MINISTRY OF SCIENCES AND EDUCATION OF UKRAINE
DONETSK NATIONAL TECHNICAL UNIVERSITY

J.F. KOSOLAPOV

LINEAR ALGEBRA AND ANALYTIC GEOMETRY
(ЛІНІЙНА АЛГЕБРА І АНАЛІТИЧНА ГЕОМЕТРІЯ)

Методичний посібник

для студентів ДонНТУ (англійською мовою)

Розглянуто на засіданні
кафедри вищої математики
Протокол № 4 від 9.11.2006 р.

Затверджено на засіданні
Вченої ради Інституту міжнародного
співробітництва ДонНТУ
Протокол № 5 від 20.03.2007 р.

ДОНЕЦЬК 2007

УДК 512.1+512.64+514.11+514.122+514.123(071)

Косолапов Ю.Ф. Linear Algebra and Analytic Geometry (Лінійна алгебра і аналітична геометрія): Методичний посібник для студентів ДонНТУ (англійською мовою)/ - Донецьк: РВА ДонНТУ, 2007. – 247 с.

Викладаються окремі розділи елементарної математики, основні поняття лінійної алгебри і аналітичної геометрії (множина дійсних чисел, відображення і функції, огляд теорії основних елементарних функцій, комплексні числа, елементи теорії многочленів, раціональні дроби; визначники і матриці, ранг матриці, системи лінійних алгебричних рівнянь, в тому числі однорідних, власні значення і власні вектори; рівняння лінії на площині, основні рівняння прямої, криві другого порядку, полярні координати, перетворення координат, способи задання кривих; рівняння поверхні, сфери, площини, просторової прямої, їх взаємне розташування). Докладно розглядаються приклади розв'язання типових задач. Вміщено англо-українсько-російські термінологічні словники до всіх розділів. Дано завдання для самостійного розв'язання.

Велику допомогу в створенні посібника надали автору студенти факультету економіки і менеджменту ДонНТУ Мамічева В., Бараненко І., Бородіна Ю. Іванова А., Гундарєва О., Драченко Л., Муза В., Прокопенко О., Серезентінов О. , Фролофф Г. (впорядкування і створення електронних версій студентських лекційних конспектів, редагування англомовного тексту, робота над термінологічними словниками). Слід особливо відзначити роботу Галі Фролофф, яка ретельно перевірила всі математичні викладки, повторно розв''язала всі приклади і допомогла значно покращити текст посібника. Значний внесок в написання посібника внесла старший викладач Слов''янського педагогічного університету Косолапова Н. В. (підготовка великого ілюстративного матеріалу, робота над термінологічними словниками).

Всім своїм помічникам автор висловлює щиру подяку.

Для студентів і викладачів технічних вузів.

УКЛАДАЧ: Косолапов Ю.Ф.

РЕЦЕНЗЕНТ: кандидат фізико-математичних наук, доцент Кочергін Є.В.

ВІДПОВІДАЛЬНИЙ ЗА ВИПУСК:
зав. кафедри вищої математики ДонНТУ,
доктор технічних наук, професор Улітін Г.М.

HIGHER MATHEMATICS, FIRST TERM

TEXTBOOKS

1. Bermant A., Aramanovich I. Mathematical analysis. A brief course for engineering students. – Moscow: Mir Publishers, 1975. - 782 p.

2. Krasnov M., Kiselev A., Makarenko G., Shikin E. Mathematical analysis for engineers. T. 1, 2. – Moscow: Mir Publishers, 1989, 1990. – 588, 685 p.

3. Piscunov N. Differential and integral calculus. – Moscow: Mir Publishers, 1969. - 895 p.

4. Yakovlev G. Higher mathematics. – Moscow: Mir Publishers, 1990, - 480 p.

5. Кремер Н.Ш. Высшая математика для экономистов. Уч. пособие для вузов.- М.: ЮНИТИ, 2004. - 471 с.

6. Общий курс высшей математики для экономистов: Учебник / Под ред. В.И.Ермакова. - М.: ИНФРА, 1999. - 656 с.

7. Пискунов Н.С. Дифференциальное и интегральное исчисления. Т.1, 2. – М.: Наука, 1978, 1985. - 456, 560 с.

8. Шнейдер В.Е., Слуцкий А.И., Шумов А.С. Краткий курс высшей математики. Тт. 1, 2. Учеб. пособие для втузов. – М.: „Высшая школа", 1978. – 384, 328 с.

PROBLEM BOOKS

1. Berman G. Problem book on mathematical analysis. – Moscow: Mir Publishers, 1977. – 482 p.

2. Берман Г.Н. Сборник задач по курсу математического анализа: Учеб. пособие для вузов.-СПб: Профессия, 2005.- 432 с.

3. Клетеник Д.В. Сборник задач по аналитической геометрии: Учеб. пособие для вузов.-СПб: Профессия, 2002.- 200 с.

4. Кремер Н.Ш. Практикум по высшей математике для экономистов: Учеб. пособие для вузов. – М.: ЮНИТИ, 2002. – 423 с.

5. Сборник задач по высшей математике для экономистов: Учеб. пособие / Под ред. В.И. Ермакова. – М.: ИНФРА-М, 2002. – 575 с.

ELEMENTS OF ELEMENTARY MATHEMATICS

LECTURE NO.1. REAL NUMBERS

POINT 1. SETS
POINT 2. SETS OF REAL NUMBERS
POINT 3. NUMBER INEQUALITIES
POINT 4. ABSOLUTE VALUE OF A REAL

POINT 1. SETS

Set is the most general mathematical notion. It hasn't definition. There are synonyms of this notion: totality [tou-], aggregate, assemblage, assembly, class, collection, ensemble, family, manifold, plurality.

Sets are denoted by capital letters (*A*, *B*, *C* …) and their elements by small letters. If *a* is an element of a set *A*, one writes $a \in A$ (*a* belongs to *A*, *A* contains *a*). If *a* isn't an element of a set *A*, one writes $a \notin A$ (*a* doesn't belong to *A*, *A* doesn't contain *a*).

Def. 1. A set having no elements is called the empty one. It's denoted by $\varnothing$.

Ex. 1. (Example 1) The set of real roots of the equation $x^2 + 1 = 0$ is empty one.

Def. 2. Every part *B* of a set *A* is called its subset ($B \subset A$, $A \supset B$, *B* is included in *A*, *A* includes *B*). The empty set is subset of any set.

Ex. 2. If $A = \{1, 2, 3, 4\}$ and $B = \{1, 2\}$ then $B \subset A$, $A \supset B$.

Def. 3. Two sets are called equal if they contain the same elements.

Def. 4. A set *C* is called a union (a sum) of sets *A* and *B*, $C = A \cup B$, if every element of *C* belongs at least to one of the sets *A*, *B*.

Def. 5. A set *D* is called an intersection (a product) of sets *A* and *B*, $D =$ $= A \cap B$, if every element of *D* belongs to both sets *A*, *B*.

Def. 6. A set E is called a difference of sets A and B, $E = A \setminus B$, if it contains all the elements of A which aren't the elements of B.

Ex. 3. Let $A = \{1, 4, 7, 10\}$, $B = \{3, 4, 8, 10\}$. Then $C = A \bigcup B = \{1, 3, 4, 7, 8, 10\}$, $D = A \bigcap B = \{4, 10\}$, $E = A \setminus B = \{1, 7\}$.

POINT 2. SETS OF REAL NUMBERS

Set of real numbers

There are natural, integer, rational, irrational, real and complex (see Lecture No. 3) numbers.

Def. 7. Numbers 1, 2, 3, 4, … are called those **natural**.

The **set of all natural numbers** is denoted by the letter ***N***,

$$\boldsymbol{N} = \{1, 2, 3, 4, \ldots\}.$$

Def. 8. Numbers $0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots$ are called those integer [integral], or simply integers.

The **set of all integers** [the set of all integer [integral] numbers] is denoted by the letter ***Z***,

$$\boldsymbol{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots\}.$$

Def. 9. A **rational number** [a **rational**] is called a number which can be represented as a ratio of an integer m to a natural number n.

The **set of all rationals** is denoted by the letter ***Q***.

The next numbers are those rational: a) all natural numbers; b) all integers; c) all common [simple] fractions; d) all finite decimals; e) all non-terminating periodic decimals.

Ex. 4. Numbers $4 = 4/1$, $3/7$, $2.43 = 243/100$, $0.(2) = 0.222... = 2/9$ are rationals.

Def. 10. An **irrational number** [an **irrational**] is called a number which can be represented by an infinite non-periodic decimal.

Ex.5. π = 3, 14159…, Euler's number $e = \lim_{n\to\infty}(1 + 1/n)^n$ =2,718281828..., $\sqrt{2}$, $\sqrt{3}$ are irrational numbers.

The **set of all irrationals** can be denoted by the letter ***I***.

Def. 11. The union of the sets of all rationals ***Q*** and irrationals ***I*** is called the **set of all real numbers** [the **set of all reals**], and every number of this latter set is called a real number [a real].

The **set of all reals** is denoted by the letter ***R***. By definition

$$\boldsymbol{R} = \boldsymbol{Q} \cup \boldsymbol{I}.$$

Geometrical representation of reals

Reals [real numbers] can be represented by points of a coordinate [number] axis.

Def. 12. A **coordinate axis** [a **number axis**] is called an infinite straight line on which are chosen: 1) a certain point O (the origin); 2) the positive direction indicated by an arrow; 3) an unit of length [of measurement] (fig. 1).

Fig. 1

Theorem 1. There exists one-to-one correspondence between the set of all reals and the set of all points of a coordinate [number] axis.

It means: a) to each real x there corresponds unique point of the coordinate axis that is the point $M(x)$ with coordinate x; b) to every point $M(x)$ of the coordinate axis there corresponds quite definite real, namely its coordinate x.

By this reason we can identify reals and corresponding points of the coordinate axis.

Multidimensional points and spaces

Def. 13. Ordered systems of two, three, n reals is called correspondingly two-, three-, n-dimensional points,

$$(x, y), (x, y, z), (x_1, x_2, x_3, ..., x_n).$$

Each two-dimensional point (x, y) is represented geometrically by the point $M(x; y)$ of the xOy-plane, and each three-dimensional point (x, y, z) is represented by the point $M(x; y; z)$ of the $Oxyz$-space.

Def. 14. The sets of all two-, three-, n-dimensional points are called correspondingly the two-, three-, n-dimensional spaces and are denoted by $\boldsymbol{R}^2$, $\boldsymbol{R}^3$, $\boldsymbol{R}^n$.

Geometrically the two-dimensional set $\boldsymbol{R}^2$ is the set of all points of the xOy-plane, and the three-dimensional set $\boldsymbol{R}^3$ is the set of all points of the $Oxyz$-space.

POINT 3. NUMBER INEQUALITIES

Def. 15. One says that a number a is greater than a number b if their difference $a - b$ is positive, $a > b$ if $a - b > 0$.

Properties of number inequalities

1. If $a > b$, then $b < a$.

2. If $a > b$, $b > c$, then $a > c$ (transitivity).

3. If $a > b$, then for any number c one has $a + c > b + c$ (regularity).

Corollary. One can transpose any term of an inequality from one side to the other taking it with opposite sign.

■ For example let $a + b > c$. Adding to both sides of the inequality the number $-b$ we get $a > c - b$. It's equivalent to transposing of b from the left to the right with opposite sign.■

4. If $a > b$, then for any **positive** number m one has $am > bm$, that is one can multiply both sides of an inequality by the same positive number without changing the sense of the inequality.

5. If $a > b$, then for any **negative** number m one has $am < bm$, that is one can multiply both sides of an inequality by the same negative number changing the sense of the inequality.

6. If $a > b$, $c > d$, then $a + c > b + d$, that is one can **termwise add** inequalities of the same sense.

7. If $a > b$, $c < d$, then $a - c > b - d$ (**termwise subtraction** of inequalities of opposite senses).

8. If a, b, c, d are positive numbers and $a > b$, $c > d$, then $ac > bd$, that is one can **termwise multiply** inequalities of the same sense with positive terms.

9. If a, b are positive numbers and $a > b$, then $1 / a < 1 / b$. If two positive numbers are connected by an inequality of one sense, then reciprocal of them are connected by the inequality of opposite sense.

10. If a, b, c, d are positive numbers and $a > b$, $c < d$, then $a / c > b / d$ (**term-wise division** of inequalities of opposite senses with positive terms).

11. If a, b are positive numbers and $a > b$, then for any natural number n one

has $a^n > b^n$, that is one can raise to any natural power both sides of an inequality with positive terms.

Corollary. If $0 \le a \le b$, then $a^n \le b^n$.

12. If a, b are positive numbers and $a > b$, then for any natural number n one has $\sqrt[n]{a} > \sqrt[n]{b}$, that is one can extract a root of any natural power of both sides of an inequality with positive terms.

Corollary. If $0 \le a \le b$, then $\sqrt[n]{a} \le \sqrt[n]{b}$.

13. If a, b are positive numbers, then their **arithmetic mean** $(a+b)/2$ is greater than or equal to their geometrical mean $\sqrt{ab}$,

$$\frac{a+b}{2} \ge \sqrt{ab}.$$

The equality holds if and only if these numbers are equal. Analogous inequality is true for arbitrary number of positive numbers,

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \ge \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}.$$

Number intervals

Let's denote by

$$\{x : P(x)\} \text{ or } \{x \mid P(x)\}$$

the set of all reals which possess a property $P(x)$.

Later on we'll use various intervals of reals. Let's enumerate some of them.

A **segment** (a closed bounded interval) $[a, b] = \{x : a \le x \le b\}$, the set of all reals not less then a and not greater then b.

An **interval** (an open bounded interval) $(a, b) = \{x : a < x < b\}$, the set of all reals which are greater then *a* and less then *b*.

Semiclosed [or semiopen] bounded intervals

$$[a,b)=\{x:a\le x<b\},(a,b]=\{x:a<x\le b\}.$$

Points *a*, *b* of all these bounded intervals are called end points, *a* is the left one and *b* is the right one.

$(-\infty,a)=\{x:x<a\}$ a set of all reals which are less then *a*.

$[b,\infty)=\{x:x\ge b\}$ a set of all reals which are greater then or equal *b*.

$R_+=\{x:x\ge 0\}$ the set of all non-negative reals;

$R_+^*=\{x:x>0\}$ the set of all positive reals;

$R_-=\{x:x\le 0\}$ the set of all non-positive reals;

$R_-^*=\{x:x<0\}$ the set of all negative reals;

$R=(-\infty,\infty)=\{x:-\infty<x<\infty\}$ the set of all reals.

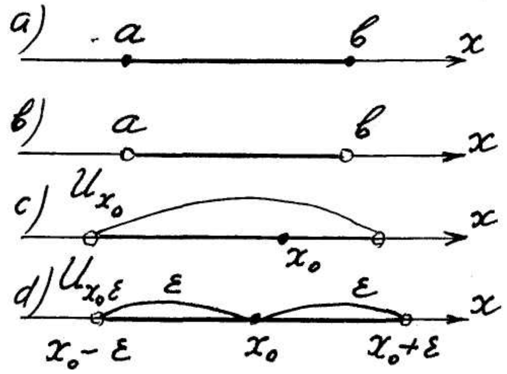


Fig. 2

All number intervals can be represented by corresponding point sets on the coordinate axis. For example a segment $[a,b]$ is represented by a segment with end points $A(a), B(b)$ including them (fig. 2 a), an interval (a,b) by the same segment excluding the points $A(a), B(b)$ (fig. 2 b).

Def. 16. A **neighbourhood** U_{x_0} of a point x_0 is called any (open) interval containing this point (fig. 2 c). In particular an ε - **neighbourhood** $U_{x_0\varepsilon}$ of a point is called an interval $U_{x_0\varepsilon}=(x_0-\varepsilon,x_0+\varepsilon)$ of the length 2ε centered at this point (fig. 2 d).

POINT 4. ABSOLUTE VALUE OF A REAL

Def. 17. Absolute value [or modulus] of a real number *x* is called the same number, if it is non-negative, and the opposite number otherwise,

$$|x| = \begin{cases} x, & if \quad x \ge 0; \\ -x, & if \quad x < 0. \end{cases}$$

Ex. 6. $|3| = 3, |-8| = -(-8) = 8, |x-5| = \begin{cases} x-5, & if \quad x-5 \ge 0, x \ge 5; \\ 5-x, & if \quad x-5 < 0, x < 5. \end{cases}$

Properties of absolute values

1. $|x| \ge 0$ (the absolute value is a non-negative quantity).

2. $|x| = |-x|$ (opposite numbers have the same absolute value).

3. $-|x| \le x \le |x|$ (every number x ranges from $-|x|$ to $|x|$).

4. $|x+y| \le |x| + |y|$ (the absolute value of a sum of two numbers doesn't outnumber the sum of their absolute values).

5. $|x-y| \ge |x| - |y|, |x-y| \ge |y| - |x|, |x-y| \ge ||x| - |y||$ (the absolute value of a difference of two numbers isn't less than the difference of their absolute values).

6. $|x \cdot y| = |x| \cdot |y|$ (the absolute value of a product of two numbers equals the product of their absolute values).

7. $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}, y \ne 0$ (the absolute value of a quotient of two numbers equals the quotient of their absolute values).

8. For any natural number n one has $|x^n| = |x|^n$ (absolute value of a natural power of a number equals the same power of the absolute value of this number).

9. For any natural number n one has $\sqrt[2n]{x^{2n}} = |x|$. In particular $\sqrt{x^2} = |x|$.

Ex. 7. $\sqrt{(x+4)^2} = |x+4| = \begin{cases} x+4 \text{ if } x+4 \ge 0, x \ge -4; \\ -x-4 \text{ if } x+4 < 0, x < -4. \end{cases}$

10. For any positive number a the next inequalities $|x| < a, -a < x < a$ are equivalent,

$$\forall a > 0 : ((|x| < a) \Leftrightarrow (-a < x < a)).$$

Corollary. For any positive ε the ε-neighbourhood $U_{x_0} = (x_0 - \varepsilon, x_0 + \varepsilon)$ of a point x_0 can be represented by the inequality $|x - x_0| < \varepsilon$,

$$\forall \varepsilon > 0 : U_{x_0} \equiv (x_0 - \varepsilon, x_0 + \varepsilon) \equiv \{x : x_0 - \varepsilon < x < x_0 + \varepsilon\} = \{x : |x - x_0| < \varepsilon\}.$$

■By virtue of the property 10 the inequalities $x_0 - \varepsilon < x < x_0 + \varepsilon, |x - x_0| < \varepsilon$ are equivalent. Indeed,

$$(x_0 - \varepsilon < x < x_0 + \varepsilon) \Leftrightarrow (-\varepsilon < x - x_0 < \varepsilon) \Leftrightarrow (|x - x_0| < \varepsilon) \blacksquare$$

11. For any positive number a an inequality $|x| > a$ is equivalent to the union of two inequalities $x < -a, x > a$,

$$\forall a > 0 : ((|x| > a) \Leftrightarrow ((x < -a) \wedge (x > a))) \equiv \left[\begin{array}{l} x < -a, \\ x > a. \end{array} \right.$$

REALS: basic terminology RUE

1. Абсолютная величина числа	Абсолютна величина числа	Ábsolute válue [mágnitude, módulus] of a númber
2. Бесконечная десятичная дробь (периодическая, непериодическая)	Нескінченний десятковий дріб (періодичний, неперіодичний)	Nòntérminàting (pèriódic, nòn-pèriódic) décimal/póint fráction
3. Быть эквивалентным [равносильным] *чему*	Бути еквівалентним [рівносильним] *чомусь*	Be equívalent to *smth*
4. Быть [являться] элементом множества ($x \in X$, x (малое) [малое x] является элементом множества X (большого) [большого X]; $x \notin X$, x не является элементом множества X)	Бути елементом множини ($x \in X$, x (мале) [мале x] є элементом множини X (великого) [великого X]; $x \notin X$, x не є элементом множини X)	Be an élément of a set ($x \in X$, small x is an élement of a set cápital X; $x \notin X$, small x isn`t an élement of cápital X)
5. Вещественное число	Дійсне число	Réal númber
6. Взаимно однозначное соответствие (между множеством всех вещественных чисел и множеством всех точек числовой [координатной] оси)	Взаємно-однозначна відповідність (між множиною всіх дійсних чисел та множиною всіх точок числової [координатної] осі)	One-to-one còrrespóndence [bijection] (betwéen the set of all réal númbers and the set of all póints of the númber [coórdinate] áxis)
7. Десятичная дробь	Десятковий дріб	Décimal/póint fráction
8. Единица длины/измерения	Одиниця довжини/вимірювання	Únit of length/measurement
9. Интервал (открытый ограниченный интервал) $(a, b) \equiv]\, a, b\, [$	Інтервал (відкритий обмежений інтервал) $(a, b) \equiv]\, a, b\, [$	Ínterval (bóunded ópen ínterval) $(a, b) \equiv]\, a, b\, [$
10.Интервал, симметричный относительно точки	Інтервал, симетричний відносно точки	Symmétric ínterval with respéct to a póint
11.Иррациональное число	Іраціональне число	Irrátional númber [irrátional]
12.Конец интервала (левый/правый)	Кінець інтервала (лівий/ правий)	(Left(-hand)/right(-hand)) end /endpoint of an ínterval
13.Конечная десятичная дробь	Скінченний десятковий дріб	Términating décimal/póint fráction
14.Координата точки	Координата точки	Coórdinate of a point
15.Координатная ось	Координатна вісь	Coórdinate áxis

16.Множество	Множина	Set
17.Множество, обладающее некоторым свойством	Множина, яка має деяку властивість	Set háving/posséssing a (certain) próperty
18.Модуль числа	Модуль числа	Módulus (*pl* móduli) of a númber
19.Натуральное число	Натуральне число	Nátural númber [nátural]
20.Начало координат	Початок координат	Órigin of coórdinates; órigin
21.Нечетное число	Непарне число	Odd númber
22.Обратное число	Обернене число	Ìnvérse (of a) númber
23.Объединение (сумма) множеств X и Y, $X \cup Y$, X или Y	Об"єднання (сума) множин X і Y, $X \cup Y$, X чи Y	Únion (sum) of (the) sets X and Y, $X \cup$ Y, X or Y
24.Обыкновенная дробь	Звичайний дріб	Cómmon/símple fráction
25.Окрестность точки	Окіл точки	Néighbourhood of a póint
26.Отрезок/сегмент (замкнутый ограниченный интервал) $[a, b]$	Відрізок/сегмент (замкнений обмежений інтервал) $[a, b]$	Ségment (bóunded clósed ínterval) $[a, b]$
27.Отрицательное число	Від"ємне число	Négative númber
28.Пересечение (произведение) множеств X и Y, $X \cap Y$, X и Y	Переріз (добуток) множин X та Y, $X \cap Y$, X і Y	Ìnterséction (próduct, còmposítion) of (the) sets X and Y, $X \cap Y$, X and Y
29.По модулю	За модулем	In módulus [módulo (*lat*)]
30.Подмножество множества. Множество X – подмножество множества Y, $X \subseteq Y$, $Y \supseteq X$	Підмножина множини. Множина X є підмножиною множини Y, $X \subseteq Y$, $Y \supseteq X$	Súbsèt of a set. The set X is the súbsèt of the set Y, $X \subseteq Y$, $Y \supseteq X$
31.Положительное направление	Додатний напрям/напрямок	Pósitive diréction
32.Положительное число	Додатне число	Pósitive númber
33.Представляться, быть представленным (в виде)	Представлятися, бути представленим (у вигляді)	Be rèpreséntеd (in the form)
34.Принадлежать множеству. $x \in X$, x (малое) [(малое) x] принадлежит множеству X (большому) [(большому) X]	Належати множині. $x \in X$, x (мале) [(мале) x] належить множині X (великому) [(великому)X]	Belóng to [be contáined in] a set. $x \in X$, (small) x belóngs to [is contáined in] a set (cápital) X; $x \notin X$, (small) x does not belong to (cápital) X)

35.Противоположное число	Протилежне число	Ópposite númber
36.Противоположный знак/смысл	Протилежний знак/сенс	Ópposite/ìnvérse/cóntrary sign/sense
37.Прямая (линия)	Пряма (лінія)	Stráight líne
38.Пустое множество	Порожня множина	Émpty set
39.Рациональное число	Раціональне число	Rátional númber [rátional]
40.Смысл неравенства	Сенс нерівності	Sense/méaning/signíficance of an ìnequálity
41.Содержать что-либо. Множество Y содержит множество X, $Y \supseteq X$	Містити щось. Множина Y містить множину X, $Y \supseteq X$	Inclúde/contáin smth. Thé set Y inclúdes/contáins the set X, $Y \supseteq X$
42.Содержаться во множестве (о множестве). Множество X содержится во множестве Y, $X \subseteq Y$	Міститися у множині (про множину). Множина X міститься в множині Y, $X \subseteq Y$	Be inclúded/contáined in a set. Thé set X is inclúded/contáined in the set Y, $X \subseteq Y$
43.Соответствовать вещественному числу (о точке числовой оси)	Відповідати дійсному числу (про точки числової осі)	Còrrespónd to a réal númber (of a póint of the númber/numérical áxis/scále)
44.Соответствовать точке числовой оси (о вещественном числе)	Відповідати точці числової осі (про дійсне число)	Còrrespónd to a póint of the númber/numérical áxis/scále (of a réal númber)
45.Ставить в соответствие каждому вещественному числу вполне определённую точку числовой оси	Ставити у відповідність кожному дійсному числу цілком певну точку числової осі	Assígn/assóciate quite défínite póint of the númber/numérical áxis/scále to éach réal númber
46.Стрелка	Стрілка	Árrow
47.Целое число	Ціле число	Ínteger/entíre (númber)
48.Чётное число	Парне число	Éven númber
49.Числовая ось	Числова вісь	Númber/numérical áxis/scále
50.Эквивалентное/равносильное неравенство	Еквівалентна/рівносильна нерівність	Equívalent ìnequálity
51.Элемент множества	Елемент множини	Élement/mémber of a set
52.Эпсилон-окрестность [ε-окрестность] точки	Епсілон-окіл [ε-окіл] точки	ε-néighbourhood of a póint

REALS: basic terminology ERU

1. Ábsolute válue [mágnitude, módulus] of a númber	Абсолютная величина числа	Абсолютна величина числа
2. Árrow	Стрелка	Стрілка
3. Assígn/assóciate quite définite póint of the númber/ numérical áxis/scále to éach réal númber	Ставить в соответствие каждому вещественному числу вполне определённую точку числовой оси	Ставити у відповідність кожному дійсному числу цілком певну точку числової осі
4. Be an élément of a set ($x \in X$, small x is an élement of a set cápital X; $x \notin X$, small x isn't an élement of cápital X)	Быть [являться] элементом множества ($x \in X$, x (малое) [малое x] является элементом множества X (большого) [большого X]; $x \notin X$, x не является элементом множества X)	Бути елементом множини ($x \in X$, x (мале) [мале x] є элементом множини X (великого) [великого X]; $x \notin X$, x не є элементом множини X)
5. Be equívalent to *smth*	Быть эквивалентным [равносильным] *чему*	Бути еквівалентним [рівносильним] *чомусь*
6. Be inclúded/contáined in a set. Thé set X is inclúded/contáined in the set Y, $X \subseteq Y$	Содержаться во множестве (о множестве). Множество X содержится во множестве Y, $X \subseteq Y$	Міститися у множині (про множину). Множина X міститься в множині Y, $X \subseteq Y$
7. Be rèpresénted (in the form)	Представляться, быть представленным (в виде)	Представлятися, бути представленим (у вигляді)
8. Belóng to [be contáined in] a set. $x \in X$, (small) x belóngs to [is contáined in] a set (cápital) X; $x \notin X$, (small) x does not belong to (cápital) X)	Принадлежать множеству. $x \in X$, x (малое) [(малое) x] принадлежит множеству X (большому) [(большому) X]	Належати множині. $x \in X$, x (мале) [(мале) x] належить множині X (великому) [(великому) X]
9. Cómmon/símple fráction	Обыкновенная дробь	Звичайний дріб
10. Coórdinate áxis	Координатная ось	Координатна вісь
11. Coórdinate of a point	Координата точки	Координата точки
12. Còrrespónd to a póint of the númber/numérical áxis/ scále (of a réal númber)	Соответствовать точке числовой оси (о вещественном числе)	Відповідати точці числової осі (про дійсне число)
13. Còrrespónd to a réal númber (of a póint of the númber/ numérical áxis/scále)	Соответствовать вещественному числу (о точке числовой оси)	Відповідати дійсному числу (про точки числової осі)
14. Décimal/póint fráction	Десятичная дробь	Десятковий дріб

15.Élement/mémber of a set	Элемент множества	Елемент множини
16.Émpty set	Пустое множество	Порожня множина
17.End/endpoint (left [left-hand], right [right-hand]) of an ínterval	Конец интервала (левый/правый)	Кінець інтервала (лівий/правий)
18.Épsilon-néighbourhood [ε-néighbourhood] of a póint	Эпсилон-окрестность [ε-окрестность] точки	Епсілон-окіл [ε-окіл] точки
19.Equívalent ìnequálity	Эквивалентное/равносильное неравенство	Еквівалентна/рівносильна нерівність
20.Éven númber	Чётное число	Парне число
21.In módulus [módulo (*lat*)]	По модулю	За модулем
22.Inclúde/contáin *smth*. Thé set Y inclúdes/contáins the set X, $Y \supseteq X$	Содержать *что-либо*. Множество Y содержит множество X, $Y \supseteq X$	Містити *щось*. Множина Y містить множину X, $Y \supseteq X$
23.Ínteger/entíre (númber)	Целое число	Ціле число
24.Ìnterséction (próduct, còmposítion) of (the) sets X and Y, $X \cap Y$, X and Y	Пересечение (произведение) множеств X и Y, $X \cap Y$, X и Y	Переріз (добуток) множин X та Y, $X \cap Y$, X і Y
25.Ínterval (bóunded ópen ínterval) $(a, b) \equiv]\, a, b\, [$	Интервал (открытый ограниченный интервал) $(a, b) \equiv]\, a, b\, [$	Інтервал (відкритий обмежений інтервал) $(a, b) \equiv]\, a, b\, [$
26.Ìnvérse (of a) númber	Обратное число	Обернене число
27.Irrátional númber	Иррациональное число	Іраціональне число
28.Módulus (*pl* móduli) of a númber	Модуль числа	Модуль числа
29.Nátural númber	Натуральное число	Натуральне число
30.Négative númber	Отрицательное число	Від"ємне число
31.Néighbourhood of a póint	Окрестность точки	Окіл точки
32.Nòntérminàting (pèriódic, nòn-pèriódic) décimal/póint fráction	Бесконечная десятичная дробь (периодическая, непериодическая)	Нескінченний десятковий дріб (періодичний, неперіодичний)
33.Númber/numérical áxis/scále	Числовая ось	Числова вісь
34.Odd númber	Нечетное число	Непарне число
35.One-to-one còrrespóndence [bijection] (betwéen the set of all réal númbers and the set of all póints of the númber [coórdinate] áxis)	Взаимно однозначное соответствие (между множеством всех вещественных чисел и множеством всех точек числовой [координатной] оси)	Взаємно-однозначна відповідність (між множиною всіх дійсних чисел та множиною всіх точок числової [координатної] осі)

36.Ópposite númber	Противоположное число	Протилежне число
37.Ópposite/ìnvérse/cóntrary sign/sense	Противоположный знак/смысл	Протилежний знак/сенс
38.Órigin of coórdinates; órigin	Начало координат	Початок координат
39.Pósitive diréction	Положительное направление	Додатний напрям/напрямок
40.Pósitive númber	Положительное число	Додатне число
41.Rátional númber	Рациональное число	Раціональне число
42.Réal númber	Вещественное число	Дійсне число
43.Ségment (bóunded clósed ínterval) $[a, b]$	Отрезок/сегмент (замкнутый ограниченный интервал) $[a, b]$	Відрізок/сегмент (замкнений обмежений інтервал) $[a, b]$
44.Sense/méaning/signíficance of an ìnequálity	Смысл неравенства	Сенс нерівності
45.Set	Множество	Множина
46.Set háving/posséssing a (certain) próperty	Множество, обладающее некоторым свойством	Множина, яка має деяку властивість
47.Stráight líne	Прямая (линия)	Пряма (лінія)
48.Súbsèt of a set. The set X is the súbsèt of the set Y, $X \subseteq Y$, $Y \supseteq X$	Подмножество мно-жества. Множество X – подмножество множества Y, $X \subseteq Y$, $Y \supseteq X$	Підмножина множини. Множина X є підмножиною множини Y, $X \subseteq Y$, $Y \supseteq X$
49.Symmétric ínterval with respéct to a póint	Интервал, симметричный относительно точки	Інтервал, симетричний відносно точки
50.Términating décimal/póint fráction	Конечная десятичная дробь	Скінченний десятковий дріб
51.Únion (sum) of (the) sets X and Y, $X \cup$ Y, X or Y	Объединение (сумма) множеств X и Y, $X \cup Y$, X или Y	Об"єднання (сума) множин X і Y, $X \cup Y$, X чи Y
52.Únit of length/measurement	Единица длины/измерения	Одиниця довжини/вимірювання

LECTURE NO. 2. MAPPINGS AND FUNCTIONS. BASIC ELEMENTARY AND ELEMENTARY FUNCTIONS

POINT 1. MAPPINGS AND FUNCTIONS

POINT 2. POWERS AND ROOTS. POWER AND EXPONENTIAL FUNCTIONS

POINT 3. LOGARITHMS. LOGARITHMIC FUNCTION

POINT 4. TRIGONOMETRICAL FUNCTIONS

POINT 5. INVERSE TRIGONOMETRICAL FUNCTIONS

POINT 6. ELEMENTARY FUNCTIONS

POINT 1. MAPPINGS AND FUNCTIONS

Def. 1. Let there be given two sets *A* and *B*. A mapping [a map] of the set *A* with values in the set *B* is called a rule [a law] according to which every element $a \in A$ is assigned to certain (and unique) element $b \in B$.

