) \le f_2(x) \tag{1}$$

and $\lim\limits_{x\to x_o} f_1(x) = \lim\limits_{x\to x_o} f_2(x) = a$ then the limit $\lim\limits_{x\to x_o} f(x)$ exists and it is equal to a.

It is clear, if the function $f(x)$ is continuous at $x = x_o$, then $a = f(x_o)$.

The inequalities (1) have a very important property of symmetry, if all functions of the inequalities are odd. We will replace in the inequalities x to $-x$

$$-f_1(x) \le -f(x) \le -f_2(x) \Rightarrow f_1(x) \ge f(x) \ge f_2(x).$$

The sings of the inequalities (1) have changed.

If the functions are even

$$f_1(-x) \le f(-x) \le f_2(-x) \Rightarrow f_1(x) \le f(x) \le f_2(x) ,$$

The sings of the inequalities (1) have not changed.

Corollary. If in the inequalities (1) $f_1(x) = xg(x)$, $f_2(x) = x\varphi(x)$ and at $x_o = 0$ $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} \varphi(x) = 1$, i.e.

$$xg(x) \le f(x) \le x\varphi(x), \tag{2}$$

then $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = 1$.

We will use the corollary for proof of the standard limits. Besides we will need in the generalization of the corollary. If $\lim_{x\to 0} g(\alpha x) = \lim_{x\to 0}\varphi(\alpha x) = 1$ and

$$\alpha x g(\alpha x) \le f(\alpha x) \le \alpha x \varphi(\alpha x), \quad (3)$$

then there $\lim_{x\to 0} f(\alpha x) = \alpha$ for any $\alpha \ne 0$.

2 The basic Inequalities for standard limits

The trigonometric and hyperbolic sine functions are satisfied to the inequalities (Fig.1).

$$\left[\begin{array}{l} xe^{-x} \le \sin x \le xe^{x}, \\ xe^{-x} \le \sinh x \le xe^{x}. \end{array}\right. \quad (4)$$

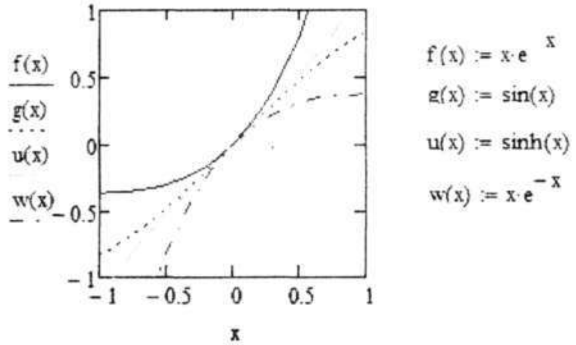

Figure 1– The graphs of the functions of the inequalities (4)

Dividing the inequalities at $x > 0$ we get

$$\left[\begin{array}{l} e^{-x} \le \dfrac{\sin x}{x} \le e^{x}, \\ e^{-x} \le \dfrac{\sinh x}{x} \le e^{x}. \end{array}\right.$$

We take a limit at $x \to 0$. Accounting that $\lim_{x\to 0} e^{-x} = \lim_{x\to 0} e^{x} = 1$ we find according to the corollary

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \quad \lim_{x\to 0}\frac{\sinh x}{x} = 1. \quad (5)$$

The first formula is called the first standard (fundamental) limit, the second represents one of the options of the second standard limit.

The formulas (5) are obtained by the assumption of $x \to +0$ although they are written for an arbitrary $x \to 0$. Evidently the equalities (5) do not change at $x \to -0$. It is clear from that all functions are between of the functions xe^{-x} and xe^{x}. The functions $\sin x$ and shx are odd. Therefore for the negative values of x we have $e^{-x} \ge \frac{\sin x}{x} \ge e^{x}$, $e^{-x} \ge \frac{\sinh x}{x} \ge e^{x}$. Taking the limit at $x \to -0$ we have the equalities (5) again.

The figure (Fig. 1) demonstrates the inequalities (4) geometrically. You can see how the inequalities (2) are changed when the point x changes the sign. All graphs of the functions of the inequalities (4) are enclosed by the graphs only two functions xe^{-x} and xe^{x} (Fig. 1.). It means that the limits (5) do not depend on $x \to +0$ or $x \to -0$.

