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DYNAMIC CONCENTRATION OF STRESSES IN ELASTIC COMPOUND SOLIDS

L. P. Vovk

Auto-Road Institute of Donetsk National Technical University

Abstract

According to a modified method of superposition the solution to the problem of harmonic vibrations of inhomogeneous areas consisting of various numbers of joined elastic areas with different elastic properties is presented. Wave field singularities in the vicinity of singular boundary points at the joint of areas are studied.

It is hardly possible to properly approximate elasticity theory problems solution and to develop an effective numerical algorithm for its finding without knowing the behavior of stress-edly-deformed state (SDS) components in the vicinity of singular points and surface lines of a considered body. This problem becomes even more important if we consider vibrational loading of structural elements when tension may undergo qualitative changes depending upon external loading frequency. Under-load behavior of structurally inhomogeneous media boundary areas was experimentally studied in terms of physical mesomechanics of materials (a field of study rapidly developing in many research organizations). These experiments revealed local stress and deformation oscillations which arise near the inner interface and have the amplitudes exceeding their average values in the volume of the material . They are determined by the internal structure of non-uniform media and depend on the elastic parameters of contacting media. At microscale level such stress oscillations generate dislocation streams, at mesoscale level they lead to the appearance of extended meso-bands of deformation and at macroscale level they result in stationary distribution of localized deformation macro-bands and destruction of the material. The obtained experimental results have important applications in the mechanics of structurally nonuniform mediums, in microelectronics, geodynamics and materials technology. Thus, it becomes obvious that in the vicinity of interface surface in a loaded inhomogeneous body there arises local stress concentration (LSC), its nature and intensity being determined by external dynamic loading and the ratio of contacting media mechanical properties.

The peculiarities of static tension distribution in the vicinity of the angular point of the boundary line of the cross-section areas of the body made up of two different prismatic bodies joined together along the side surface had been considered earlier [1-3]. For example [1], an elastic-static plane problem about two diverse wedges with random opening angles was considered in the following way: the solution was plotted in Mellin transformants, and after the conjugation conditions had been met it was possible to study (taking into account stresses in wedge vertices) the local singularity parameter (LSP) dependence on the opening angles and elastic constants combinations. The method presented in [4] allows defining the nature of the specified features without boundary problem direct solution. The dynamic aspects were considered in [5-7] and, in particular, the notion of "boundary" resonance (generalization of well studied edge resonance) [8] was introduced.

Some SDS peculiarities in the vicinity of the vertex of a compound body made of three rigidly attached diverse wedges with vertex angles α, β and γ, were studied in [9]. The numerical analysis is done for a compound body with total opening angle equal to $\alpha+\beta+\gamma=\pi$. A special emphasis in [10] was made on studying the influence of wedges opening angles on LSP.

The problem stated below is that of defining qualitative and quantitative character of the wave field singularity arising in the vicinity of the angular point of the joint of two, three and four diverse rectangular areas. Similar problems are connected with calculating the strength parameters of welded and soldered joints, including fillet welds [10]. The presented technique is universal enough and can be used to analyze LSC intensity in compound bodies of any structure. With its help it is possible not only to study LSC intensity factors, but also to plot a solution in the whole area of a compound body section.

1. Interface of two different media

1.1. Problem Statement. Let us consider wave motions completely characterized by a two-dimensional field in a plane $\alpha_1 O\alpha_2$ in a prism V which is infinite along axis $O\alpha_3$. We assume that the prism section occupies region $D=\{(\alpha_1,\alpha_2):|\alpha_1|\le a,|\alpha_2|\le b\}$ in the plane $\alpha_1 O\alpha_2$. Suppose $S_{\pm}^{(m)}$ $(m=1,2,...,N)$ are straight lines which have equations $\alpha_1=\pm a_m$ in a chosen coordinate system and divide section D into regions $D^{(1)}$ and $D^{(m)}=D_{-}^{(m)}\cup D_{+}^{(m)}$ $(m=2,3,...,N)$; and $a_1<a_2<...<a_N=a.$ Regions $D_{-}^{(m)}$ and $D_{+}^{(m)}$ are arranged symmetrically with respect to the coordinates origin, have identical thickness $h^{(m)}=a_m-a_{m-1}$ and are defined as $D_{-}^{(m)}=\{(\alpha_1,\alpha_2):\alpha_1\in[-a_m,-a_{m-1}],|\alpha_2|\le b\}$, $D_{+}^{(m)}=\{(\alpha_1,\alpha_2):\alpha_1\in[a_{m-1},a_m],|\alpha_2|\le b\}$. The region $D^{(1)}$ is an internal rectangular simply connected region with the centroid in the origin of coordinates: $D^{(1)}=\{(\alpha_1,\alpha_2):|\alpha_1|\le a_1,|\alpha_2|\le b\}$.

Each of the listed N regions $D^{(m)}$ is occupied by its homogeneous and isotropic elastic material with shear modulus $\mu^{(m)}$, Poisson's ratio $\nu^{(m)}$ and density $\rho^{(m)}$. We assume that the wave field in the region of section D is excited on the boundaries $\alpha_1=\pm a,\alpha_2=\pm b$ by normal, harmonically changing in time with the frequency ω, self-counterbalanced loads of intensity $q_1(\alpha_2)$ and $q_2(\alpha_1)$ respectively. In subsequent calculations time multiplier $e^{i\omega t}$ common for all wave field characteristics is dropped. Taking into account the symmetry of the region, it is possible to consider the wave field of a part of prism section area located in the first quarter.

Amplitude displacements and stresses of points of elastic media of regions $D^{(m)}$ we designated with $u_{\beta}^{(m)}$ and $\tau_{\gamma\beta}^{(m)}$ respectively ($\beta=1,2;\gamma=1,2$). The general boundary problem of defining particular frequencies and mode forms of the piecewise inhomogeneous region D is formulated in the following dimensionless form. In all regions $D^{(m)}$ we introduce dimensionless local coordinates $x^{(m)}$ and search for the functions $U_{\beta}^{(m)}(x^{(m)},y)$, satisfying the equations of motion.

$$\Delta U_{\beta}^{(m)}+(C_{12}^{(m)}+1)(U_{1,1}^{(m)}+U_{2,2}^{(m)})_{,\beta}+(\Omega^{(m)})^2U_{\beta}^{(m)}=0, \qquad (1)$$

where $U_{\beta}^{(m)}=u_{\beta}^{(m)}/a,$ $C_{12}^{(m)}=C_{11}^{(m)}-2,$ $C_{11}^{(m)}=2(1-\nu^{(m)})/(1-2\nu^{(m)}),$ $f_{,1}^{(m)}=\dfrac{\partial f^{(m)}}{\partial x^{(m)}},$ $f_{,2}^{(m)}=\dfrac{\partial f^{(m)}}{\partial y},$ $\Omega^{(m)}=a\omega\sqrt{\dfrac{\rho^{(m)}}{\mu^{(m)}}},$ $x=\alpha_1/a,\delta_m=a_m/a,\delta_0=0,$ $y=\alpha_2/a,$ $\overline{h}^{(m)}=\delta_m-\delta_{m-1},$ $x^{(m)}\in[0,\overline{h}^{(m)}],y\in[0,\eta],$ $\eta=b/a,$ $m=1,2,...,N$.

Dimensionless amplitude components of the stress tensor $\sigma_{\gamma\beta}^{(m)}=\tau_{\gamma\beta}^{(m)}/\mu^{(m)}$ are calculated according to the proportions of Hooke's law. On straight lines $S_{+}^{(m)}$ the conditions of contact between areas with various elastic properties should be satisfied. In particular, right on the joint of all regions $\overline{D}^{(m)}=\{(x^{(m)},y):0\le x^{(m)}\le\overline{h}^{(m)},0\le y\le\eta\}$ and $\overline{D}^{(m+1)}$ $(m=1,2,...,N-1)$ the conditions of rigid adhesion are considered as satisfied

$$\sigma_{1\gamma}^{(m)}(\overline{h}^{(m)},y)=r_m\sigma_{1\gamma}^{(m+1)}(0,y),\ U_{\gamma}^{(m)}(\overline{h}^{(m)},y)=U_{\gamma}^{(m+1)}(0,y),\ r_m=\mu^{(m+1)}/\mu^{(m)} \qquad (2)$$

On the outer boundary of the section we accept the following force boundary conditions $(q_{\gamma}^{(m)}=q_{\gamma}/\mu^{(m)})$

$$\sigma_{11}^{(N)}(1-\delta_{N-1},y)=q_1^{(N)}(y)\,,\sigma_{22}^{(m)}(x^{(m)},\eta)=q_2^{(m)}(x^{(m)})\,,\sigma_{12}^{(N)}(1-\delta_{N-1},y)=\sigma_{12}^{(m)}(x^{(m)},\eta)=0\,, \qquad (3)$$