If we denote a mapping by a letter *f*, we can write

$$b = f(a) \text{ or } f \mid a \to b.$$

The latter designation is read as follows: a mapping *f* assigns an element $b \in B$ to an element $a \in A$.

Def. 2. The element $b = f(a) \in B$ is called an **image** of the element $a \in A$ in the mapping *f*, and the element $a \in A$ is called a preimage of the element $b = f(a)$.

Def. 3. A set of images of all the elements $a \in A$ is called an **image of the set** *A* and is denoted $f(A)$.

Def. 4. If the image $f(A)$ of the set *A* is a proper part of the set *B* ($f(A) \subset B$ but $f(A) \neq B$), the mapping *f* is called that of *A* **in [into]** *B*. If the image $f(A)$ coincides with the whole set *B*, the mapping *f* is called that of *A* **on** [**onto**] *B*.

Ex. 1. Let a mapping *f* of a circle *A* **in** a straight line *B* be defined with the help of rays radiating from a point *M* (fig. 1), and *MC*, *MD* be the tangents to the circle with tangency points P, Q. We have a mapping of the circle **on** the segment CD of B. Every point $b \in CD$, distinct from C and D, is an image of two preimages a_1, a_2, and the points C, D have only one preimage, P, Q respectively.

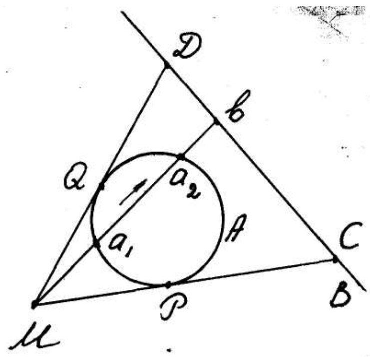


Fig. 1

Let's consider an important case of a mapping of a set A on a set B when it sets **one-to-one correspondence between** the sets A and B. In this case not only every element of A has unique image in B but also every element of B has unique preimage in A. It's useful to call such the mapping as **one-to-one mapping.**

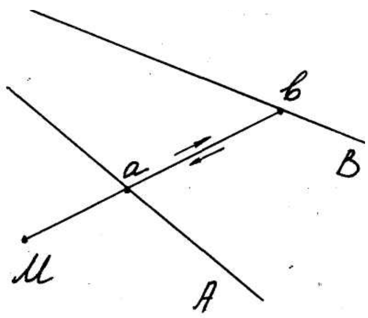


Fig. 2

Ex. 2. A mapping of a straight line A on a straight line B, determined by rays radiating from a point M (fig. 2), sets one-to-one correspondence between A and B and is the one-to-one mapping.

Def. 5. Let there be given an one-to-one mapping *f* of a set A on a set B (that is a mapping which sets one-to-one correspondence between A and B). A mapping assigning an element $a \in A$ to an element $b \in B$, for which $b = f(a)$, is called an **inverse mapping** for *f* and is often denoted by f^{-1}.

By the definition of the inverse mapping

$$f^{-1}(b) = a \text{ if } f(a) = b; \quad f^{-1}(f(a)) = a, \, f(f^{-1}(b)) = b.$$

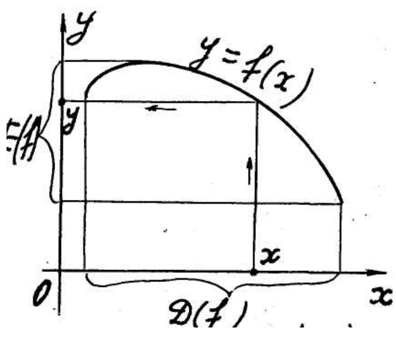


Fig. 3

Ex. 3. An inverse mapping for the mapping of Ex. 2 is determined by the same rays: every point $b \in B$ is assigned to certain point $a \in A$ with the help of corresponding ray.

Def. 6. A **function** (a number function) *f* with a domain of definition $D(f)$ and a set of values $E(f) \subset \boldsymbol{R} = (-\infty, \infty)$ is called a mapping *f* of the set $D(f)$ on the set $E(f)$ (fig. 3).

The domain of definition of a function can be some part: a) of the set of all reals ***R***; b) of the two-, three-, n-dimensional space. Respectively we deal with a function of one, two, three, n variables

$$y = f(x),\ y = f(x, y), u = f(x, y, z), u = f(x_1, x_2, \ldots, x_n).$$

The set of values of a function of any variables is every some part of the set of all reals ***R*** or the whole ***R*** ($E(f) \subseteq \boldsymbol{R}$).

Def. 7. A function $y = f(x)$ of one variable x is called increasing, non-decreasing, non-increasing, decreasing on an interval $U \subseteq D(f)$ if for any two values of the variable $x_1 \in U, x_2 \in U$, such that $x_1 < x_2$, one has correspondingly

$$f(x_1) < f(x_2),\ f(x_1) \leq f(x_2),\ f(x_1) \geq f(x_2), f(x_1) > f(x_2).$$

Theorem 2 (sufficient condition of existing of an inverse function for a function of one variable). If a function $y = f(x)$ of one variable increases or decreases on some interval U of its domain of definition ($U \subseteq D(f) \subseteq Ox$), then it has an inverse function $x = f^{-1}(y)$ which is defined on the image $f(U) \subseteq Oy$ of the interval U and correspondingly increases or decreases on $f(U)$.

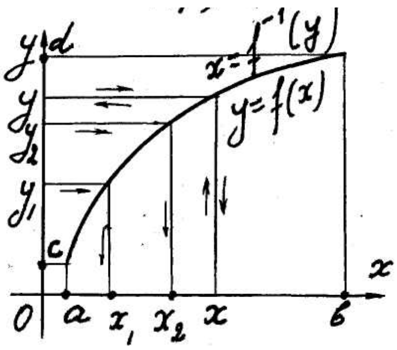


Fig. 4

■ For example let a function $y = f(x)$ increase on an interval $U = (a, b)$ (see fig. 4). Being a mapping it sets one-to-one correspondence between the interval $U = (a, b)$ and its image $f(U) = (c, d)$ and therefore possesses an inverse mapping $x = f^{-1}(y)$ of $f(U) = (c, d)$ on $U = (a, b)$. This mapping is the inverse function for $f(x)$. Increase of the inverse function is obvious: if $y_1 < y_2$ for two arbitrary values $y_1 \in f(U), y_2 \in f(U)$, then $x_1 = f^{-1}(y_1) < x_2 = f^{-1}(y_2)$.■

For a function $y = f(x)$ and its inverse one $x = f^{-1}(y)$ we have

$$D(f) = E(f^{-1}),\ E(f) = D(f^{-1}),\ f(f^{-1}(y)) = y,\ f^{-1}(f(x)) = x.$$

Functions $y = f(x)$ and $x = f^{-1}(y)$ have the same graph (fig. 5). But if we substitute *x* by *y* and *y* by *x* in the relation $x = f^{-1}(y)$, the graphs of the functions $y = f(x)$ and $y = f^{-1}(x)$ be symmetric with respect to the straight line $y = x$ (fig. 5).

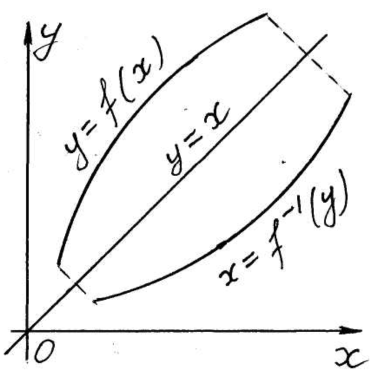


Fig. 5

Def. 8. Let a domain of definition of a function *f* be the set of all natural numbers ***N***. Such the function is called that of natural argument. The set of all its values is called a **number sequence** and is denoted as follows

$x_1 = f(1), x_2 = f(2), \ldots, x_n = f(n), \ldots$ or $\{x_n = f(n)\}$.

Numbers $x_1, x_2, \ldots, x_n, \ldots$ are called terms of the sequence, and the number x_n is called its **general term**.

A number sequence is the mapping of the set ***N*** of all natural numbers in the set of all reals ***R***.

Def. 9. A number sequence $\{x_n\}$ is called increasing, non-decreasing, non-increasing, decreasing if for any two natural numbers n_1, n_2 the inequality $n_1 < n_2$ implies respectively the inequalities

$$x_{n_1} < x_{n_2},\ x_{n_1} \le x_{n_2},\ x_{n_1} \ge x_{n_2},\ x_{n_1} > x_{n_2}.$$

POINT 2. POWERS AND ROOTS. POWER AND EXPONENTIAL FUNCTIONS

Powers

Def. 9. If *n* is some natural number ($n \in$ ***N***) then

$$a^n = a \cdot a \cdot a \cdot \ldots \cdot a \ (n \text{ times}).$$

Def. 10. Zero power of a real number:

$$a^0 = 1, a \ne 0.$$

Def. 11. A power with negative index:

$$a^{-n} = \frac{1}{a^n},\ a \ne 0,\ n \in \boldsymbol{N}.$$

Def. 12. A power with rational index. For any rational number $m/n \in \boldsymbol{R}$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}, a > 0.$$

Definition of an irrational power of a number $a > 0$ is given in complete courses of mathematical analysis.

Properties of powers

1. $a^m \cdot a^n = a^{m+n}$ 2. $a^m : a^n = a^{m-n}$ 3. $\left(a^m\right)^n = a^{mn}$ 4. $a^m \cdot b^m = (ab)^m$ 5. $a^m : b^m = \left(\dfrac{a}{b}\right)^m$

6. If $0 \le a < b$ then for any natural number n $a^n < b^n$.

Remarkable formulas

1. $(a+b)^2 = a^2 + 2ab + b^2, (a-b)^2 = a^2 - 2ab + b^2$.
2. $a^2 - b^2 = (a+b)(a-b)$.
3. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a+b)$,

   $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a-b)$.
4. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$, $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

Roots

Def. 13. A number *b* is called the *n*th root of a number *a*, namely

$$\sqrt[n]{a} = b,$$

if $b^n = a$.

Properties of roots

1. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ 2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$. 3. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$. 4. $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ 5. $\sqrt[nk]{a^{mk}} = \sqrt[n]{a^m}$

6. $\sqrt[2n]{a^{2n}} = |a|, \sqrt{a^2} = |a|$ 7. $\sqrt[2n-1]{-a} = -\sqrt[2n-1]{a}$

8. If $0 \le a < b$ then for any natural number n $\sqrt[n]{a} < \sqrt[n]{b}$.

A power funcion

Def. 14. A function $y = x^\alpha$, where α is a real number is called a power one.

Properties and the graph of a power function depend on the value of the index of a power. Graphs of power functions for $\alpha = 3, \alpha = 4, \alpha = -2, \alpha = -3$ are represented on the figure 6.

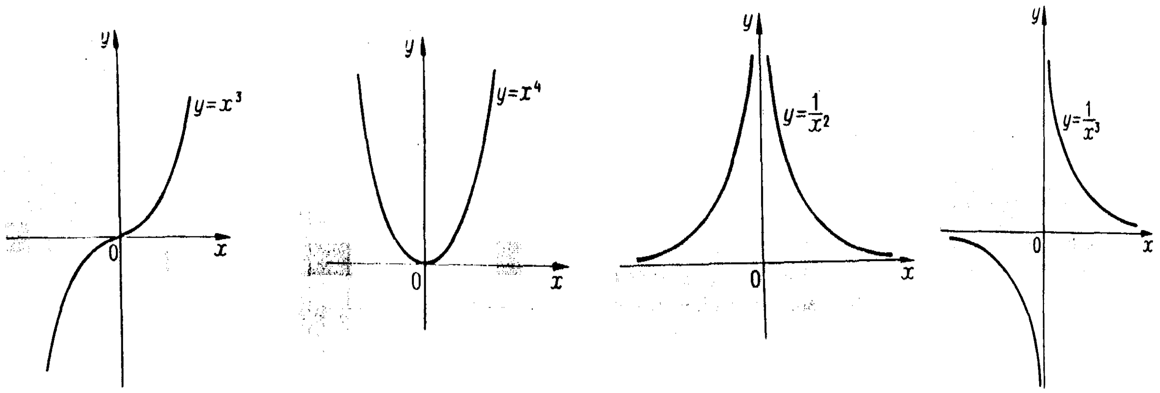


Fig. 6. Power functions for $\alpha = 3, \alpha = 4, \alpha = -2, \alpha = -3$

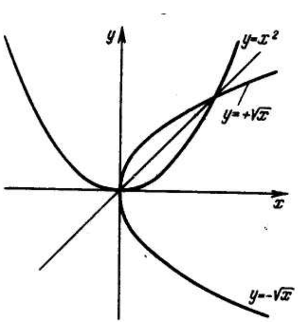


Fig. 7

Let's study the case $\alpha = 2$ that is the function $y = x^2$ (see fig. 7). Its domain of definition $D(y) = \Re = (-\infty, \infty)$ is the set of all reals and the set of values $E(y) = \Re_+ = [0, \infty)$ is the set of all nonnegative reals. The function takes on the value 0 at the point $x = 0$, decreases on the interval $(-\infty, 0]$, increases on the interval $(-\infty, 0]$, has a local minimum 0 at the point $x = 0$.

The graph of the function (it's called a quadratic parabola) passes through the origin $O(0; 0)$, is descending one for $x < 0$ and ascending one for $x > 0$, is concave one for all *x*, has the lowest point $O(0; 0)$, hasn't inflexion points.

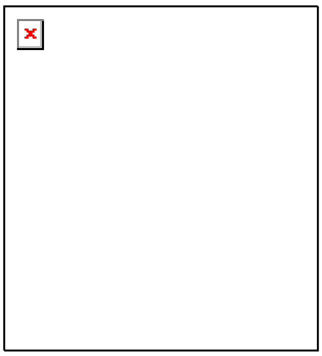


Fig. 8

The function has two inverse functions $x = \sqrt{y}, x = -\sqrt{y}$ with domains of definition

$$D_1(x) = \Re_+ = [0, \infty) \subset Oy, D_2(x) = \Re_- = (-\infty, 0] \subset Oy$$

respectively and the same set of values $E_1(x) = E_2(x) = [0, \infty)$. The first inverse function increases, and the second one decreases. Changing *x* by *y* and *y* by *x* we get the functions $y = \pm\sqrt{x}$ the graphs of which are symmetric to the graph of the function $y = x^2$ with respect to the straight line $y = x$ (fig. 7). The function $y = \sqrt{x}$ is a power one with $\alpha = 1/2$.

By analogous way one can formulate properties of power functions for $\alpha = 3$, $\alpha = 4$, $\alpha = -2$, $\alpha = -3$ and properties of their graphs. Do this yourselves.

The graphs of a power function $y = x^{\frac{2}{3}}$ (for $\alpha = 2/3$) and two its inverse $y = x^{\frac{3}{2}}$, $y = -x^{\frac{3}{2}}$ are represented on the fig. 8. State yourselves properties of these graphs and corresponding properties of the functions. Let's remark that the graph of the function $y = x^{\frac{2}{3}}$ has a cuspidal point namely the origin O(0; 0), and the graphs of three functions haven't inflexion points. The graphs of the first and third functions are those convex, and the graph of the second function is concave one.

An exponential function

Def. 15. A function $y = a^x$, where *a* is a real number such that $0 < a \neq 1$, is called an exponential one.

The domain of definition of an exponential function $D(y) = \Re = (-\infty, \infty)$ is the set of all reals and the set of values $E(y) = \Re_+^* = (0, \infty)$ is the set of all positive reals. The function takes on the value 1 at the point $x = 0$, increases for $a > 1$ and decreases for $0 < a < 1$.

The graph of the function lies in the upper semi-plane $y > 0$, passes through the point $(0; 1)$, is ascending one for $a > 1$, descending for $0 < a < 1$, concave in both cases (see fig. 9a, 9b).

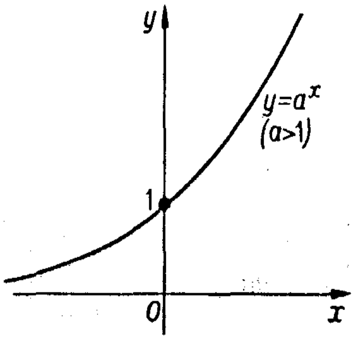


Fig. 9 a

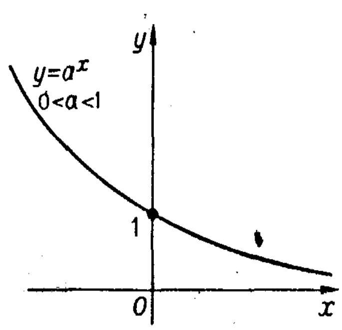


Fig. 9 b

POINT 3. LOGARITHMS. A LOGARITHMIC FUNCTION

Logarithms

Let

$$a^c = b \qquad (1)$$

A number *c* is the index of a power to which it's necessary to raise a number *a* to get a number *b*.

Def. 16. Logarithm of a number *b* to a base *a* is called the index of a power to which it's necessary to raise the base *a* to get the number *b*. It's denoted as $log_a b$.

On the base of the definition 8 and the equality (1) we can write

$$c = log_a b. \qquad (2)$$

Ex. 4. $log_2 8 = 3$ because of $2^3 = 8$; $log_3 \frac{1}{9} = -2$ because of $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$.

We'll always suppose that a base of a logarithm satisfies the next conditions

$$0 < a \neq 1. \qquad (3)$$

In such the supposition logarithms of positive numbers always exist, but the zero and negative numbers don't possess logarithms.

Def. 17. Logarithm to the base 10 is called the decimal one and is denoted *lg*.

Ex. 5. $lg\,100 = log_{10} 100 = 2$ because of $10^2 = 100$, $lg\,0.003 = -3$ because of $10^{-3} = 0.001$.

Def. 18. Logarithm to the base $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.718281828459045...$ is called the natural one and is denoted by *ln*: $ln\,b = log_e b$.

Ex. 6. Solve an equation $log_5\left(5^x + x^2 - 7x + 6\right) = x$.

Solution. On the base of the definition of a logarithm

$$5^x + x^2 - 7x + 6 = 5^x,\ x^2 - 7x + 6 = 0,\ x_1 = 1,\ x_2 = 6.$$

Properties of logarithms

1. The basic [principal, fundamental] logarithmic identity follows from the equalities (1), (2) namely

$$a^{log_a b} = b. \qquad (4)$$

Ex. 7.

$$e^{ln b} = b, 10^{lg b} = b, 3^{5 log_3 2} = \left(3^{log_3 2}\right)^5 = 2^5 = 32, 4 = 2^{log_2 4} = 7^{log_7 4} = e^{ln 4} = 10^{lg 4}$$

2. Logarithm of the unity equals zero,

$$log_a 1 = 0,$$

because of $a^0 = 1$ for any $a\ (0 < a \neq 1)$.

3. Logarithm of a base equals the unity,

$$log_a a = 1,$$

because of $a^1 = a$.

4. Logarithm of a product of two positive numbers equals the sum of their logarithms,

$$log_a bc = log_a b + log_a c \quad (b > 0, c > 0).$$

■Let $log_a b = m, log_a c = n, log_a bc = p$, that is $b = a^m, c = a^n, bc = a^p$. We get $bc = a^{m+n}$ and $bc = a^p$, so $a^{m+n} = a^p, p = m + n, log_a bc = log_a b + log_a c$.■

Ex. 8. Solve an equation $lg\, x + lg(x - 4) = lg\, 5$.

Solution. It's must be $\begin{cases} x > 0, \\ x - 4 > 0, \end{cases}$ $x > 4$. By the property 4

$$lg\, x(x-4) = lg\, 5, x(x-4) = 5, x^2 - 4x - 5 = 0, x_1 = -1 < 0, x_2 = 5.$$

The check shows that the number 5 is the solution of the equation.

5. Logarithm of a quotient of two positive numbers equals the difference of their logarithms,

$$log_a \frac{b}{c} = log_a b - log_a c \quad (b > 0, c > 0).$$

Prove this property yourselves.

Ex. 9. Solve an equation $lg(50x + 25) - lg\, x = 2$.

Solution. Let's notice that $2 = lg\, 100$. By the fifth property

$$lg\frac{50x+25}{x}=lg100,\ \frac{50x+25}{x}=100,\ 50x+25=100x,\ 50x=25,\ x=\frac{25}{50}=\frac{1}{2}.$$

Ex. 10. Prove that for any even natural number $2n$ and positive a, b ($0<a\neq1$)

$$log_a^{2n}\frac{1}{b}=log_a^{2n}b.$$

$$■\ log_a^{2n}\frac{1}{b}=\left(log_a\frac{1}{b}\right)^{2n}=\left(log_a1-log_ab\right)^{2n}=\left(-log_ab\right)^{2n}=log_a^{2n}b\ ■$$

6. Logarithm of a power of a positive number equals the product of the index of the power and the logarithm of the base of the power,

$$log_ab^n=n\,log_ab\quad(b>0).$$

■If $log_ab=p,\ log_ab^n=q$, then

$$b=a^p,\ b^n=a^{np},\ b^n=a^q,\ a^{np}=a^q,\ q=np,\ log_ab^n=n\,log_ab\ ■$$

Ex. 11.

$$log_525^{10}=10\,log_525=10\cdot2=20;\quad 3log_7\sqrt[3]{49}=log_7\left(\sqrt[3]{49}\right)^3=log_749=2.$$

Ex. 12. Prove that for any positive numbers a, b, c ($0<a\neq1$) the equality

$$b^{log_ac}=c^{log_ab}$$

is true.

■It's sufficient to prove that logarithms to the base a of the left and right sides of the equality are equal. We have

$$log_ab^{log_ac}=log_ac\cdot log_ab,\ log_ac^{log_ab}=log_ab\cdot log_ac=log_ac\cdot log_ab\ ■$$

7. Logarithm of a root of a positive number equals the logarithm of this number divided by the index of the root,

$$log_a\sqrt[n]{b}=\frac{log_ab}{n}=\frac{1}{n}log_ab\quad(b>0).$$

■It's sufficient to notice that $\sqrt[n]{b}=b^{\frac{1}{n}}$ and to use the preceding property.■

Ex. 13. $log_3\sqrt[8]{81}=\frac{1}{8}log_381=\frac{1}{8}\cdot4=\frac{1}{2};\quad \frac{1}{9}log_416^9=log_4\sqrt[9]{16^9}=log_416=2.$

Ex. 14. Take a logarithm to the base *a* of the next expression $x = \frac{b^3 c^5}{d^2 \sqrt{e}}$.

Solution.

$$log_a x = log_a\left(b^3 c^5\right) - log_a\left(d^2 \sqrt{e}\right) = log_a\left(b^3\right) + log_a\left(c^5\right) - \left(log_a\left(d^2\right) + log_a\left(\sqrt{e}\right)\right) =$$

$$= 3 log_a b + 5 log_a c - 2 log_a d - \frac{1}{2} log_a e.$$

Ex. 15. Exponentiate the next expression (that is find an expression which logarithm is known) $log_a x = 4 log_a b - 7 log_a c + \frac{1}{4} log_a d - 5 log_a e$.

Solution.

$$log_a x = log_a b^4 + log_a \sqrt[4]{d} - \left(log_a c^7 + log_a e^5\right) = log_a\left(b^4 \sqrt[4]{d}\right) - log_a\left(c^7 e^5\right) =$$

$$= log_a \frac{b^4 \sqrt[4]{d}}{c^7 e^5} \Rightarrow x = \frac{b^4 \sqrt[4]{d}}{c^7 e^5}.$$

8. It's possible to pass to a new base of logarithms by the next formula,

$$log_a b = \frac{log_c b}{log_c a}, \qquad (5)$$

according to which the logarithm of a number *b* to the base *a* equals the logarithm of *b* to the new base *c* divided by the logarithm of the preceding base to the new base.

■Let $log_a b = m, log_c b = n.$ Then

$$b = a^m = c^n, log_c a^m = log_c c^n, m log_c a = n log_c c, log_a b log_c a = log_c b,$$

whence it follows the equality (5).■

Ex. 16. Passing to the base 2 we obtain $log_{64} 32 = \frac{log_2 32}{log_2 64} = \frac{5}{6}$.

Ex. 17. Prove that for any numbers *a*, *b* such that $0 < a \neq 1, 0 < b \neq 1$

$$log_a b = \frac{1}{log_b a}.$$

■ It's sufficient to pass to the base *b*:

$$log_a b = \frac{log_b b}{log_b a} = \frac{1}{log_b a}$$ ■

Ex. 18. Prove that for any positive numbers a, b, $0 < a \neq 1$, and two arbitrary numbers n, m

$$log_{a^n} b^n = log_a b, log_{a^n} b^m = \frac{m}{n} log_a b .$$

Some corollaries.

1. If functions $f(x), g(x)$ have the same sign, then

$$log_a(f(x)g(x)) = log_a|f(x)| + log_a|g(x)|; \; log_a \frac{f(x)}{g(x)} = log_a|f(x)| - log_a|g(x)| .$$

2. If $2n$ be an even natural number, then

$$log_a f^{2n}(x) = 2n \cdot log_a|f(x)| .$$

Ex. 19. Solve an equation $log_3(x+2)^6 + log_3|x+2| = 14$.

Solution.

$$6 log_5|x+2| + log_5|x+2| = 14, 7 log_5|x+2| = 14, log_5|x+2| = 2, |x+2| = 25, x+2 = \pm 25,$$

$$x_1 = -27, x_2 = 23 .$$

3. If $2n-1$ be an odd natural number, then for any positive function $f(x)$

$$log_a f^{2n-1}(x) = (2n - 1) \cdot log_a f(x).$$

4. For any even natural numbers $2n, 2m$

$$log_{f^{2n}(x)} g^{2n}(x) = log_{|f(x)|}|g(x)|, log_{f^{2n}(x)} g^{2m}(x) = \frac{m}{n} log_{|f(x)|}|g(x)| .$$

A logarithmic function

Exponential function $y = a^x$, being increasing (for $a > 1$) or decreasing (for $0 < a < 1$) on its domain of definition $D(a^x) = \Re = (-\infty, \infty)$, possesses an inverse function, namely the logarithmic function $x = log_a y$. Replacing, as usually, by places x and y we get the function $y = log_a x$ with the graph symmetric to that of the function $y = a^x$ about the straight line $y = x$ (fig. 10).

The domain of definition of logarithmic function $D(log_a x) = \Re_+^* = (0, \infty)$ coincides with the set of values $E(a^x)$ of corresponding exponential function a^x, and the set of values $E(log_a x) = \Re = (-\infty, \infty)$ coincides with the domain of definition $D(a^x)$ of the function a^x. The function takes on the value 0 at the point $x = 1$, increases for $a > 1$ and decreases for $0 < a < 1$.

The graph of the function $y = log_a x$ lies in the right semi-plane $x > 0$ passes

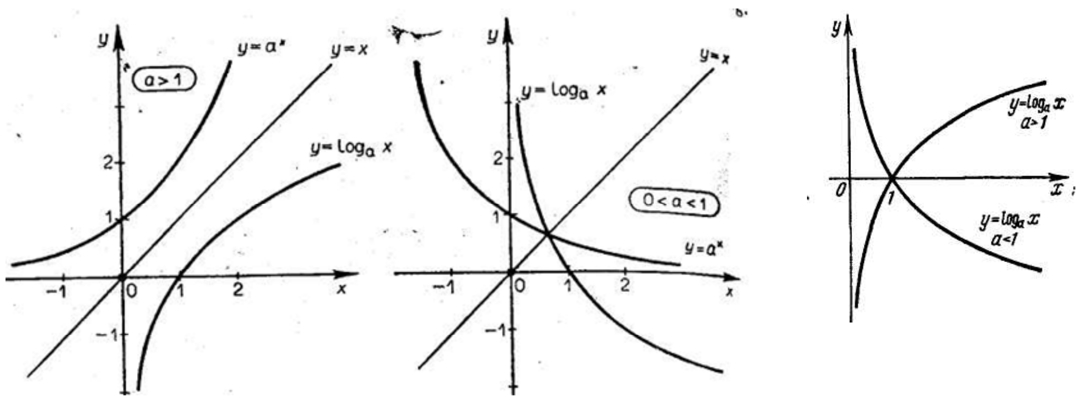


Fig. 10

through the point $(1; 0)$, is ascending and convex one for $a > 1$, descending and concave one for $0 < a < 1$ (fig. 10).

POINT 4. TRIGONOMETRIC FUNCTIONS

Trigonometric functions of an acute angle (in a right triangle)

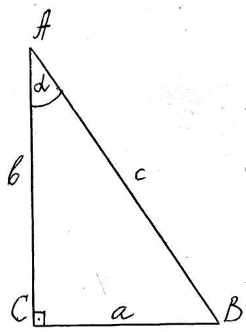


Fig. 11

Let's consider a right triangle ABC with the right angle C and an acute angle $\alpha = \angle BAC$ (fig. 11). A leg $b = AC$ is called **adjacent** one and a leg $a = BC$ **opposite** one to the angle α.

Def. 19. Sine of the angle α is called the ratio of the opposite leg to the hypotenuse,

$$sin\alpha = \frac{BC}{AB} = \frac{a}{c}.$$

Def. 20. Cosine of the angle α is called the ratio of the adjacent leg to the hypotenuse,

$$\cos\alpha = \frac{AC}{AB} = \frac{b}{c}.$$

Def. 21. Tangent of the angle α is called the ratio of the opposite leg to adjacent one,

$$\tan\alpha = tg\alpha = \frac{BC}{AC} = \frac{a}{b}.$$

Def. 22. Cotangent of the angle α is called the ratio of the adjacent leg to opposite one,

$$\cot\alpha = ctg\alpha = \frac{AC}{BC} = \frac{b}{a}$$

Trigonometric functions of an arbitrary number argument

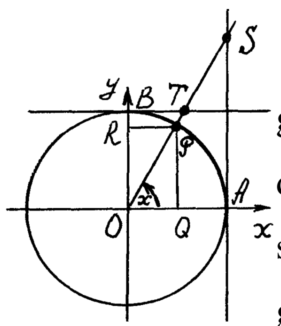


Fig. 12

We'll always consider so-called **standard position** of an angle *x* when its vertex *O* (fig. 12) coincides with the origin of coordinates, an initial side *OA* lies on the positive semi-axis of abscissas and a terminal side *OP* is the result of a rotation about the origin *O*.

Let's take into consideration so-called **trigonometric(al) circle** bounded by the unit circle [unit circumference] centered at the origin *O* (fig. 12). The tangent *AS* to the circle is called the **tangent line** (or the **tangent axis**), and the tangent *BT* the **cotangent line** (or the **cotangent axis**). The radius *OA* of the trigonometric circle (the initial side of all angles) is called **fixed** one, and the radius *OP* (the terminal side of an angle *x*) is called **mobile** one. The prolongation of the mobile radius *OP* crosses the tangent line at the point *S* and the cotangent line at the point *T*.

Def. 23. Sine of the angle *x* ($\sin x$) is called the ordinate of the end point *P* of the mobile radius.

Def. 24. Cosine of the angle *x* ($\cos x$) is called the abscissa of the end point *P* of the mobile radius.

Def. 25. Tangent of the angle *x* ($\tan x$ or tgx) is called the ordinate of the point *S*, that is the ordinate of the intersect point of the prolongation of the mobile radius *OP* with the tangent line.