You can see from the figure also how instead of inequalities (4) we may work with the inequalities

$$xe^{-x} \le \sin x \le \sinh x \le xe^{x}, \qquad (6)$$

from which also the limits (5) are followed. Here we have one more thing. The inequalities (4) need to proof of four inequalities but the inequalities (6) only of three. For this reason the inequalities (6) are more favorable. The proof of the inequalities is given in the appendix.

3 Corollaries from the fundamental limits

The corollaries of the fist fundamental limit are well-known and they are described in many books. We represent the corollaries in the table (Tab. 1) without of a detailed proof.

Table 1 – The corollaries from the first and second fundamental limits

First standard limit $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$		Second standard limit $\lim\limits_{x\to 0}\frac{\sinh x}{x}=1$	
Replacement or changing of the variable	Result	Replacement or changing of the variable	Result
$x=\arcsin y$	$\lim\limits_{x\to 0}\frac{\arcsin x}{x}=1$	$x=a\sinh y$	$\lim\limits_{x\to 0}\frac{a\sinh x}{x}=1$
$\tan x=\frac{\sin x}{\cos x}$, $\lim\limits_{x\to 0}\cos x=1$	$\lim\limits_{x\to 0}\frac{\tan x}{x}=1$	$\tanh x=\frac{\sinh x}{\cosh x}$, $\lim\limits_{x\to 0}\cosh x=1$	$\lim\limits_{x\to 0}\frac{\tanh x}{x}=1$
$x=\arctan y$	$\lim\limits_{x\to 0}\frac{\arctan x}{x}=1$	$x=a\tanh y$	$\lim\limits_{x\to 0}\frac{a\tanh x}{x}=1$

In the table the corollaries from the second limit are represented also. They look like the corresponding corollaries from the first limit. These corollaries are new. Therefore we will consider the proof of the corollaries in details:

$$\lim_{x\to 0}\frac{\tanh x}{x}=\lim_{x\to 0}\frac{\sinh x}{x}\lim_{x\to 0}\frac{1}{\cosh x}=1, \text{ where } \lim_{x\to 0}\frac{1}{\cosh x}=1.$$

In the limit $\lim\limits_{x\to 0}\frac{\sinh x}{x}=1$ we substitute $x=\sinh y$. Then, $y\to 0$ at $x\to 0$ and we have $\lim\limits_{x\to 0}\frac{a\sinh x}{x}=\lim\limits_{y\to 0}\frac{y}{\sinh y}=1/\lim\limits_{y\to 0}\frac{\sinh y}{y}=1$.

In the equality $\lim\limits_{x\to 0}\frac{\tanh x}{x}=1$ we replace $x=\tanh y$. Then $y\to 0$ at $x\to 0$, we get $\lim\limits_{x\to 0}\frac{a\tanh x}{x}=\lim\limits_{y\to 0}\frac{y}{\tanh y}=1/\lim\limits_{y\to 0}\frac{\tanh y}{y}=1$.

To find the connection of $\lim\limits_{x\to 0}\frac{\sinh x}{x}=1$ with the known corollaries from the second fundamental limit we may do it at least two ways. For example, we can use the definition of the function $\sinh x=\frac{e^{x}-e^{-x}}{x}$: $\lim\limits_{x\to 0}\frac{\sinh x}{x}=\frac{1}{2}\lim\limits_{x\to 0}\frac{e^{x}-e^{-x}}{x}=\frac{1}{2}\lim\limits_{x\to 0}\frac{e^{x}-1}{x}-\frac{1}{2}\lim\limits_{x\to 0}\frac{e^{-x}-1}{x}$. In the second limit we replace x by $-y$. As a result we have $\lim\limits_{x\to 0}\frac{e^{-x}-1}{x}=-\lim\limits_{y\to 0}\frac{e^{y}-1}{y}$. From where

$$\lim_{x\to 0}\frac{\sinh x}{x}=\lim_{x\to 0}\frac{e^x-1}{x}. \tag{7}$$

The second way is based on the representation $a\tanh x=\frac{1}{2}\ln\frac{x+1}{1-x}$ [5].