1.2. General Solution Construction. According to the algorithm of superposition method [7,8,11], general solution $U_\beta^{(m)}$, satisfying equations set (1) within region $\overline{D}^{(m)}$ is presented as the sum of two particular solutions to this equation set, each describing symmetric fluctuations of infinite strips $G_1^{(m)}=\{(x^{(m)},y):0\le x^{(m)}\le \overline{h}^{(m)},|y|<\infty\}$ and $G_2^{(m)}=\{(x^{(m)},y):|x^{(m)}|<\infty,|y|\le\eta\}$, which cross and form region $\overline{D}^{(m)}$ at their crossing. Besides, we should take into account that on the coordinate $x^{(m)}$ functions $U_\beta^{(m)}(x^{(m)},y)$ are general functions. As a result we will have general solution in regions $\overline{D}^{(m)}(m=2,3,...,N)$ in the form

$$U_1^{(m)}=(\overline{H}^{(m)}sh(t^{(m)}x^{(m)})+\overline{Q}^{(m)}ch(t^{(m)}x^{(m)}))\cos\alpha(y-\eta)+\overline{R}^{(m)}ch(r^{(m)}y)\sin(\chi^{(m)}(x^{(m)}-\overline{h}^{(m)}))\,,$$

$$U_2^{(m)}=(H^{(m)}ch(t^{(m)}x^{(m)})+Q^{(m)}sh(t^{(m)}x^{(m)}))\sin\alpha(y-\eta)+R^{(m)}sh(r^{(m)}y)\cos(\chi^{(m)}(x^{(m)}-\overline{h}^{(m)})) \qquad (4)$$

For the region $D^{(1)}$ the solution is considered as in the case with symmetric fluctuations of a homogeneous rectangle [5,8]

$$U_1^{(1)}=\overline{H}^{(1)}sh(t^{(1)}x^{(1)})\cos\alpha(y-\eta)+\overline{R}^{(1)}ch(r^{(1)}y)\sin\chi^{(1)}(x^{(1)}-\delta_1)\,,$$

$$U_2^{(1)}=H^{(1)}ch(t^{(1)}x^{(1)})\sin\alpha(y-\eta)+R^{(1)}sh(r^{(1)}y)\cos\chi^{(1)}(x^{(1)}-\delta_1) \qquad (5)$$

The set of constants $\overline{H}^{(m)},H^{(m)},\overline{Q}^{(m)},...,R^{(m)}$ in formulas (4), (5) provides necessary degree of arbitrariness for satisfying boundary and interface conditions in a considered compound area. As values $\alpha,\chi^{(m)}$ we should choose such sequences of numbers $\alpha_k,\chi_j^{(m)}$ which will make the systems of respective functions full and orthogonal in the intervals $|y|\le\eta$ and $x^{(m)}\in[0,\overline{h}^{(m)}]$, i.e.,

$$\alpha_k=k\pi/\eta\,,\ \chi_j^{(m)}=j\pi/\overline{h}^{(m)},\ k=1,2,\ldots;\ j=1,2, \qquad (6)$$

After substitution of expressions (4) and (5) in equations (1), for each value *k* and *j* we obtain sets of linear homogeneous equations concerning coefficients $\overline{H}^{(m)}$ and $H^{(m)}$,…, $\overline{R}^{(m)}$ and $R^{(m)}$. From the existence condition of these sets nontrivial solution we define the values of parameters $t^{(m)}$ and $r^{(m)}$

$$(t_{\beta k}^{(m)})^2=\alpha_k^2-(l_\beta^{(m)})^2,(r_{\beta j}^{(m)})^2=(\chi_j^{(m)})^2-(l_\beta^{(m)})^2,l_1^{(m)}=\Omega^{(m)}/\sqrt{C_{11}^{(m)}}\,,l_2^{(m)}=\Omega^{(m)},\beta=1,2 \qquad (7)$$

and the connection between the above coefficients. It determines completely the problem general solution in all regions $\overline{D}^{(m)}$ and allows satisfying the interface conditions (2) and force boundary conditions (3).

1.3. Auxiliary Problems Formulation and Solution. Similarly to [7, 11] we introduce for consideration auxiliary boundary problems.

Regions $\overline{D}^{(m)}(m=2,...,N-1,N)$:

$$U_1^{(m)}(\overline{h}^{(m)},y)=f_1^{(m)}(y)\,,\sigma_{12}^{(m)}(\overline{h}^{(m)},y)=\varphi_1^{(m)}(y)\,,\varphi_1^{(N)}(y)=0\,,U_2^{(m)}(x^{(m)},\eta)=f_2^{(m)}(x^{(m)})\,,$$

$$\sigma_{12}^{(m)}(x^{(m)},\eta)=0\,,U_1^{(m)}(0,y)=f_1^{(m-1)}(y),\ \sigma_{12}^{(m)}(0,y)=r_m^{-1}\varphi_1^{(m-1)}(y)\,. \qquad (8)$$

Region $\overline{D}^{(1)}$:

$$U_1^{(1)}(\delta_1,y)=f_1^{(1)}(y)\,,\ \sigma_{12}^{(1)}(\delta_1,y)=\varphi_1^{(1)}(y)\,,\ U_2^{(1)}(x^{(1)},\eta)=f_2^{(1)}(x^{(1)})\,,\ \sigma_{12}^{(1)}(x^{(1)},\eta)=0\,. \qquad (9)$$

Here $f_1^{(m)}(y), \varphi_1^{(m)}(y), f_2^{(m)}(x^{(m)})$ are unknown auxiliary functions. We expand these (*3N-1*) functions into Fourier series in respective intervals and, using the problem general solution we make conditions (8) and (9). Then the conditions on vertical parts of the boundary $x = \delta_m$ will give, at each value of index *k,* $(4N-2)$, linear equations for defining the same number of coefficients $H_{\gamma k}^{(1)}, H_{\gamma k}^{(2)}, Q_{\gamma k}^{(2)}, ..., H_{\gamma k}^{(N)}, Q_{\gamma k}^{(N)} (\gamma = 1,2)$ in the problem general solution. The remaining $2N$ coefficients $R_{\gamma j}^{(m)}$ at each value j will be defined from $2N$ boundary conditions if $y = \eta$. Obtained sets of linear systems presuppose an analytical solution and make possible to explicitly express wave field characteristics in the whole compound region of the section through Fourier coefficients $f_{10}^{(m)}, f_{1k}^{(m)}, f_{20}^{(m)}, f_{2j}^{(m)}, \varphi_{1k}^{(m)}$ of the introduced auxiliary functions.

1.4. Derivation of Integrated Equations Set and Its Asymptotic Analysis. For the purpose of defining the introduced auxiliary functions let us take into consideration (*3N-1*) unused boundary and interface conditions from (2), (3), namely

$$\sigma_{11}^{(m)}(\bar{h}^{(m)}, y) = r_m \sigma_{11}^{(m+1)}(0, y),\ U_2^{(m)}(\bar{h}^{(m)}, y) = U_2^{(m+1)}(0, y),\ \sigma_{22}^{(m)}(x^{(m)}, \eta) = q_2^{(m)}(x^{(m)})\ (m = 1 \div N-1),$$

$$\sigma_{11}^{(N)}(1-\delta_{N-1}, y) = q_1^{(N)}(y),\ \sigma_{22}^{(N)}(x^{(N)}, \eta) = q_2^{(N)}(x^{(N)}) \tag{10}$$

As all the wave field components appearing in (10) are expressed through Fourier coefficients of auxiliary functions, these conditions represent a set of integral equations (SIE) for their definition. In order to develop an effective algorithm of its solution and further successful selection of coordinate functions in the asymptotical method we investigate wave field singularities in the vicinity of irregular boundary points. In the considered problem such points are angular points of the areas joint $A_m(\bar{h}^{(m)}, \eta)$ and the external angular point of section $B(1-\delta_{N-1}, \eta)$. Taking into account the mechanical meaning of auxiliary functions we suppose that their singularities in points A_m are defined by formulas (*m = 1,2, ..., N-1*)

$$\varphi_1^{(m)}(\xi) = \Phi_1^{(m)}(\eta-\xi)^{p_m-1},\ f_1^{(m)/}(\xi) = F_1^{(m)}(\eta-\xi)^{p_m-1} \text{ at } \xi \to \eta;$$

$$f_2^{(m)/}(\xi) = F_2^{(m)}(\bar{h}^{(m)}-\xi)^{p_m-1} \text{ at } \xi \to \bar{h}^{(m)},\ f_2^{(m+1)/}(\xi) = \bar{F}_2^{(m+1)}\xi^{p_m-1} \text{ at } \xi \to 0.$$