Def. 26. Cotangent of the angle x ($cot\,x$ or $ctgx$) is called the abscissa of the point *T*, that is the abscissa of the intersect point of the prolongation of the mobile radius *OP* with the cotangent line.

An angle x terminated in the first quadrant is represented on the fig. 12. For it

$$sin\,x = OR = QP,\ cos\,x = OQ,\ tan\,x = AS,\ cot\,x = BT\,.$$

Angles which terminate in the second, third, fourth quadrants and corresponding trigonometric functions with their signs are represented on the fig. 13 – 15.

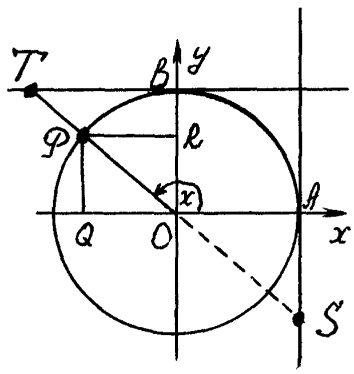


Fig. 13

$\sin x = QP > 0,$

$\cos x = -OQ < 0,$

$tan\,x = -AS < 0,$

$cot\,x = -BT < 0$

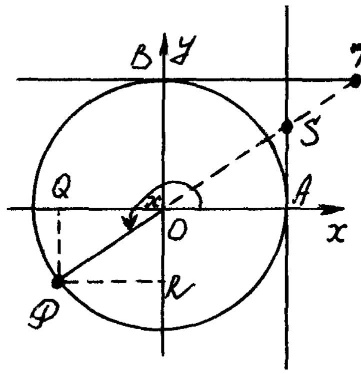


Fig. 14

$\sin x = -QP < 0,$

$\cos x = -OQ < 0,$

$tan\,x = AS > 0,$

$cot\,x = BT > 0$

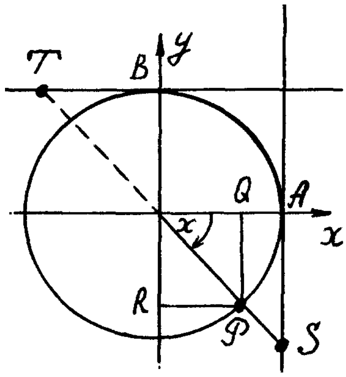


Fig. 15

$\sin x = -QP < 0,$

$\cos x = OQ > 0,$

$tan\,x = -AS < 0,$

$cot\,x = -BT < 0$

All properties of trigonometric functions can be stated with the help of the trigonometric circle. For example sine and cosine are **periodic** functions with the least positive **period** [or simply period] 2π, and tangent and cotangent are periodic with the period π. It means that

$$sin(x+2\pi) = sin\,x,\quad cos(x+2\pi) = cos\,x,\quad tan(x+\pi) = tan\,x,\quad cot(x+\pi) = cot\,x\,.$$

On the other hand cosine is **even** function, and the other functions are **odd**, that is

$$cos(-x) = cos\,x,\quad sin(-x) = -sin\,x,\quad tan(-x) = -tan\,x,\quad cot(-x) = -cot\,x\,.$$

The trigonometric circle allows to get formulas for solving so-called simplest trigonometric equations

$$\sin x = a; \quad \cos x = a; \quad tgx = a; \quad ctgx = a.$$

To begin with the next particular cases

$$\sin x = 0,\ x = \pi n;\ \sin x = 1,\ x = \frac{\pi}{2} + 2\pi n;\ \sin x = -1,\ x = -\frac{\pi}{2} + 2\pi n \quad (n \in Z)$$

$$\cos x = 0,\ x = \frac{\pi}{2} + \pi n;\ \cos x = 1,\ x = 2\pi n;\ \cos x = -1,\ x = \pi + 2\pi n \quad (n \in Z)$$

$$tgx = 0,\ x = \pi n;\ ctgx = 0,\ x = \frac{\pi}{2} + \pi n \quad (n \in Z)$$

In general case we have

$$1)\ \sin x = a\ (a \neq 0, a \neq \pm 1),$$

$$x = \arcsin a + 2\pi n \text{ or } x = \pi - \arcsin a + 2\pi n\ (n \in Z),$$

or simply

$$x = (-1)^k \arcsin a + \pi k \quad (k \in Z),$$

where $\arcsin a$ (arcsine a) is an angle of the segment $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, sine of which equals the number a.

$$2)\ \cos x = a\ (a \neq 0, a \neq \pm 1),$$

$$x = \pm \arccos a + 2\pi n\ (n \in Z),$$

where $\arccos a$ (arccosine a) is an angle of the segment $[0, \pi]$, cosine of which equals the number a.

$$3)\ tgx = a\ (a \neq 0),$$

$$x = arctga + \pi n\ (n \in Z),$$

where $arctan\, a$ (arctangent a) is an angle of the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, tangent of which equals a.

$$4)\ ctgx = a\ (a \neq 0),$$

$$x = arc\cot a + \pi n\ (n \in Z),$$

where $arc\cot a$ (arccotangent a) is an angle of the interval $(0, \pi)$, cotangent of which equals a.

One can also say that $\arccos a$ (corr. $arcctga$) is the least positive angle, cosine (corr. cotangent) of which equals a; $\arcsin a$ (corr. $arctan\, a$) is the least in modulus angle, sine (corr. tangent) of which equals a.

FUNDAMENTAL TRIGONOMETRIC IDENTITIES

1. RELATIONS BETWEEN TRIGONOMETRIC(AL) FUNCTIONS OF THE SAME ARGUMENT

$$1)\ sin^2 x + cos^2 x = 1;\ 2)\ 1 + tan^2 x = sec^2 x = \frac{1}{cos^2 x};$$

$$3)\ 1 + cot^2 x = cosec^2 x = \frac{1}{sin^2 x};\ 4)\ tan\, x = \frac{sin\, x}{cos\, x};\ 5)\ cot\, x = \frac{cos\, x}{sin\, x};$$

$$6)\ \sin x = \tan x \cdot \cos x\ ;\quad 7)\ \cos x = \cot x \cdot \sin x\ ;\quad 8)\ \tan x \cdot \cot x = 1$$

2. ADDITION FORMULAS

$$1)\ \sin(x \pm y) = \sin x \cdot \cos y \pm \cos x \cdot \sin y;\ 2)\ \cos(x \pm y) = \cos x \cdot \cos y \mp \sin x \cdot \sin y\ ;$$

$$3)\ \tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \cdot \tan y}$$

3. DOUBLE-ARGUMENT FORMULAE

$$1)\ \sin 2x = 2\sin x \cdot \cos x\ ;\quad 2)\ \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x\ ;$$

$$3)\ \tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

4. HALF-ARGUMENT FORMULAS

$$1)\ \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}\ ;\quad 2)\ \cos\frac{x}{2} \pm \sqrt{\frac{1+\cos x}{2}}\ ;\quad 3)\ 1 - \cos x = 2\sin^2\frac{x}{2}\ ;$$

$$4)\ 1 + \cos x = 2\cos^2\frac{x}{2}\ ;\quad 5)\ \tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

5. POWER REDUCTION FORMULAE

1) $\sin^2 x = \dfrac{1-\cos 2x}{2}$ 2) $\cos^2 x = \dfrac{1+\cos 2x}{2}$

6. TRANSFORMATION FORMULAS OF A SUM/DIFFERENCE OF (CO)SINES (IN)TO A PRODUCT.

1) $\sin x + \sin y = 2\sin\dfrac{x+y}{2}\cdot\cos\dfrac{x-y}{2}$; 2) $\sin x - \sin y = 2\cos\dfrac{x+y}{2}\cdot\sin\dfrac{x-y}{2}$;

3) $\cos x + \cos y = 2\cos\dfrac{x+y}{2}\cdot\cos\dfrac{x-y}{2}$

4) $\cos x - \cos y = 2\sin\dfrac{x+y}{2}\cdot\sin\dfrac{y-x}{2} = -2\sin\dfrac{x+y}{2}\cdot\sin\dfrac{x-y}{2}$.

7. REDUCTION FORMULAS

1) Functions of $\pi \pm x$ doesn't change their names. Functions of $\dfrac{\pi}{2}\pm x, \dfrac{3\pi}{2}\pm x$ are changed by cofunctions.

2) The sign + or – to the right is defined by the sign of the function to the left if one supposes the angle x as acute one.

8. SINE THEOREM. Sides a, b, c of an arbitrary triangle are proportional to the sines of opposite angles α, β, γ, that is

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R,$$

where R is the circumradius of the triangle.

9. COSINE THEOREM. Square of a side c of an arbitrary triangle equals sum of squares of the other sides minus double product of these sides and the cosine of the angle γ between them, that is

$$c^2 = a^2 + b^2 - 2ab\sin\gamma.$$

Graphs of trigonometric functions are represented on fig. 16 – 19. Enumerate yourselves properties of these functions and properties of their graphs.

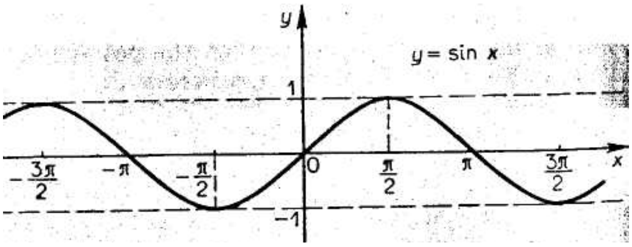


Fig. 16

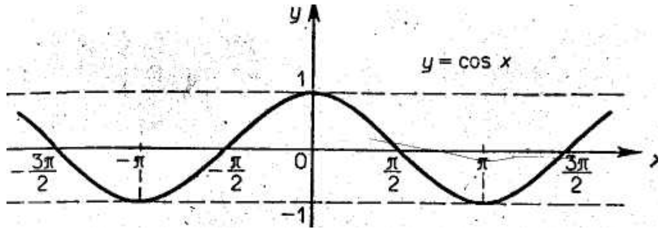


Fig. 17

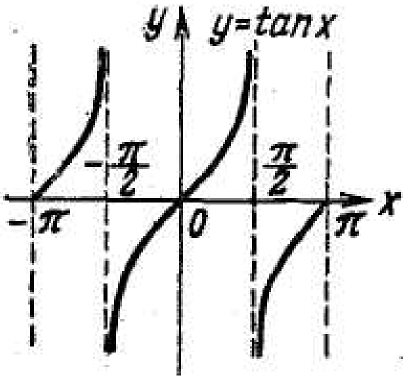


Fig. 18

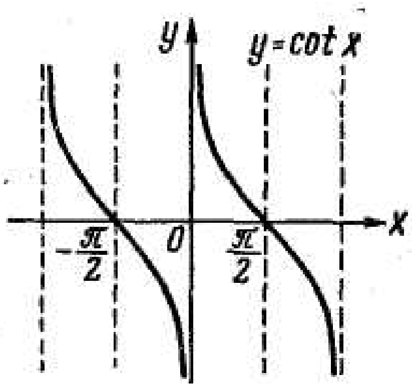


Fig. 19

POINT 5. INVERSE TRIGONOMETRIC FUNCTIONS

Function **arc sine.** The function $y = sin\,x$ increases on the segment $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and has one-to-one image $[-1, 1]$ of this segment. Therefore it possesses the inverse function $x = arcsin\,y$, or (after interchanging by places x and y) $y = arcsin\,x$.

The function $y = arcsin\,x$ has the domain of definition $D(y) = [-1, 1]$, the set of values $E(y) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, increases on $[-1, 1]$. Its graph (fig. 20, 21) rises from left to right, is convex over the interval $(-1, 0)$, concave over the interval $(0, 1)$, has an inflection point, namely the origin $O(0; 0)$.

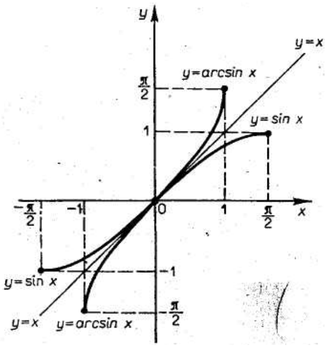


Fig. 20

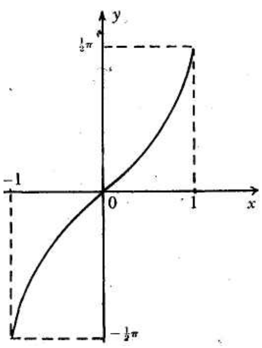


Fig. 21

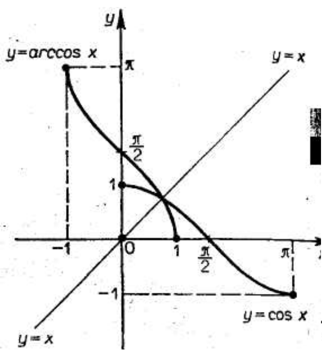


Fig. 22

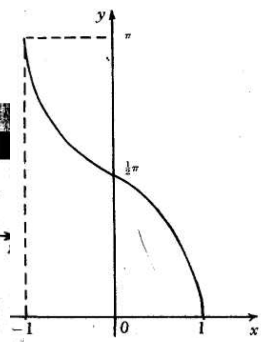


Fig. 23

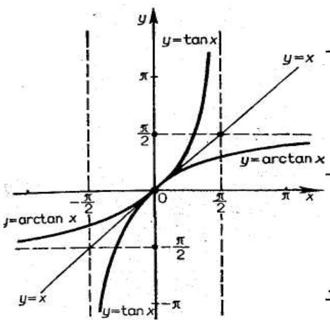


Fig. 24

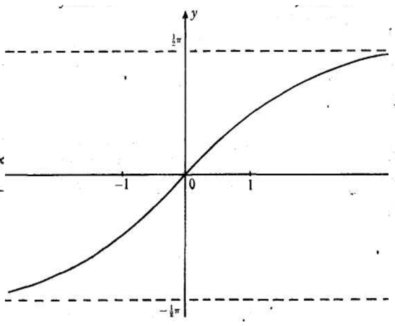


Fig. 25

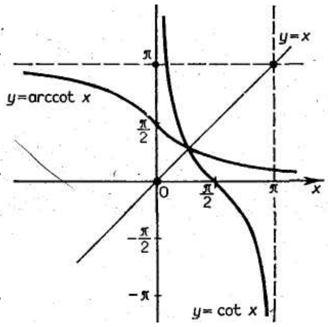


Fig. 26

Function $y = arccos\, x$ (**arc cosine**, fig. 22, 23) is introduced by analogous way (do it yourselves). Its domain of definition and set of values are the segments $D(y) = [-1, 1]$, $E(y) = [0, \pi]$, it decreases on the domain of definition. The graph of the function is concave over the interval $(-1, 0)$, convex over $(0, 1)$, has an inflection point $O(0; 0)$.

Function **arc tangent.** The function $y = tan\, x$ is increasing one on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and establishes correspondence this interval and the set of all reals $\Re$. Hence it possesses the inverse function $x = arctan\, y$, or (after substitution *y* by *x* and *x* by *y*) $y = arctan\, x$.

The function $y = arctan\, x$ has the domain of definition $D(y) = \Re = (-\infty, \infty)$, the set of values $E(y) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, increases on the domain of derinition. Its graph (fig. 24, 25) rises from left to right, is concave over the interval $(-\infty, 0)$, convex over the interval $(0, \infty)$, has the origin $O(0; 0)$ as an inflection point.

Introduce yourselves the function $y = arc\, cot\, x$ **(arc cotangent**, fig. 26**)**, enumerate its properties and properties of its graph.

POINT 6. ELEMENTARY FUNCTIONS. WAYS OF DEFINITION OF A FUNCTION

Def. 27. A power function, an exponential function, a logarithmic function, trigonometric functions, inverse trigonometric functions and a constant function $y = C$ (C - const) form the class of **basic elementary functions**.

Def. 28. Let the set of values $E(\varphi)$ of a function $u = \varphi(x)$ be a subset of the domain of definition $D(f)$ of a function $y = f(u)$. The function $y = f(\varphi(x))$ is called the **composite** one or the **function of a function**, or the **superposition** of functions $y = f(u)$ and $u = \varphi(x)$. The function $u = \varphi(x)$ is called **inner** [interior] one or an **intermediate** argument.

Ex. 20. Let $y = f(u) = \ln u,\ u = \varphi(x) = \sin x,\ \sin x \geq 0$. Then the superposition of $y = f(u)$ and $u = \varphi(x)$ is $y = f(\varphi(x)) = \ln \sin x$.

Def. 29. Elementary function is called a function which is a basic elementary one or can be represented by means of finite number of arithmetical operations and superpositions on basic elementary functions.

Ex. 21. Polynomial of the n-th degree

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n,\ a_n \neq 0,$$

where $a_0, a_1, a_2, a_3, \ldots, a_n$ are coefficients, a_n is a leading [highest, highest degree] coefficient, a_0 (the coefficient of x^0) is usually called a free [a constant, an absolute] term.

Ex. 22. A rational function [a rational fraction] that is a function which can be represented as the ratio of two polynomials.

Ex. 23. Hyperbolic functions: a) hyperbolic sine

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ (fig. 27)},$$

b) hyperbolic cosine

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ (fig. 27)},$$

c) hyperbolic tangent

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \text{ (fig. 28)},$$

d) hyperbolic cotangent

$$coth\,x = \frac{cosh\,x}{sinh\,x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \text{ (fig. 29).}$$

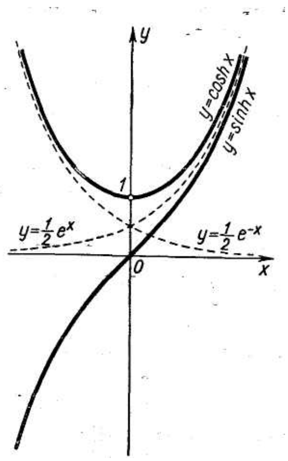


Fig. 27

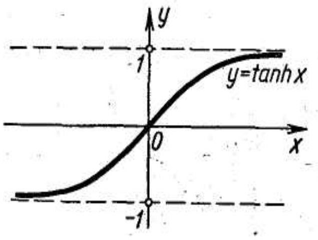


Fig. 28

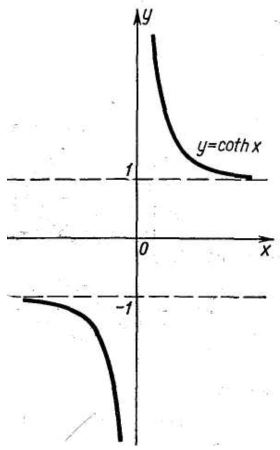


Fig. 29

Enumerate yourselves properties of these functions and properties of their graphs.

There are many non-elementary functions. Some of them we'll study in integral calculus.

Ways of definition of a function:

1. **Analytical way**: with the help of some formula.

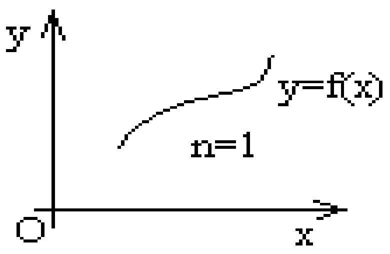


Fig. 30

Ex. 24. $y = x^2$, $y = x_1^{\ 2} + x_2^{\ 2}$, $y = x_1^{\ 2} + x_2^{\ 2} + x_3^{\ 2}$.

2. **Graphical way** (for n = 1, 2): with the help of some graph(ic).

All is clear for n = 1 (see fig. 30).

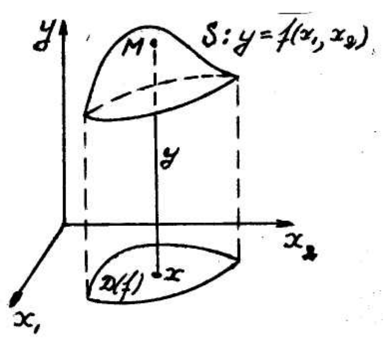


Fig. 31

Let n = 2 that is

$$y = f(x) = f(x_1, x_2).$$

For any $(x_1, x_2) \in D(f)$ we get the point $M(x_1, x_2, y)$ of the space Ox_1x_2y, where $y = f(x_1, x_2)$. Set of all such the points M forms some surface S which is called the graph of the function $y = f(x_1, x_2)$ (fig. 31).

A function of two variables can be geometrically represented by so-called **level lines** [level curves, equiscalar lines]

$$f(x_1, x_2) = C, C - const$$

that is lines along which the function takes on constant values.

Ex.25. Level lines of a function

$$y = x_1^2 + x_2^2$$

are determined by the equation

$$x_1^2 + x_2^2 = C; C \geq 0.$$

For $C = 0$ we have $x_1 = x_2 = 0$ that is one point (the origin) $O(0; 0)$. If $C > 0$, the level lines are circles with radii $R = \sqrt{C}$ centered at the origin $O(0,0)$.

A function of three variables can be geometrically represented by so-called **level surfaces** [equiscalar surfaces]

$$f(x_1, x_2, x_3) = C, C - const.$$

Ex. 26. The level surfaces of a function

$$y = x_1^2 + x_2^2 + x_3^2$$

are the next

$$x_1^2 + x_2^2 + x_3^2 = C; C \geq 0.$$

For $C = 0$ we have the origin $O(0;0;0)$. For $C > 0$ the level surfaces are spheres centered at the origin $O(0,0,0)$ with radii $R = \sqrt{C}$.

3. **Tabular way** (for n = 1, 2): with the help of some table.

For n = 1 see for example tables of trigonometrical functions, of logarithms etc. There are double entry tables [two-input tables] for n = 2, three-entry tables [three-input tables] for n = 3 etc.

4. **Description way** (with the help of some description).

Ex. 27. Definition of trigonometric functions of arbitrary argument with the help of trigonometric circle.

5. **Algorithmic way** (with the help of a program for a computer).

FUNCTIONS: basic terminology RUE

1. Аналитический способ задания [представления] функции	Аналітичний спосіб задання [визначення, подання, представлення] функції	Ànalу́tic(al) way [méthod, mode] of representátion [dèfinítion, detèrminátion, rèpreséntіng] a fúnction
2. Аргумент	Арґумент	Árgument
3. Арккосинус	Арккосинус	Àrc cósine
4. Арккотаненс	Арккотанґенс	Àrc còtángent
5. Арксинус	Арксинус	Àrc sine
6. Арктангенс	Арктанґенс	Àrc tángent
7. Внешняя функция	Зовнішня функція	Óuter [extérior] fúnction
8. Внутренняя функция	Внутрішня функція	Ínner [intérior] fúnction
9. Вогнутый график функции	Угнутий [вгнутий] графік функції	Còncáve graph of a fúnction
10.Возрастать (о функции)	Зростати (про функцію)	Incréase (of a fúnction)
11.Возрастающая функция	Зростаюча функція	Incréasing fúnction
12.Восходящий (слева направо) график	Висхідний (зліва направо) графік	Ascénding graph (from left to right)
13.Выпуклый график функции	Опуклий графік функції	Cònvéx graph of a fúnction
14.Гиперболическая функция	Гіперболічна функція	Hyperbólic fúnction
15.График функции	Графік функції	Graph of a fúnction, plótted fúnction
16.Графический способ задания [определения, представления] функции	Графічний спосіб задання [визначення, подання, представлення] функції	Gráphic(al) way [méthod, mode] of detèrminátion [dèfinítion, representátion, rèpreséning] of a fúnction
17.Зависимая переменная	Залежна змінна	Depéndent váriable
18.Значение аргумента	Значення арґументу	Válue of an árgument
19.Значение функции в точке *a*	Значення функції в точці *a*	Válue of a fúnction at a póint *a*
20.Косинус	Косинус	Cósine
21.Косинусоида	Косинусоїда	Cósinusoid, cósine cúrve
22.Котангенс	Котанґенс	Còtángent
23.Котангенсоида	Котанґенсоїда	Còtángent cúrve
24.Кофункция	Кофункція	Cofúnction
25.Коэффициент	Коефіцієнт	Còefficient
26.Логарифмическая функция	Логарифмічна функція	Lògaríthmic fúnction
27.Максимум функции	Максимум функції	Máximum of a fúnction

28.Минимум функции	Мінімум функції	Mínimum of a fúnction
29.Многочлен (n-й степени)	Многочлен (n-го степеня)	Pòlynómial (of degrée n, n-th degrée pòlynómial)
30.Множество значений функции	Множина значень функції	Set of válues of a fúnction
31.Монотонная функция	Монотонна функція	Mónotone [mónotònic] fúnction
32.Невозрастающая функция	Незростаюча функція	Non-incréasing fúnction
33.Независимая переменная	Незалежна змінна	Ìndepéndent váriable
34.Неубывающая функция	Неспадна функція	Non-decréasing fúnction
35.Нечетная функция	Непарна функція	Odd [unéven] fúnction
36.Нечетность функции	Непарність функції	Óddness of a fúnction
37.Нисходить/опускаться (слева направо) (о графике, функции)	Спадати/опускатися/спускатися (зліва направо) (про графік, криву)	Descénd [drop, come dówn] (from left to right) (about a graph/cúrve)
38.Нисходящий (слева направо) график	Низхідний (зліва направо) графік	Descénding graph (from left to right)
39.Область значений функции	Область значень функції	Domáin of válues of a fúnction
40.Область определения функции	Область визначення функції	Domáin/ránge of dèfinítion of a fúnction
41.Обозначать [обозначить] (функцию)	Позначити [позначати] (функцію)	Denóte [désignate] (a fúnction)
42.Обозначаться	Позначатися	To be denóted, to be désignated
43.Обозначение (функции)	Позначення (функції)	Notátion [dèsignátion, sign, sýmbol] (of a fúnction)
44.Образ элемента	Образ елемента	Ímage of an élement
45.Обратная тригонометрическая функция, арк-функция	Обернена тригонометрична функція, арк-функція	Ìnvérse [recíprocal, cónverse] trìgonométric(al) function
46.Обратная функция	Обернена функція	Ìnvérse [recíprocal, cónverse] fúnction
47.Обратное отображение	Обернене відображення	Ìnvérse [recíprocal, cónverse] mapping [map]
48.Общий член числовой последовательности	Загальний член числової послідовності	Géneral term of a nùmérical/númber séquence
49.Основная элементарная функция	Основна елементарна функція	Básic èleméntary fúnction
50.Отображение	Відображення	Mápping [map]

51.Отображение множества X **в/на** множество Y	Відображення множини X **в/на** множину Y	Mápping of a set X **in(to)/ onto** a set Y
52.Переменная	Змінна	Váriable
53.Период функции	Період функції	Périod of a fúnction
54.Периодическая функция	Періодична функція	Pèriódic(al) fúnction
55.Периодичность функции	Періодичність функції	Pèriódícity of a fúnction
56.Подниматься (слева направо) (о графике/кривой)	Підніматися (зліва направо) (про графік/криву)	Ascénd (from left to right) (about a graph/cúrve)
57.Подниматься, восходить (слева направо) (о кривой, о графике)	Сходити/підійматися (зліва направо) (про криву, про графік)	Ríse [ascénd, come up] (from left to right) (about a cúrve/graph)
58.Показательная функция	Показникова функція	Éxponéntial fúnction
59.Постоянная функция	Стала функція	Cónstant fúnction
60.Принимать значение b в точке a (о функции)	Набувати значення b в точці a (про функцію)	Take on the value b at the point a (of a fúnction)
61.Промежуточный аргумент	Проміжний арґумент	Íntermédiate árgument
62.Прообраз элемента	Прообраз елемента	Preímage of an élement
63.Прямая (линия)	Пряма (лінія)	Straight [right] line
64.Равносильный/эквивалентный термин	Рівносильний/еквівалентний термін	Equívalent term
65.Расположение кривой	Розташування кривої	Position [dìsposítion] of a cúrve
66.Рациональная функция [рациональная дробь]	Раціональна функція [раціональний дріб]	Rátional fúnction [rational fráction]
67.Свободный член	Вільний член	Cónstant [ábsolute, free] term
68.Синус	Синус	Síne
69.Синусоида	Синусоїда	Sínusoid, sine cúrve
70.Сложная функция	Складена функція	Cómposite [compósed, cómplicated] fúnction
71.Способ задания [определения, представления] функции	Спосіб задання [визначення, подання, представлення] функції	Way [méthod, mode] of detèrminátion [dèfinítion, rèpresentátion, representting] a fúnction
72.Ставить в соответствие элементу $x \in X$ эле-	Ставити у відповідність елементу $x \in X$ елемент	Assígn/assóciate the element $y \in Y$ to the élement

мент $y \in Y$	$y \in Y$	$x \in X$
73.Старший коэффициент	Старший коефіцієнт	Léading [híghest (degrée)] còeffícient
74.Степенная функция	Степенева функція	Pówer fúnction
75.Суперпозиция фунций	Суперпозиція функцій	Sùperposítion [composítion] of fúnctions
76.Таблица значений функции	Таблиця значень функції	Táble of válues of a fúnction
77.Табличный способ [метод] задания [определения, представления] функции	Табличний спосіб задання [визначення, подання, представлення] функції	Tábular way [méthod, mode] of detèrminátion [dèfinítion, rèpresentátion, rèpreséntíng] a fúnction
78.Тангенс	Танґенс	Tángent
79.Тангенсоида	Танґенсоїда	Tángent cúrve
80.Точка перегиба кривой, графика функции	Точка перегину кривої, графіка функції	Infléction/infléxion póint, póint of infléction/infléxion of a curve, of a graph of a fúnction
*81.*Точка пересечения кривой, графика функции с *чем-либо*	Точка перетину кривої, графіка функції з *чимсь*	Intersection point [crosspoint, cross point, intercept, intersection] (of a curve, of a graph of a fúnction) with *smth*
82.Тригонометрическая функция	Тригонометрична функція	Trìgonométric(al) fúnction
83.Убывать (о функции)	Спадати (про функцію)	Decréase (of a fúnction)
84.Убывающая функции	Спадна функція	Decréasing fúnction
85.Функция	Функція	Fúnction
86.Функция от функции	Функція від функції	Fúnction of fúnction
87.Четная функция	Парна функція	Éven fúnction
88.Четность функции	Парність функції	Évenness of a fúnction
89.Числовая последовательность	Числова послідовність	Nùmérical [númber] séquence
90.Элемент $y \in Y$ соответствует элементу $x \in X$	Елемент $y \in Y$ відповідає елементу $x \in X$	Élement $y \in Y$ còrrespónds to an element $x \in X$
91.Элементарная функция	Елементарна функція	Èleméntary fúnction
92.Элементу $x \in X$ ставится в соответствие элемент $\mathrm{y} \in \mathrm{Y}$	Елементу $x \in X$ ставиться у відповідність елемент $\mathrm{y} \in \mathrm{Y}$	Élement $y \in Y$ is assigned to an élement $x \in X$

FUNCTIONS: basic terminology ERU

1. Ànalýtic(al) way [méthod, mode] of representátion [dèfinítion, detèrminátion, rèpresénting] a fúnction	Аналитический способ задания [представления] функции	Аналітичний спосіб задання [визначення, подання, представлення] функції
2. Àrc cósine	Арккосинус	Арккосинус
3. Àrc còtángent	Арккотаненс	Арккотанґенс
4. Àrc sine	Арксинус	Арксинус
5. Àrc tángent	Арктангенс	Арктанґенс
6. Árgument	Аргумент	Арґумент
7. Ascénd (from left to right) (about a graph/cúrve)	Подниматься (слева направо) (о графике/кривой)	Підніматися (зліва направо) (про графік/криву)
8. Ascénding graph (from left to right)	Восходящий (слева направо) график	Висхідний (зліва направо) графік
9. Assígn/assóciate the element $y \in Y$ to the élement $x \in X$	Ставить в соответствие элементу $x \in X$ элемент $y \in Y$	Ставити у відповідність елементу $x \in X$ елемент $y \in Y$
10. Básic èleméntary fúnction	Основная элементарная функция	Основна елементарна функція
11. Còeffícient	Коэффициент	Коефіцієнт
12. Cofúnction	Кофункция	Кофункція
13. Cómposite [compósed, cómplicated] fúnction	Сложная функция	Складена функція
14. Còncáve graph of a fúnction	Вогнутый график функции	Угнутий [вгнутий] графік функції
15. Cónstant [ábsolute, free] term	Свободный член	Вільний член
16. Cónstant fúnction	Постоянная функция	Стала функція
17. Cònvéx graph of a fúnction	Выпуклый график функции	Опуклий графік функції
18. Cósine	Косинус	Косинус
19. Cósinusoid, cósine cúrve	Косинусоида	Косинусоїда
20. Còtángent	Котангенс	Котанґенс
21. Còtángent cúrve	Котангенсоида	Котанґенсоїда
22. Decréase (of a fúnction)	Убывать (о функции)	Спадати (про функцію)
23. Decréasing fúnction	Убывающая функции	Спадна функція
24. Denóte [désignate] (a fúnction)	Обозначать [обозначить]	Позначити [позначати] (функцію)
25. Depéndent váriable	Зависимая переменная	Залежна змінна