$$\lim_{x\to 0}\frac{a\tanh x}{x}=\frac{1}{2}\lim_{x\to 0}\frac{1}{x}\ln\frac{x+1}{1-x}.=\frac{1}{2}\lim_{x\to 0}\frac{\ln(1+x)}{x}-\frac{1}{2}\lim_{x\to 0}\frac{\ln(1-x)}{x}.$$

In the last equality we replace $x\to -y$, we get $\lim_{x\to 0}\frac{\ln(1-x)}{x}=-\lim_{y\to 0}\frac{\ln(1-y)}{y}$. From where

$$\lim_{x\to 0}\frac{a\tanh x}{x}=\lim_{x\to 0}\frac{\ln(1+x)}{x}. \tag{8}$$

New representation of second fundamental limit we will find from the representation $a\sinh x=\ln\left(x+\sqrt{x^2+1}\right)$ [5].

$$\lim_{x\to 0}\frac{a\sinh x}{x}=\lim_{x\to 0}\frac{1}{x}\ln\left(x+\sqrt{x^2+1}\right)=\lim_{x\to 0}\ln\left(x+\sqrt{x^2+1}\right)^{1/x}=1.$$ Where from

$$\lim_{x\to 0}\frac{1}{x}\ln\left(x+\sqrt{x^2+1}\right)=1,\ \lim_{x\to 0}\left(x+\sqrt{x^2+1}\right)^{1/x}=e. \tag{9}$$

In the last equality we replace $x=1/y$, we get

$$\lim_{x\to\infty}\left(\frac{1+\sqrt{x^2+1}}{x}\right)^x=e. \tag{10}$$

The formulas (9) and (10) are new in mathematical analysis.

Other corollaries from the second fundamental limit follow by corresponding substitutions in the formulas (7) and (8). The results are represented in the tab.2.

Table 2 – The corollaries from the second fundamental limit $\lim_{x\to 0}\frac{\sinh x}{x}=1$

An option of the second standard limit	Substitution or transformation	Corollary
$\lim_{x\to 0}\frac{\sinh x}{x}=1$	$\lim_{x\to 0}\frac{\sinh x}{x}=\lim_{x\to 0}\frac{e^x-e^{-x}}{2x}=$ $=\lim_{x\to 0}\frac{e^{2x}-1}{2x}\lim_{x\to 0}\frac{1}{e^x}=1$	$\lim_{x\to 0}\frac{e^x-1}{x}=1$
	$\sinh x=\frac{e^x-e^{-x}}{2}$	$\lim_{x\to 0}\frac{e^x-e^{-x}}{2x}=1$
$\lim_{x\to 0}\frac{e^x-1}{x}=1$	$x=\ln(1+x)$	$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$
$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$	$\lim_{x\to 0}\ln(1+x)^{1/x}=1$	$\lim_{x\to 0}(1+x)^{1/x}=e$
	$\lim_{x\to 0}\ln(1+x)^{1/x}=1,\ x=1/y$	$\lim_{x\to\infty}(1+1/x)^x=e$
	$x\to x\ln a$	$\lim_{x\to 0}\frac{\ln(1+x\ln a)}{x}=\ln a$
$\lim_{x\to 0}\frac{\ln(1+x\ln a)}{x}=\ln a$	$\lim_{x\to 0}\ln(1+x\ln a)^{1/x}=\ln a$	$\lim_{x\to 0}(1+x\ln a)^{1/x}=a$