In the vicinity of point *B* of the area $\bar{D}^{(N)}$:

$$f_1^{(N)/}(\xi) = F_1^{(N)}(\eta-\xi)^{p_N-1} \text{ at } \xi \to \eta;\ f_2^{(N)/}(\xi) = F_2^{(N)}(1-\delta_{N-1}-\xi)^{p_{N-1}} \text{ at } \xi \to 1-\delta_{N-1}$$

Here p_m stands for LSP which define the character of required functions discontinuity in the specified points and $\Phi_1^{(m)}, ..., F_2^{(N)}$ stand for arbitrary constants. Making integration in these formulas, we define the asymptotics of auxiliary functions Fourier coefficients in the vicinity of all irregular points of the boundary. It allows investigating the behaviour of the obtained SIE at the approach to irregular points. The required limitation of the left sides of the equation set allows obtaining an equation set for defining parameters p_m for each irregular point. The components containing Fourier coefficients of functions $f_1^{(m)}, \varphi_1^{(m)}, f_2^{(m)}, f_2^{(m+1)}$ will be asymptotically significant in points A_m. In the vicinity of an external angular point *B* the behavior of wave field characteristics will be defined by functions $f_1^{(N)}, f_2^{(N)}$. After re-designation of the constants we come to the following sets of homogeneous equations.

In points $A_m (m = 1,2,...,N-1)$:

$$-s_m \sin\frac{\pi p_m}{2}\Phi_1^{(m)} + 2(n_m + r_m n_{m+1})\sin\frac{\pi p_m}{2}F_1^{(m)} + 2n_m p_m F_2^{(m)} + 2n_{m+1}p_m r_m \overline{F}_2^{(m+1)} = 0\,,$$

$$(t_m + r_m t_{m+1})\sin\frac{\pi p_m}{2}\Phi_1^{(m)} + 2s_m \sin\frac{\pi p_m}{2}F_1^{(m)} - -2(1-n_m p_m)F_2^{(m)} - 2(1-n_{m+1}p_m)\overline{F}_2^{(m+1)} = 0\,, \quad (11)$$

$$(n_m^{-1} + p_m)\Phi_1^{(m)} + 2p_m F_1^{(m)} + 2\sin\frac{\pi p_m}{2}F_2^{(m)} = 0\,,$$

$$r_m^{-1}(n_{m+1}^{-1} + p_m)\Phi_1^{(m)} + 2p_m F_1^{(m)} + 2\sin\frac{\pi p_m}{2}\overline{F}_2^{(m+1)} = 0\,,$$

where $s_m = (C_{11}^{(m)})^{-1} + (C_{11}^{(m+1)})^{-1}$, $n_m = (C_{11}^{(m)} - 1)/C_{11}^{(m)}$, $t_m = (C_{11}^{(m)} + 1)/C_{11}^{(m)}$.

In point B:

$$\sin\frac{\pi p_N}{2}F_1^{(N)} + p_N F_2^{(N)} = 0\,,\ p_N F_1^{(N)} + \sin\frac{\pi p_N}{2}F_2^{(N)} = 0 \quad (12)$$

It is obvious that constants $F_1^{(N)}, F_2^{(N)}$ in the system (12) will not be equal to zero if the parameter p_N satisfies the equation

$$\sin^2\frac{\pi p_N}{2} - p_N^2 = 0 \quad (13)$$

Equation (13) corresponds to the equation obtained in [12] for a single wedge with loose sides and opening angle 90^0. As we see, the character of mechanical field singularity in point B does not depend on elastic constants of regions $\overline{D}^{(m)}$. Naturally, proceeding from the mechanical meaning of functions $f_1^{(N)}(\xi), f_2^{(N)}(\xi)$, we consider only the real root $p_N = 1$ of the equation (13) and countable set of complex roots [7] with a positive real component.

Parameters p_m, characterizing wave peculiarities in inner angular points of a compound region, do not depend on the frequency and geometrical parameters η and δ_m and are determined only by shear modulus and Poisson's ratio of joined areas. They can be obtained from the existence condition of nontrivial solution (11)

$$\Delta(p_m, \mu^{(m)}, \nu^{(m)}, \mu^{(m+1)}, \nu^{(m+1)}) = 0 \quad (14)$$

Equation (14) is symmetric with respect to the elastic parameters of the neighboring regions $\overline{D}^{(m)}, \overline{D}^{(m+1)}$ and will not change if values $\mu^{(m)}, \nu^{(m)}$ are replaced with $\mu^{(m+1)}, \nu^{(m+1)}$ and vice versa. It can be easily proved by means of elementary transformations of rows and columns of the determinant (11). At certain proportions of the mechanical properties of joined areas materials equation (14) has a real root $0 < p_m < 1$ and it characterizes the occurrence of local singularities in stress values in points A_m.

It is possible to construct the asymptotics of parameter p_m at high values of r_m. Searching for the roots of equation (14) in the form of expansion into series by the small parameter: $\varepsilon = r_m^{-1}$: $p_m = p_{m0} + \varepsilon p_{m1} + \ldots$, we can easily obtain a sequence of equations for defining $p_{m0}, p_{m1}, \ldots$. For example, the first expansion term satisfies the equation

$$(\sin^2(\pi p_{m0}/2) - p_{m0}{}^2)(p_{m0}{}^2 - (3 - 4\nu^{(m)})\sin^2(\pi p_{m0}/2) - 4(1-\nu^{(m)})^2) = 0 \quad (15)$$

The first factor in equation (15) coincides with the left side of equation (13). The numerical analysis shows that the second factor has roots $0 < p_{m0} < 1$ practically at any value of Poisson's ratio $\nu^{(m)}$.

Then Bubnov-Galerkin method is applied for solving SIE. This method takes into account the character of the solution features when coordinate functions are being selected. As a result we come to an infinite set of algebraic equations with known asymptotics of unknowns defined by the roots of equations (13), (14).

2. Interface of Three Different Media

We assume that the section of a piecewise inhomogeneous elastic prism infinite along axis α_3 occupies (in the coordinate system $\alpha_1 O \alpha_2$) region $D_3 = G^{(1)} \cup G^{(2)} \cup G^{(3)}$, where regions $G^{(m)}$ are stuck together and are defined by inequalities

$$G^{(1)} = \{(\alpha_1, \alpha_2) : |\alpha_1| \le c; \alpha_2 \in [-b, -d] \cup [d, b]\},\ G^{(2)} = \{(\alpha_1, \alpha_2) : \alpha_1 \in [-a, -c] \cup [c, a]; |\alpha_2| \le d\},$$

$$G^{(3)} = \{(\alpha_1, \alpha_2) : \alpha_1 \in [-a, -c] \cup [c, a]; \alpha_2 \in [-b, -d] \cup [d, b]\}$$

Suppose, on the outer sides of the section $\alpha_1 = \pm a$, $\alpha_2 = \pm b$, there is a vibration load of variable intensity $q_1(\alpha_2), q_2(\alpha_1)$ respectively, harmonically changing in time with frequency ω, and the inner boundary of the section is free from loadings. Taking into account the symmetry of region D, it is possible to consider the wave field of the part of the region located in the first quarter. This part of the region is presented in Fig. 1 in dimensionless coordinates $x = \alpha_1 / a, y = \alpha_2 / a$.

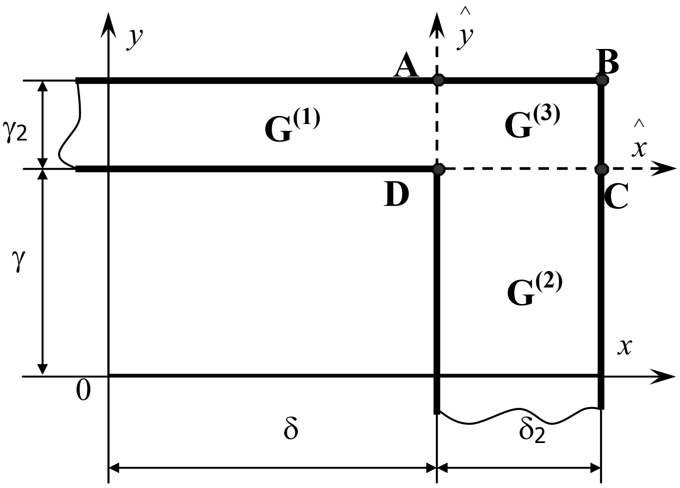

Figure 1. Coupling of three different media

To make it easier we introduced local dimensionless coordinates $\hat{x} = (\alpha_1 - c)/a, \hat{y} = (\alpha_2 - d)/a$ and dimensionless geometrical parameters $\eta = b/a$, $\delta = c/a$, $\gamma = d/a$, $\delta_2 = 1 - \delta, \gamma_2 = \eta - \gamma$ in the section area. Boundary conditions of the problem include force conditions of loading on external and internal boundaries of the section and the condition of rigid adhesion of regions $G^{(m)}$.