26. Descénd [drop, come dówn] (from left to right) (about a graph/cúrve)	Нисходить/опускаться (слева направо) (о графике, функции)	Спадати/опускатися/спускатися (зліва направо) (про графік, криву)
27. Descénding graph (from left to right)	Нисходящий (слева направо) график	Низхідний (зліва направо) графік
28. Domáin of válues of a fúnction	Область значений функции	Область значень функції
29. Domáin/ránge of dèfinítion of a fúnction	Область определения функции	Область визначення функції
30. Élement $y \in Y$ còrrespónds to an element $x \in X$	Элемент $y \in Y$ соответствует элементу $x \in X$	Елемент $y \in Y$ відповідає елементу $x \in X$
31. Élement $y \in Y$ is assigned to an élement $x \in X$	Элементу $x \in X$ ставится в соответствие элемент $y \in Y$	Елементу $x \in X$ ставиться у відповідність елемент $y \in Y$
32. Èleméntary fúnction	Элементарная функция	Елементарна функція
33. Equívalent term	Равносильный/эквивалентный термин	Рівносильний/еквівалентний термін
34. Éven fúnction	Четная функция	Парна функція
35. Évenness of a fúnction	Четность функции	Парність функції
36. Éxponéntial fúnction	Показательная функция	Показникова функція
37. Fúnction	Функция	Функція
38. Fúnction of fúnction	Функция от функции	Функція від функції
39. Géneral term of a nùmérical/númber séquence	Общий член числовой последовательности	Загальний член числової послідовності
40. Graph of a fúnction, plótted fúnction	График функции	Графік функції
41. Gráphic(al) way [méthod, mode] of detèrminátion [dèfinítion, representtátion, rèpreséning] a fúnction	Графический способ задания [определения, представления] функции	Графічний спосіб задання [визначення, подання, представлення] функції
42. Hyperbólic fúnction	Гиперболическая функция	Гіперболічна функція
43. Ímage of an élement	Образ элемента	Образ елемента
44. Incréase (of a fúnction)	Возрастать (о функции)	Зростати (про функцію)
45. Incréasing fúnction	Возрастающая функция	Зростаюча функція
46. Ìndepéndent váriable	Независимая переменная	Незалежна змінна
47. Infléction/infléxion póint, póint of infléction/infléxion of a curve, of a graph of a fúnction	Точка перегиба кривой, графика функции	Точка перегину кривої, графіка функції

48.Ínner [intérior] fúnction	Внутренняя функция	Внутрішня функція
49.Íntermédiate árgument	Промежуточный аргумент	Проміжний арґумент
50.Intersection point [crospoint, cross point, intercept, intersection] (of a curve, of a graph of a fúnction) with *smth*	Точка пересечения кривой, графика функции с *чем-либо*	Точка перетину кривої, графіка функції з *чимсь*
51.Ìnvérse [recíprocal, cónverse] fúnction	Обратная функция	Обернена функція
52.Ìnvérse [recíprocal, cónverse] trìgonométric(al) function	Обратная тригонометрическая функция, аркфункция	Обернена тригонометрична функція, аркфункція
53.Ìnvérse [recíprocal, cónverse] mapping [map]	Обратное отображение	Обернене відображення
54.Léading [híghest (degrée)] còeffícient	Старший коэффициент	Старший коефіцієнт
55.Lògaríthmic fúnction	Логарифмическая функция	Логарифмічна функція
56.Mápping [map]	Отображение	Відображення
57.Mápping of a set *X* **in(to)/onto** a set *Y*	Отображение множества *X* **в/на** множество *Y*	Відображення множини *X* **в/на** множину *Y*
58.Máximum of a fúnction	Максимум функции	Максимум функції
59.Mínimum of a fúnction	Минимум функции	Мінімум функції
60.Mónotone [mónotònic] fúnction	Монотонная функция	Монотонна функція
61.Non-decréasing fúnction	Неубывающая функция	Неспадна функція
62.Non-incréasing fúnction	Невозрастающая функция	Незростаюча функція
63.Notátion [dèsignátion, sign, sýmbol] (of a fúnction)	Обозначение (функции)	Позначення (функції)
64.Nùmérical [númber] séquence	Числовая последовательность	Числова послідовність
65.Odd [unéven] fúnction	Нечетная функция	Непарна функція
66.Óddness of a fúnction	Нечетность функции	Непарність функції
67.Óuter [extérior] fúnction	Внешняя функция	Зовнішня функція
68.Périod of a fúnction	Период функции	Період функції
69.Pèriódic(al) fúnction	Периодическая функция	Періодична функція

70.Pèriódícity of a fúnction	Периодичность функции	Періодичність функції
71.Pòlynómial (of degrée *n*, *n*-th degrée pòlynómial)	Многочлен (*n*-й степени)	Многочлен (*n*-го степеня)
72.Position [dìsposítion] of a cúrve	Расположение кривой	Розташування кривої
73.Pówer fúnction	Степенная функция	Степенева функція
74.Preímage of an élement	Прообраз элемента	Прообраз елемента
75.Rátional fúnction [rational fráction]	Рациональная функция [рациональная дробь]	Раціональна функція [раціональний дріб]
76.Ríse [ascénd, come up] (from left to right) (about a cúrve/graph)	Подниматься, восходить (слева направо) (о кривой, о графике)	Сходити/підійматися (зліва направо) (про криву, про графік)
77.Set of válues of a fúnction	Множество значений функции	Множина значень функції
78.Síne	Синус	Синус
79.Sínusoid, sine cúrve	Синусоида	Синусоїда
80.Straight [right] line	Прямая (линия)	Пряма (лінія)
81.Sùperposítion [composítion] of fúnctions	Суперпозиция фунций	Суперпозиція функцій
82.Táble of válues of a fúnction	Таблица значений функции	Таблиця значень функції
83.Tábular way [méthod, mode] of detèrminátion [dèfinítion, rèpresentátion, rèpreséntíng] a fúnction	Табличный способ [метод] задания [определения, представления] функции	Табличний спосіб задання [визначення, подання, представлення] функції
84.Take on the value b at the point a (of a fúnction)	Принимать значение b в точке a (о функции)	Набувати значення b в точці a (про функцію)
85.Tángent	Тангенс	Танґенс
86.Tángent cúrve	Тангенсоида	Танґенсоїда
87.To be denóted, to be désignated	Обозначаться	Позначатися
88.Trìgonométric(al) fúnction	Тригонометрическая функция	Тригонометрична функція
89.Válue of a fúnction at a póint a	Значение функции в точке a	Значення функції в точці a
90.Válue of an árgument	Значение аргумента	Значення арґументу
91.Váriable	Переменная	Змінна
92.Way [méthod, mode] of detèrminátion [dèfinítion, rèpresentátion, representting] a fúnction	Способ задания [определения, представления] функции	Спосіб задання [визначення, подання, представлення] функції

LECTURE NO. 3. COMPLEX NUMBERS

POINT 1. COMPLEX NUMBERS IN ALGEBRAIC FORM

POINT 2. GEOMETRIC REPRESENTATION, TRIGONOMETRIC AND EXPONENTIAL FORMS OF A COMPLEX NUMBER

POINT 3. ELEMENTS OF POLYNOMIAL THEORY

POINT 4. RATIONAL FRACTIONS

POINT 1. COMPLEX NUMBERS IN ALGEBRAIC FORM

Non-sufficiency [deficiency] of the set of all reals for solving many problems can be illustrated by the next example.

Ex. 1. An equation $x^2+1=0$ hasn't real roots.

It's necessary to extend the set of reals. We'll introduce a new number set, the set of complex numbers, introducing one new element, namely an imaginary unit.

Def. 1. The **imaginary unit** i is defined by the next equality:

$$i^2=-1. \qquad (1)$$

Def. 2. A **complex number** (an imaginary number) is called the next expression:

$$z=x+iy\ , \qquad (2)$$

where x, y are real numbers and i is the imaginary unit.

Def. 3. The number x is called a **real part** of the complex number (2) and is denoted by

$$x=Re\,z=Re(x+iy).$$

Def. 4. The number y is called an **imaginary part** of the complex number (2) and is denoted by

$$y=Im\,z=Im(x+iy).$$

Every real number x can be written as a complex one,

$$x=x+0\cdot i\,.$$

It means that the set of all reals is a subset of the set of all complex numbers.

Def. 5. Modulus of the complex number (2) is called the next real number

$$|z| = |x + iy| = \sqrt{x^2 + y^2} \qquad (3)$$

Ex. 2. $|3 - 2i| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$.

Def. 6. A complex number

$$\bar{z} = \overline{x + iy} = x - iy \qquad (4)$$

is called that **conjugate** for the complex number (2).

Ex. 3. $\overline{3 - 2i} = 3 + 2i$.

Arithmetical operations on complex numbers

Def. 7. Operations of addition, subtraction and multiplication on complex numbers are defined similarly to corresponding operations on polynomials, that is if

$$z_1 = x_1 + iy_1,\, z_2 = x_2 + iy_2$$

be two complex numbers, then

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2);$$

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = x_1 + iy_1 - x_2 - iy_2 = (x_1 - x_2) + i(y_1 - y_2);$$

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 + ix_1y_2 + ix_2y_1 + i^2 y_1y_2 = \left|i^2 = -1\right| =$$
$$= x_1x_2 + (-1)y_1y_2 + i(x_1y_2 + x_2y_1) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

Ex. 4.

$$(4 - 5i) + (-3 + 6i) = 4 - 5i - 3 + 6i = 1 + i;$$

$$(-7 + 3i) - (6 - 5i) = -7 + 3i - 6 + 5i = -13 + 8i;$$

$$(-2 - 4i)(5 + i) = -10 - 2i - 20i - 4i^2 = \left|i^2 = -1\right| = -10 - 4 \cdot (-1) - 22i = -6 - 22i.$$

Ex. 5. $\sqrt{-1} = \pm i$, because of $(\pm i)^2 = (\pm 1)^2 \cdot i^2 = 1 \cdot (-1) = -1$. Therefore the equation

$$x^2 + 1 = 0$$

has two complex roots, namely $x_{1,2} = \pm i$.

Ex. 6. By the way of Ex. 5 prove yourselves that $\sqrt{-4}=\pm 2i,\ \sqrt{-7}=\pm i\sqrt{7}$.

Ex. 7. Solve a quadratic equation $x^2+2x+14=0$ (with negative discriminant $D=-52=-4\cdot 13$).

$$x_{1,\,2}=\frac{-2\pm\sqrt{-4\cdot 13}}{2}=\frac{-2\pm 2i\sqrt{13}}{2}=-1\pm i\sqrt{13}.$$

Ex. 8. Product of two **mutual conjugate numbers** is equal to the square of the modulus of each of them. Indeed,

$$z\cdot\bar{z}=(x+iy)\cdot\overline{(x+iy)}=(x+iy)\cdot(x-iy)=x^2-(iy)^2=\left|i^2=-1\right|=x^2+y^2=|z|^2=|\bar{z}|^2.$$

$$z\cdot\bar{z}=|z|^2=|\bar{z}|^2. \qquad (5)$$

For example, $(-4+2i)(-4-2i)=|-4+2i|^2=|-4-2i|^2=16+4=20$.

Def. 8. Division of complex numbers by definition is reduced to multiplication, namely

$$\frac{z_1}{z_2}=\frac{z_1\cdot\bar{z}_2}{z_2\cdot\bar{z}_2}=\frac{z_1\cdot\bar{z}_2}{|z_2|^2}. \qquad (6)$$

Ex. 9.

$$\frac{3+4i}{-5-6i}=\frac{(3+4i)(-5+6i)}{(-5-6i)(-5+6i)}=\frac{-15+18i-20i+24i^2}{|-5-6i|^2}=\frac{-39-2i}{25+36}=\frac{-39-2i}{61}=-\frac{39}{61}-\frac{2}{61}i$$

POINT 2. GEOMETRIC REPRESENTATION, TRIGONOMETRIC AND EXPONENTIAL FORMS OF A COMPLEX NUMBER

Every complex number $z=x+iy$ can be represented by a point $M(x;\,y)$ or by a vector $\overline{OM}$ on so-called **complex plane** xOy (see fig. 1 - 4). The Ox-axis of this plane is called the **real axis**, and the Oy-axis the **imaginary axis**. Real numbers x are represented by points of the real axis, and pure imaginary numbers iy by points of the imaginary axis.

A vector $\overline{OM}$, representing a complex number $z=x+iy$, forms an angle φ with positive direction of the real axis which is called an **argument** $(Argz)$ of this number.

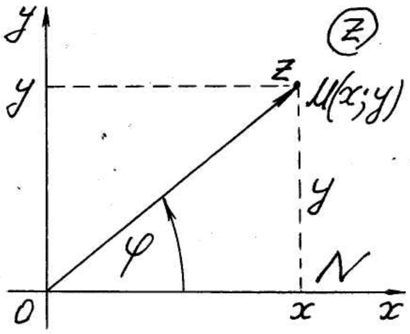


Fig. 1

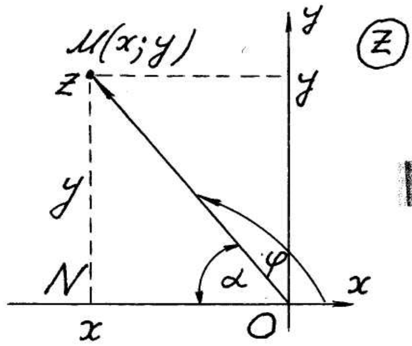


Fig. 2

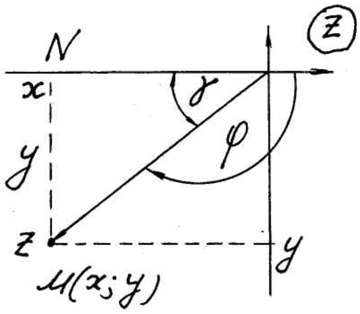


Fig. 3

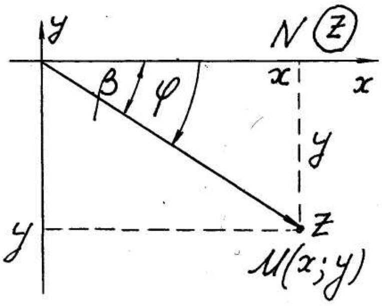


Fig. 4

Any complex number has infinitely many arguments which differ one from another by an integer number of 2π. To fix the idea we'll always consider so-called **principal value** $\varphi = arg\, z$ of an argument which is contained in the interval $(-\pi, \pi]$[1].

If a point $M(x;\, y)$, that is a geometrical representation of a complex number $z = x + iy$, is in the first quadrant (fig. 1), then

$$\varphi = arg\, z = arctan\frac{y}{x}.$$

If $M(x;\, y)$ lies in the second [third, fourth] quadrant (fig. 2 [corresp. 3, 4]), then respectively

$$\varphi = arg\, z = \pi + arctan\frac{y}{x}\left[corresp.\, \varphi = arg\, z = -\pi + arctan\frac{y}{x},\, \varphi = arg\, z = arctan\frac{y}{x}\right].$$

■a) For $\pi/2 < \varphi < \pi$ (fig. 2) one has

$$\varphi = \pi - \alpha,$$

[1] One can also suppose that $\varphi \in [0, 2\pi)$.

where α is an acute angle in a right triangle OMN, and

$$tan\,\alpha = MN/ON = y/|x| = y/(-x) = -y/x\,,\ \alpha = arctan(-y/x) = -arctan(y/x),,$$

$$\varphi = \pi - \alpha = \pi + arctan(y/x).$$

b) If $\varphi \in (-\pi, -\pi/2)$ (fig. 3), then

$$\varphi = -\pi + \gamma\,,\ tan\,\gamma = |y|/|x| = y/x\,,\ \gamma = arctan(y/x),\ \varphi = -\pi + arctan(y/x).$$

c) For $\varphi \in (-\pi/2, 0)$ (fig. 4) $\varphi = -\beta,\ tan\,\beta = |y|/x = -y/x\,,\ \beta = arctan(-y/x) =$
$= -arctan(y/x),\ \varphi = -\beta = arctan(y/x)$.■

Thus, we can write the next general formula for the principal value of an argument of a complex number $z = x + iy$:

$$\varphi = arg\,z = \begin{cases} arctan\dfrac{y}{x},\ \text{if } -\pi/2 < \varphi < \pi/2, \\ \pi + arctan\dfrac{y}{x},\ \text{if } \pi/2 < \varphi < \pi, \\ -\pi + arctan\dfrac{y}{x},\ \text{if } -\pi < \varphi < -\pi/2. \end{cases} \tag{7}$$

We can see (fig. 1 - 4), that the length of the vector $\overline{OM}$ equals the modulus of a complex number $z = x + iy$,

$$\left|\overline{OM}\right| = OM = \sqrt{x^2 + y^2} = |z|,$$

$$x = |z|\,cos\,\varphi,\quad y = |z|\,sin\,\varphi\,,$$

and therefore this number can be expressed in the next form (**trigonometric form** of a complex number):

$$z = |z|(cos\,\varphi + i\,sin\,\varphi). \tag{8}$$

Let's take for granted (as definition) the next famous formula (Euler[1] formula)

$$e^{ix} = cos\,x + i\,sin\,x \tag{9}$$

By virtue of Euler formula we can write

$$cos\,\varphi + i\,sin\,\varphi = e^{i\varphi} \tag{10}$$

[1] Euler, L. (1707 - 1783), a great scientist (a Swiss by birth). He spent most of his life in Russia and died in St. Petersburg. L. Euler contributed many outstanding results to mathematical analysis, celestial mechanics, shipbuilding and other divisions of science.

and introduce so-called **exponential form** of a complex number

$$z = |z| e^{i\varphi} . \qquad (11)$$

Ex. 10. Represent in trigonometric and exponential forms the next numbers:

$$z_1 = 5,\, z_2 = -4,\, z_3 = 2i,\, z_4 = -3i,\, z_5 = -2 + 3i \; .$$

It is evident that moduli of the numbers z_1, z_2, z_3, z_4 are

$$|z_1| = 5, |z_2| = 4, |z_3| = 2, |z_4| = 3,$$

and their arguments respectively equal

$$\varphi_1 = arg\, z_1 = 0,\, \varphi_2 = arg\, z_2 = \pi,\, \varphi_3 = arg\, z_3 = \pi/2,\, \varphi_4 = arg\, z_4 = -\pi/2 .$$

Therefore,

$$z_1 = 5(cos\, 0 + i\, sin\, 0) = 5e^{0 \cdot i},\, z_2 = 4(cos\, \pi + i\, sin\, \pi) = 4e^{\pi i},$$
$$z_3 = 2(cos\, \pi/2 + i\, sin\, \pi/2) = 2e^{\pi/2 \cdot i},\, z_4 = 3(cos(-\pi/2) + i\, sin(-\pi/2)) = 3e^{(-\pi/2) \cdot i} .$$

At least let $z_5 = -2 + 3i$. The modulus of the number $|z| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$. A point $M(-2; 3)$ of the complex plane, corresponding to the number, lies in the second quadrant. Hence by the formula (7)

$$arg\, z = \pi + arctan(3/(-2)) = \pi - arctan(3/2),$$

$$z_5 = -2 + 3i = \sqrt{13}(cos(\pi - arctan(3/2)) + i\, sin(\pi - arctan(3/2))) = \sqrt{13} e^{i(\pi - arctan(3/2))}$$

Let there be given two complex numbers represented in trigonometric or exponential form

$$z_1 = |z_1|(cos\, \varphi_1 + i\, sin\, \varphi_1) = |z_1| e^{i\varphi_1}, \quad z_2 = |z_2|(cos\, \varphi_2 + i\, sin\, \varphi_2) = |z_2| e^{i\varphi_2} .$$

Their multiplication and division can by fulfilled in correspondence with the next rule.

To multiply two complex numbers it’s sufficient to multiply their moduli and add arguments. To divide these numbers it’s sufficient to divide their moduli and subtract arguments. By this rule

$$z_1 \cdot z_2 = |z_1||z_2|(cos(\varphi_1 + \varphi_2) + i\, sin(\varphi_1 + \varphi_2)) = |z_1||z_2| e^{i(\varphi_1 + \varphi_2)}, \qquad (12)$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}(cos(\varphi_1 - \varphi_2) + i\, sin(\varphi_1 - \varphi_2)) = \frac{|z_1|}{|z_2|} e^{i(\varphi_1 - \varphi_2)} . \qquad (13)$$

■We'll prove, for example, the formula (12) ((13) prove yourselves).

$$z_1 \cdot z_2 = |z_1||z_2|(\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 + i\sin\varphi_2) =$$

$$= |z_1||z_2|((\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) + i(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2)) =$$

$$= |z_1||z_2|(cos(\varphi_1 + \varphi_2) + i\,sin(\varphi_1 + \varphi_2))$$■

Ex. 11. For numbers z_2, z_3 from Ex. 10

$$z_2 z_3 = 8(cos(\pi + \pi/2) + i\,sin(\pi + \pi/2)) = 8(cos\,3\pi/2 + i\,sin\,3\pi/2) = 8e^{3\pi/2\cdot i},$$
$$\frac{z_2}{z_3} = \frac{4}{2}(cos(\pi - \pi/2) + i\,sin(\pi - \pi/2)) = 8(cos\,\pi/2 + i\,sin\,\pi/2) = 8e^{\pi/2\cdot i}.$$

Ex. 12. Let

$$z = |z|(cos\,\varphi + i\,sin\,\varphi) = |z|e^{i\varphi}.$$

Then

$$z^2 = |z|(cos\,\varphi + i\,sin\,\varphi)|z|(cos\,\varphi + i\,sin\,\varphi) = |z|^2(cos\,2\varphi + i\,sin\,2\varphi) = |z|^2 e^{2i\varphi},$$

$$z^3 = z^2 \cdot z = |z|^2(\cos 2\varphi + i\sin 2\varphi)|z|(\cos\varphi + i\sin\varphi) = |z|^3(\cos 3\varphi + i\sin 3\varphi) = |z|^3 e^{3i\varphi}$$

$$z^4 = z^2 \cdot z^2 = |z|^2(\cos 2\varphi + i\sin 2\varphi)|z|^2(\cos 2\varphi + i\sin 2\varphi) = |z|^4(\cos 4\varphi + i\sin 4\varphi) = |z|^4 e^{4i\varphi}$$

....

In general, as can be proved by induction, the next formula (Moivre[1] formula)

$$z^n = |z|^n(cos\,n\varphi + i\,sin\,n\varphi) = |z|^n e^{in\varphi} \qquad (14)$$

is valid.

Ex. 13. For the number z_5 from Ex. 10 Moivre formula gives

$$z_5^4 = (-2+3i)^4 = (\sqrt{13})^4(\cos 4(\pi - \arctan(3/2)) + i\sin 4(\pi - \arctan(3/2))) =$$
$$= 169(\cos 4(\pi - \arctan(3/2)) + i\sin 4(\pi - \arctan(3/2))) = 169e^{4i(\pi - \arctan(3/2))}$$

With the help of Moivre formula it can be proved that the *n*-th root from a complex number

$$z = |z|(\cos\varphi + i\sin\varphi) = |z|e^{i\varphi}$$

equals

[1] Moivre, A. (1667 - 1754), an English mathematician

$$\sqrt[n]{z}=\sqrt[n]{|z|}\left(cos\frac{\varphi+2\pi k}{n}+i\,sin\frac{\varphi+2\pi k}{n}\right)=\sqrt[n]{|z|}e^{i\frac{\varphi+2\pi k}{n}},\quad k=0,1,2,...,n-1. \qquad (15)$$

On the base of (15) we see that n-th root of a complex number has n values.

Ex. 15. For the number $z_2=4(cos\,\pi+i\,sin\,\pi)=4e^{\pi i}$ (Ex. 10) the formula (15) gives $\sqrt[3]{z_2}=\sqrt[3]{4}\left(cos\frac{\pi+2\pi k}{3}+i\,sin\frac{\pi+2\pi k}{3}\right)=4e^{i\frac{\pi+2\pi k}{3}},\quad k=0,1,2.$

Putting $k=0,1,2$ we successively get

$$\left(\sqrt[3]{z_2}\right)_0=\sqrt[3]{4}\left(cos\frac{\pi}{3}+i\,sin\frac{\pi}{3}\right)=4e^{i\frac{\pi}{3}};\quad \left(\sqrt[3]{z_2}\right)_1=\sqrt[3]{4}(cos\,\pi+i\,sin\,\pi)=4e^{\pi i};$$

$$\left(\sqrt[3]{z_2}\right)_2=\sqrt[3]{4}\left(cos\frac{5\pi}{3}+i\,sin\frac{5\pi}{3}\right)=4e^{i\frac{5\pi}{3}}.$$

POINT 3. ELEMENTS OF POLYNOMIAL THEORY

We'll study here polynomials with (in general) complex coefficients.

Theorem 1 (division of polynomials with remainder)**.** For any n-th degree polynomial

$$P_n(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+...+a_1x+a_0\ (a_n\neq 0)$$

and any m-th degree polynomial

$$Q_m(x)=b_mx^m+b_{m-1}x^{m-1}+b_{m-2}x^{m-2}+...+b_1x+b_0\ (b_m\neq 0)$$

there are two polynomials $S(x), R_k(x)\,(k<m)$ such that the next identity holds:

$$P_n(x)=Q_m(x)S(x)+R_k(x). \qquad (16)$$

The polynomial $P_n(x)$ is called the **dividend**, $Q_m(x)$ the divisor, $S(x)$ the quotient, $R_k(x)$ the remainder [the residue]. One can fulfill division by the well known method (division by corner)

Ex. 16. Let

$$P_4(x)=2x^4-7x^3+4x^2-x+7,\, Q_3(x)=x^3-3x^2+5x-6.$$

Then

$$S(x) = 2x - 1,\ R(x) = -9x^2 + 16x + 1,$$

and

$$2x^4 - 7x^3 + 4x^2 - x + 7 = (x^3 - 3x^2 + 5x - 6)(2x - 1) + (-9x^2 + 16x + 1).$$

If $R_k(x) \equiv 0$, it means that $P_n(x)$ is divided by $Q_m(x)$[or $P_n(x)$ contains $Q_m(x)$], and therefore $P_n(x)$ is represented as a product

$$P_n(x) = Q_m(x)S(x) \tag{17}$$

Ex. 17. $x^3 + a^3$ is divided by $x + a$ and by $x^2 - ax + a^2$, that is

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2).$$

Let a divisor be a linear binomial

$$Q_1(x) = x - x_0. \tag{18}$$

Dividing $P_n(x)$ by $Q_1(x) = x - x_0$ we obtain

$$P_n(x) = (x - x_0)S_{n-1}(x) + R, \tag{19}$$

where a quotient

$$S_{n-1}(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \ldots + b_1x + b_0 \tag{20}$$

is the (n -1)-th degree polynomial, and a remainder R is some number.

Theorem 2 (Bezout[1] theorem). Remainder R of division of a polynomial $P_n(x)$ by a linear binomial $x - x_0$ equals the value of the polynomial at the point x_0.

■Putting $x = x_0$ in (19) we get $P_n(x_0) = R$ ■

Def. 9. A number x_0 is called a **root** of a polynomial $P_n(x)$ if $P_n(x_0) = 0$, that is if the value of the polynomial at the point x_0 equals zero.

Corollary. A number x_0 is a root of a polynomial $P_n(x)$ if and only if the polynomial is divided by $x - x_0$.

■a) If x_0 is a root, $P_n(x_0) = 0$, then R = 0 in (19), and the polynomial is divided by $x - x_0$.

b) If the polynomial is divided by $x - x_0$, then R = 0, whence it follows that

[1] Bézout, E. (1730 - 1783), a French mathematician

$$P_n(x) = (x - x_0)S_{n-1}(x) \text{ and } P_n(x_0) = 0 .\blacksquare$$

It can be easy proved (see below) that the coefficients of the quotient $S_{n-1}(x)$ and the remainder R in (19) can be determined by so-called Horner[1] scheme

	a_n	a_{n-1}	a_{n-2}	$\dots$	a_1	a_0
x_0	$b_{n-1} = a_n$	$b_{n-2} = x_0 b_{n-1} + a_{n-1}$	$b_{n-3} = x_0 b_{n-2} + a_{n-2}$	$\dots$	$b_0 = x_0 b_1 + a_1$	$R = x_0 b_0 + a_0$

Ex. 18. Find the values of a polynomial $P(x) = x^3 - 3x^2 + 5x - 6$ at the points $x = 1, x = -2$ and its roots.

a) At first, on the base of Bezout theorem, we'll find remainders of division of the polynomial by $x - 1, x - (-2) = x + 2$. Using twice Horner scheme, we have

	1	-3	5	-6
1	1	$1 \cdot 1 + (-3) = -2$	$1 \cdot (-2) + 5 = 3$	$R = 1 \cdot 3 + (-6) = -3 = P(1)$
-2	1	$(-2) \cdot 1 + (-3) = -5$	$(-2) \cdot (-5) + 5 = 15$	$R = (-2) \cdot 15 + (-6) = -36 = P(-2)$

Thus, $P(1) = -3, P(-2) = -36$.

b) With the help of the same scheme we find numbers such that to have $R = 0$. Testing the numbers – 1, 2, 3, -3, 6, - 6 (divisors of the constant term of the polynomial; divisors 1 and – 2 were already tested, and they aren't roots) we obtain

	1	-3	5	-6
2	1	$2 \cdot 1 + (-3) = -1$	$2 \cdot (-1) + 5 = 3$	$R = P(2) = 2 \cdot 3 + (-6) = 0$

$P(2) = 0$, an so the first root $x_1 = 2$. To find the rest of roots of the polynomial we represent it in the form

$$P(x) = x^3 - 3x^2 + 5x - 6 = (x - 2)(x^2 - x + 3)$$

(coefficients of the second factor equal 1, -1, 3 by Horner scheme). Solving the quadratic

$$x^2 - x + 3 = 0 ,$$

[1] Horner, W.G. (1786 - 1837), an English mathematician

we get

$$x_{2,3}=\frac{1\pm\sqrt{-11}}{2}=\frac{1\pm i\sqrt{11}}{2}=\frac{1}{2}\pm i\frac{\sqrt{11}}{2}.$$

Theorem 3 (**main theorem** of the algebra)**.** Every n-th degree polynomial

$$P_n(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\ldots+a_1x+a_0 \quad (a_n\neq 0) \qquad (21)$$

for $n > 0$ possesses at least one (generally complex) root.

Proving of this theorem is very difficult. We'll confine ourselves to some important corollaries.

Some corollaries of the main theorem

1. A polynomial (21) can be represented as a product of linear binomials

$$P_n(x)=a_n(x-x_1)(x-x_2)\ldots(x-x_n). \qquad (22)$$

■Having a root x_1, the polynomial can be represented in the form

$$P_n(x)=(x-x_1)Q_{n-1}(x).$$

If $n-1>0$, a polynomial $Q_{n-1}(x)$ has a root x_2, hence

$$Q_{n-1}(x)=(x-x_2)Q_{n-2}(x) \text{ and } P_n(x)=(x-x_1)(x-x_2)Q_{n-2}(x).$$

If $n-2>0$, we get by the same way

$$P_n(x)=(x-x_1)(x-x_2)(x-x_3)Q_{n-3}(x),$$

an so on. Finally we obtain the expansion (22).■

2. Some factors in (22) can coincide. We'll collect all them together and come to expansion of the next form

$$P_n(x)=a_n(x-x_1)^{k_1}(x-x_2)^{k_2}\ldots(x-x_i)^{k_i}\ldots(x-x_s)^{k_s}, \qquad (23)$$

where

$$k_1+k_2+\ldots+k_i+\ldots+k_s=n. \qquad (24)$$

Def. 10. A root x_i in the expansion (23) is called **simple** one if $k_i=1$ and **multiple** (k_i-tuple [k_i-fold, k_i-th]) root otherwise.

3. It follows from (24) that every n-th degree polynomial has n roots if one takes into account k_i times the k_i-tuple root.

4. If two polynomials are equal (that is identically equal), then their coefficients of the same powers of x coincide.

5. Every polynomial with **real coefficients** can be factorized into linear and quadratique (with negative discriminants) factors, namely

$$P_n(x) = a_n(x - x_1)^{k_1} \ldots (x - x_s)^{k_s} (x^2 + p_1 x + q_1)^{l_1} \ldots (x^2 + p_1 x + q_1)^{l_m} . \qquad (25)$$

Linear factors correspond to real roots of the polynomial, and those quadratique to imaginary roots.

Ex. 19. Factorize the next polynomial $x^4 + x^2 + 4$.