$\lim\limits_{x\to0}(1+x\ln a)^{1/x}=a$	$x=1/y$	$\lim\limits_{x\to\infty}(1+1/x\ln a)^{x}=a$
$\lim\limits_{x\to0}\frac{a\tanh x}{x}=1$	$a\tanh x=\frac{1}{2}\ln\frac{x+1}{1-x}$	$\lim\limits_{x\to0}\frac{1}{x}\ln\frac{x+1}{1-x}=2$
$\lim\limits_{x\to0}\frac{1}{x}\ln\frac{x+1}{1-x}=2$	$\lim\limits_{x\to0}\ln\left(\frac{x+1}{1-x}\right)^{1/x}=2$	$\lim\limits_{x\to0}\left(\frac{x+1}{1-x}\right)^{1/x}=e^2$
$\lim\limits_{x\to0}\ln\left(\frac{x+1}{1-x}\right)^{1/x}=2$	$x=1/y$	$\lim\limits_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x}=e^2$
	$\lim\limits_{x\to0}\frac{1}{\log_a e}\log_a\left(\frac{x+1}{1-x}\right)^{1/x}=2$	$\lim\limits_{x\to0}\log_a\left(\frac{x+1}{1-x}\right)^{1/x}=2\log_a e$
	$\lim\limits_{x\to0}\frac{1}{\log_a e}\log_a\left(\frac{x+1}{1-x}\right)^{1/x}=2,\ x=1/y$	$\lim\limits_{x\to\infty}\log_a\left(\frac{x+1}{x-1}\right)^{x}=2\log_a e$
$\lim\limits_{x\to0}\frac{a\sinh x}{x}=1$	$a\sinh x=\ln\left(x+\sqrt{x^2+1}\right)$	$\lim\limits_{x\to0}\frac{1}{x}\ln\left(x+\sqrt{x^2+1}\right)=1$
$\lim\limits_{x\to0}\frac{1}{x}\ln\left(x+\sqrt{x^2+1}\right)=1$	$\lim\limits_{x\to0}\ln\left(x+\sqrt{x^2+1}\right)^{1/x}=1$	$\lim\limits_{x\to0}\left(x+\sqrt{x^2+1}\right)^{1/x}=e$
$\lim\limits_{x\to0}\left(x+\sqrt{x^2+1}\right)^{1/x}=e$	$x=1/y$	$\lim\limits_{x\to\infty}\left(\frac{1+\sqrt{x^2+1}}{x}\right)^{x}=e$
$\lim\limits_{x\to0}\frac{1}{x}\ln\left(x+\sqrt{x^2+1}\right)=1$	$\lim\limits_{x\to0}\frac{1}{x\log_a e}\log_a\left(x+\sqrt{x^2+1}\right)=1$	$\lim\limits_{x\to0}\frac{1}{x}\log_a\left(x+\sqrt{x^2+1}\right)=\log_a e$
$\lim\limits_{x\to0}\frac{\tanh x}{x}=1$	$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}}$	$\lim\limits_{x\to0}\frac{1}{x}\frac{e^x-e^{-x}}{e^x+e^{-x}}=1$

3 The elementary generalization of the theory

The system of the inequalities (6) can be generalized by the replacement $x\to\alpha\cdot x$. For definiteness we put $\alpha>0$, although the theory will be fair for any $\alpha:\neq0$

$$\left[\begin{array}{l} xe^{-\alpha x}\le\sin\alpha x\le\alpha xe^{\alpha x}, \\ \alpha xe^{-\alpha x}\le\sinh\alpha x\le\alpha xe^{\alpha x}. \end{array}\right. \tag{11}$$

Applying the corollary from the theorem (3) to the inequality (11), accounting that $\lim\limits_{x\to0}e^{-x}=\lim\limits_{x\to0}e^{x}=1$, we have

$$\lim_{x\to0}\frac{\sin\alpha x}{x}=\alpha,\quad \lim_{x\to0}\frac{\sinh\alpha x}{x}=\alpha. \tag{12}$$

The first formula can be called the fist general fundamental limit, the second is an option of the second general limit. Using the results of the table 1 we get immediately

$$\begin{array}{l} \lim\limits_{x\to0}\frac{\sin\alpha x}{x}=\alpha,\quad \lim\limits_{x\to0}\frac{\arcsin\alpha x}{x}=\alpha,\quad \lim\limits_{x\to0}\frac{\tan\alpha x}{x}=\alpha,\quad \lim\limits_{x\to0}\frac{\arctan\alpha x}{x}=\alpha, \\ \lim\limits_{x\to0}\frac{\sinh\alpha x}{x}=\alpha,\quad \lim\limits_{x\to0}\frac{a\sinh\alpha x}{x}=\alpha,\quad \lim\limits_{x\to0}\frac{\tanh\alpha x}{x}=\alpha,\quad \lim\limits_{x\to0}\frac{a\tanh\alpha x}{x}=\alpha. \end{array} \tag{13}$$

For the second limit we choose $\alpha=\ln a$. Then we will get new equalities (Tab. 3).