Then we start to construct the general solution. Similarly to the previous one, the general solution $U_\beta^{(m)}$, satisfying the equations system of movement within regions $G^{(m)}$ is developed according to superposition method. Here we should take into account that functions

$U_\beta^{(1)}(x,\hat{y}), U_\beta^{(3)}(\hat{x},\hat{y})$ on the coordinate $\hat{y}$, and functions $U_\beta^{(2)}(\hat{x},y), U_\beta^{(3)}(\hat{x},\hat{y})$ on the coordinate $\hat{x}$ are general functions. Thus, the problem general solution in regions $G^{(m)}$ will be as follows

$$U_1^{(1)} = H_1^{(1)} sh(t^{(1)}x)\cos\alpha^{(1)}(\hat{y}-\gamma_2) + (R_1^{(1)} sh(r^{(1)}\hat{y}) + S_1^{(1)} ch(r^{(1)}\hat{y}))\sin\chi^{(1)}(x-\delta)$$

$$U_2^{(1)} = H_2^{(1)} ch(t^{(1)}x)\sin\alpha^{(1)}(\hat{y}-\gamma_2) + (R_2^{(1)} sh(r^{(1)}\hat{y}) + S_2^{(1)} ch(r^{(1)}\hat{y}))\cos\chi^{(1)}(x-\delta)$$

$$U_1^{(2)} = (H_1^{(2)} sh(t^{(2)}\hat{x}) + Q_1^{(2)} ch(t^{(2)}\hat{x}))\cos\alpha^{(2)}(y-\gamma) + R_1^{(2)} ch(r^{(2)}y)\sin\chi^{(2)}(\hat{x}-\delta_2)$$

$$U_2^{(2)} = (H_2^{(2)} sh(t^{(2)}\hat{x}) + Q_2^{(2)} ch(t^{(2)}\hat{x}))\sin\alpha^{(2)}(y-\gamma) + R_2^{(2)} sh(r^{(2)}y)\cos\chi^{(2)}(\hat{x}-\delta_2)$$

$$U_1^{(3)} = (H_1^{(3)} sh(t^{(3)}\hat{x}) + Q_1^{(3)} ch(t^{(3)}\hat{x}))\cos\alpha^{(1)}(\hat{y}-\gamma_2) + (R_1^{(3)} sh(r^{(3)}\hat{y}) + S_1^{(3)} ch(r^{(3)}\hat{y}))\sin\chi^{(2)}(\hat{x}-\delta_2)$$

$$U_2^{(3)} = (H_2^{(3)} sh(t^{(3)}\hat{x}) + Q_2^{(3)} ch(t^{(3)}\hat{x}))\sin\alpha^{(1)}(\hat{y}-\gamma_2) + (R_2^{(3)} sh(r^{(3)}\hat{y}) + S_2^{(3)} ch(r^{(3)}\hat{y}))\cos\chi^{(2)}(\hat{x}-\delta_2)$$

It is expedient to choose

$\alpha_k^{(1)} = k\pi/\gamma_2$, $\alpha_k^{(2)} = k\pi/\gamma$, $\chi_j^{(1)} = j\pi/\delta$, $\chi_j^{(2)} = j\pi/\delta_2$, $k = 1,2,\ldots$; $j = 1,2,\ldots$

as values $\alpha^{(\beta)}, \chi^{(\beta)}$.

According to the basic technique we formulate auxiliary boundary problems which presuppose an analytical solution. In this case the boundary conditions of auxiliary problems will be much more complicated as compared to those considered above (see (8), (9)) due to the presence of two inner lines of the section of areas $G^{(m)}$ and the inner boundary of the section. They will be as follows [13]

$$G^{(1)} = \{|x| \le \delta; 0 \le \hat{y} \le \gamma_2\}: \; U_1^{(1)}(\delta,\hat{y}) = f_1(\hat{y}), \sigma_{12}^{(1)}(\delta,\hat{y}) = \varphi_1(\hat{y})$$

$$U_2^{(1)}(x,\gamma_2) = f_2(x), \sigma_{12}^{(1)}(x,\gamma_2) = \sigma_{12}^{(1)}(x,0) = 0, U_2^{(1)}(x,0) = f_3(x) \;\; G^{(2)} = \{0 \le \hat{x} \le \delta_2; |y| \le \gamma\}:$$

$$U_1^{(2)}(\delta_2, y) = f_4(y), \sigma_{12}^{(2)}(\delta_2, y) = \sigma_{12}^{(2)}(0,y) = 0, U_1^{(2)}(0,y) = f_5(y),$$

$$U_2^{(2)}(\hat{x},\gamma) = f_6(\hat{x}), \sigma_{12}^{(2)}(\hat{x},\gamma) = \varphi_2(\hat{x}) \tag{16}$$

$$G^{(3)} = \{0 \le x \le \delta_2; 0 \le \hat{y} \le \gamma_2\}: U_1^{(3)}(\delta_2,\hat{y}) = f_7(\hat{y}), \sigma_{12}^{(3)}(\delta_2,\hat{y}) = 0, U_1^{(3)}(0,\hat{y}) = f_1(\hat{y}); \sigma_{12}^{(3)}(0,\hat{y}) = r_{13}\varphi_1(\hat{y})$$

$$U_2^{(3)}(\hat{x},\gamma_2) = f_8(\hat{x}), \sigma_{12}^{(3)}(\hat{x},\gamma_2) = 0, U_2^{(3)}(\hat{x},0) = f_6(\hat{x}), \sigma_{12}^{(3)}(\hat{x},0) = r_{23}\varphi_2(\hat{x})$$

Here unknown auxiliary functions are designated through $f_1(\hat{y}), \varphi_1(\hat{y}), \ldots f_8(\hat{x})$. It should be noted that the choice of auxiliary problem boundary conditions in the form of (16) allows satisfying automatically those boundary conditions of the initial boundary problem which concern normal displacements and shear stresses on external and internal boundaries of the region.

After replacing the initial boundary problem with the auxiliary one, some boundary and interface conditions remained unsatisfied. They represent a SIE concerning unknown auxiliary functions. As the character of singularities in section singular points *A*, *B* and *C* (Fig. 1) was investigated earlier, we set the problem of defining wave field singularity in the internal point $D(\delta,\gamma)$ of the joint of three regions. For this purpose, we assume that the auxiliary functions asymptotically significant in the vicinity of this point have singularities of the following form

$$f_i'(\xi) = F_i^D \xi^{\alpha-1}, \; \varphi_j(\xi) = \Phi_j^D \xi^{\alpha-1} \;\; (i = 1,6; j = 1,2) \text{ at } \xi \to 0;$$

$$f_3'(\xi) = F_3^D(\delta-\xi)^{\alpha-1} \text{ at } \xi \to \delta; \;\; f_5'(\xi) = F_5^D(\gamma-\xi)^{\alpha-1} \text{ at } \xi \to \gamma.$$

In these formulas α stands for LSP in point D, and $F_i^D, \Phi_j^D (i = 1,3,5,6; j = 1,2)$ – for arbitrary constants. We define the asymptotics of Fourier coefficients of considered functions and write down the boundary conditions not used in auxiliary problems and the interface conditions of $G^{(m)}$ in the point D vicinity at limiting values of the arguments. These conditions can be recorded as follows

$$\sigma_{11}^{(1)}(\delta,\hat{y}) = r_{31}\sigma_{11}^{(3)}(0,\hat{y}),\ \hat{y} \to 0;\ \sigma_{22}^{(2)}(\hat{x},\gamma) = r_{32}\sigma_{22}^{(3)}(\hat{x},0),\ \hat{x} \to 0;\ U_2^{(1)}(\delta,\hat{y}) = U_2^{(3)}(0,\hat{y}),\ \hat{y} \to 0;$$

$$U_1^{(2)}(\hat{x},\gamma) = U_1^{(3)}(\hat{x},0),\ \hat{x} \to 0;\ \sigma_{11}^{(2)}(0,y) = 0;\ \sigma_{22}^{(1)}(x,0) = 0,\ x \to \delta \qquad (17)$$