Solution. $x^4 + x^2 + 4 = x^4 + 4x^2 + 4 - 3x^2 = (x^2 + 2)^2 - (\sqrt{3}x)^2 =$

$$= (x^2 + 2 + \sqrt{3}x)(x^2 + 2 - \sqrt{3}x) = (x^2 + \sqrt{3}x + 2)(x^2 - \sqrt{3}x + 2).$$

Ex. 20. Prove Horner's scheme.

■Let

$$P_n(x) = (x - x_0)S_{n-1}(x) + R,$$

where

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0,$$

$$S_{n-1}(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \ldots + b_1 x + b_0 .$$

Therefore,

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0 =$$
$$= (x - x_0)(b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + b_{n-3}x^{n-3} + \ldots + b_1 x + b_0) + R;$$

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0 =$$

$$= b_{n-1}x^n + (b_{n-2} - x_0 b_{n-1})x^{n-1} + (b_{n-3} - x_0 b_{n-2})x^{n-2} + \ldots + (b_0 - x_0 b_1)x + (R - x_0 b_0).$$

Equating the coefficients of the same powers of x in two last polynomials we get

$$a_n = b_{n-1}, a_{n-1} = b_{n-2} - x_0 b_{n-1}, a_{n-2} = b_{n-3} - x_0 b_{n-2}, \ldots, a_1 = b_0 - x_0 b_1, a_0 = R - x_0 b_0,$$

whence it follows that

$$b_{n-1} = a_n, b_{n-2} = x_0 b_{n-1} + a_{n-1}, b_{n-3} = x_0 b_{n-2} + a_{n-2}, \ldots, b_0 = x_0 b_1 + a_1, R = x_0 b_0 + a_0 .■$$

POINT 4. RATIONAL FRACTIONS

As we know, a rational fraction is a ratio of two polynomials

$$R(x)=\frac{P_n(x)}{Q_m(x)}=\frac{a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+...+b_1x+b_0},\ a_n\neq 0, b_m\neq 0. \qquad (26)$$

Def. 11. Rational fraction (26) is called **proper** one if $n<m$ and **improper** otherwise ($n\geq m$).

Theorem 4 (extraction of integer part of an improper rational fraction)**.** Every improper rational fraction can be represented as a sum of some polynomial (so-called **integer part**) and a proper rational fraction.

■Let $n\geq m$. Dividing the numerator $P_n(x)$ by the denominator $Q_m(x)$ we get

$$P_n(x)=Q_m(x)S(x)+r(x),\quad R(x)=\frac{Q_m(x)S(x)+r(x)}{Q_m(x)}=S(x)+\frac{r(x)}{Q_m(x)}$$

where polynomials $S(x), r(x)$ are a quotient and a remainder respectively and

$$r(x)/Q_m(x)$$

is a proper rational fraction.■

Ex. 21. Extract an integer part of an improper rational fraction

$$R(x)=\frac{x^2}{x+1}$$

a) The first (theoretical) way. After division of x^2 by $x+1$ we get

$$x^2=(x+1)(x-1)+1,\ S(x)=x-1,\ r(x)=1,$$

$$\frac{x^2}{x+1}=\frac{(x+1)(x-1)+1}{x+1}=x-1+\frac{1}{x+1}$$

b) The second way. Subtracting and adding 1 in the numerator we'll have

$$\frac{x^2}{x+1}=\frac{x^2-1+1}{x+1}=\frac{(x^2-1)+1}{x+1}=x-1+\frac{1}{x+1}$$

There are partial [simplest, elementary] fractions of 1- 4 types.

1. $\dfrac{A}{ax+b}$;

2. $\dfrac{A}{(ax+b)^k},\ k\in \boldsymbol{N}$;

3. $\dfrac{Ax+B}{ax^2+bx+c}\quad (D=b^2-4ac<0)$;

4. $\dfrac{Ax+B}{(ax^2+bx+c)^k},\ k\in \boldsymbol{N}\quad (D=b^2-4ac<0)$.

Theorem 5 (a partial decomposition of a proper rational fraction)**.** Every proper rational fraction can be represented as a linear combination of partial fractions.

For example

$$\frac{Ax+B}{(ax+b)(cx+d)}=\frac{P}{ax+b}+\frac{Q}{cx+d};$$
$$\frac{Ax^2+Bx+C}{(ax+b)(cx^2+dx+c)}=\frac{P}{ax+b}+\frac{Qx+R}{cx^2+dx+c};$$
$$\frac{Ax^3+Bx^2+Cx+D}{(ax+b)^2(cx^2+dx+c)}=\frac{P}{(ax+b)^2}+\frac{Q}{ax+b}+\frac{Rx+S}{cx^2+dx+c}.$$

Here P, Q, R, S are some unknown numbers (undetermined coefficients), which one can find by so-called **method of undetermined coefficients**.

Ex. 22. Do a partial decomposition of a proper rational fraction

$$\frac{x^2-27x+14}{(x-3)(4-8x-x^2)}.$$

$$\frac{x^2-27x+14}{(x-3)(4-8x-x^2)}=\frac{A}{x-3}+\frac{Bx+C}{4-8x-x^2}\ \Big|\cdot(x-3)(4-8x-x^2),$$

$$x^2-27x+14=A(4-8x-x^2)+(Bx+C)(x-3). \qquad (**)$$

Assigning three arbitrary values to x in (**), for example $x=3,\ x=0,\ x=1$, we get a system of linear equations in A, B, C,

$$\begin{array}{l} x=3 \\ x=0 \\ x=1 \end{array}\left|\left\{\begin{array}{l} -58=-29A, \\ 14=4A-3C, \\ -12=-5A-2B-2C; \end{array}\right.\right. \quad \begin{array}{l} A=2, \\ C=-2, \\ B=3; \end{array} \Rightarrow \frac{x^2-27x+14}{(x-3)(4-8x-x^2)}=\frac{2}{x-3}+\frac{3x-2}{4-8x-x^2}.$$

COMPLEX NUMBERS AND POLYNOMIALS: basic terminology RUE

Complex numbers

1. Алгебраическая форма комплексного числа	Алгебрична форма комплексного числа	Àlgebráic form [algebraic rèpresentátion] of a cómplex númber
2. Аргумент комплексного числа	Арґумент комплексного числа	Árgument of a complex númber
3. Вещественная ось	Дійсна вісь	Réal áxis, áxis of réals
4. Вещественная часть комплексного числа	Дійсна частина комплексного числа	Réal part of a cómplex númber
5. Взаимно сопряжённые комплексные числа	Взаємно спряжені комплексні числа	Mútual cónjugate cómplex númbers
6. Возведение в степень комплексного числа	Піднесення до степеня [степенювання] комплексного числа	Ráising of a cómplex númber to a pówer
7. Возвести в степень комплексное число	Піднести до степеня [степенювати] комплексне число	Ráise a cómplex númber to a pówer
8. Вычитание комплексных чисел	Віднімання комплексних чисел	Subtráction of complex númbers
9. Деление комплексных чисел	Ділення комплексних чисел	Divísion of cómplex númbers
10.Изображение комплексного числа (на комплексной плоскости)	Зображення комплексного числа (на комплексній площині)	Rèpresentátion of a cómplex númber (on the cómplex pláne)
11.Изобразить комплексное число (на комплексной плоскости)	Зобразити комплексне число (на комплексній площині)	Rèpresént a cómplex númber on the complex pláne
12.Комплексная плоскость	Комплексна площина	Cómplex pláne
13.Комплексное число	Комплексне число	Cómplex númber
14.Комплексно-сопряжённое число (комплексного числа)	Комплексно-спряжене число (комплексного числа)	Cómplex cónjugate númber (of a cómplex númber)
15.Мнимая единица	Уявна одиниця	Imáginary unit, unit imáginary number
16.Мнимая ось	Уявна вісь	Imáginary áxis
17.Мнимая часть комплексного числа	Уявна частина комплексного числа	Imáginary part of a cómplex númber
18.Мнимое число	Уявне число	Imáginary númber
19.Модуль комплексного числа	Модуль комплексного числа	Módulus (*pl* móduli) of a cómplex númber

20. Показательная форма комплексного числа	Показникова форма комплексного числа	Èxponéntial form [rèpresentátion] of a cómplex númber
21. Сложение комплексных чисел	Додавання комплексних чисел	Addítion of cómplex númbers
22. Сопряжённое комплексное число	Спряжене комплексне число	Cónjugate cómplex númber
23. Тригонометрическая форма комплексного числа	Тригонометрична форма комплексного числа	Trìgonométric(al) form [rèpresentátion] of a complex númber
24. Умножение комплексных чисел	Множення комплексних чисел	Mùltiplicátion of cómplex númbers
25. Формула Муавра, Эйлера	Формула Муавра, Ейлера	Moivre(´s), Euler(´s) fórmula
26. Чисто мнимое число	Чисто уявне число	Púre imáginary númber

Polynomials

1. Алгебраическое уравнение первой [второй, третьей, *n*-ой] степени	Алгебричне рівняння першого [другого, третього, *n*-го] степеня	Àlgebráic equátion of the first [second, third, *n*th] degrée. First- [second-, third-, *n*-th] degrée àlgebráic equátion
2. Быть разложимым в произведение многочленов степени не выше второй (о многочлене)	Бути розкладним в добуток многочленів степеня не вище другого (про многочлен)	Be fáctorable into [can be written as] a próduct of pòlynómials of degrée not hígher than two (of a pòlynómial)
3. Делитель (свободного члена)	Дільник (вільного члена)	Divísor (of the cónstant [ábsolute, free] term)
4. Дискриминант квадратного уравнения	Дискримінант квадратного рівняння	Discríminant of a quadrátic equátion [of a quadrátic]
5. Квадратное уравнение [квадратное неравенство]	Квадратне рівняння [квадратна нерівність]	Quadrátic equátion; quadrátic; quadrátic ìnequálity
6. Квадратный трёхчлен	Квадратний тричлен	Quadrátic trinómial
7. Корень [нуль] многочлена	Корінь [нуль] многочлена	Róot [zéro] of a pòlynómial
8. Кратный корень	Кратний корінь	Múltiple [repéated] root [zéro]
9. Многочлен первой [второй, третьей, *n*-ой] степени	Многочлен першого [другого, третього, *n*-го] степеня	Polynomial of the first [second, third, *n*th] degrée. First- [second-, third-, *n*-

		th] degrée pòlynómial
10.Многочлен с единичным старшим коэффициентом	Многочлен з одиничним старшим коефіцієнтом	Monic [nórmalized] pòlynómial
11.Многочлен с целыми коэффициентами	Многочлен з цілими коефіцієнтами	Pòlynómial with ínteger còeffícients; íntegral pòlynómial
12.Простой [однократный] корень [нуль]	Простий [однократний, одноразовий] корінь [нуль]	Símple [únrepéated, single] root [zéro]
13.Разложение многочлена на множители	Розкладання многочлена на множники	Fàctorizátion/fáctoring/ expánsion of a pòlynómial (into fáctors)
14.Разложить многочлен на линейные и квадратичные (с отрицательными дискриминантами) множители	Розкласти многочлен на лінійні й квадратичні (з від"ємними дискримінантами) множники	Fáctor [fáctorize, expánd] the pòlynómial into línear and quadrátique (with négative discríminants) fáctors
15.Разложить на множители	Розкласти на множники	Fáctor [fáctorize]; expánd [dècompóse] into fáctors [compónents]; présent in a factor form
16.Старший коэффициент	Старший коефіцієнт	Léading [highest (degrée)] còeffícient
17.Старший член (многочлена)	Старший, головний [основний] член (многочлена)	Léading [highest (degrée)] term (of a pòlynómial)
18.Угадать корень	Вгадати корінь	Guess [tell] a róot
19.Целый корень многочлена с целыми коэффицентами	Цілий корінь многочлена з цілими коефіцієнтами	Ínteger róot of a pòlynómial with ínteger còeffícients [of an íntegral pòlynómial]
20.Эн-кратный (*n*-кратный) корень [*n*-кратный нуль]	Ен-кратний (*n*-кратний) корінь [*n*-кратний нуль]	*n*-th root/zéro, *n*-tuple root/zéro, *n*-fold root/zéro

Rational functions [rational fractions]

1. Выделение целой части неправильльной рациональной дроби	Виділення цілої частини неправильного раціонального дробу	Extráction of an íntegral part of an imprópеr fráction
2. Выделить целую часть неправильльной рациональной дроби	Виділити цілу частину неправильного раціонального дробу	Extráct an íntegral part of the imprópеr fráction

3. Давать/приписывать (частные) значения аргументу	Давати/надавати/приписувати (частинні) значення арґументу	Give/assígn (partícular) válues to the árgument
4. Дробно-рациональная функция	Дробово-раціональна функція	Fráctional rátional fúnction
5. Метод неопределенных коэффициентов	Метод невизначених коефіцієнтів	Méthod of úndetérmined còefficients
6. Неправильная рациональная дробь	Неправильний раціональний дріб	Impróper rátional fráction
7. Отношение двух многочленов	Відношення двох многочленів	Rátio of two pòlynómials
8. Порождать *k* (простейших) дробей вида… (о множителе знаменателя правильной рациональной дроби)	Породжувати *k* (найпростіших) дробів вигляду... (про множник знаменника правильного раціонального дробу)	Génerate [give, rise to, indúce] *k* (simplest) fráctions of the form … (of a fáctor of a denóminator of a próper rátional fráction)
9. Порождать простейшую рациональную дробь вида … (о множителе знаменателя правильной рациональной дроби)	Породжувати найпростіший дріб вигляду... (про множник знаменника правильного раціональнго дробу)	Génerate [give, rise to, indúce] the símplest fráction of the form … (of a fáctor of a denominator of a próper rátional fráction)
10.Правильная рациональная дробь	Правильний раціональний дріб	Próper rátional fráction
11.Приведение дробей к общему знаменателю	Приведення/зведення дробів до спільного знаменника	Redúction of fractions to the cómmon/sáme denóminator
12.Привести дроби к общему знаменателю	Привести/звести дроби до спільного знаменника	Redúce the fráctions to the cómmon denóminator
13.Приравнивание (числителей, коэффициентов при одинаковых степенях *x*)	Прирівнювання (чисельників, коефіцієнтів при однакових степенях *x*)	Équalize [equáting, équalizátion, sétting équal, idèntificátion, idéntifying] (the numerators, the còefficients of équal (of the same) degrées/pówers (in like pówers) of *x*)
14.Приравнять (числители, коэффициенты при одинаковых степенях *x*)	Прирівняти (чисельники, коефіцієнти при однакових степенях *x*)	Equáte [set équal, idéntify] (the númerators, the còefficients of équal (of the same) degrées/pówers (in like pówers) of *x*)
15.Простейшая (элементарная) рациональная дробь первого (второго,	Найпростіший (елементарний) раціональний дріб першого (другого,	Pártial/símplest/èleméntary rátional fráction of the first (second, third, fourth)

третьего, четвертого) типа	третього, четвертого) типу	type/kind [first/second/ third/fourth type ~]
16.Разложение правильной рациональной дроби в сумму простейших (дробей)	Розвинення правильного раціонального дробу на найпростіші/елементарні (в суму найпростіших /елементарних дробів)	Rèsolútion [expánsion, devéloppement, dècomposítion] of a próper rátional fráction into (a sum of) pártial/símplest/èleméntary fráctions [partial fráction dècomposítion of...]
17.Разложить правильную рациональную дробь в сумму простейших (дробей)	Розвинути правильний раціональний дріб в суму найпростіших (дробів)	Resólve [expánd, devélop, dècompóse] a próper rational fráction into pártial/símplest/èleméntary fráctions [fulfill partial fráction dècomposítion]
18.Рациональная дробь	Раціональний дріб	Rátional fráction
19.Рациональная функция	Раціональна функція	Rátional fúnction
20.Степень числителя (не) меньше степени знаменателя	Степінь чисельника (не) менше степеня знаменника	The degrée of the númerator is (not) less than the degrée of the denóminator
21.Умножить обе части равенства/тождества на общий знаменатель	Помножити обидві частини рівності/тотожності на спільний знаменник	Múltiply two members of the equálity/idéntity by the cómmon denóminator
22.Целая рациональная функция (многочлен)	Ціла раціональна функція, многочлен	Entíre [integrál] rational fúnction, rátional integrál fúnction; pòlynómial
23.Целая часть неправильной рациональной дроби	Ціла частина неправильного раціонального дробу	Íntegral part of the impróper rátional fráction

COMPLEX NUMBERS AND POLYNOMIALS: basic terminology ERU

1. Addítion of cómplex númbers	Сложение комплексных чисел	Додавання комплексних чисел
2. Àlgebráic form [algeb-raic rèpresentátion] of a cómplex númber	Алгебраическая форма комплексного числа	Алгебрична форма ком-плексного числа
3. Árgument of a comp-lex númber	Аргумент комплексного числа	Арґумент комплексного числа
4. Cómplex cónjugate númber (of a cómplex númber)	Комплексно-сопряжён-ное число (комплексного числа)	Комплексно-спряжене число (комплексного чи-сла)
5. Cómplex númber	Комплексное число	Комплексне число
6. Cómplex pláne	Комплексная плоскость	Комплексна площина
7. Cónjugate cómplex númber	Сопряжённое комплекс-ное число	Спряжене комплексне число
8. Divísion of cómplex númbers	Деление комплексных чисел	Ділення комплексних чисел
9. Èxponéntial form [rè-presentátion] of a cómplex númber	Показательная форма комплексного числа	Показникова форма ком-плексного числа
10. Imáginary áxis	Мнимая ось	Уявна вісь
11. Imáginary númber	Мнимое число	Уявне число
12. Imáginary part of a cómplex númber	Мнимая часть комплекс-ного числа	Уявна частина комплек-сного числа
13. Imáginary unit, unit imáginary number	Мнимая единица	Уявна одиниця
14. Módulus (*pl* móduli) of a cómplex númber	Модуль комплексного числа	Модуль комплексного числа
15. Moivre(´s), Euler(´s) fór-mula	Формула Муавра, Эйле-ра	Формула Муавра, Ейле-ра
16. Mùltiplicátion of cóm-plex númbers	Умножение комплекс-ных чисел	Множення комплексних чисел
17. Mútual cónjugate cóm-plex númbers	Взаимно сопряжённые комплексные числа	Взаємно спряжені комп-лексні числа
18. Púre imáginary númber	Чисто мнимое число	Чисто уявне число
19. Ráise a cómplex núm-ber to a pówer	Возвести в степень ком-плексное число	Піднести до степеня [степенювати] комплек-сне число
20. Ráising of a cómplex númber to a pówer	Возведение в степень комплексного числа	Піднесення до степеня [степенювання] compле-ксного числа
21. Réal áxis, áxis of réals	Вещественная ось	Дійсна вісь

22. Réal part of a cómplex númber	Вещественная часть комплексного числа	Дійсна частина комплексного числа
23. Rèpresént a cómplex númber on the complex pláne	Изобразить комплексное число (на комплексной плоскости)	Зобразити комплексне число (на комплексній площині)
24. Rèpresentátion of a cómplex númber (on the cómplex pláne)	Изображение комплексного числа (на комплексной плоскости)	Зображення комплексного числа (на комплексній площині)
25. Subtráction of complex númbers	Вычитание комплексных чисел	Віднімання комплексних чисел
26. Trìgonométric(al) form [rèpresentátion] of a complex númber	Тригонометрическая форма комплексного числа	Тригонометрична форма комплексного числа

Polynomials

1. Àlgebráic equátion of the first [second, third, *n*th] degrée. First- [second-, third-, *n*-th] degrée àlgebráic equátion	Алгебраическое уравнение первой [второй, третьей, *n*-ой] степени	Алгебричне рівняння першого [другого, третього, *n*-го] степеня
2. Be fáctorable into [can be written as] a próduct of pòlynómials of degrée not hígher than two (of a pòlynómial)	Быть разложимым в произведение многочленов степени не выше второй (о многочлене)	Бути розкладним в добуток многочленів степеня не вище другого (про многочлен)
3. Discríminant of a quadrátic equátion [of a quadrátic]	Дискриминант квадратного уравнения	Дискримінант квадратного рівняння
4. Divísor (of the cónstant [ábsolute, free] term)	Делитель (свободного члена)	Дільник (вільного члена)
5. Fáctor [fáctorize, expánd] the pòlynómial into línear and quadrátique (with négative discríminants) fáctors	Разложить многочлен на линейные и квадратичные (с отрицательными дискриминантами) множители	Розкласти многочлен на лінійні й квадратичні (з від"ємними дискримінантами) множники
6. Fáctor [fáctorize]; expánd [dècompóse] into fáctors [compónents]; présent in a factor form	Разложить на множители	Розкласти на множники
7. Fàctorizátion/fáctoring/ expánsion of a pòlynómial (into fáctors)	Разложение многочлена на множители	Розкладання многочлена на множники
8. Guess [tell] a róot	Угадать корень	Вгадати корінь

9. Íntegerróot of a pòly-nómial with íntegercòeffí-cients [of an íntegral pò-lynómial]	Целый корень многочле-на с целыми коэффицен-тами	Цілий корінь многочле-на з цілими коефіцієн-тами
10.Léading [highest (de-grée)] còeffícient	Старший коэффициент	Старший коефіцієнт
11.Léading [highest (de-grée)] term (of a pòlynó-mial)	Старший член (много-члена)	Старший, головний [ос-новний] член (многочле-на)
12.Monic [nórmalized] pòlynómial	Многочлен с единичным старшим коэффициен-том	Многочлен з одиничним старшим коефіцієнтом
13.Múltiple [repéated] root [zéro]	Кратный корень	Кратний корінь
14.*n*-th root/zéro, *n*-tuple root/zéro, *n*-fold root/zéro	Эн-кратный (*n*-кратный) корень [*n*-кратный нуль]	Ен-кратний (*n*-кратний) корінь [*n*-кратний нуль]
15.Pòlynómial with ínte-ger còeffícients; íntegral pòlynómial	Многочлен с целыми ко-эффициентами	Многочлен з цілими кое-фіцієнтами
16.Polynomial of the first [second, third, *n*th] degrée. First- [second-, third-, *n*-th] degrée pòlynómial	Многочлен первой [вто-рой, третьей, *n*-ой] сте-пени	Многочлен першого [другого, третього, *n*-го] степеня
17.Quadrátic equátion; quadrátic; quadrátic ìn-equálity	Квадратное уравнение [квадратное неравенст-во]	Квадратне рівняння [квадратна нерівність]
18.Quadrátic trinómial	Квадратный трёхчлен	Квадратний тричлен
19.Róot [zéro] of a pòly-nómial	Корень [нуль] многочле-на	Корінь [нуль] многочле-на
20.Símple [únrepéated, single] root [zéro]	Простой [однократный] корень [нуль]	Простий [однократний, одноразовий] корінь [нуль]

Rational functions [rational fractions]

1. Entíre [integrál] ratio-nal fúnction, rátional inte-grál fúnction; pòlynómial	Целая рациональная функция (многочлен)	Ціла раціональна функ-ція, многочлен
2. Équalize [equáting, équalizátion, sétting équal, idèntificátion, idéntifying] (the numerators, the còef-fícients of équal (of the same) degrées/pówers (in	Приравнивание (числи-телей, коэффициентов при одинаковых степе-нях *x*)	Прирівнювання (чисель-ників, коефіцієнтів при однакових степенях *x*)

like pówers) of *x*)		
3. Equáte [set équal, idéntify] (the númerators, the còeffícients of équal (of the same) degrées/pówers (in like pówers) of *x*)	Приравнять (числители, коэффициенты при одинаковых степенях *x*)	Прирівняти (чисельники, коефіцієнти при однакових степенях *x*)
4. Extráct an íntegral part of the impróper fráction	Выделить целую часть неправильльной рациональной дроби	Виділити цілу частину неправильного раціонального дробу
5. Extráction of an íntegral part of an impróper fráction	Выделение целой части неправильльной рациональной дроби	Виділення цілої частини неправильного раціонального дробу
6. Fráctional rátional fúnction	Дробно-рациональная функция	Дробово-раціональна функція
7. Génerate [give, rise to, indúce] *k* (simplest) fráctions of the form … (of a fáctor of a denóminator of a próper rátional fráction)	Порождать *k* (простейших) дробей вида… (о множителе знаменателя правильной рациональной дроби)	Породжувати *k* (найпростіших) дробів вигляду... (про множник знаменника правильного раціонального дробу)
8. Génerate [give, rise to, indúce] the símplest fráction of the form … (of a fáctor of a denominator of a próper rátional fráction)	Порождать простейшую рациональную дробь вида … (о множителе знаменателя правильной рациональной дроби)	Породжувати найпростіший дріб вигляду... (про множник знаменника правильного раціональнго дробу)
9. Give/assígn (partícular) válues to the árgument	Давать/приписывать (частные) значения аргументу	Давати/надавати/приписувати (частинні) значення арґументу
10. Impróper rátional fráction	Неправильная рациональная дробь	Неправильний раціональний дріб
11. Íntegral part of the impróper rátional fráction	Целая часть неправильной рациональной дроби	Ціла частина неправильного раціонального дробу
12. Méthod of úndetérmined còeffícients	Метод неопределенных коэффициентов	Метод невизначених коефіцієнтів
13. Múltiply two members of the equálity/idéntity by the cómmon denóminator	Умножить обе части равенства/тождества на общий знаменатель	Помножити обидві частини рівності/тотожності на спільний знаменник
14. Pártial/símplest/èleméntary rátional fráction of the first (second, third, fourth) type/kind [first/second/third/fourth type ~]	Простейшая (элементарная) рациональная дробь первого (второго, третьего, четвертого) типа	Найпростіший (елементарний) раціональний дріб першого (другого, третього, четвертого) типу
15. Próper rátional fráction	Правильная рациональная дробь	Правильний раціональний дріб

16.Rátio of two pòlynómials	Отношение двух многочленов	Відношення двох многочленів
17.Rátional fráction	Рациональная дробь	Раціональний дріб
18.Rátional fúnction	Рациональная функция	Раціональна функція
19.Redúce the fráctions to the cómmon denóminator	Привести дроби к общему знаменателю	Привести/звести дроби до спільного знаменника
20.Redúction of fractions to the cómmon/sáme denóminator	Приведение дробей к общему знаменателю	Приведення/зведення дробів до спільного знаменника
21.Rèsolútion [/expánsion, devéloppement, dècomposítion] of a próper rátional fráction into (a sum of) pártial/símplest/èleméntary fráctions [partial fráction dècomposítion of...]	Разложение правильной рациональной дроби в сумму простейших (дробей)	Розвинення правильного раціонального дробу на найпростіші/елементарні (в суму найпростіших/елементарних дробів)
22.Resólve [expánd, devélop, dècompóse] a próper rational fráction into pártial/símplest/èleméntary fráctions [fulfill partial fráction dècomposítion]	Разложить правильную рациональную дробь в сумму простейших (дробей)	Розвинути правильний раціональний дріб в суму найпростіших (дробів)
23.The degrée of the númerator is (not) less than the degrée of the denóminator	Степень числителя (не) меньше степени знаменателя	Степінь чисельника (не) менше степеня знаменника

LINEAR ALGEBRA

LECTURE NO. 4. DETERMINANTS AND SYSTEMS OF LINEAR EQUATIONS

POINT 1. DETERMINANTS AND THEIR PROPERTIES

POINT 2. SYSTEMS OF N LINEAR EQUATIONS IN n UNKNOWNS. CRAMER RULE

POINT 3. ARBITRARY SYSTEMS OF LINEAR EQUATIONS. GAUSS AND JORDAN-GAUSS METHODS

POINT 4. SYSTEMS OF LINEAR HOMOGENEOUS EQUATIONS

POINT 1. DETERMINANTS AND THEIR PROPERTIES

Def(inition) 1. The second order determinant is called the next expression:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \qquad (1)$$

Here: a_{ij} is an element of the *i*-th row and *j*-th column; a_{11}, a_{12} are the elements of the first row (briefly the first row), a_{21}, a_{22} are those of the second row (the second row), a_{11}, a_{21}, a_{12}, a_{22} are those of the first, correspondingly second column (the first and second columns); a_{11}, a_{22} are the elements of leading [main, principal] diagonal (leading [main, principal] diagonal), a_{12}, a_{21} are those of secondary diagonal (secondary diagonal).

Ex. 1.

$$\begin{vmatrix} 2 & 3 \\ -4 & 5 \end{vmatrix} = 2 \cdot 5 - 3 \cdot (-4) = 10 + 12 = 22.$$

Def. 2. The third order determinant is called the next expression:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{23}a_{32}a_{11} \qquad (2)$$

Summands of the right side of the formula (2) can be calculated with the help of the next scheme

«+» «-»

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \qquad \begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Ex. 2. Calculate the third order determinant

$$\Theta = \begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 4 & -2 \end{vmatrix} = \left\{ \begin{matrix} -1 & 0 & 3 & -1 & 0 \\ 2 & 1 & 0 & 2 & 1 \\ 0 & 4 & -2 & 0 & 4 \end{matrix} \right\} = (-1)\cdot 1\cdot(-2) + 0\cdot 0\cdot 0 + 3\cdot 2\cdot 4 -$$

$$-3\cdot 1\cdot 0 - (-1)\cdot 0\cdot 4 - 0\cdot 2\cdot(-2) = 2 + 0 + 24 - 0 - 0 - 0 = 26$$

By special rule one can define n-th order determinant (n = 4, 5,…).

Def. 3. Transposition of a determinant is called its transform(ation) such that every its row becomes a column with the same number (and vice versa).

Result of transposition of a determinant D we'll denote by D^{T}.

Ex. 3 (see Ex. 2).

$$\Theta^T = \begin{vmatrix} -1 & 2 & 0 \\ 0 & 1 & 4 \\ 3 & 0 & -2 \end{vmatrix} = \left\{ \begin{matrix} -1 & 2 & 0 & -1 & 0 \\ 0 & 1 & 4 & 0 & 4 \\ 3 & 0 & -2 & 3 & -2 \end{matrix} \right\} = (-1)\cdot 1\cdot(-2) + 2\cdot 4\cdot 3 + 0\cdot 0\cdot(-2) -$$

$$-0\cdot 1\cdot 3 - (-1)\cdot 4\cdot 0 - 0\cdot 0\cdot(-2) = 2 + 24 + 0 - 0 - 0 - 0 = 26$$

Def. 4. Minor M_{ij} of the dement a_{ij} of a determinant is called a determinant obtained from given one by crossing out of the i-th row and the j-th column.

Ex. 4. Minors of all the elements of the determinant Θ (see Ex. 2)

$$\Theta = \begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 4 & -2 \end{vmatrix} \quad \begin{array}{lll} M_{11} = \begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} = -2 & M_{12} = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 & M_{13} = \begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} = 8 \\ M_{21} = \begin{vmatrix} 0 & 3 \\ 4 & -2 \end{vmatrix} = -12 & M_{22} = \begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix} = 2 & M_{23} = \begin{vmatrix} -1 & 0 \\ 0 & 4 \end{vmatrix} = -4 \\ M_{31} = \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} = -3 & M_{32} = \begin{vmatrix} -1 & 3 \\ 2 & 0 \end{vmatrix} = -6 & M_{33} = \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} = -1 \end{array}$$

Def. 5. Cofactor [signed minor, algebraic complement, algebraic supplement, algebraic adjunct] A_{ij} of the element a_{ij} of a determinant is called the minor of this element taken with the sign $(-1)^{i+j}$ that with the same sign if $i+j$ is even number and with opposite sign otherwise

$$A_{ij} = (-1)^{i+j} M_{ij} = \begin{cases} +M_{ij} \text{ if } i+j \text{ is even number,} \\ -M_{ij} \text{ if } i+j \text{ is odd number} \end{cases}$$

Ex. 5. Cofactors of all the elements of the same determinant Θ are

$$\Theta = \begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 4 & -2 \end{vmatrix} \quad \begin{matrix} A_{11} = M_{11} = -2 & A_{12} = -M_{12} = 4 & A_{13} = M_{13} = 8 \\ A_{21} = -M_{21} = 12 & A_{22} = M_{22} = 2 & A_{23} = -M_{23} = 4 \\ A_{31} = M_{31} = -3 & A_{32} = -M_{32} = 6 & A_{33} = M_{33} = -1 \end{matrix}$$

PROPERTIES OF DETERMINANTS

1. Determinant D doesn't change after transposition that is $D^{\mathrm{T}} = D$.

For example we've seen in examples 2, 3 that $\Theta^T = \Theta$.

The property states equal status of rows and columns of a determinant. It means that every property of a row is valid for a column and vice versa. By this reason we'll name a row or a column of a determinant by common term a **series**.