Table 3 – The corollaries from the limit $\lim\limits_{x\to 0}\frac{\sinh(\ln a\cdot x)}{x}=\ln a$

An option of the second standard limit	Substitution or transformation	Corollary
$\lim\limits_{x\to 0}\frac{sh(\alpha x)}{x}=\alpha\,,\ \alpha=\ln a$	$\lim\limits_{x\to 0}\frac{sh(\ln a\cdot x)}{x}=\lim\limits_{x\to 0}\frac{a^x-a^{-x}}{2x}=$ $=\lim\limits_{x\to 0}\frac{a^{2x}-1}{2x}\lim\limits_{x\to 0}\frac{1}{a^x}=\ln a$	$\lim\limits_{x\to 0}\frac{a^x-1}{x}=\ln a$
	$shx=\frac{e^{\alpha x}-e^{-\alpha x}}{2}\,,\ \alpha=\ln a$	$\lim\limits_{x\to 0}\frac{a^x-a^{-x}}{2x}=\ln a$
$\lim\limits_{x\to 0}\frac{a^x-1}{x}=\ln a$	$x=\log_a(1+x)$	$\lim\limits_{x\to 0}\frac{\log_a(1+x)}{x}=\ln a$
$\lim\limits_{x\to 0}\frac{arth(\alpha x)}{x}=\alpha$	$arth\alpha x=\frac{1}{2}\ln\frac{\alpha x+1}{1-\alpha x}\,,\ \alpha=\ln a$	$\lim\limits_{x\to 0}\frac{1}{x}\ln\frac{x\cdot\ln a+1}{1-x\cdot\ln a}=2\ln a$
$\lim\limits_{x\to 0}\frac{1}{x}\ln\frac{x\cdot\ln a+1}{1-x\cdot\ln a}=2\ln a$	$\lim\limits_{x\to 0}\ln\left(\frac{x\cdot\ln a+1}{1-x\cdot\ln a}\right)^{1/x}=2\cdot\ln a$	$\lim\limits_{x\to 0}\left(\frac{x\cdot\ln a+1}{1-x\cdot\ln a}\right)^{1/x}=a^2$
$\lim\limits_{x\to 0}\ln\left(\frac{x\cdot\ln a+1}{1-x\cdot\ln a}\right)^{1/x}=2\cdot\ln a$	$x=1/y$	$\lim\limits_{x\to\infty}\left(\frac{x+\ln a}{x-\ln a}\right)^{x}=a^2$
	$\lim\limits_{x\to 0}\frac{1}{\log_a e}\log_a\left(\frac{x\cdot\ln a+1}{1-x\cdot\ln a}\right)^{1/x}=2\cdot\ln a$	$\lim\limits_{x\to 0}\log_a\left(\frac{x\cdot\ln a+1}{1-x\cdot\ln a}\right)^{1/x}=2$
$\lim\limits_{x\to 0}\log_a\left(\frac{x\cdot\ln a+1}{1-x\cdot\ln a}\right)^{1/x}=2$	$x=1/y$	$\lim\limits_{x\to\infty}\log_a\left(\frac{\ln a+x}{x-\ln a}\right)^{x}=2$
$\lim\limits_{x\to 0}\frac{arsh\alpha x}{x}=\alpha$	$arshx=\ln\left(\alpha x+\sqrt{x^2\alpha^2+1}\right),$ $\alpha=\ln a$	$\lim\limits_{x\to 0}\frac{1}{x}\ln\left(x\cdot\ln a+\sqrt{x^2\ln^2 a+1}\right)=$ $=\ln a$
	$\lim\limits_{x\to 0}\ln\left(\alpha x+\sqrt{x^2\alpha^2+1}\right)^{1/x}=\alpha\,,$ $\alpha=\ln a$	$\lim\limits_{x\to 0}\left(x\cdot\ln a+\sqrt{x^2\ln^2 a+1}\right)^{1/x}=$ $=a$
$\lim\limits_{x\to 0}\left(x\cdot\ln a+\sqrt{x^2\ln^2 a+1}\right)^{1/x}=$ $=a$	$x=1/y$	$\lim\limits_{x\to\infty}\left(\frac{\ln a+\sqrt{x^2+\ln^2 a}}{x}\right)^{x}=a$
$\lim\limits_{x\to 0}\ln\left(x\cdot\ln a+\sqrt{x^2\ln^2 a+1}\right)^{1/x}=$ $=\ln a$	$\lim\limits_{x\to 0}\frac{1}{x\log_a e}\log_a\left(x\cdot\ln a+\sqrt{x^2\ln^2 a+1}\right)=$ $=\ln a=1/\log_a e$	$\lim\limits_{x\to 0}\frac{1}{x}\log_a\left(x\cdot\ln a+\sqrt{x^2\ln^2 a+1}\right)=$ $=1$
$\lim\limits_{x\to 0}\frac{th\alpha x}{x}=\alpha$	$thx=\frac{e^x-e^{-x}}{e^x+e^{-x}}\,,\ \alpha=\ln a$	$\lim\limits_{x\to 0}\frac{1}{x}\frac{a^x-a^{-x}}{a^x+a^{-x}}=\ln a$