As a result of SIE asymptotical analysis we come to a homogeneous set of equations which define the specified constants

$$-m_{13}\sin\frac{\pi\alpha}{2}\Phi_1 + r_{21}(1+\alpha n^{(3)})\Phi_2 - 2(n^{(1)} + r_{31}n^{(3)})\sin\frac{\pi\alpha}{2}F_1 - 2n^{(1)}\alpha F_3 - 2r_{31}n^{(3)}\alpha F_6 = 0$$

$$r_{12}(1+\alpha n^{(3)})\Phi_1 - m_{23}\sin\frac{\pi\alpha}{2}\Phi_2 - 2r_{32}n^{(3)}\alpha F_1 - 2n^{(2)}\alpha F_5 + 2(n^{(2)} - r_{32}n^{(3)})\sin\frac{\pi\alpha}{2}F_6 = 0$$

$$-(s^{(1)} + r_{13}s^{(3)})\sin\frac{\pi\alpha}{2}\Phi_1 + r_{23}n^{(3)}\alpha\Phi_2 + 2m_{13}\sin\frac{\pi\alpha}{2}F_1 + 2(1-\alpha n^{(1)})F_3 + 2(1-\alpha n^{(3)})F_6 = 0$$

$$r_{13}n^{(3)}\alpha\Phi_1 - (s^{(2)} + r_{23}s^{(3)})\sin\frac{\pi\alpha}{2}\Phi_2 + 2(1-\alpha n^{(3)})F_1 + 2(1-\alpha n^{(2)})F_5 + 2m_{23}\sin\frac{\pi\alpha}{2}F_6 = 0$$

$$\left(\frac{1}{n^{(2)}} + \alpha\right)\Phi_2 - 2\sin\frac{\pi\alpha}{2}F_5 - 2\alpha F_6 = 0,\ \left(\frac{1}{n^{(1)}} + \alpha\right)\Phi_1 - 2\alpha F_1 - 2\sin\frac{\pi\alpha}{2}F_3 = 0, \qquad (18)$$

where $n^{(m)} = \dfrac{1}{2(1-\nu^{(m)})}$, $s^{(m)} = \dfrac{3-4\nu^{(m)}}{2(1-\nu^{(m)})}$, $m_{ij} = \dfrac{2-3(\nu^{(i)}+\nu^{(j)})+4\nu^{(i)}\nu^{(j)}}{2(1-\nu^{(i)})(1-\nu^{(j)})}$

$\Phi_\beta = -2\Phi_\beta^D\Gamma(\alpha)\sin\dfrac{\pi\alpha}{2}$, $F_k = 2F_k^D\Gamma(\alpha)\sin\dfrac{\pi\alpha}{2}$, $\Gamma(\alpha)$ is a gamma function, $k = 1,3,5,6$.

It is possible to define LSP α from the existence condition of system (18) nontrivial solution

$$\Delta_1(\alpha, \mu^{(m)}, \nu^{(m)}) = 0 \qquad (19)$$

Let us define the asymptotics at high values of shear modulus $\mu^{(3)}$ of angular area $\mu^{(3)}$ For this purpose we introduce small dimensionless parameters $\varepsilon_j = \mu^{(j)}/\mu^{(3)} = r_{3j}^{-1} (j = 1,2)$ and search for the solution to equation (19) in the form of a series by the powers of these parameters $\alpha = \alpha_0 + \varepsilon_1\alpha_{11} + \varepsilon_2\alpha_{12} + \ldots$. For example, the first expansion term satisfies the equation

$$(\sin^2(\pi\alpha_0/2) - \alpha_0{}^2)\prod_{i=1}^{2}(\alpha_0{}^2 - (3-4\nu^{(i)})\sin^2(\pi\alpha_0/2) - 4(1-\nu^{(i)})^2) = 0 \qquad (20)$$

The first multiplier in equation (20) coincides with the left side of equation (13) which defines the singularity of stress tensor components at the vertex of a homogeneous wedge with the angle opening of 90^0. The second and third multipliers have the form of the second multiplier of the asymptotic equation (15) which corresponds to the case of two media interface. The presence of an additional multiplier in equation (20) allows varying the values of Poisson's ratio of two areas $G^{(1)}$ and $G^{(2)}$ for LSP value control.

3. Interface of Four Different Media

Let the section of an inhomogeneous elastic prism infinite in the axis α_3 direction occupy region $D_4 = G^{(1)} \cup G^{(2)} \cup G^{(3)} \cup G^{(4)}$ where regions $G^{(m)}$ are stuck together and defined by inequalities

$$G^{(1)} = \{|\alpha_1| \le c; \alpha_2 \in [-b,-d] \cup [d,b]\};\ G^{(2)} = \{\alpha_1 \in [-a,-c] \cup [c,a]; |\alpha_2| \le d\};$$

$$G^{(3)} = \{\alpha_1 \in [-a,-c] \cup [c,a]; \alpha_2 \in [-b,-d] \cup [d,b]\}; G^{(4)} = \{|\alpha_1| \le c; |\alpha_2| \le d\}.$$

In each region $G^{(m)}$ we consider Lame motion equations recorded in dimensionless displacements $U_\beta^{(m)}\ (\beta = 1,2)$ and dimensionless coordinates $x = \alpha_1 / a, y = \alpha_2 / a$. Again, proceeding from the symmetry of region D_4, we can consider the wave field of a part of the region located in the first quarter (Fig. 2).

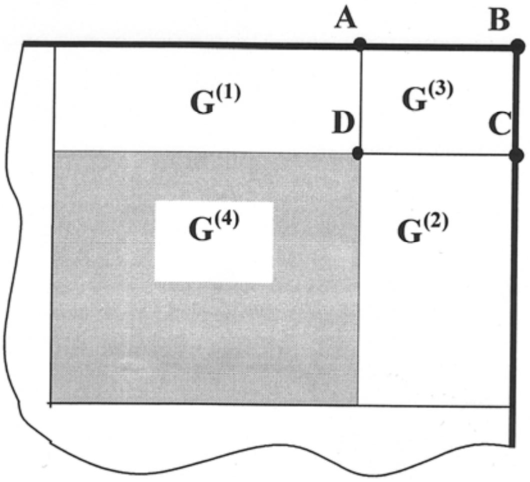

Figure 2. Coupling of four different media

Coordinate systems and section dimensions are similar to those presented in Fig. 1. Boundary conditions of the problem include loading force conditions on the external boundary of the section and the condition of rigid adhesion of the regions. This problem had not been studied earlier in such a form. The task complexity is caused by the following factors. First, in order to define the singularities of LSC in the vicinity of an irregular point at the joint of four different media we should take into account the behavior of wave field components at the approach to a singular point in four directions. Second, in the given case the nature of the wave field undergoes considerable changes due to the checkerboard pattern of inhomogeneity. It is conditioned by the presence of four singular points in the section of an item. On the one hand the research into LSC singularities in the vicinity of the interface point of four media is relevant by itself, on the other hand it is important for complete representation of the wave field. Third, most likely we should expect the interference of LSC singularities in irregular points of the section boundary, especially if the angular region is small.

To construct the general solution we use the superposition method and write down the problem general solution in regions $G^{(m)}$ in the following form

$$U_1^{(1)} = H_1^{(1)} sh(t^{(1)} x) \cos \alpha^{(1)} (\hat{y} - \gamma_2) + (R_1^{(1)} sh(r^{(1)} \hat{y}) + S_1^{(1)} ch(r^{(1)} \hat{y})) \sin \chi^{(1)} (x - \delta)$$

$$U_2^{(1)} = H_2^{(1)} ch(t^{(1)} x) \sin \alpha^{(1)} (\hat{y} - \gamma_2) + (R_2^{(1)} sh(r^{(1)} \hat{y}) + S_2^{(1)} ch(r^{(1)} \hat{y})) \cos \chi^{(1)} (x - \delta)$$

$$U_1^{(2)} = (H_1^{(2)} sh(t^{(2)} \hat{x}) + Q_1^{(2)} ch(t^{(2)} \hat{x}))\cos\alpha^{(2)}(y-\gamma) + R_1^{(2)} ch(r^{(2)} y)\sin\chi^{(2)}(\hat{x}-\delta_2)$$

$$U_2^{(2)} = (H_2^{(2)} sh(t^{(2)} \hat{x}) + Q_2^{(2)} ch(t^{(2)} \hat{x}))\sin\alpha^{(2)}(y-\gamma) + R_2^{(2)} sh(r^{(2)} y)\cos\chi^{(2)}(\hat{x}-\delta_2)$$