2. Determinant with zero series (that is with zero row or column) equals zero.

For example

$$\begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & 4 \\ 3 & 1 & 6 \end{vmatrix} = \begin{Bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 1 & 4 \\ 3 & 1 & 6 & 3 & 6 \end{Bmatrix} = 0\cdot 2\cdot 6 + 0\cdot 4\cdot 3 + 0\cdot 1\cdot 6 - 0\cdot 2\cdot 3 - 0\cdot 4\cdot 1 - 0\cdot 1\cdot 6 = 0$$

3. Interchanging of two series of a determinant changes its sign but its modulus.

Let's for example interchange the second and the third rows in the determinant Θ from the example 2:

$$\begin{vmatrix} -1 & 0 & 3 \\ 0 & 4 & -2 \\ 2 & 1 & 0 \end{vmatrix} = \begin{Bmatrix} -1 & 0 & 3 & -1 & 0 \\ 0 & 4 & -2 & 0 & 4 \\ 2 & 1 & 0 & 2 & 1 \end{Bmatrix} = (-1)\cdot 4\cdot 0 + 0\cdot(-2)\cdot 2 + 3\cdot 0\cdot 1 -$$

$$-3\cdot4\cdot2-(-1)\cdot(-2)\cdot1-0\cdot0\cdot0=0+0+0-24-2-0=-26=-\Theta$$

4. Determinant with two equal series is equal to zero.

For example a determinant with two equal columns

$$\begin{vmatrix}3&2&3\\1&5&1\\4&1&4\end{vmatrix}=\begin{Bmatrix}3&2&3&3&2\\1&5&1&1&5\\4&1&4&4&1\end{Bmatrix}=3\cdot5\cdot4+2\cdot1\cdot4+3\cdot1\cdot1-3\cdot5\cdot4-3\cdot1\cdot1-2\cdot1\cdot4=0$$

5. If all elements of some series of a determinant have a common factor then it can be taken out the determinant sign

For example

$$\begin{vmatrix}-4&0&3\\8&1&0\\0&4&-2\end{vmatrix}=\begin{vmatrix}4\cdot(-1)&0&3\\4\cdot2&1&0\\4\cdot0&4&-2\end{vmatrix}=104\quad\text{and}\quad4\cdot\begin{vmatrix}-1&0&3\\2&1&0\\0&4&-2\end{vmatrix}=4\cdot\Theta=4\cdot26=104.$$

6. If all elements of some series of determinant are sums of two summands, then this determinant equals the sum of two determinants: the first contains the first summands, the second – the seconds summands in corresponding series and the other elements of both determinants are the same as in the given determinant.

For example

$$\begin{vmatrix}2+(-3)&3+5&-4+1\\3&6&-2\\0&4&1\end{vmatrix}=\begin{vmatrix}-1&8&-3\\3&6&-2\\0&4&1\end{vmatrix}=-74,\ \begin{vmatrix}2&3&-4\\3&6&-2\\0&4&1\end{vmatrix}+\begin{vmatrix}-3&5&1\\3&6&-2\\0&4&1\end{vmatrix}=-74$$

7. If we add all elements of one series of a determinant multiplied by any number to corresponding elements of the other series, the determinant will not be changed.

For example

$$\begin{vmatrix}2&-3&1\\5&6&8\\4&-6&3\end{vmatrix}=27\ \text{and}\ \begin{vmatrix}2&-3&1\\5&6&8\\4+2\cdot(-2)&-6+(-3)\cdot(-2)&3+1\cdot(-2)\end{vmatrix}=\begin{vmatrix}2&-3&1\\5&6&8\\0&0&1\end{vmatrix}=27$$

We've added elements of the first row multiplied by (-2) to corresponding elements of the third row.

This property is often called "creator of zero".

8 (expansion of a determinant with respect to a series). Determinant equals sum of products of elements of any series by their cofactors.

Expansion of the n-th order determinant D

$$D=\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

with respect to the i-th row is

$$D=a_{i1}\cdot A_{i1}+a_{i2}\cdot A_{i2}+a_{i3}\cdot A_{i3}+\ldots+a_{in}\cdot A_{in}=\sum_{k=1}^{n}a_{ik}\ .$$

Its expansion with respect to the j-th column is

$$D=a_{1j}\cdot A_{1j}+a_{2j}\cdot A_{2j}+a_{3j}\cdot A_{3j}+\ldots+a_{nj}\cdot A_{nj}=\sum_{k=1}^{n}a_{kj}$$

Ex. 6. Expansions of the determinant Θ (Ex. 2, 5) with respect to the first and second rows and with respect to the second and third columns

$$\Theta=\begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 0 \\ 0 & 4 & -2 \end{vmatrix}$$

$$\Theta=(-1)\cdot A_{11}+0\cdot A_{12}+3\cdot A_{13}=(-1)\cdot(-2)+0\cdot 4+3\cdot 8=26$$
$$\Theta=2\cdot A_{21}+1\cdot A_{22}+0\cdot A_{23}=2\cdot 12+1\cdot 2+0\cdot 4=26$$
$$\Theta=0\cdot A_{12}+1\cdot A_{22}+4\cdot A_{32}=0\cdot 4+1\cdot 2+4\cdot 6=26$$
$$\Theta=3\cdot A_{13}+0\cdot A_{23}+(-2)\cdot A_{33}=3\cdot 8+0\cdot 4+(-2)\cdot(-1)=26$$

Ex. 7. Calculation of determinant which we've said in the property 7 about

$$\begin{vmatrix} 2 & -3 & 1 \\ 5 & 6 & 8 \\ 4 & -6 & 3 \end{vmatrix}=\begin{vmatrix} 2 & -3 & 1 \\ 5 & 6 & 8 \\ 4+2\cdot(-2) & -6+(-3)\cdot(-2) & 3+1\cdot(-2) \end{vmatrix}=\begin{vmatrix} 2 & -3 & 1 \\ 5 & 6 & 8 \\ 0 & 0 & 1 \end{vmatrix}=$$
$$=0\cdot A_{31}+0\cdot A_{32}+1\cdot A_{33}=A_{33}=M_{33}=\begin{vmatrix} 2 & -3 \\ 5 & 6 \end{vmatrix}=27\ .$$

9 (Lagrange[1] theorem). Sum of products of element of one series by cofactors of corresponding elements of the other series is equal to zero.

[1] Lagrange, J.L. (1736 - 1813), an outstanding French mathematician and astronomer

For example for the determinant Θ (Ex. 2, 5, 6)

$$a_{11}\cdot A_{12}+a_{21}\cdot A_{22}+a_{31}\cdot A_{32}=(-1)\cdot A_{12}+2\cdot A_{22}+0\cdot A_{32}=(-1)\cdot 4+2\cdot 2+0\cdot 6=0.$$

Ex. 8. Calculate the fourth order determinant

$$D=\begin{vmatrix} 2 & -2 & 0 & 1 \\ 2 & 3 & 1 & -3 \\ 3 & 4 & -1 & 2 \\ 1 & 3 & 1 & -1 \end{vmatrix}.$$

The first step. We transform the determinant so that to annul three elements in some its row or column. For example we'll add the fourth column multiplied by – 2 to the first column and multiplied by 2 to that second. By virtue of the property 7 of determinants

$$D=\begin{vmatrix} 0 & 0 & 0 & 1 \\ 8 & -3 & 1 & -3 \\ -1 & 8 & -1 & 2 \\ 3 & 1 & 1 & -1 \end{vmatrix}.$$

The second step. We expand the transformed determinant with respect to the first row which contains three zeros, namely

$$D=0\cdot A_{11}+0\cdot A_{12}+0\cdot A_{13}+1\cdot A_{14}=A_{14}=-M_{14}=$$

$$=-\begin{vmatrix} 8 & -3 & 1 \\ -1 & 8 & -1 \\ 3 & 1 & 1 \end{vmatrix}=\left\{\begin{matrix} 8 & -3 & 1 & 8 & -3 \\ -1 & 8 & -1 & -1 & 8 \\ 3 & 1 & 1 & 3 & 1 \end{matrix}\right\}=-53.$$

It's better to calculate $A_{14}=-M_{14}$ generating zeros in some row or column. For example,

$$A_{14}=-M_{14}=-\begin{vmatrix} 8 & -3 & 1 \\ -1 & 8 & -1 \\ 3 & 1 & 1 \end{vmatrix}=-\begin{vmatrix} 8-1 & -3+8 & 1-1 \\ -1 & 8 & -1 \\ 3-1 & 1+8 & 1-1 \end{vmatrix}=-\begin{vmatrix} 7 & 5 & 0 \\ -1 & 8 & -1 \\ 2 & 9 & 0 \end{vmatrix}=-(-1)A_{23}=$$

$$=-M_{23}=-\begin{vmatrix} 7 & 5 \\ 2 & 9 \end{vmatrix}=-53.$$

POINT 2. SYSTEMS OF n LINEAR EQUATIONS IN n UNKNOWNS. CRAMER RULE.

Let there be given a system of *m* linear algebraical equations (SLAE) in *n* unknowns $x_1, x_2, x_3, \ldots, x_n$ (SLAE $m \times n$)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m, \end{cases} \quad \text{or} \quad \sum_{j=1}^{n} a_{ij}x_j = b_i,\, i = \overline{1, m} \qquad (7)$$

Here $a_{ij},\, i = \overline{1,m},\ j = \overline{1,n}$ be coefficients, $b_i,\, i = \overline{1,m}$ be free [absolute] terms.

Def. 6. Solution of a system of linear algebraical equations (SLAE) in *n* unknowns is called an ordered set of *n* numbers $(x_{10}, x_{20}, \ldots, x_{n0})$ which satisfies every equation of the system that is turns each its equation into exact equality.

In the case of two unknowns x_1, x_2 we say about ordered pair (x_{10}, x_{20}) of numbers and in the case of three unknowns x_1, x_2, x_3 about ordered triplet of numbers (x_{10}, x_{20}, x_{30}).

Def. 7. SLAE is called compatible [consistent, solvable, decidable] one if it possesses at least one solution and non-compatible [inconsistent, non-solvable, undecidable] otherwise.

Def. 8. Compatible SLAE is called that determined [determinate, defined] one if it has only one solution and undetermined [indeterminate, undefined] otherwise.

In this point we'll study only the case *m* = *n* when the number of equations coincides with the number of unknowns. The general case see in the next points.

Let to fix the idea *m* = *n* = 3 that is we deal with a system (SLAE 3×3)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3. \end{cases} \quad \begin{array}{|c|c|c} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{array} \qquad (8)$$

Let's introduce the next determinants: principal [main, system] determinant, which is formed by coefficients of unknowns

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot A_{11} + a_{21} \cdot A_{21} + a_{31} \cdot A_{31}, \qquad (9)$$

and three auxiliary determinants

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = b_1 \cdot A_{11} + b_2 \cdot A_{21} + b_3 \cdot A_{31}; \qquad (10)$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}; \ \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}. \qquad (11)$$

One gets Δ_1 (Δ_2, Δ_3) substituting the first (second, third) column by the column of free terms in Δ. We've expanded Δ and Δ_1 with respect to the first column to prove one of cases of the next theorem.

Theorem 1 (Cramer's[1] rule). If the main determinant Δ isn't zero then the system (8) is compatible and determined one and it's solution is given by the next formulas:

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad x_3 = \frac{\Delta_3}{\Delta} \qquad (12)$$

■ (Proving). Let's multiply the equations of the system (8) by A_{11}, A_{21}, A_{31} correspondingly and termwise add:

$$\left(\underbrace{a_{11} \cdot A_{11} + a_{21} \cdot A_{21} + a_{31} \cdot A_{31}}_{=\Delta}\right) \cdot x_1 + \left(\underbrace{a_{12} \cdot A_{11} + a_{22} \cdot A_{21} + a_{32} \cdot A_{31}}_{=0}\right) \cdot x_2 +$$

$$+\left(\underbrace{a_{13} \cdot A_{11} + a_{23} \cdot A_{21} + a_{33} \cdot A_{31}}_{=0}\right) \cdot x_3 = \underbrace{b_1 \cdot A_{11} + b_2 \cdot A_{21} + b_3 \cdot A_{31}}_{=\Delta_1}$$

In this equality the coefficient of x_1 is equal to Δ and the right member equals Δ_1 by the 8-th property of determinants. Coefficients of x_1 and x_2 are equal to zero by the 9-th property. Therefore, we get (two last equalities are obtained analogically)

[1] Cramer, G. (1704 - 1752), a Swiss mathematician

$$\Delta\cdot x_1=\Delta_1,\quad \Delta\cdot x_2=\Delta_2,\quad \Delta\cdot x_3=\Delta_3 \qquad (13)$$

In the case $\Delta\neq 0$ we get the formulas (12) which mean that the solution of the system is unique one if it exists. By testing we can prove that the formulas (12) indeed represent the solution. ■ (which it was required to be proved; q.e.d. - quod erat demonstrandum *lat*).

Ex. 8. Solve a SLAE 3×3

$$\begin{cases} 6x_1+2x_2-5x_3=1, \\ x_1+3x_2+2x_3=2, \\ x_1+4x_2-x_3=5. \end{cases}$$

Solving. Principal and auxiliary determinants are equal to

$$\Delta=\begin{vmatrix} 6 & 2 & -5 \\ 1 & 3 & 2 \\ 1 & 4 & -1 \end{vmatrix}=\begin{vmatrix} 6+(-6)\cdot 1 & 2+(-6)\cdot 3 & -5+(-6)\cdot 2 \\ 1 & 3 & 2 \\ 1+(-1)\cdot 1 & 4+(-1)\cdot 3 & -1+(-1)\cdot 2 \end{vmatrix}=\begin{vmatrix} 0 & -16 & -17 \\ 1 & 3 & 2 \\ 0 & 1 & -3 \end{vmatrix}=$$

$$=0\cdot A_{11}+1\cdot A_{21}+0\cdot A_{31}=A_{21}=-M_{21}=-\begin{vmatrix} -16 & -17 \\ 1 & -3 \end{vmatrix}=-65$$

$$\Delta_1=\begin{vmatrix} 1 & 2 & -5 \\ 2 & 3 & 2 \\ 5 & 4 & -1 \end{vmatrix}=48;\quad \Delta_2=\begin{vmatrix} 6 & 1 & -5 \\ 1 & 2 & 2 \\ 1 & 5 & -1 \end{vmatrix}=84;\quad \Delta_3=\begin{vmatrix} 6 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 4 & 5 \end{vmatrix}=37.$$

Therefore

$$x_1=\frac{\Delta_1}{\Delta}=-\frac{48}{65};\quad x_2=\frac{\Delta_2}{\Delta}=\frac{84}{65};\quad x_3=\frac{\Delta_3}{\Delta}=\frac{37}{65}.$$

Ex. 9. Investigate a system of equations

$$\begin{cases} 3x_1+4x_2-x_3=-3, \\ 2x_1-x_2+ax_3=11, \\ 5x_1+3x_2+3x_3=b \end{cases}$$

depending on values of parameters *a*, *b*.

Principal determinant of the system equals $\Delta=11(a-4)$, and the system has unique solution if $a\neq 4$.

For $a=4$ the system takes on the form

$$\begin{cases} 3x_1 + 4x_2 - x_3 = -3, \\ 2x_1 - x_2 + 4x_3 = 11, \\ 5x_1 + 3x_2 + 3x_3 = b \end{cases}$$

with auxiliary determinants $\Delta_1 = 15(b-8), \Delta_2 = 14(b-8), \Delta_3 = 11(8-b)$. If $b \neq 8$ the system is non-compatible one because of the equalities (13). For $b = 8$ (and $a = 4$) we have the system

$$\begin{cases} 3x_1 + 4x_2 - x_3 = -3, \\ 2x_1 - x_2 + 4x_3 = 11, \\ 5x_1 + 3x_2 + 3x_3 = 8. \end{cases}$$

Its third equation can be omitted because of it is the sum of two first equations, and we get

$$\begin{cases} 3x_1 + 4x_2 - x_3 = -3, \\ 2x_1 - x_2 + 4x_3 = 11, \end{cases} \text{ or } \begin{cases} 3x_1 + 4x_2 = x_3 - 3, \\ 2x_1 - x_2 = 11 - 4x_3, \end{cases}$$

whence

$$\Delta = \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = -11, \Delta_1 = \begin{vmatrix} x_3 - 3 & 4 \\ 11 - 4x_3 & -1 \end{vmatrix} = 15x_3 - 41, \Delta_2 = \begin{vmatrix} 3 & x_3 - 3 \\ 2 & 11 - 4x_3 \end{vmatrix} = 39 - 14x_3,$$

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{41 - 15x_3}{11}, x_2 = \frac{\Delta_2}{\Delta} = \frac{14x_3 - 39}{11}.$$

Assigning to x_3 arbitrary values $x_3 = c$, we find infinitely many solutions of the system, namely $\left(\frac{41 - 15c}{11}, \frac{14c - 39}{11}, c \right)$.

Thus the system is compatible determined for $a \neq 4$, compatible undetermined for $a = 4, b = 8$ and non-compatible for $a = 4, b \neq 8$.

Ex. 10. A company produces items of three types A_1, A_2, A_3 utilizing raw stuff of three kinds S_1, S_2, S_3. It's known norms for consumption of every kind of raw stuff per one item of every type and the general consumption of raw stuff of every kind per diem (see the table 1). Find the daily volume of output of every type of items.

Let the company produces daily x_1, x_2, x_3 items of the first, second, third types respectively. By conditions of the problem we get a system of linear equations in unknowns x_1, x_2, x_3

$$\begin{cases} 5x_1 + 3x_2 + 4x_3 = 2700, \\ 2x_1 + x_2 + x_3 = 900, \\ 3x_1 + 2x_2 + 2x_3 = 1600. \end{cases}$$

Table 1

Kind of a raw stuff	Norms for consumption of a raw stuff per one item (in c.u.[1])			Consumption of a raw stuff per diem (in c.u.)
	A_1	A_2	A_3	
S_1	5	3	4	2700
S_2	2	1	1	900
S_3	3	2	2	1600

Principal and auxiliary determinants of the system equal

$$\Delta = 1, \Delta_1 = 200, \Delta_2 = 300, \Delta_3 = 200,$$

and therefore $x_1 = \frac{\Delta_1}{\Delta} = 200$, $x_2 = \frac{\Delta_2}{\Delta} = 300$, $x_3 = \frac{\Delta_3}{\Delta} = 200$.

POINT 3. ARBITRARY SYSTEMS OF LINEAR EQUATIONS. GAUSS AND JORDAN-GAUSS METHODS

Let there be given a system (7) of m linear algebraic equations in n unknowns. There are two general methods of its solving, namely those of Gauss and Jordan-Gauss.

Gauss[2] **method** (method of **successive exclusion** of unknowns).

1. The first step (exclusion of one unknown). Let $a_{11} \neq 0$ (otherwise we can interchange two equations or unknowns). We divide the first equation by a_{11} and get

---

[1] In c.u. – in conventional units

[2] Gauss, K.F. (1777 - 1855), a great German mathematician, astronomer, physicist, and land-surveyor

$$\begin{cases} x_1 + a'_{12}x_2 + a'_{13}x_3 + \ldots + a'_{1n}x_n = b'_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m, \end{cases}$$

where $a'_{12} = a_{12}/a_{11}$, $a'_{13} = a_{13}/a_{11}, \ldots, a'_{1n} = a_{1n}/a_{11}$, $b'_1 = b_1/a_{11}$. Then we add the first equation, multiplied by $-a_{21}, -a_{31}, \ldots, -a_{m1}$, to the second, third, …, *n*th equations respectively excluding x_1 in these equations and obtaining the system of the form

$$\begin{cases} x_1 + a'_{12}x_2 + a'_{13}x_3 + \ldots + a'_{1n}x_n = b'_1, \\ a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a'_{m2}x_2 + a'_{m3}x_3 + \ldots + a'_{mn}x_n = b'_m. \end{cases}$$

Here $a'_{i2} = a_{i2} - a_{i1}a'_{12}$, $a'_{i3} = a_{i3} - a_{i1}a'_{13}, \ldots, a'_{in} = a_{in} - a_{i1}a'_{1n}$, $b'_i = b_i - a_{i1}b'_1$, $i = \overline{2, m}$.

2. The second step (exclusion of the other unknown). Let $a'_{22} \neq 0$ (otherwise after interchanging two equations or unknowns). By the same way we exclude x_2 from the third, fourth, …, *m*th equations obtaining a system of the form

$$\begin{cases} x_1 + a'_{12}x_2 + a'_{13}x_3 + \ldots + a'_{1n}x_n = b'_1, \\ x_2 + a''_{23}x_3 + \ldots + a''_{2n}x_n = b''_2, \\ a''_{33}x_3 + \ldots + a''_{3n}x_n = b''_{32}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a''_{m3}x_3 + \ldots + a''_{mn}x_n = b''_m, \end{cases}$$

and so on.

After some steps we can get one of three cases.

1) Some equation takes on the form $0 = d_k$, $d_k \neq 0$. It means that the system (7) is non-compatible one

2) System (7) is reduced to a triangular form

$$\begin{cases} x_1 + a'_{12}x_2 + a'_{13}x_3 + \ldots + a'_{1n}x_n = b'_1, \\ x_2 + a''_{23}x_3 + \ldots + a''_{2n}x_n = b''_2, \\ x_3 + \ldots + a'''_{3n}x_n = b'''_{32}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n = d_n. \end{cases}$$

In this case we step by step find $x_n, x_{n-1}, x_{n-2},..., x_3, x_2, x_1$, and the system (7) is compatible determined one.

3) System (7) is reduced to a trapeziumform

$$\begin{cases} x_1 + a'_{12}x_2 + a'_{13}x_3 + ... + a'_{1k}x_k + a'_{1,k+1}x_{k+1} + ... + a'_{1n}x_n = b'_1, \\ x_2 + a''_{23}x_3 + ... + a''_{2k}x_k + a''_{2,k+1}x_{k+1} + ... + a''_{2n}x_n = b''_2, \\ x_3 + ... + a'''_{3k}x_k + a'''_{3,k+1}x_{k+1} + ... + a'''_{3n}x_n = b'''_{32}, \\ .... \\ x_k + c_{k,k+1}x_{k+1} + ... + c_{kn}x_n = d_k. \end{cases}$$

In this case the system is compatible undetermined one. We regard $x_{k+1},..., x_n$ as free unknowns, assign them arbitrary values ($x_{k+1} = \lambda_{k+1},..., x_n = \lambda_n$) and write the system in the form

$$\begin{cases} x_1 + a'_{12}x_2 + a'_{13}x_3 + ... + a'_{1k}x_k = b'_1 - \left(a'_{1,k+1}\lambda_{k+1} + ... + a'_{1n}\lambda_n\right), \\ x_2 + a''_{23}x_3 + ... + a'_{2k}x_k = b''_2 - \left(a'_{2,k+1}\lambda_{k+1} + ... + a''_{2n}\lambda_n\right), \\ x_3 + ... + a'_{3k}x_k = b'''_{32} - \left(a'_{3,k+1}\lambda_{k+1} + ... + a'''_{3n}\lambda_n\right), \\ .... \\ x_k = d_k - \left(c_{k,k+1}\lambda_{k+1} + ... + c_{kn}\lambda_n\right). \end{cases}$$

Now we step by step express $x_k, x_{k-1}, x_{k-2},..., x_3, x_2, x_1$ in terms of λ_{k+1}, $\lambda_{k+2},..., \lambda_n$.

Jordan[1]-Gauss method

According to this method, which is a simple modification of Gauss method, we exclude unknowns x_i from all equations excepting the *i*th one.

If $a_{11} \neq 0$, the first step of the method gives the same system as in Gauss one. If $a'_{22} \neq 0$, the second step leads to the system of the form

$$\begin{cases} x_1 \qquad + a''_{13}x_3 + ... + a''_{1n}x_n = b''_1, \\ x_2 + a''_{23}x_3 + ... + a''_{2n}x_n = b''_2, \\ a''_{33}x_3 + ... + a''_{3n}x_n = b''_{32}, \\ .... \\ a''_{m3}x_3 + ... + a''_{mn}x_n = b''_m, \end{cases}$$

[1] Jordan, M.E.C. (1838 - 1922), a French mathematician

with excluded x_2 in all equations excepting the second one, and so on. As result of several steps we get one of three cases which are represented in very simple form.

1) The case of incompatibility when some one of obtained equations has a form $0 = d_k$, $d_k \neq 0$.

2) The case of compatibility and definiteness when the system is reduces to the triangular form

$$\begin{cases} x_1 \qquad\qquad\qquad = d_1, \\ \qquad x_2 \qquad\qquad = d_2, \\ \qquad\qquad x_3 \qquad = d_3, \\ \dots\dots\dots\dots\dots\dots\dots \\ \qquad\qquad\qquad x_n = d_n. \end{cases}$$

We at once obtain unique solution of the system (7), namely $(d_1, d_2, ..., d_n)$.

3) The case of compatibility and undeterminedness when we come to a system of the trapezoidal form

$$\begin{cases} x_1 \qquad\qquad\qquad + c_{1,k+1}x_{k+1} + ... + c_{1n}x_n = d_1, \\ \qquad x_2 \qquad\qquad + c_{2,k+1}x_{k+1} + ... + c_{2n}x_n = d_2, \\ \qquad\qquad x_3 \qquad + c_{3,k+1}x_{k+1} + ... + c_{3n}x_n = d_3, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \qquad\qquad\qquad x_k + c_{k,k+1}x_{k+1} + ... + c_{kn}x_n = d_k. \end{cases}$$

Putting $x_{k+1} = \lambda_{k+1}, ..., x_n = \lambda_n$ we at once find the values of $x_1, x_2, ..., \quad x_k$ in terms of $\lambda_{k+1}, \lambda_{k+2}, ..., \lambda_n$.

Note 1. We can begin Gauss (or Jordan-Gauss) method from any unknown. Its coefficient is called **leading** [solving, resolving] one.

Note 2. All operations of both methods concern only to coefficients and absolute terms. Consequently we can write only those (using corresponding tables).

Ex. 11. Solve a system of linear equations by Jordan-Gauss method

$$\begin{cases} 3x_1 - x_2 + 2x_3 = -4, \\ 6x_1 + x_2 - 3x_3 = 2, \\ 5x_1 - 2x_2 + x_3 = 4. \end{cases}$$

We represent the system and process of its solving by the next sequence of tables (with corresponding comments below).

x_1	x_2	x_3	
3	**-1**	2	-4
6	1	-3	2
5	-2	1	4

x_1	x_2	x_3	
3	-1	2	-4
9	0	-1	-2
-1	0	-3	12

x_1	x_2	x_3	
0	-1	-7	32
0	0	-28	106
-1	0	-3	12

x_1	x_2	x_3	
0	1	7	-32
0	0	**1**	$-53/14$
1	0	3	-12

x_1	x_2	x_3	
0	1	0	$-11/2$
0	0	1	$-53/14$
1	0	0	$-9/14$

1. The first step. We choose the coefficient $a_{12} = -1$ as leading one in the first table, and we add the elements of the first row multiplied by 1, -2 to the second and third rows respectively. As result we get the second table.

2. The second step. In the second table we take the leading coefficient $a_{31} = -1$ and add the third row multiplied by 3, 9 to the first and second rows respectively. The result is in the third table or (after multiplying of the first and third its rows by -1 and dividing the second row by -28) in the fourth table.

3. The third step. In the fourth table we take the leading coefficient $a_{23} = 1$ and add the second row multiplied by -7, -3 to the first and third rows respectively. The last fifth table represents the solution of the system, namely

$$x_1 = -9/11, x_2 = -11/2, x_3 = -53/14.$$

Ex. 12. Solve a system of linear equations by Gauss and Jordan-Gauss methods

$$\begin{cases} 2x - 2y \quad + t & = & -3, \\ 2x + 3y + z - 3t & = & -6, \\ 3x + 4y - z + 2t & = & 0, \\ x + 3y + z - t & = & 2. \end{cases}$$

1. Solving the system by Gauss method.

The first step. We choose the coefficient $a_{41}=1$ as leading one in the table 1, and we add the elements of the fourth row multiplied by - 2 to the first and second rows and multiplied by - 3 to the third row. We get the table 2. Then we add the elements of the second row multiplied by - 3 to the first row and obtain the table 3.

1.

x	*y*	*z*	*t*	
2	-2	0	1	-3
2	3	1	-3	-6
3	4	-1	2	0
1	3	1	-1	2

2.

x	*y*	*z*	*t*	
0	-8	-2	3	-7
0	-3	-1	-1	-10
0	-5	-4	5	-6
1	3	1	-1	2

3.

x	*y*	*z*	*t*	
0	**1**	1	6	23
0	3	1	1	10
0	-5	-4	5	-6
1	3	1	-1	2

The second step. Choosing as the leading coefficient $a_{12}=1$ in the table 3 we add the elements of the first row multiplied by - 3 and 5 to the second and third rows respectively (see the table 4).

The third step. We add the elements of the third row multiplied by 2 to the second row (the leading coefficient is $a_{33}=1$). As result we get the table 5 or, after division of the third row by 53, the table 6.

4.

x	*y*	*z*	*t*	
0	1	1	6	23
0	0	-2	-17	-59
0	0	**1**	35	109
1	3	1	-1	2

5.

x	*y*	*z*	*t*	
0	1	1	6	23
0	0	0	53	159
0	0	1	35	109
1	3	1	-1	2

6.

x	*y*	*z*	*t*	
0	1	1	6	23
0	0	0	1	3
0	0	1	35	109
1	3	1	-1	2

The final form of the system is

$$\begin{cases} y+z+6t=23, \\ t=3, \\ z+35t=109, \\ x+3y+z-t=2, \end{cases}$$

whence it follows that

$$t=3, z=109-35t=4, y=23-z-6t=1, \quad x=2-3y-z+t=-2.$$

Answer: $\{-2; 1; 4; 3\}$.

2. Now we'll solve the system by Jordan-Gauss method.

The first step is the same as in Gauss method.

The second step. We fulfil preceding operations of Gauss method and in addition add the elements of the first row multiplied by - 3 to the fourth row. We get the table 4 of the other form then above.

The third step. Using the table 4 we add the elements of the third row multiplied by - 1 to the first row and multiplied by 2 to the second and fourth rows (the leading coefficient is $a_{33} = 1$). We obtain the table 5 or, after division of the second row by 53, the table 6.

4

x	y	z	t	
0	1	1	6	23
0	0	-2	-17	-59
0	0	**1**	35	109
1	0	-2	-19	-67

5

x	y	z	t	
0	1	0	-29	-86
0	0	0	53	159
0	0	1	35	109
1	0	0	51	151

6

x	y	z	t	
0	1	0	-29	-86
0	0	0	**1**	3
0	0	1	35	109
1	0	0	51	151

The fourth step. Adding the elements of the second row multiplied by 29, - 35, - 51 respectively to the first, third and fourth rows we come to the last table

x	y	z	t	
0	1	0	0	1
0	0	0	1	3
0	0	1	0	4
1	0	0	0	-2

which gives $x = -2, y = 1, z = 4, t = 3$ that is the same result as in Gauss method.

Ex. 13. Solve the system of linear algebraic equations

$$\begin{cases} 2x + y - x - t + u & = & 1, \\ x - y + z + t - 2u & = & 0, \\ 3x + 3y - 3z - 3t + 4u & = & 2, \\ 4x + 5y - 5z - 5t + 7u & = & 3. \end{cases}$$

The first step. The system in question is represented in the first of three tables. We add the second row multiplied by – 2, - 3, - 4 to the first, third, fourth rows correspondingly and obtain the second table. Dividing the third and fourth rows by 2, 3 respectively we get the third table with the same first, third and fourth rows.

The second step. Adding the first row multiplied by – 1 to the third and fourth rows we annul them and reduce the given system to that with two equations

x	y	z	t	u	
2	1	-1	-1	1	1
1	-1	1	1	-2	0
3	3	-3	-3	4	2
4	5	-5	-5	7	3

x	y	z	t	u	
0	3	-3	-3	5	1
1	-1	1	1	-2	0
0	6	-6	-6	10	2
0	9	-9	-9	15	3

x	y	z	t	u	
0	3	-3	-3	5	1
1	-1	1	1	-2	0
0	3	-3	-3	5	1
0	3	-3	-3	5	1

$$\begin{cases} 3y-3z-3t+5u = 1, \\ x-y+z+t-2u = 0. \end{cases}$$

In this latter we consider z, t, u as free unknowns and transpose them to the right,

$$\begin{cases} 3y = 1+3z+3t-5u, \\ x-y = -z-t+2u. \end{cases}$$

Finely we express y from the first equation and then x from the second one in terms of the free unknowns,

$$y=\frac{1+3z+3t-5u}{3},\ x=y-z-t+2u=\frac{1+u}{3}.$$

Answer. The system has infinitely many solutions

$$x=\frac{1+u}{3},\ y=\frac{1+3z+3t-5u}{3},\ \text{where } z, t, u \text{ are arbitrary numbers}.$$

POINT 4. SYSTEMS OF LINEAR HOMOGENEOUS EQUATIONS

Def. 9. If in a system (7) all absolute terms are equal to zero $\left(b_i=0, i=\overline{1,m}\right)$, then it's called a **system of linear homogeneous equations** (SLHE) or homogeneous system of linear equations,

$$\begin{cases} a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots+a_{1n}x_n=0, \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\ldots+a_{2n}x_n=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+\ldots+a_{mn}x_n=0, \end{cases} \qquad (8)$$

System of linear homogeneous equations (8) is always compatible one because of it always has the trivial [zero] solution $x_1 = x_2 = x_3 = \ldots = x_n = 0$. The main problem consists in follows: in what conditions there are nontrivial solutions of the system.