APPENDIX 1. The proof of the inequalities (4) $x\geq 0$

$$\left[\begin{array}{l} xe^{-x}\leq\sin x\leq xe^{x}, \\ xe^{-x}\leq\sinh x\leq xe^{x}. \end{array}\right.$$

To the first inequality we apply the well-known inequality $\sin x\leq x$. Then $\sin x\leq x\leq xe^{x}$. The last inequality is obvious because $x\leq xe^{x}\Rightarrow e^{x}\geq 1$.

For proving the inequality $xe^{-x} \le \sin x$ we replace x by $-x$. we have $-xe^{x} \le \sin(-x)$. We use the property $\sin(-x) = -\sin x$. Multiplying the inequality $-xe^{x} \le -\sin x$ by the factor -1 we have the previous inequality $\sin x \le xe^{x}$.

To the second inequality (4) we apply the inequality $e^{x} - 1 \ge x$. The second inequality (4) is proved like the first, only instead of the inequality $\sin x \le x$ we will use the inequality $e^{x} - 1 \ge x$.

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \le xe^{x} \Rightarrow 2xe^{-2x} \le 2x \le e^{2x} - 1.$$

we have the obvious inequality $2xe^{-2x} \le 2x \Rightarrow e^{-2x} \le 1$. For the proving of the inequality $\sinh x \le xe^{x}$ we replace x by $-x$. We have $\sinh(-x) \le -xe^{-x}$. We use the property $\sinh(-x) = -\sinh x$. Multiplying the inequality $-\sinh x \le -xe^{-x}$ by the factor -1 we return to the proved inequality $\sinh x \ge xe^{-x}$.

APPENDIX 2. The proof of the inequalities $\sin x \le \sinh x$.

We replace the left part of the inequality by a value x that is more then $\sin x$, we get $x \le \sinh x$ or $2x \le e^{x} - e^{-x}$. We rewrite the inequality in the form $e^{2x} - 2xe^{x} - 1 \ge 0$. The roots of the equation are $e^{x} = x \pm \sqrt{x^2 + 1}$. The negative root we omit because $e^{x} > 0$. Geometrically we have a parabola $y = e^{2x} - 2xe^{x} - 1$ respect to e^{x}. The parabola is touched the axis x only at the point $x = 0$, the branches of the parabola are directed up, i.e. $y > 0$. Therefore the inequality $e^{2x} - 2xe^{x} - 1 \ge 0$ is fulfilled for any x.

Corollary. $e^{x} \ge 1 + x + \frac{x^2}{2}$. Here it is used the inequality $\sqrt{1 + x^2} \le 1 + x^2/2$ which is checked by raising both parts of the inequality into square.

Results

1) It is developed a new effective method of the proof of the first and second fundamental limits in the classical limit theory.
2) The approach generalizes usual theory of the fundamental limits and from our point of view the theory is more simple and effective.
3) The approach is generalized that leads to many equivalent forms of the limits (from the first limit we have got 3 corollaries, from the second limits 34). The most of them are new.
4) The proposed method improves the classical theory of the fundamental limits.

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