$$U_1^{(3)} = (H_1^{(3)} sh(t^{(3)} \hat{x}) + Q_1^{(3)} ch(t^{(3)} \hat{x}))\cos\alpha^{(1)}(\hat{y}-\gamma_2) + (R_1^{(3)} sh(r^{(3)} \hat{y}) + S_1^{(3)} ch(r^{(3)} \hat{y}))\sin\chi^{(2)}(\hat{x}-\delta_2),$$

$$U_2^{(3)} = (H_2^{(3)} sh(t^{(3)} \hat{x}) + Q_2^{(3)} ch(t^{(3)} \hat{x}))\sin\alpha^{(1)}(\hat{y}-\gamma_2) + (R_2^{(3)} sh(r^{(3)} \hat{y}) + S_2^{(3)} ch(r^{(3)} \hat{y}))\cos\chi^{(2)}(\hat{x}-\delta_2);$$

$$U_1^{(4)} = H_1^{(4)} sh(t^{(4)} x)\cos\alpha^{(2)}(y-\gamma) + R_1^{(4)} ch(r^{(4)} y)\sin\chi^{(1)}(x-\delta),$$

$$U_2^{(4)} = H_2^{(4)} ch(t^{(4)} x)\sin\alpha^{(2)}(y-\gamma) + R_2^{(4)} sh(r^{(4)} y)\cos\chi^{(1)}(x-\delta).$$

Here $\alpha_k^{(1)} = k\pi/\gamma_2$, $\alpha_k^{(2)} = k\pi/\gamma$, $\chi_j^{(1)} = j\pi/\delta$, $\chi_j^{(2)} = j\pi/\delta_2$, $k = 1,2,\ldots$; $j = 1,2,\ldots$

When formulating auxiliary boundary problems we use the principles applied above. Namely, we bring into the formulation of auxiliary problems "cross" boundary conditions which affect shearing stresses and normal displacements on respective parts of the boundary of a compound section. In this case it is necessary to take into account the fact that the section boundary consists of eight pieces (Fig. 2). Thus, boundary conditions of auxiliary boundary problems are formulated

$$G^{(1)} : U_1^{(1)}(\delta,\hat{y}) = f_1(\hat{y}), \sigma_{12}^{(1)}(\delta,\hat{y}) = \varphi_1(\hat{y}),$$

$$U_2^{(1)}(x,\gamma_2) = f_2(x), \sigma_{12}^{(1)}(x,\gamma_2) = 0, U_2^{(1)}(x,0) = f_3(x), \sigma_{12}^{(1)}(x,0) = \varphi_4(x);$$

$$U_1^{(2)}(\delta_2,y) = f_4(y), \sigma_{12}^{(2)}(\delta_2,y) = 0, U_1^{(2)}(0,y) = f_5(y), \sigma_{12}^{(2)}(0,y) = \varphi_3(y),$$

$$U_2^{(2)}(\hat{x},\gamma) = f_6(\hat{x}), \sigma_{12}^{(2)}(\hat{x},\gamma) = \varphi_2(\hat{x});$$

$$G^{(3)} : U_1^{(3)}(\delta_2,\hat{y}) = f_7(\hat{y}), \sigma_{12}^{(3)}(\delta_2,\hat{y}) = 0, U_1^{(3)}(0,\hat{y}) = f_1(\hat{y});$$

$$\sigma_{12}^{(3)}(0,\hat{y}) = r_{13}\varphi_1(\hat{y}), U_2^{(3)}(\hat{x},\gamma_2) = f_8(\hat{x}), \sigma_{12}^{(3)}(\hat{x},\gamma_2) = 0,$$

$$G^{(4)} : U_1^{(4)}(\delta,y) = f_5(y), \sigma_{12}^{(4)}(\delta,y) = r_{24}\varphi_3(y),\ U_2^{(4)}(x,\gamma) = f_3(x), \sigma_{12}^{(4)}(x,\gamma) = r_{14}\varphi_4(x).$$

Here $f_1(\hat{y}), \varphi_1(\hat{y}), \ldots f_8(\hat{x})$ are unknown auxiliary functions.

Let us take into consideration 12 unused boundary conditions and interface conditions of the initial boundary problem which represent a SIE for defining the introduced auxiliary functions, namely

$$\sigma_{11}^{(1)}(\delta,\hat{y}) = r_{31}\sigma_{11}^{(3)}(0,\hat{y}), \sigma_{22}^{(2)}(\hat{x},\gamma) = r_{32}\sigma_{22}^{(3)}(\hat{x},0), U_2^{(1)}(\delta,\hat{y}) = U_2^{(3)}(0,\hat{y}), U_1^{(2)}(\hat{x},\gamma) = U_1^{(3)}(\hat{x},0),$$

$$\sigma_{11}^{(2)}(0,y) = r_{42}\sigma_{11}^{(4)}(\delta,y), \sigma_{22}^{(1)}(x,0) = r_{41}\sigma_{22}^{(4)}(x,\gamma), U_2^{(2)}(0,y) = U_2^{(4)}(\delta,y), U_1^{(1)}(x,0) = U_1^{(4)}(x,\gamma),$$

$$\sigma_{22}^{(1)}(x,\gamma_2) = q_2^{(1)}, \sigma_{11}^{(2)}(\delta_2,y) = q_1^{(2)}, \sigma_{11}^{(3)}(\delta_2,\hat{y}) = q_1^{(3)}, \sigma_{22}^{(3)}(\hat{x},\gamma_2) = q_2^{(3)}, q^{(m)} = q/\mu^{(m)}.$$

Then we study the wave field singularities in the vicinity of irregular points of the section boundary. The suggested algorithm of the modified superposition method allows studying local singularities of wave field characteristics in all singular points using common technique. For this purpose it is necessary to conduct the asymptotic analysis of all equations (21) at the approach to each irregular point of the boundary. It should be mentioned that unknown functions of SIE will have different local singularities in the vicinity of points *A, B, C, D.* Taking into account the previous results we perform the asymptotic analysis only for point *D* of the joint of four heterogeneous media. For this purpose it is necessary to conduct the asymptotic analysis of the first eight equations of SIE (21) at limiting values of the arguments, namely

$$\sigma_{11}^{(1)}(\delta,\hat{y}) = r_{31}\sigma_{11}^{(3)}(0,\hat{y}), U_2^{(1)}(\delta,\hat{y}) = U_2^{(3)}(0,\hat{y}),\ \hat{y}\to 0;$$

$$\sigma_{22}^{(1)}(x,0) = r_{41}\sigma_{22}^{(4)}(x,\gamma), U_1^{(1)}(x,0) = U_1^{(4)}(x,\gamma),\ x\to\delta;$$

$$\sigma_{11}^{(2)}(0,y) = r_{42}\sigma_{11}^{(4)}(\delta,y), U_2^{(2)}(0,y) = U_2^{(4)}(\delta,y), \qquad (22)$$

$$\sigma_{22}^{(2)}(\hat{x},\gamma) = r_{32}\sigma_{22}^{(3)}(\hat{x},0), U_1^{(2)}(\hat{x},\gamma) = U_1^{(3)}(\hat{x},0),\ \hat{x}\to 0.$$

We assume that the singularities of auxiliary functions in point *D* are defined by formulas

$$\varphi_\beta(\xi) = \overline{\Phi}_\beta^D(\xi)^{\alpha-1},\ f_i^/(\xi) = \overline{F}_i^D(\xi)^{\alpha-1}, (\beta=1,2; i=1,6), \xi\to 0;$$

$$\varphi_3(\xi) = \overline{\Phi}_3^D(\gamma-\xi)^{\alpha-1}, f_5^/(\xi) = \overline{F}_5^D(\gamma-\xi)^{\alpha-1}, \xi\to\gamma;$$

$$\varphi_4(\xi) = \overline{\Phi}_4^D(\delta-\xi)^{\alpha-1}, f_3^/(\xi) = \overline{F}_3^D(\delta-\xi)^{\alpha-1}, \xi\to\delta.$$