Theorem 2. If in a system of linear homogeneous equations (8) the number n of unknowns is greater then a number m of equations then the system always has non-trivial solutions.

■There always exist free unknowns in this case. Therefore we can get infinitely many nontrivial solutions by Gauss (Jordan-Gauss) method.■

Theorem 3. If in a system of linear homogeneous equations (8) the number n of unknowns and the number m of equations coincide $(n = m)$, then the system has nontrivial solutions if and only if its principal determinant equalf zero $(\Delta = 0)$.

■1 (necessity). All auxiliary determinants of a system (8) equal zero, and in the case $\Delta \neq 0$ the system has only the trivial solution. Therefore nontrivial solutions can exist only if $\Delta = 0$.

2 (sufficiency). This part will be proved below.■

Such the situation we've met in Ex. 9 for the case $a = 4, b = 8$.

DETERMINANTS AND SYSTEMS OF LINEAR EQUATIONS: basic terminology RUE

1. Алгебраическое дополнение элемента определителя	Алгебричне доповнення елемента визначника	Cofáctor [sígned mínor, àlgebráic(al) cómplement [determinant, súpplement, ádjunct]] of an élement of a detérminant
2. Вспомогательный определитель системы линейных алгебраических уравнений	Допоміжний визначник системи лінійних алгебричних рівнянь	Auxíliary detérminant of a sýstem of línear àlgebráic equátions
3. Вычеркивание (столбца и строки, в которых находится элемент)	Викреслювання (рядка і стовпця, в яких знаходиться елемент)	Stríking out [delétion, deléting, cróssing out] (of the row and column in which an élement stands [is, is fóund])
4. Вычеркнуть (столбец и строку, в которых находится элемент)	Викреслити (рядок і стовпець, в яких знаходиться елемент)	Strike out [deléte, cross out] (the row and column in which an élement stands [is, is fóund])
5. Вычислить (определитель), используя, с помощью …	Обчислити (визначник), використовуючи, за допомоги, з допомогою…	Cálculate [eváluate, compúte] (a detérminant) úsing [by méans of, with the help of, making úse of, applýing, útilizing] …
6. Главная диагональ определителя	Головна діагональ визначника	Principal [main, léading, pósitive] diágonal of a detérminant
7. Главный определитель системы *n* линейных алгебраических уравнений с *n* неизвестными	Головний визначник системи *n* лінійних алгебричних рівнянь з *n* невідомими	Príncipal [main, system] detérminant of a sýstem of *n* línear àlgebráic equátions in *n* ùnknówns
8. Индекс	Індекс	Índex (*pl* índeces, índexes)
9. Коэффициент при неизвестном	Коефіцієнт (при невідомому)	Còeffícient (of an ùnknówn)
10.Метод (последовательного исключения неизвестных) Гаусса для решения системы линейных алгебраических ура-	Метод (послідовного виключення невідомих) Ґаусса для розв"язання системи лінійних алгебричних рівнянь	Gaussian (succéssive exclúsion of ùnknówns) méthod [méthod of Gauss] of solution [of sólving] a sýstem of línear àlgeb-

внений		ráic(al) equátions
11.Минор (первого [второго, третьего, n-го] порядка)	Мінор (першого [другого, третього, n-го] порядку)	Minor (of the first [second, third, n-th] order); (first-[second-, third-,n-th] órder) mínor
12.Минор элемента определителя	Мінор елемента визначника	Mínor of an élement of a detérminant
13.Неизвестное	Невідоме	Ùnknówn
14.Неопределенная система уравнений	Невизначена система рівнянь	Undetermined [indeterminate, undefíned] sýstem of equátions
15.Несовместная система уравнений	Несумісна система рівнянь	Nòn-compátible [inconsístent, nònsólvable, undecídable] sýstem of equátions
16.Нетривиальное [ненулевое] решение	Нетривіальний [ненульовий] розв"язок	Nòn-trívial [nòn-zéro] solútion
17.Нижний индекс	Нижній індекс	Lówer índex (*pl* índeces, índexes)
18.Обращать (каждое) уравнение в верное равенство	Перетворювати (кожне) рівняння у вірну рівність	Turn [convért, chánge, transfórm] (éach) equátion ínto exáct [correct, precíse] equálity
19.Однородная система линейный уравнений	Однорідна система лінійних рівнянь	Hòmogéneous sýstem of línear equátions
20.Определённая система уравнений	Визначена система рівнянь	Detérmined [detérminate, defíned] sýstem of equátions
21.Определитель (первого [второго, третьего, n-го] порядка)	Визначник (першого [другого, третього, n-го] порядку)	Determinant (of the first [second, third, n-th] órder); (first- [second-, third-, n-th] órder) determinant
22.Побочная диагональ определителя	Побічна діагональ (визначника)	Sécondary [négative] diágonal (of a detérminant)
23.Правило Крамера для решения системы n линейных алгебраических уравнений с n неизвестными	Правило Крамера для розв"язання системи n лінійних алгебричних рівнянь з n невідомими	Cramer´s rúle, rúle of Cramer for solution [for sólving] a sýstem of n línear àlgebráic equátions in n ùnknówns
24.Разложение определителя по элементам первой [второй, третьей, i-й] строки и первого	Розвинення [розклад] визначника за елементами першого [другого, третього, i-го] рядка та	Expánsion of a deter-minant with respéct to the first [second, third, i-th] rów [column]; first [se-

[второго, третьего, *i*-го] столбца	першого [другого, третього, *i*-го] стовпця	cond, third, *i*-th] rów [column] expánsion of a detérminant
25.Разложить определитель по элементам первой [второй, третьей, *i*-й] строки, первого [второго, третьего, *i*-го] столбца	Розвинути [розкласти] визначник за елементами першого [другого, третього, *i*-го] рядка, першого [другого, третього, *i*-го] стовпця	Expánd a detérminant with respéct to the first [second, third, *i*-th] rów [cólumn]
26.Решение [разрешение] системы линейных алгебраических уравнений	Розв"язання [розв"язування] системи лінійних алгебричних рівнянь	Solútion [sólving] a sýstem of línear algebraic(-al) equátions (action)
27.Решение системы линейных алгебраических уравнений (результат)	Розв"язок системи лінійних алгебричних рівнянь (результат)	Solútion of a sýstem of línear algebraic equations (result)
28.Решить систему линейных алгебраических уравнений	Розв"язати систему лінійних алгебричних рівнянь	Solve a sýstem of línear algebraic(al) equátions
29.Ряд определителя	Ряд визначника	Séries of a detérminant
30.Свободный член уравнения	Вільний член рівняння	Cónstant [free, ábsolute] member [term] of an equátion
31.Свойство определителя	Властивість визначника	Próperty of a detérminant
32.Система *m* линейных алгебраических уравнений с *n* неизвестными	Система *m* лінійних алгебричних рівнянь з *n* невідомими	Sýstem of *m* línear àlgebráic(al) equátions in *n* ùnknówns
33.Система линейных однородных уравнений	Система лінійних однорідних рівнянь	Sýstem of línear homogéneous equátions
34.Совместная система	Сумісна система	Compátible [consístent, sólvable, decídable] sýstem
35.Столбец определителя	Стовпець визначника	Cólumn of a detérminant
36.Строка определителя	Рядок визначника	Rów of a detérminant
37.Тривиальное [нулевое] решение	Тривіальний [нульовий] розв"язок	Trívial [null, zéro] solútion
38.Элемент первой [второй, третьей, *i*-й] строки и первого [второго, третьего, *j*-го] столбца	Елемент першого [другого, третього, *i*–го] рядка і першого [другого, третього, *j*–го] стовпця	Élement of the first [second, third, *i*-th] rów and the first [second, third, *j*-th] cólumn

DETERMINANTS AND SYSTEMS OF LINEAR EQUATIONS: basic terminology ERU

1. Auxíliary detérminant of a sýstem of línear àlgebráic(al) equátions	Вспомогательный определитель системы линейных алгебраических уравнений	Допоміжний визначник системи лінійних алгебричних рівнянь
2. Cálculate [eváluate, compúte] (a detérminant) úsing [by méans of, with the help of, making úse of, applýing, útilizing] …	Вычислить (определитель), используя, с помощью …	Обчислити (визначник), використовуючи, за допомоги, з допомогою…
3. Còefficient (of an ùnknówn)	Коэффициент (при неизвестном)	Коефіцієнт (при невідомому)
4. Cofáctor [sígned mínor, àlgebráic(al) cómplement [determinant, súpplement, ádjunct]] of an élement of a detérminant	Алгебраическое дополнение элемента определителя	Алгебричне доповнення елемента визначника
5. Cólumn of a detérminant	Столбец определителя	Стовпець визначника
6. Compátible [consístent, sólvable, decídable] sýstem	Совместная система	Сумісна система
7. Cónstant [free, ábsolute] member [term] of an equátion	Свободный член уравнения	Вільний член рівняння
8. Cramer´s rúle, rúle of Cramer for solution [sólving] a sýstem of *n* línear àlgebráic equátions in *n* ùnknówns	Правило Крамера для решения системы *n* линейных алгебраических уравнений с *n* неизвестными	Правило Крамера для розв"язання системи *n* лінійних алгебричних рівнянь з *n* невідомими
9. Determinant (of the first [second, third, *n*-th] órder); (first- [second-, third-, *n*-th] órder) determinant	Определитель (первого [второго, третьего, *n*-го] порядка)	Визначник (першого [другого, третього, *n*-го] порядку)
10. Detérmined [detérminate, defíned] sýstem of equátions	Определённая система уравнений	Визначена система рівнянь
11. Élement of the first [second, third, *i*-th] rów and the first [second, third, *j*-	Элемент первой [второй, третьей, *i*-й] строки и первого [второго, треть-	Елемент першого [другого, третього, *i*–го] рядка і першого [другого,

th] cólumn	его, *j*-го] столбца	третього, *j*–го] стовпця
12.Expánd a detérminant with respéct to the first [second, third, *i*-th] rów [cólumn]	Разложить определитель по элементам первой [второй, третьей, *i*-й] строки, первого [второго, третьего, *i*-го] столбца	Розвинути [розкласти] визначник за елементами першого [другого, третього, *i*-го] рядка, першого [другого, третього, *i*-го] стовпця
13.Expánsion of a determinant with respéct to the first [second, third, *i*-th] rów [column]; first [second, third, *i*-th] rów [column] expánsion of a detérminant	Разложение определителя по элементам первой [второй, третьей, *i*-й] строки, первого [второго, третьего, *i*-го] столбца	Розвинення [розклад] визначника за елемен-тами першого [другого, третього, *i*-го] рядка, першого [другого, третього, *i*-го] стовпця]
14.Gaussian (succéssive exclúsion of ùnknówns) méthod [méthod of Gauss] of solution [of sólving] a sýstem of línear àlgebráic equátions	Метод (последовательного исключения неизвестных) Гаусса для решения системы линейных алгебраических уравнений	Метод (послідовного виключення невідомих) Ґаусса для розв"язання системи лінійних алгебричних рівнянь
15.Hòmogéneous sýstem of línear equátions	Однородная система линейный уравнений	Однорідна система лінійних рівнянь
16.Índex (*pl* índeces, índexes)	Индекс	Індекс
17.Lówer índex (*pl* índeces, índexes)	Нижний индекс	Нижній індекс
18.Minor (of the first [second, third, *n*-th] order); (first-[second-, third-,*n*-th] órder) mínor	Минор (первого [второго, третьего, *n*-го] порядка)	Мінор (першого [другого, третього, *n*-го] порядку)
19.Mínor of an élement of a detérminant	Минор элемента определителя	Мінор елемента визначника
20.Nòn-compátible [inconsístent, nònsólvable, undecídable] sýstem of equátions	Несовместная система уравнений	Несумісна система рівнянь
21.Nòn-trívial [nòn-zéro] solútion	Нетривиальное [ненулевое] решение	Нетривіальний [ненульовий] розв"язок
22.Principal [main, léading, pósitive] diágonal of a detérminant	Главная диагональ определителя	Головна діагональ визначника
23.Prínсipal [main, system] detérminant of a sýstem of *n* línear àlgebráic	Главный определитель системы *n* линейных алгебраических уравнений	Головний визначник системи *n* лінійних алгебричних рівнянь з *n* не-

equátions in *n* ùnknówns	с *n* неизвестными	відомими
24.Próperty of a detérminant	Свойство определителя	Властивість визначника
25.Rów of a detérminant	Строка определителя	Рядок визначника
26.Sécondary [négative] diágonal (of a detérminant)	Побочная диагональ определителя	Побічна діагональ (визначника)
27.Séries of a detérminant	Ряд определителя	Ряд визначника
28.Solútion [sólving] a sýstem of línear algebraic equátions (action)	Решение [разрешение] системы линейных алгебраических уравнений	Розв"язання [розв"язування] системи лінійних алгебричних рівнянь
29.Solútion of a sýstem of línear algebraic equations (result)	Решение системы линейных алгебраических уравнений (результат)	Розв"язок системи лінійних алгебричних рівнянь (результат)
30.Solve a sýstem of línear algebraic equátions	Решить систему линейных алгебраических уравнений	Розв"язати систему лінійних алгебричних рівнянь
31.Strike out [deléte, cross out] (the row and column in which an élement stands [is, is fóund])	Вычеркнуть (столбец и строку, в которых находится элемент)	Викреслити (рядок і стовпець, в яких знаходиться елемент)
32.Stríking out [delétion, deléting, cróssing out] (of the row and column in which an élement stands [is, is fóund])	Вычеркивание (столбца и строки, в которых находится элемент)	Викреслювання (рядка і стовпця, в яких знаходиться елемент)
33.Sýstem of línear homogéneous equátions	Система линейных однородных уравнений	Система лінійних однорідних рівнянь
34.Sýstem of *m* línear àlgebráic(al) equátions in *n* ùnknówns	Система *m* линейных алгебраических уравнений с *n* неизвестными	Система *m* лінійних алгебричних рівнянь з *n* невідомими
35.Trívial [null, zéro] solútion	Тривиальное [нулевое] решение	Тривіальний [нульовий] розв"язок
36.Turn [convért, chánge, transfórm] (éach) equátion ínto exáct [correct, precíse] equálity	Обращать (каждое) уравнение в верное равенство	Перетворювати (кожне) рівняння у вірну рівність
37.Undetermined [indeterminate, undefíned] sýstem of equátions	Неопределенная система уравнений	Невизначена система рівнянь
38.Ùnknówn	Неизвестное	Невідоме

LECTURE NO 5. MATRICES

POINT 1. MATRICES AND OPERATIONS ON THEM

POINT 2. INVERSE MATRIX

POINT 3. MATRIX EQUATIONS. MATRIX METHOD FOR SOLVING SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS $n \times n$

POINT 4. ADDITIONAL QUESTIONS

POINT 1. MATRICES AND OPERATIONS ON THEM

Def. 1. A matrix of the dimension $m \times n$ is called a rectangular table containing *m* rows and *n* columns.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \left(a_{ij}\right)_{m\times n} = \left(a_{ij}\right), i = \overline{1, m}, \ j = \overline{1, n}. \qquad (1)$$

We'll denote matrices [matrixes] by capital letters and their elements by small letters with two inferior indeces [subindeces]. As a rule we'll consider number matrixes (with number elements).

Ex. 1. A matrix

$$\left(a_1, a_2, \dots, a_n\right)$$

is called a **row matrix** [one-by-*n* row matrix, single-row matrix (with *n* elements)] (of dimension $1 \times n$). It can be also called as ***n*-dimensional row vector**.

Ex. 2. A matrix

$$\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

is called a **column matrix** [one-by-*m* column matrix, single column matrix (with *m* elements)] (of dimension $m \times 1$). It can be called as *m*-**dimensional column vector**.

Ex. 3. A matrix $(0)_{m\times n}$ with zero elements is called a zero [null] matrix.

Def. 2. If a number *m* of rows coincides with a number *n* of columns ($m = n$), a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \left(a_{ij}\right)_{n\times n} = \left(a_{ij}\right), i = \overline{1, n},\ j = \overline{1, n}, \qquad (2)$$

is called the *n*th order **square matrix** (with principal diagonal $a_{11}, a_{22}, \dots, a_{nn}$ and secondary diagonal $a_{1n}, a_{2,n-1}, \dots, a_{n1}$). It possesses a determinant denoted by $|A|$ or $\det A$.

Ex. 4. A square matrix

$$E = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \qquad (3)$$

is called the **unit** [the **unity**] **matrix**. Its determinant is equal to 1,

$$|E| = \det E = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{vmatrix} = 1. \qquad (4)$$

Def. 3. A square matrix is called **regular** one if its determinant doesn't equal zero and **singular** one otherwise.

Linear operations on matrices

1. Multiplication of a matrix by a number *k*, when every its element is multiplied by *k*.

Ex. 5. $k\cdot\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} k\cdot a & k\cdot b & k\cdot c \\ k\cdot d & k\cdot e & k\cdot f \end{pmatrix}.$

2. Addition of matrices of the same dimension, when one adds all corresponding elements of these matrices.

Ex. 6. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} m & n \\ p & q \end{pmatrix} = \begin{pmatrix} a+m & b+n \\ c+p & d+q \end{pmatrix}$.

Properties of linear operations on matrices are analogous to those on real numbers. For example

$$A+B=B+A, (A+B)+C=A+(B+C), (kl)A=k(lA)=l(kA),$$
$$k(A+B)=kA+kB, (k+l)A=kA+lA.$$

Multiplication of matrices

Def. 4. Let there be given two matrices $A=(a_{ik})_{m\times n}$, $B=(b_{kj})_{n\times p}$ such that a number of columns of the matrix A coincides with a number of rows of the matrix B. A product AB of these matrices is called a matrix C, for which an element c_{ij} of the ith row and the jth column is calculated by the next rule:

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{ik}b_{kj}+\ldots+a_{in}b_{nj}=\sum_{k=1}^{n}a_{ik}b_{kj}, \qquad (5)$$

or schematically

$$C=AB=\begin{pmatrix} \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{i1} \quad a_{i2} \quad \ldots \quad a_{ik} \quad \ldots \quad a_{in} \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{pmatrix}\begin{pmatrix} \ldots & b_{1j} & \ldots \\ \ldots & b_{2j} & \ldots \\ \cdots & \cdots & \cdots \\ \ldots & b_{kj} & \ldots \\ \cdots & \cdots & \cdots \\ \ldots & b_{nj} & \ldots \end{pmatrix}=\begin{pmatrix} \cdots\cdots\cdots \\ \ldots \quad c_{ij} \quad \ldots \\ \cdots\cdots\cdots \end{pmatrix}= \qquad (6)$$

$$=\begin{pmatrix} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \ldots \quad a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{ik}b_{kj}+\ldots+a_{in}b_{nj} \quad \ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{pmatrix}==\begin{pmatrix} \cdots\cdots\cdots\cdots \\ \ldots \quad \sum_{k=1}^{n}a_{ik}b_{kj} \quad \ldots \\ \cdots\cdots\cdots\cdots \end{pmatrix}.$$

Thus, an element c_{ij} of the ith row and the jth column of the matrix C equals the sum of products of corresponding elements of the ith row of the matrix A and the jth column of the matrix B.

Ex. 7. Let $A=\begin{pmatrix}1 & 0\\ -2 & 3\end{pmatrix}$, $B=\begin{pmatrix}-2 & 3\\ 0 & -1\end{pmatrix}$. Find the products AB, BA and compare them.

$$AB=\begin{pmatrix}1 & 0\\ -2 & 3\end{pmatrix}\begin{pmatrix}-2 & 3\\ 0 & -1\end{pmatrix}=\begin{pmatrix}1\cdot(-2)+0\cdot 0 & 1\cdot 3+0\cdot(-1)\\ (-2)\cdot(-2)+3\cdot 0 & (-2)\cdot 3+3\cdot(-1)\end{pmatrix}=\begin{pmatrix}-2 & 3\\ 4 & -9\end{pmatrix},$$

$$BA=\begin{pmatrix}-2 & 3\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ -2 & 3\end{pmatrix}=\begin{pmatrix}(-2)\cdot 1+3\cdot(-2) & (-2)\cdot 0+3\cdot 3\\ 0\cdot 1+(-1)\cdot(-2) & 0\cdot 0+(-1)\cdot 3\end{pmatrix}=\begin{pmatrix}-8 & 9\\ 2 & -3\end{pmatrix}\neq AB.$$

Properties of matrix product

1. $AB \neq BA$ generally speaking (anticommutativity).
2. $(AB)C = A(BC)$ (associativity).
3. $(A+B)C = AC+BC,\ A(B+C)=AB+AC$ (distributivity).
4. $k(AB)=(kA)B=A(kB)$ for any real number k
5. $AE=EA=A$ if these products AE, EA exist.
6. For two square matrices A, B

$$|A\cdot B| = |A|\cdot|B|,$$

that is the determinant of a product of two square matrices is equal to the product of their determinants.

All these properties can be verified on examples. We've tested the first property in Ex. 7. Let's verify the property 6 with the help of the same matrices A and B. Namely,

$$|AB|=\begin{vmatrix}-2 & 3\\ 4 & -9\end{vmatrix}=6,\ |A|=\begin{vmatrix}1 & 0\\ -2 & 3\end{vmatrix}=3,\ |B|=\begin{vmatrix}-2 & 3\\ 0 & -1\end{vmatrix}=2 \Rightarrow |A\cdot B|=6=|A|\cdot|B|.$$

Test yourselves the other properties.

POINT 2. INVERSE MATRIX

Def. 5. A matrix A^{-1} is called that inverse for a matrix A if the next equality

$$A\cdot A^{-1}=A^{-1}\cdot A=E \qquad (7)$$

holds.

Theorem 1. The inverse matrix of a regular square matrix (2) is given by the next formula

$$A^{-1}=\frac{1}{|A|}\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn}\end{pmatrix}^T=\frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn}\end{pmatrix} \qquad (8)$$

■Let's prove the theorem for the second order square matrix

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}.$$

We have to prove that the matrix

$$A^{-1}=\frac{1}{|A|}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}^T=\frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22}\end{pmatrix}$$

is an inverse one, that is

$$AA^{-1}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\left(\frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22}\end{pmatrix}\right)=E,\quad A^{-1}A=\left(\frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22}\end{pmatrix}\right)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}=E.$$

We'll prove the validity of the equality $AA^{-1}=E$. By virtue of the properties 8, 9 of determinants

$$AA^{-1}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\left(\frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22}\end{pmatrix}\right)=\frac{1}{|A|}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22}\end{pmatrix}=$$

$$=\frac{1}{|A|}\begin{pmatrix} a_{11}A_{11}+a_{12}A_{12} & a_{11}A_{21}+a_{12}A_{22} \\ a_{21}A_{11}+a_{22}A_{12} & a_{21}A_{21}+a_{22}A_{22}\end{pmatrix}=\frac{1}{|A|}\begin{pmatrix} |A| & 0 \\ 0 & |A|\end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}=E.$$

Prove yourselves the validity of the second equality $A^{-1}A=E$.■

Rule for finding an inverse matrix

To find the inverse matrix for a given square regular matrix A we:

1) calculate the determinant $|A|$ of the matrix A;

2) substitute all the elements of the matrix A by their cofactors and transpose

the obtained matrix;

3) multiply this latter matrix by $1/|A|$.

Ex. 8. Find the inverse matrix for a given third order matrix

$$A=\begin{pmatrix}2 & -1 & -1\\ 3 & 4 & -2\\ 3 & -2 & 4\end{pmatrix}$$

Solution. 1) The determinant of the matrix

$$|A|=\begin{vmatrix}2 & -1 & -1\\ 3 & 4 & -2\\ 3 & -2 & 4\end{vmatrix}=60\,.$$

2) Cofactors of all the elements of the matrix A

$$A_{11}=M_{11}=\begin{vmatrix}4 & -2\\ -2 & 4\end{vmatrix}=12 \qquad A_{12}=-M_{12}=-\begin{vmatrix}3 & -2\\ 3 & 4\end{vmatrix}=-18 \qquad A_{13}=M_{13}=\begin{vmatrix}3 & 4\\ 3 & -2\end{vmatrix}=-18$$

$$A_{21}=-M_{21}=-\begin{vmatrix}-1 & -1\\ -2 & 4\end{vmatrix}=6 \qquad A_{22}=M_{22}=\begin{vmatrix}2 & -1\\ 3 & 4\end{vmatrix}=11 \qquad A_{23}=-M_{23}=-\begin{vmatrix}2 & -1\\ 3 & -2\end{vmatrix}=1$$

$$A_{31}=M_{31}=\begin{vmatrix}-1 & -1\\ 4 & -2\end{vmatrix}=6 \qquad A_{32}=-M_{32}=-\begin{vmatrix}2 & -1\\ 3 & -2\end{vmatrix}=1 \qquad A_{33}=M_{33}=\begin{vmatrix}2 & -1\\ 3 & 4\end{vmatrix}=11$$

3) The inverse matrix is

$$A^{-1}=\frac{1}{|A|}\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}^T=\frac{1}{60}\begin{pmatrix}12 & -18 & -18\\ 6 & 11 & 1\\ 6 & 1 & 11\end{pmatrix}^T=\frac{1}{60}\begin{pmatrix}12 & 6 & 6\\ -18 & 11 & 1\\ -18 & 1 & 11\end{pmatrix}.$$

POINT 3. MATRIX EQUATIONS. MATRIX METHOD FOR SOLVING SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS $n\times n$

Let there be given a matrix equation

$$AXB=C\,, \qquad (9)$$

where A, B, C are known matrices, and X is that unknown.

1) At first we suppose that the equation (9) possesses a solution. Let's multiply both sides of (9) by A^{-1} from the left and by B^{-1} from the right. Making use of the properties of matrix product we get

$$A^{-1}(AXB)B^{-1} = A^{-1}CB^{-1},\ (A^{-1}A)X(BB^{-1}) = A^{-1}CB^{-1},\ EXE = A^{-1}CB^{-1},$$

$$X = A^{-1}CB^{-1}. \qquad (10)$$

Thus, if an equation (9) has a solution, then it is represented by the formula (10) and so is unique one.

2) To prove existence of the solution of the equation (9) we'll substitute the expression (10) in (9) and get

$$A(A^{-1}CB^{-1})B = (AA^{-1})C(BB^{-1}) = ECE = C,\ C = C.$$

Therefore the matrix equation (9) possesses the unique solution (10).

Ex. 9. Solve a matrix equation

$$\begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix} X \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 4 & -1 \end{pmatrix}.$$

Solving. Here

$$A = \begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix},\ B = \begin{pmatrix} 3 & -2 \\ 5 & 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 6 \\ 4 & -1 \end{pmatrix};\ |A| = 14,\ |B| = 10;$$

$$A^{-1} = \frac{1}{14}\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix},\ B^{-1} = \frac{1}{10}\begin{pmatrix} 0 & 2 \\ -5 & 3 \end{pmatrix};$$

$$X = A^{-1}CB^{-1} = \left(\frac{1}{14}\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}\right)\begin{pmatrix} 1 & 6 \\ 4 & -1 \end{pmatrix}\left(\frac{1}{10}\begin{pmatrix} 0 & 2 \\ -5 & 3 \end{pmatrix}\right) =$$

$$= \frac{1}{140}\left(\begin{pmatrix} 1 & -4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 4 & -1 \end{pmatrix}\right)\begin{pmatrix} 0 & 2 \\ -5 & 3 \end{pmatrix} = \frac{1}{140}\begin{pmatrix} -15 & 10 \\ 11 & 16 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -5 & 3 \end{pmatrix} = \frac{1}{140}\begin{pmatrix} -50 & 0 \\ -80 & 60 \end{pmatrix},$$

Answer. $X = \frac{1}{14}\begin{pmatrix} -5 & 0 \\ -8 & 6 \end{pmatrix}.$

Theory of matrix equations can be applied to systems of linear algebraic equations with the same number of equations and unknowns.

Let's consider a system of n linear equations in n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n. \end{cases} \qquad (11)$$

We can reduce it to a matrix equation introducing the system matrix (the matrix of coefficients of unknowns) and column matrices (or column vectors) of unknowns and constants terms

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{pmatrix}. \qquad (12)$$

It is obviously, that the system (11) is equivalent to a matrix equation

$$AX = B, \qquad (13)$$

which has the unique solution

$$X = A^{-1}B \qquad (14)$$

(prove it yourselves!).

Ex. 10. Solve by the matrix method a system of equations

$$\begin{cases} 2x_1 - x_2 - x_3 = 4, \\ 3x_1 + 4x_2 - 2x_3 = 11, \\ 3x_1 - 2x_2 + 4x_3 = 11. \end{cases}$$

The system matrix, column matrices (column vectors) of unknowns and constants terms are

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, B = \begin{pmatrix} 4 \\ 11 \\ 11 \end{pmatrix}; A^{-1} == \frac{1}{60}\begin{pmatrix} 12 & 6 & 6 \\ -18 & 11 & 1 \\ -18 & 1 & 11 \end{pmatrix} \text{ (Ex. 8)}$$

the equivalent matrix equation and its solution are

$$AX = B, X = A^{-1}B,$$

hence

$$X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\left(\frac{1}{60}\begin{pmatrix}12&6&6\\-18&11&1\\-18&1&11\end{pmatrix}\right)\begin{pmatrix}4\\11\\11\end{pmatrix}=\frac{1}{60}\begin{pmatrix}12&6&6\\-18&11&1\\-18&1&11\end{pmatrix}\begin{pmatrix}4\\11\\11\end{pmatrix}=\frac{1}{60}\begin{pmatrix}180\\60\\60\end{pmatrix}=\begin{pmatrix}3\\1\\1\end{pmatrix}.$$

Answer. $(3,1,1)$.

POINT 4. ADDITIONAL QUESTIONS

Linear transformations of unknowns and matrices

We'll limit ourselves to two unknowns. In general case reasonings are analogous.

Def. 6. A linear transformation of unknowns is called a change [overpass, transition] from unknowns x_1, x_2 to unknowns y_1, y_2 in which old unknowns are linearly represented through new unknowns by formulas

$$\begin{cases}x_1=a_{11}y_1+a_{12}y_2,\\x_2=a_{21}y_1+a_{22}y_2.\end{cases}\tag{15}$$

If we'll introduce matrices *X*, *Y* of unknowns and a matrix (transformation matrix) *A*, namely

$$X=\begin{pmatrix}x_1\\x_2\end{pmatrix},\ Y=\begin{pmatrix}y_1\\y_2\end{pmatrix},\ A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix},\tag{16}$$

we'll can represent the transformation (15) in a matrix form

$$X=AY.\tag{17}$$

Let's fulfill a linear transformation from y_1, y_2 to z_1, z_2 with transformation matrix *B* after the linear transformation (15) (or (17)) from x_1, x_2 to y_1, y_2, that is

$$\begin{cases}y_1=b_{11}z_1+b_{12}z_2,\\y_2=b_{21}z_1+b_{22}z_2,\end{cases}\ B=\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix},\ Z=\begin{pmatrix}z_1\\z_2\end{pmatrix},\ Y=BZ.\tag{18}$$

In this case we can overpass from x_1, x_2 to z_1, z_2. Indeed, by virtue of the formulas (17), (18) we have

$$X=AY=A(BZ)=(AB)Z,\tag{19}$$

that is a linear transformation with the matrix

$$C=\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}=AB=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix},\ \begin{cases} x_1=c_{11}z_1+c_{12}z_2, \\ x_2=c_{21}z_1+c_{22}z_2. \end{cases} \quad (20)$$

Def. 7. Result of successive fulfillment of two linear transformations is called a product of these transformations.

A product of two linear transformations with transformation matrices *A* and *B* is also a linear transformation with transformation matrix *C* = *AB*.