Here $\alpha=\alpha(D)$ stands for the parameter characterizing the singularities of required functions in point *D*, and $\overline{\Phi}_\beta^D,...,\overline{F}_3^D$ stand for arbitrary constants. The asymptotic analysis of expressions (22) leads to the following set of homogeneous equations for defining the mentioned constants

$$-m_{13}\sin(0{,}5\pi\alpha)\Phi_1^D + r_{21}(1+n^{(3)}\alpha)\Phi_2^D - (1+n^{(1)}\alpha)\Phi_4^D - 2(n^{(1)}+r_{31}n^{(3)})\sin(0{,}5\pi\alpha)F_1^D - 2n^{(1)}\alpha F_3^D -$$
$$-2r_{31}n^{(3)}\alpha F_6^D = 0,$$

$$r_{12}(1+n^{(3)}\alpha)\Phi_1^D - m_{23}\sin(0{,}5\pi\alpha)\Phi_2^D - (1+n^{(2)}\alpha)\Phi_3^D - 2r_{32}n^{(3)}\alpha F_1^D - 2n^{(1)}\alpha F_5^D -$$
$$-2(n^{(2)}+r_{32}n^{(3)})\sin(0{,}5\pi\alpha)F_6^D = 0,$$

$$-(s^{(1)}+r_{13}s^{(3)})\sin(0{,}5\pi\alpha)\Phi_1^D + r_{23}n^{(3)}\alpha\Phi_2^D - n^{(1)}\alpha\Phi_4^D + 2m_{13}\sin(0{,}5\pi\alpha)F_1^D + 2(1-n^{(1)}\alpha)F_3^D +$$
$$+2(1-n^{(3)}\alpha)F_6^D = 0,$$

$$r_{13}n^{(3)}\alpha\Phi_1^D - (s^{(2)}+r_{23}s^{(3)})\sin(0{,}5\pi\alpha)\Phi_2^D - n^{(2)}\alpha\Phi_3^D + 2(1-n^{(3)}\alpha)F_1^D + 2(1-n^{(2)}\alpha)F_5^D +$$
$$+2m_{23}\sin(0{,}5\pi\alpha)F_6^D = 0,$$

$$(1+n^{(2)}\alpha)\Phi_2^D + m_{24}\sin(0{,}5\pi\alpha)\Phi_3^D - r_{12}(1+n^{(4)}\alpha)\Phi_4^D - 2r_{42}n^{(4)}\alpha F_3^D -$$
$$-2(n^{(2)}+r_{42}n^{(4)})\sin(0{,}5\pi\alpha)F_5^D - 2n^{(2)}\alpha F_6^D = 0, \qquad (23)$$

$$(1+n^{(1)}\alpha)\Phi_1^D - r_{21}(1+n^{(4)}\alpha)\Phi_3^D + m_{14}\sin(0{,}5\pi\alpha)\Phi_4^D - 2n^{(1)}\alpha F_1^D -$$
$$-2(n^{(1)}+r_{41}n^{(4)})\sin(0{,}5\pi\alpha)F_3^D - 2r_{41}n^{(4)}\alpha F_5^D = 0,$$

$$-n^{(2)}\alpha\Phi_2^D - (s^{(2)}+r_{24}s^{(4)})\sin(0{,}5\pi\alpha)\Phi_3^D + r_{14}n^{(4)}\alpha\Phi_4 - 2(1-n^{(4)}\alpha)F_3^D - 2m_{24}\sin(0{,}5\pi\alpha)F_5^D -$$
$$-2(1-n^{(2)}\alpha)F_6^D = 0,$$

$$-n^{(1)}\alpha\Phi_2^D + r_{42}n^{(4)}\alpha\Phi_3^D - (s^{(1)} + r_{14}s^{(4)})\sin(0{,}5\pi\alpha)\Phi_4^D - 2(1-n^{(1)}\alpha)F_1^D - 2m_{14}\sin(0{,}5\pi\alpha)F_3^D -$$
$$-2(1-n^{(4)}\alpha)F_5^D = 0.$$

From the condition of equality to zero of the determinant of the system (23)

$$\Delta_D(r_{ij,}\nu^{(m)},\alpha) = 0 \qquad (24)$$

we obtain a characteristic equation for defining LSP in point *D*. The singularity order by stresses remains equal to $|\mathrm{Re}\,\alpha - 1|$.

It should be noted that the obtained system (23) passes into defining system (18) for defining LSP in the point of the joint of three regions. For this purpose it is necessary to set $\varphi_3(y) = \varphi_4(x) = \sigma_{11}^{(4)}(\delta, y) = \sigma_{22}^{(4)}(x,\gamma) \equiv 0$ and to remove the seventh and eighth interface conditions.

4. Some Results of Numerical Research

Numerical investigation of the obtained solutions is started with the analysis of LSP values in the singular point of the joint of two media defined by equation (14). In expressions from 1 we assume $N = 3$ and that will correspond to the inner region $D^{(1)}$ joined with two outer surfaces $D_-^{(2)} \cup D_+^{(2)}$. Fig. 3 presents the diagrams of LSP dependence on the ratio of joined media rigidities. Steel, lead and tungsten were taken as materials of the internal area, and the elastic characteristics of the surfacing varied with the change of parameter $r_{2S} = \mu^{(2)} / \mu^{(St)}$, where $\mu^{(St)}$ is the shear modulus of steel. The value of Poisson's ratio of surfacing is assumed to be fixed and equal $\nu^{(2)} = 0.29$.

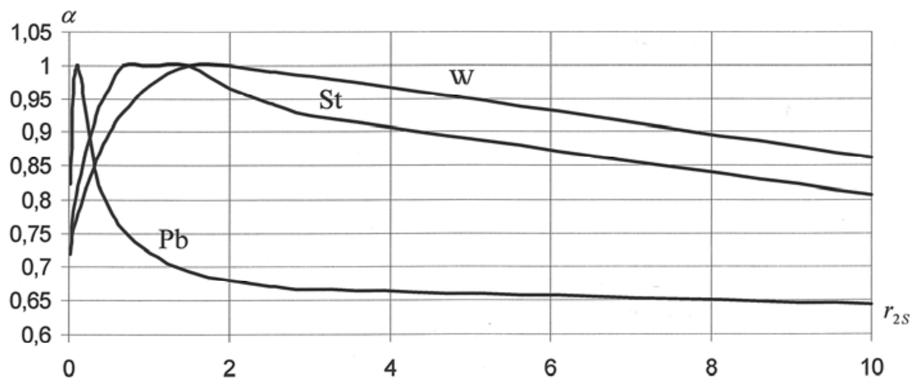

Figure 3. Dependence of LFP α = α(r_{2s}) from a proportion of rigidity of joined media ($r_{2s}=\mu^{(2)}/\mu^{(st)}$, $\nu^{(2)}=0.29$).

The analysis of the results of equation (14) numerical solution and the data from Fig. 3 leads to the following: 1) LSP $\alpha = p_1$ depends essentially on the elastic parameters of the internal region. 2). Equation (14) complex roots with the positive real part less then 1 were not obtained for any of the considered combinations of joined regions elastic constants. 3). Value $\alpha = 1$ is the root of equation (14) at any combination of the materials of joined areas. For the majority of combinations, however, this root turns out to be the second-large positive root of this equation. 4). In case when the joint regions are made of the same material LSC is absent, LSP α is always equal to 1. 5). At some values of parameter r_{2S} the singularity disappears even if different materials are joined. 6). The variation of Poisson's ratio of the surfacing material makes the minimum impact on LSP practically at any ratio of joined regions shear moduli. 7). If the values of the surfacing shear modulus are enough small or enough high the value of LSP α becomes stable, tending to a certain value.

The final conclusion can be specified if we study the roots of the second multiplier of asymptotic equation (15). For all actual materials the roots of this multiplier satisfy inequality $0 < p_{m0} < 1$. Table 1 provides the values of these roots for various materials of the first region.

Table 1. The asymptotic values of LFP corresponding to the infinitely high value of shear modulus of a section region

Al,Mg	W	Au	Cu	Mo	Ni	Sn	Pt	Pb	Ag	Ti	Zn
0,680	0,718	0,638	0,692	0,781	0,705	0,692	0,656	0,633	0,662	0,711	0,857

Thus, there is a possibility of obtaining the highest possible LSP value through the change of Poisson's ratio of one of the regions if the distinctions in values of shear modulus of joined regions are great.

Let us begin the numerical analysis of LSP values in the angular point D of the joint of three section regions presented in Fig. 1. It should be mentioned that in this case there is no possibility of introducing compact parameters analogous to Danders parameters [1,6], which are, in some way, an LSP analogue.

The results of the numerical analysis lead to the following conclusions: 1). The values of LSP in the internal angular point *D* of the joint of three media for any combination of materials are much smaller than the values of LSP in points *A* and *C* analyzed above. 2). For any combination of materials there is a root of the solving equation (19) which is less than 1, i.e. stresses will always have a singularity in the angular point of the joint of three media. 3). If three identical materials are joined LSP is always equal to $\alpha = 0.544$. It testifies to the essential intensity of LSC in point *D*. 4). Unlike the case of two media interface for many combinations of materials there are two (but not one) roots less than 1. This fact is important when the asymptotics of SIE solution in point D is being considered. Table 2 provides the least real roots of equation (19) for the case of different materials of an angular region. In calculations it is assumed that joined regions are made of steel.

Table 2. The least values of LFP for various materials of angular region

Кор-ни	Материал угловой области														
	St	Al	W	Fe	Au	Mg	Cu	Mo	Ni	Sn	Pt	Pb	Ag	Ti	Zn
1	0.544	0.430	0.607	0.542	0.475	0.370	0.492	0.549	0.545	0.386	0.556	0.286	0.449	0.484	0.409
2	0.909	0.657	1.000	0.910	0.667	0.552	0.786	1.000	0.899	0.593	0.859	0.364	0.664	0.793	0.741
3	1.000	1.000		1.000	1.000	1.000	1.000		1.000	1.000	1.000	1.000	1.000	1.000	1.000

In order to confirm the results of the analysis of LSP dependence on the rigidities of the joined regions we checked the following conditions. First, the values of LSP should not change when the materials of regions $G^{(1)}$ and $G^{(2)}$ are interchanged. Second, when one of shear moduli of joined regions tends to zero, the parameter α must tend to the asymptotic value of LSP in the joint point of two regions. Third, if the angular region shear modulus $\mu^{(3)}$ is too high LSP must tend to its asymptotic values defined by equation (20).

For example, Fig. 4 provides the diagrams of the mentioned dependence for the case when the materials of regions $G^{(1)}$ and $G^{(2)}$ are identical and fixed ($\mu^{(1)} = \mu^{(2)}, \nu^{(1)} = \nu^{(2)}$), while the elastic characteristics of angular region $G^{(3)}$ vary.

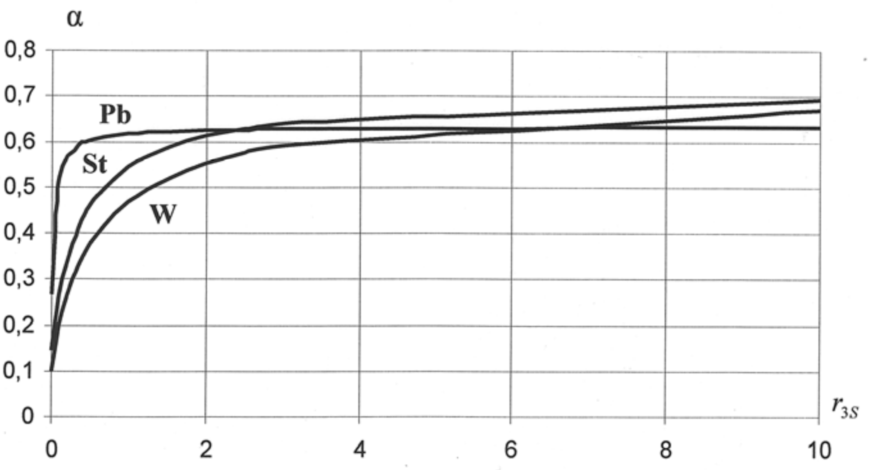

Figure 4. Dependence of LFP from a parameter rigidity for a case when materials of regions $G^{(1)}$ and $G^{(2)}$ are identical

The results of the numerical analysis lead to the following conclusions.

1). When $\mu^{(3)} \to 0$ the section loses bearing capacity and that agrees with the mechanical meaning of the problem.

2). When $\mu^{(3)}$ tends to infinity LSP α tends to its asymptotic value defined by equation (20) for α_0, and this value will be the highest in the whole range of parameter r_{3S} variation.

3). The less rigid the material of regions $G^{(1)}$ and $G^{(2)}$ is, the smaller values r_{3S} are required for parameter α to coincide with its asymptotics.

Possible recommendations in the considered combination of regions $G^{(m)}$ materials are quite obvious: the material of the angular region should be as rigid as possible with value $\nu^{(3)}$ high enough. The materials of regions $G^{(1)}$ and $G^{(2)}$ should be selected from the materials with minimum possible Poisson's ratio.

Numerical analysis of equation (24) which defines LSP by stresses in the joint point *D* of four different media (Fig. 2) leads to the following conclusions.

First, for the majority of combinations of materials there is a positive root of this equation less than 1. Second, practically for all considered combinations of materials LSP values exceed the values of LSP at the interface of three media. Third, when four regions with identical elastic characteristics are joined LSP $\alpha = 1$. Fourth, the value of LSP $\alpha = 1$ is reached for some combinations of materials even if different regions are joined. Earlier this property had only been typical of two different regions interface. Fifth, for many combinations of materials (not in the case when two media are joined) there are several roots less than 1. Sixth, LSP does not change after the mutual replacement of elastic characteristics of crosswise regions and after their circular permutation. While describing the considered combinations of materials we will list them in circular order beginning with the material of region $G^{(1)}$. For example, combination **St-Pb-St-Pb** means that the material of regions $G^{(1)}$ and $G^{(2)}$ is steel, and the material of regions $G^{(3)}$ and $G^{(4)}$ is lead. If, for example, some elastic parameter of region $G^{(2)}$ varies, the combination will be written down as **St-Pb-G$^{(2)}$–Pb**.

Table 3. Value of LSP for various materials of two adjacent regions in case when the material of two joined regions is steel

	Al	W	Fe	Au	Mg	Cu	Pb
Al	0.710	0.910	0.841	0.757	0.642	0.779	0.562
W	0.910	0.811	0.917	1	0.829	1	0.932
Fe	0.841	0.917	1	0.905	0.793	0.934	0.674
Au	0.757	1	0.905	0.821	0.681	0.913	0.586
Mg	0.642	0.829	0.793	0.681	0.585	0.710	0.504
Cu	0.779	1	0.934	0.913	0.710	0.857	0.623
Pb	0.562	0.932	0.674	0.586	0.504	0.623	0.428

Table 3 presents the values of the least positive root of equation (24) for the case when the elastic parameters of regions $G^{(1)}$ and $G^{(2)}$ are fixed and correspond to steel, and the elastic parameters of regions $G^{(3)}$ and $G^{(4)}$ vary. In particular the table shows that combination **St-Pb-St-Pb** will be the least preferable. While analyzing the problem of LSP dependence on joined media rigidities ratio let us study the case when two neighboring regions are of identical materials, the third region is of another material, and the shear modulus of the fourth region varies. The results of numerical calculations for some combinations of materials are presented in Fig. 5.

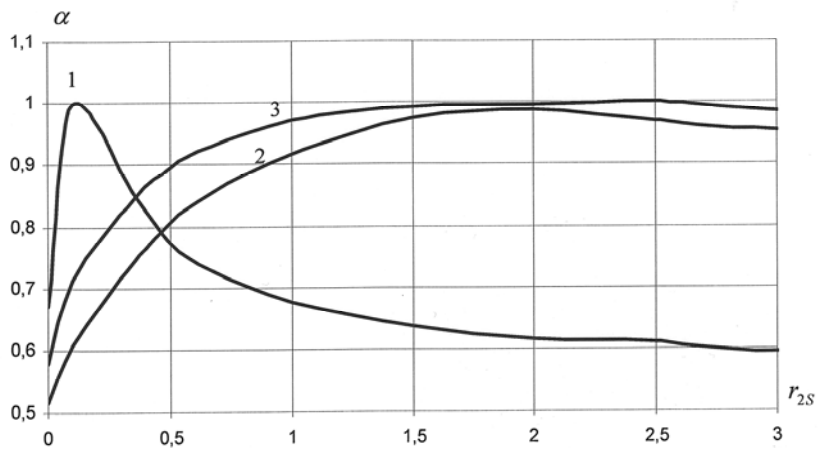

Figure 5. Dependence of LFP in a point of joint of four different media from rigidity parameter **(St-Pb-G$^{(2)}$-St** – curve 1;**St-W-G$^{(2)}$-St** – curve 2; **Pb-W-G$^{(2)}$-Pb** – curve 3)

From these results it follows: 1). The increase in rigidity of the material of region $G^{(3)}$ displaces the area of extremely high values of LSP (close to 1) to the area of higher values of parameter r_{2S} and vice versa. 2). When parameter r_{2S} tends to zero the increase in rigidity of the material of region $G^{(3)}$ reduces the minimum value of LSP. 3). At very high values of parameter r_{2S} the increase in rigidity of the material of region $G^{(3)}$ increases the limiting value of LFP.

Conclusion

Taking into account the laws of LSP change helps to optimize the computational algorithm of wave characteristics construction in the entire non-uniform region on the one hand, and gives the possibility to define stress concentration coefficients and [14, 15] resonance frequency spectrum singularities and eigenvibrations in the vicinity of an irregular joint point of two, three and four media on the other hand.

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