Ex. 11. Product of linear transformations

$$\begin{cases} x_1=3y_1-y_2, \\ x_2=y_1+5y_2, \end{cases} A=\begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix}, X=AY; \begin{cases} y_1=z_1+z_2, \\ y_2=4z_1+2z_2, \end{cases} B=\begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}, Y=BZ;$$

is a linear transformation with the matrix

$$C=AB=\begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}=\begin{pmatrix} -1 & 1 \\ 21 & 11 \end{pmatrix}, X=CZ, \begin{cases} x_1=-z_1+z_2, \\ x_2=21z_1+11z_2. \end{cases}$$

Def. 8. Proceeding of a linear transformation (15) from x_1, x_2 to y_1, y_2, we can fulfill the passage from y_1, y_2 to x_1, x_2. Such the passage is called an **inverse transformation** for the given one.

Inverse transformation for a linear transformation with a matrix *A* is also a linear one with the matrix A^{-1}.

$$\blacksquare\ Y=A^{-1}X, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\left(\frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_2 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{cases} y_1=A_{11}/|A|x_1+A_{21}/|A|x_2, \\ y_2=A_{12}/|A|x_1+A_{22}/|A|x_2. \end{cases}\ \blacksquare$$

Ex. 12. The inverse linear transformation for the transformation

$$\begin{cases} x_1=3y_1-y_2, \\ x_2=y_1+5y_2, \end{cases} A=\begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix}, X=AY$$

is

$$Y=\mathrm{A}^{-1}X=\left(\frac{1}{|A|}\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_2 \end{pmatrix}\right)X=\frac{1}{16}\left(\begin{pmatrix} 5 & 1 \\ -1 & 3 \end{pmatrix}X\right),$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} 5/16 & 1/16 \\ -1/16 & 3/16 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{cases} y_1=5/16x_1+1/16x_2, \\ y_2=-1/16x_1+3/16x_2. \end{cases}$$

Quadratic forms and matrices

Def. 9. Quadratic form of two, three, n unknowns are called the next expressions

$$F(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2; \qquad (21)$$

$$F(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3; \qquad (22)$$

$$F(x_1, x_2, ..., x_n) = \sum_{i,j=1}^{n} a_{ij}x_ix_j, \; a_{ij} = a_{ji}. \qquad (23)$$

Forms (21), (22) are particular cases of a quadratic form (23).

Quadratic form is completely determined by its matrix A.

For the form (21) it is the next one

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \; a_{12} = a_{21} \text{ or simply } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}; \qquad (24)$$

for the form (22) it is

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \begin{matrix} a_{12} = a_{21}, \\ a_{13} = a_{31}, \\ a_{23} = a_{32}, \end{matrix} \text{ or simply } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}; \qquad (25)$$

In general case of the quadratic form (23) it is a matrix symmetric with respect to the principal diagonal

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ .... & .... & .... & ..... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}, \begin{matrix} a_{12} = a_{21} \\ a_{13} = a_{31}, \\ .... \\ a_{ij} = a_{ji}, \end{matrix} \text{ or simply } A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{12} & a_{22} & ... & a_{2n} \\ .... & .... & .... & ..... \\ a_{1n} & a_{2n} & ... & a_{nn} \end{pmatrix}. \qquad (26)$$

It is easy to prove that

$$F(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = (x_1 \quad x_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad (27)$$

$$F(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 =$$

$$= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad (28)$$

$$F(x_1, x_2, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{12} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{1n} & a_{2n} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}. \qquad (29)$$

If we denote respectively

$$\text{a) } X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, X^T = \begin{pmatrix} x_1 & x_2 \end{pmatrix}; \text{ b) } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, X^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix};$$

$$\text{c) } X = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}, X^T = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}$$

than a quadratic form of any number of unknowns can be represented in a matrix form as follows

$$F = X^T A X \qquad (30)$$

Ex. 13. Let $F(x_1, x_2) = 3x_1^2 + 12a_{12}x_1x_2 + x_2^2$. Then

$$a_{11} = 3, a_{22} = 1, 2a_{12} = 12, a_{12} = a_{21} = 6,$$

and the matrix of the quadratic form is

$$A = \begin{pmatrix} 3 & 6 \\ 6 & 1 \end{pmatrix} \Rightarrow F(x_1, x_2) = 3x_1^2 + 12a_{12}x_1x_2 + x_2^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Def. 10. Quadratic form (23) is called **positive (negative) definite** if it takes on only positive (negative) values for any values of $x_1, x_2, \ldots, x_n$ (with the exception of the case $x_1 = x_2 = \ldots = x_n = 0$).

Def. 11. Principal minors of the matrix (26) of the quadratic form (23) are called its diagonal minors

$$\Delta_1 = a_{11},\ \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \ldots,\ \Delta_k = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ a_{k1} & a_{k2} & \ldots & a_{kk} \end{vmatrix}, \ldots,\ \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

Theorem 2 (Sylvester[1])**.** Quadratic form (23) is **positive definite** if and only if all its principal minors are positive,

$$\Delta_1 > 0, \Delta_2 > 0, \Delta_3 > 0, \ldots, \Delta_n > 0. \qquad (31)$$

It is **negative definite** if and only if these minors have alternating signs in the next manner

$$\Delta_1 < 0, \Delta_2 > 0, \Delta_3 < 0, \Delta_4 > 0, \ldots.$$

Ex. 14. Quadratic form $F(x_1, x_2, x_3) = 5x_1^2 + x_2^2 + 5x_3^2 + 4x_1x_2 - 8x_1x_3 - 4x_2x_3$ with a matrix

$$A = \begin{pmatrix} 5 & 2 & -4 \\ 2 & 1 & -2 \\ -1 & -2 & 5 \end{pmatrix}$$

is positive definite because of the principal minors

$$\Delta_1 = 5 > 0,\ \Delta_2 = \begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} = 1 > 0,\ \Delta_3 = \begin{vmatrix} 5 & 2 & -4 \\ 2 & 1 & -2 \\ -1 & -2 & 5 \end{vmatrix} = 1 > 0$$

are positive.

Ex. 15. Quadratic form $F(x_1, x_2) = 3x_1^2 + 12a_{12}x_1x_2 + x_2^2$ of Ex. 13 is neither positive nor negative definite because of signs of the principal minors, namely

$$\Delta_1 = 3 > 0,\ \Delta_2 = \begin{vmatrix} 3 & 6 \\ 6 & 1 \end{vmatrix} = -33 < 0.$$

[1] Sylvester J.J. (1814 - 1897), an English mathematician

LECTURE NO. 6. MATRIX RANK AND SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

POINT 1. MATRIX RANK

POINT 2. KRONECKER – CAPELLI THEOREM

POINT 3. SYSTEMS OF LINEAR HOMOGENEOUS EQUATIONS

POINT 4. EIGENVALUES AND EIGENVECTORS OF A MATRIX

POINT 1. RANK OF A MATRIX

Def. 1. Let's take any *k* rows and *k* columns of some matrix *A*. Elements, lying on the intersection of these rows and columns, form a determinant, which is called the *k*-th order minor of the matrix *A*.

Ex. 1. Principal minors of a quadratic form (see Lecture No. 5, Point 4).

Def. 2. Rank of a non-zero matrix A ($RankA$) is called the highest order of its non-zero minors.

Ex. 2. Find the rank of a matrix

$$A = \begin{pmatrix} 2 & 3 & 5 & 4 \\ 4 & 1 & -1 & 0 \\ 2 & -2 & -6 & -4 \\ -8 & -2 & 2 & 0 \end{pmatrix}.$$

The matrix has many non-zero minors of the first and second order, for example

$$M_1 = a_{11} = 2 \neq 0, M_2 = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = -10 \neq 0.$$

All the third order minors of the matrix equal zero. For example

$$M_{3_1} = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & -1 \\ 2 & -2 & -6 \end{vmatrix} = 0, M_{3_2} = \begin{vmatrix} 2 & 3 & 4 \\ 4 & 1 & 0 \\ 2 & -2 & -4 \end{vmatrix} = 0, M_{3_3} = \begin{vmatrix} 3 & 5 & 4 \\ 1 & -1 & 0 \\ 2 & -6 & -4 \end{vmatrix} = 0.$$

Test yourselves that the rest of the third order minors equal zero. As result the fourth order minor of the matrix, that is $M_4 = |A|$ equals zero (what one can easy see if he'll expand the minor with respect to any its row or column).

Therefore $RankA = 2$.

As the rule we find matrix ranks with the help of so-called elementary transformations.

Def. 3. Elementary transformations of a matrix are called:

1) multiplying of any its series (that is of all elements of any row or column) by non-zero number;

2) interchanging any two series;

3) addition (of all elements) of any series, multiplied by arbitrary non-zero number, to (corresponding elements of) any other series;

4) casting-out [deletion, dropping] of null-series [null-rows, null-columns]

Theorem. Elementary transformations don't change the matrix rank.

Ex. 3. Find the rank of the same matrix of Ex. 2.

$$A = \begin{pmatrix} 2 & 3 & 5 & 4 \\ 4 & 1 & -1 & 0 \\ 2 & -2 & -6 & -4 \\ -8 & -2 & 2 & 0 \end{pmatrix} \cong \begin{pmatrix} 2 & 3 & 5 & 4 \\ 4 & 1 & -1 & 0 \\ 4 & 1 & -1 & 0 \\ -8 & -2 & 2 & 0 \end{pmatrix} \cong \begin{pmatrix} 2 & 3 & 5 & 1 \\ 4 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & -2 & 2 & 0 \end{pmatrix} \cong$$

$$\cong \begin{pmatrix} 0 & 0 & 0 & 1 \\ 4 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & -2 & 2 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 0 & 1 \\ 4 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The rank of the latter matrix equals 2, and therefore $RankA = 2$.

We've used the next elementary transformations: 1) addition of the first row to the third one; 2) addition of the second row, multiplied by -1, to the third row and division of the fourth column by 4; 3) addition of the fourth column, multiplied by -2, -3, -5, to the first, second, third columns correspondingly; 4) addition of the second row, multiplied by 2, to the fourth row; 5) addition of the second column, multiplied by -4, 1, to the first, third columns respectively and casting-out the third and fourth zero-rows; 6) dropping the first and third zero-columns. The sign $\cong$ means hear the equality of ranks of matrices.

POINT 2. KRONECKER-CAPELLI THEOREM

Let's suppose that we study a system of *m* linear algebraic equations in *n* unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m. \end{cases} \quad (1)$$

Let's introduce the next two matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{pmatrix}.$$

The first of them is the system [leading, principal, coefficient] matrix, and the second one is the so-called extended [augmented, dilated] matrix. It is obviously that the rank of this latter can't be greater than the rank of the system matrix.

Theorem (Kronecker[1]-Cappelli[2]). A system of linear algebraic equations (1) is compatible if and only if the rank of the system matrix *A* equals the rank of the extended matrix $\widetilde{A}$.

We'll omit the proving of this theorem and concentrate upon its applications.

Starting the solving a system (1) of linear algebraic equations we at first find the ranks of its system and extended matrices. If $RancA < Ranc\widetilde{A}$, the system is non-compatible one.

Let's consider the main case, namely $RancA = Ranc\widetilde{A} = k$. By virtue of the definition of a matrix rank there is at least one *k*-th order non-zero minor M_k of the matrix *A* (and therefore of the extended matrix $\widetilde{A}$), and all *l*-th order minors of both matrices (for $l > k$) are equal to zero. This minor M_k, which we can call the **basic** one,

---

[1] Kronecker, L. (1823 - 1891), a German mathematician
[2] Capelli, A. (1855 - 1910), an Italian mathematician

defines equations and unknowns of the system, which we also can call those basic. If, for example,

$$M_k = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{vmatrix} \neq 0,$$

the first k equations and k unknowns ($x_1, x_2, ..., x_k$) be those basic.

The number m of equations of the system (1) isn't less then the common rank k of the matrices A, $\widetilde{A}$. If $m > k$, we leave k basic equations and throw away the other $m - k$ non-basic equations, because of, as it follows from the theory, the base equations are independent, and the other equations are their corollaries and must be deleted.

Thus we get a system of k equations in n unknowns, and $n \geq k$. Two cases can occur here.

If $n = k$, we have a system of k equations in k unknowns with non-zero determinant M_k. Such the system possesses a **unique solution**.

If $n > k$, we leave k basic unknowns from the left and transpose the rest $n - k$ unknowns to the right. We consider those transposed as arbitrary numbers [or **free unknowns**] and express the basic unknowns in terms of free unknowns. In this case the system (1) is compatible undetermined, and we get its so-called **general solution**.

Remark. All positions of the aforesaid theory can be fixed by Gauss method (see Lecture No. 4, Point 2).

If $RancA < Ranc\widetilde{A}$, Gauss method leads to at least one equation of the form $0 = d_k$, where $d_k \neq 0$.

If $RancA = Ranc\widetilde{A} = k, m \geq k, k = n$, it leads to a triangular system, what means that a system in question has a unique solution (it's compatible determined).

If $RancA = Ranc\widetilde{A} = k, m \geq k, k < n$, it leads to a system of a trapeziumform, and so the given system has infinitely many solutions (it's compatible undetermined).

Ex. 4. Investigate for compatibility and solve a system of linear algebraic equations

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 + 4x_4 & = & 6, \\ 4x_1 + x_2 - x_3 & = & 1, \\ 2x_1 - 2x_2 - 6x_3 - 4x_4 & = & -5, \\ -8x_1 - 2x_2 + 2x_3 & = & -2. \end{cases}$$

The system and extended matrices be

$$A = \begin{pmatrix} 2 & 3 & 5 & 4 \\ 4 & 1 & -1 & 0 \\ 2 & -2 & -6 & -4 \\ -8 & -2 & 2 & 0 \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} 2 & 3 & 5 & 4 & 6 \\ 4 & 1 & -1 & 0 & 1 \\ 2 & -2 & -6 & -4 & -5 \\ -8 & -2 & 2 & 0 & -2 \end{pmatrix}.$$

The rank of the first one equals 2 (see Ex. 2, 3), the second has the same rank (verify yourselves!), so $Rank A = Rank \widetilde{A} = 2$, and the system of equations is compatible undetermined one.

We choose $M_2 = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = -10 \neq 0$ as the basic second order minor and therefore the first and second equations and the first and second unknowns as those basic. Deleting the third and fourth equations and transposing x_3, x_4 to the right we get

$$\begin{cases} 2x_1 + 3x_2 & = & 6 - 5x_3 - 4x_4, \\ 4x_1 + x_2 & = & 1 + x_3. \end{cases}$$

If we add the second equation, multiplied by - 3, to the first one, we'll get

$$\begin{cases} -10x_1 & = & 3 - 8x_3 - 4x_4, \\ 4x_1 + x_2 & = & 1 + x_3, \end{cases}$$

whence it follows that

$$x_1 = -0.3 + 0.8x_3 + 0.4x_4, \; x_2 = 1 + x_3 - 4x_1 = 2.2 - 2.2x_3 - 1.6x_4.$$

Answer: the general solution of the system in question is

$$\left(-0.3 + 0.8x_3 + 0.4x_4, 2.2 - 2.2x_3 - 1.6x_4, x_3, x_4\right),$$

where x_3, x_4 are arbitrary numbers.

Note. One can put $x_3 = \alpha,\, x_4 = \beta$ and represent the general solution of the system in the next form

$$(-0.3 + 0.8\alpha + 0.4\beta,\, 2.2 - 2.2\alpha - 1.6\beta,\, \alpha,\, \beta),$$

where α, β are arbitrary numbers.

Ex. 5. Investigate for compatibility a system of equations[1]

$$\begin{cases} 3x_1 + 4x_2 - x_3 = -3, \\ 2x_1 - x_2 + ax_3 = 11, \\ 5x_1 + 3x_2 + 3x_3 = b \end{cases}$$

The system and extended matrices are

$$A = \begin{pmatrix} 3 & 4 & -1 \\ 2 & -1 & a \\ 5 & 3 & 3 \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} 3 & 4 & -1 & -3 \\ 2 & -1 & a & 11 \\ 5 & 3 & 3 & b \end{pmatrix}.$$

The matrix A has the second order non-zero minor

$$M_2 = \begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} = -11 \neq 0.$$

Its single third order minor (which is simultaneously the principal determinant Δ of the system)

$$M_3 = \Delta = \begin{vmatrix} 3 & 4 & -1 \\ 2 & -1 & a \\ 5 & 3 & 3 \end{vmatrix} = 11(a-4).$$

If $a \neq 4$, $\Delta \neq 0 \Rightarrow RankA = Rank\widetilde{A} = 3$, and the system has unique solution.

For $a = 4$

$$A = \begin{pmatrix} 3 & 4 & -1 \\ 2 & -1 & 4 \\ 5 & 3 & 3 \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} 3 & 4 & -1 & -3 \\ 2 & -1 & 4 & 11 \\ 5 & 3 & 3 & b \end{pmatrix}$$

the third order minor M_3 of the matrix A equals 0, hence $RancA = 2$. It's necessary to investigate the extended matrix $\widetilde{A}$ for its rank. We'll do it with the help of elementary transformations (explain the next transformations yourselves!).

[1] We have already studied this system (see Lecture No. 4, Point 2)

$$\widetilde{A}=\begin{pmatrix}3&4&-1&-3\\2&-1&4&11\\5&3&3&b\end{pmatrix}\cong\begin{pmatrix}0&0&-1&0\\14&15&4&-1\\14&15&3&b-9\end{pmatrix}\cong\begin{pmatrix}0&0&1&0\\1&1&0&-1\\1&1&0&b-9\end{pmatrix}\cong$$
$$\cong\begin{pmatrix}0&0&1&0\\1&1&0&1\\0&0&0&b-8\end{pmatrix}\cong\begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&b-8\end{pmatrix}\cong\begin{pmatrix}0&1&0\\1&0&0\\0&0&b-8\end{pmatrix}.$$

For the last matrix of this chine of matrices (of the same rank!)

$$M_2=\begin{vmatrix}0&1\\1&0\end{vmatrix}=-1\neq 0,\quad M_3=\begin{vmatrix}0&1&0\\1&0&0\\0&0&b-8\end{vmatrix}=8-b=\begin{cases}0 & \text{if } b=8,\\ 8-b\neq 0 & \text{if } b\neq 8.\end{cases}$$

Thus if $b\neq 8$ (and $a=4$), then $Rank\widetilde{A}=3>RankA$, and the system is non-compatible one. It's compatible (and undetermined) for $a=4, b=8$.

POINT 3. SYSTEMS OF LINEAR HOMOGENEOUS EQUATIONS

In the case of a systems of linear homogeneous equations

$$\begin{cases}a_{11}x_1+a_{12}x_2+a_{13}x_3+...+a_{1n}x_n=0,\\ a_{21}x_1+a_{22}x_2+a_{23}x_3+...+a_{2n}x_n=0,\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ a_{m1}x_1+a_{m2}x_2+a_{m3}x_3+...+a_{mn}x_n=0,\end{cases}\qquad (2)$$

we have

$$A=\begin{pmatrix}a_{11}&a_{12}&\dots&a_{1n}\\a_{21}&a_{22}&\dots&a_{2n}\\ \dots&\dots&\dots&\dots\\a_{m1}&a_{m2}&\dots&a_{mn}\end{pmatrix},\quad \widetilde{A}=\begin{pmatrix}a_{11}&a_{12}&\dots&a_{1n}&0\\a_{21}&a_{22}&\dots&a_{2n}&0\\ \dots&\dots&\dots&\dots&\dots\\a_{m1}&a_{m2}&\dots&a_{mn}&0\end{pmatrix},$$

whence $Rank\widetilde{A}=RankA=k$, and the system is compatible one. Let M_k be the mentioned above *k*-th order non-zero minor of the system matrix *A*, and non-basic equations are deleted (see Point 2 of this lecture).

If $n = k$, the system has only a trivial solution because of its principal determinant is distinct from zero.

If $n > k$, the system has infinitely many solutions. One can find its general solution by the method of preceding Point. But homogeneity of the system permits to go somewhat further because of properties of its solutions.

Properties of solutions of a system of linear homogeneous equations

1. If $(x_{11}, x_{21}, \ldots, x_{n1})$, $(x_{12}, x_{22}, \ldots, x_{n2})$ be two solutions of the system (2), then their sum, that is $(x_{11}+x_{12}, x_{21}+x_{22}, \ldots, x_{n1}+x_{n2})$, is also solution.

2. If $(x_{10}, x_{20}, \ldots, x_{n0})$ is a solution of the system (2), then its product by any number k, namely $(kx_{10}, kx_{20}, \ldots, kx_{n0})$, is also solution.

By virtue of these properties a corresponding theory says that all solutions of a system of linear homogeneous equations can be obtained from so-called **fundamental system of solutions.** One gets it if he assigns successively the values

$$(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1)$$

to free unknowns in the general solution of the system.

Ex. 6. Find the fundamental system of solutions of the next system of linear homogeneous equations

$$\begin{cases} 3x_1 + 4x_2 - 5x_3 + 7x_4 & = & 0, \\ 2x_1 - 3x_2 + 3x_3 - 2x_4 & = & 0, \\ 4x_1 + 11x_2 - 13x_3 + 16x_4 & = & 0, \\ 7x_1 - 2x_2 + x_3 + 3x_4 & = & 0. \end{cases}$$

A system matrix of the system

$$\begin{pmatrix} 3 & 4 & -5 & 7 \\ 2 & -3 & 3 & -2 \\ 4 & 11 & -13 & 16 \\ 7 & -2 & 1 & 3 \end{pmatrix}$$

has the rank $k = 2$ (verify!), we can choose as the basic minor the next one:

$$M_2 = \begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix} = -17 \neq 0,$$

and so the basic equations and unknowns are those first and second. We delete the third and fourth equations and transpose x_3, x_4 to the right, hence

$$\begin{cases} 3x_1 + 4x_2 & = & 5x_3 - 7x_4, \\ 2x_1 - 3x_2 & = & -3x_3 + 2x_4. \end{cases}$$

The general solution of the system is $\left(\frac{3}{17}x_3 - \frac{13}{17}x_4, \frac{19}{17}x_3 - \frac{20}{17}x_4, x_3, x_4\right)$. Putting successfully $x_3 = 1, x_4 = 0; x_3 = 0, x_4 = 1$ in the general solution we get the fundamental system of solutions

$$\left(\frac{3}{17}, \frac{19}{17}, 1, 0\right), \quad \left(-\frac{13}{17}, -\frac{20}{17}, 0, 1\right).$$

POINT 4. EIGENVALUES AND EIGENVECTORS OF A MATRIX

Let there be given a set M of all n-dimensional column vectors (that is[1] the set of all column matrices of dimension $n \times 1$) and a mapping f of M in M determined by the n-th order square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

It means that to every vector

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \in M$$

there corresponds a unique vector

[1] See the beginning of the Lecture 5, Ex. 2

$$Y=\begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix}\in M$$

such that

$$Y=f(X)=AX\,. \tag{3}$$

The next interesting problem arises: is there a non-zero vector X (**eigenvector**) for which

$$f(X)=\lambda X\,,\ \text{or}\ AX=\lambda X \tag{4}$$

where λ is some number (**eigenvalue**)?

Let an eigenvalue λ and an eigenvector X exist, and the question consists in finding them. On the base of (4) we have

$$AX-\lambda X=0,\ AX-\lambda EX=0,$$

$$(A-\lambda E)X=0\,, \tag{5}$$

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots\dots\dots & \dots\dots\dots & \dots & \dots\dots\dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \tag{5 a}$$

or in extensive form

$$\begin{cases} (a_{11}-\lambda)x_1+a_{12}x_2+\dots+a_{1n}x_n & = & 0, \\ a_{21}x_1+(a_{22}-\lambda)x_2+\dots+a_{2n}x_n & = & 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \dots & \dots. \\ a_{n1}x_1+a_{n2}x_2+\dots+(a_{nn}-\lambda)x_n & = & 0. \end{cases} \tag{6}$$

For existing non-trivial solutions of the system (in $x_1, x_2, \dots, x_n$) its principal determinant must be equal zero,

$$|A-\lambda E|=Det(A-\lambda E)=\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & \dots & a_{2n} \\ \dots\dots\dots & \dots\dots\dots & \dots & \dots\dots\dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-\lambda \end{vmatrix}=0\,. \tag{7}$$

Opening the determinant we get the n-th degree equation in an eigenvalue λ.

Let $\lambda = \lambda_0$ be a root of the equation (7). We substitute it in (6) (or in (5 a)) and obtain the system

$$\begin{cases} (a_{11}-\lambda_0)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & 0, \\ a_{21}x_1 + (a_{22}-\lambda_0)x_2 + \ldots + a_{2n}x_n & = & 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots & \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + (a_{nn}-\lambda_0)x_n & = & 0 \end{cases} \tag{8}$$

to find coordinates of one or several eigenvectors corresponding to the eigenvalue λ_0. It's useful to take into account that the rank of the system matrix of the system (8) is less than n.

By the same way we do as to other eigenvalues.

Ex. 7. Find eigenvalues and eigenvectors of a matrix

$$A = \begin{pmatrix} 7 & -4 & -2 \\ -2 & 5 & -2 \\ 0 & 0 & 9 \end{pmatrix}.$$

Solving the problem.

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \lambda E = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, A - \lambda E = \begin{pmatrix} 7-\lambda & -4 & -2 \\ -2 & 5-\lambda & -2 \\ 0 & 0 & 9-\lambda \end{pmatrix},$$

$$|A-\lambda E| = \mathrm{Det}(A-\lambda E) = \begin{vmatrix} 7-\lambda & -4 & -2 \\ -2 & 5-\lambda & -2 \\ 0 & 0 & 9-\lambda \end{vmatrix} = (9-\lambda)\begin{vmatrix} 7-\lambda & -4 \\ -2 & 5-\lambda \end{vmatrix} = (9-\lambda)^2(\lambda-3).$$

On the base of (7) we solve the equation

$$(9-\lambda)^2(\lambda-3) = 0 \Rightarrow \lambda_{1,2} = 9, \lambda_3 = 3.$$

Let at first $\lambda = 9$. Substituting the value 9 in (5 a) instead λ we get

$$\begin{pmatrix} -2 & -4 & -2 \\ -2 & -4 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{cases} -2x_1 - 4x_2 - 2x_3 & = & 0, \\ -2x_1 - 4x_2 - 2x_3 & = & 0, \\ 0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 & = & 0. \end{cases}$$

It is obvious that the system of equations in x_1, x_2, x_3 reduces to one equation

$$x_1 + 2x_2 + x_3 = 0$$

with the basic unknown x_1 and free unknowns x_2, x_3. It directly follows also from the theory, because of (verify!) the rank of the system matrix equals 1,

$$Ranc\begin{pmatrix} -2 & -4 & -2 \\ -2 & -4 & -2 \\ 0 & 0 & 0 \end{pmatrix} = 1,$$

because of all its second order and unique third order minors are equal to zero.

Putting $x_2 = 1, x_3 = 0$ and then $x_2 = 0, x_3 = 1$ we obtain the fundamental system of solutions $(-2, 1, 0)$, $(-1, 0, 1)$ of the system and therefore two eigenvectors

$$X_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Now let $\lambda = 3$. For this value a system (5 a) takes on the form

$$\begin{pmatrix} 4 & -4 & -2 \\ -2 & 2 & -2 \\ 0 & 0 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{cases} 4x_1 - 4x_2 - 2x_3 & = & 0, \\ -2x_1 + 2x_2 - 2x_3 & = & 0, \\ 0 \cdot x_1 + 0 \cdot x_2 + 6x_3 & = & 0. \end{cases}$$

The rank of its coefficient matrix equals 2[1]:

$$Rank\begin{pmatrix} 4 & -4 & -2 \\ -2 & 2 & -2 \\ 0 & 0 & 6 \end{pmatrix} = Rank\begin{pmatrix} 2 & -2 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 6 \end{pmatrix} = Rank\begin{pmatrix} 2 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= Rank\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Rank\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Rank\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2.$$

Therefore the fundamental system of solutions of the system contains only one solution (for $x_2 = 1$), namely $(1, 1, 0)$, and so we get only one eigenvector

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

[1] We find it with the help of elementary transformations. Evaluate it making use of matrix minors.

Answer. $X_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ for $\lambda = 9$, $X_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ for $\lambda = 3$.

Ex. 8. The same problem for a matrix

$$A = \begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix}.$$

Solution.

$$A - \lambda E = \begin{pmatrix} 5-\lambda & -3 & 2 \\ 6 & -4-\lambda & 4 \\ 4 & -4 & 5-\lambda \end{pmatrix}, |A - \lambda E| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & -3 & 2 \\ 6 & -4-\lambda & 4 \\ 4 & -4 & 5-\lambda \end{vmatrix} =$$
$$= -\left(\lambda^3 - 6\lambda^2 + 11\lambda - 6\right) = -(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$$

For $\lambda = 1$ we must solve the next system of equations with the rank 2 of the system matrix (verify it yourselves), and so

$$\begin{cases} 4x_1 - 3x_2 + 2x_3 = 0, \\ 6x_1 - 5x_2 + 4x_3 = 0, \\ 4x_1 - 4x_2 + 4x_3 = 0; \end{cases} \begin{cases} x_1 - x_2 + x_3 = 0, \\ 4x_1 - 3x_2 + 2x_3 = 0; \end{cases} \begin{cases} x_1 - x_2 = -x_3, \\ 4x_1 - 3x_2 = -2x_3. \end{cases}$$

Putting $x_3 = 1$, we get $x_1 = 1, x_2 = 2$ and the eigenvector

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

For $\lambda = 2$ we have by analogy

$$\begin{cases} 3x_1 - 3x_2 + 2x_3 = 0, \\ 6x_1 - 6x_2 + 4x_3 = 0, \\ 4x_1 - 4x_2 + 3x_3 = 0; \end{cases} \begin{cases} 3x_1 - 3x_2 + 2x_3 = 0, \\ 4x_1 - 4x_2 + 3x_3 = 0; \end{cases} \begin{cases} 3x_1 + 2x_3 = 3x_2, \\ 4x_1 + 3x_3 = 4x_2. \end{cases}$$

Putting $x_2 = 1$, we get $x_1 = 1, x_3 = 0$ and the eigenvector

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

At least for $\lambda = 3$ we obtain by the same way

$$\begin{cases} 2x_1 - 3x_2 + 2x_3 &= 0, \\ 6x_1 - 7x_2 + 4x_3 &= 0, \\ 4x_1 - 4x_2 + 2x_3 &= 0; \end{cases} \quad \begin{cases} 2x_1 - 3x_2 + 2x_3 &= 0, \\ 6x_1 - 7x_2 + 4x_3 &= 0; \end{cases} \begin{cases} 2x_1 - 3x_2 &= -2x_3, \\ 6x_1 - 7x_2 &= -4x_3. \end{cases}$$

Putting $x_3 = 1$, we get the values of the other unknowns ($x_1 = 1/2$, $x_2 = 1$) and the eigenvector

$$X_3 = \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix}.$$

In all three cases the rank of corresponding system matrix equals 2, and the fundamental system of solutions contains only one solution.

Answer.

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \text{ for } \lambda = 1,\ X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ for } \lambda = 2,\ X_3 = \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix} \text{ for } \lambda = 3.$$

MATRIX AND SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS: basic terminology RUE

1. Сложение матриц (одного и того же размера)	Додавання матриць (одного й того ж розміру)	Addítion of mátrices (of the sáme dimension [order, extént])
2. Расширенная матрица (системы линейных уравнений)	Розширена матриця (системи лінійних рівнянь)	Augménted [dilated, exténded] mátrix (of a sýstem of línear equátions)
3. Базисный/главный минор	Базисний/головний мінор	Báse/básic/príncipal mínor
4. Базисные строки/столбцы главной матрицы [матрицы системы, матрицы коэффициентов при неизвестных]	Базисні рядки/стовпці головної матриці [матриці системи, матриці коефіцієнттів при невідомих]	Báse/básic rо́ws/cólumns of a príncipal/léading mátrix [sýstem mátrix, mátrix of còeffícients of ùnknо́wns]
5. Базисная неизвестная	Базисна невідома	Báse/básic ùnknо́wn
6. Матрица коэффициентов (при неизвестных)	Матриця коэфіцієнтів (при невідомих)	Còeffícient mátrix; mátrix of còeffícients (of ùnknо́wns)
7. Матрица-столбец свободных членов	Матриця-стовпець вільних членів	Cólumn mátrix of áb-solute [free,cónstant] terms
8. Матрица-столбец неизвестных	Матриця-стовпець невідомих	Cólumn mátrix of ùnknо́wns
9. Матрица-столбец (m×1, с m элементами)	Матриця-стовпець (m×1, з m елементами)	Column matrix [one-by-m cólumn mátrix, single-cólumn mátrix] (with m élements)
10.Определитель квадратной матрицы	Визначник квадратної матриці	Detérminant of a square [quadrátic] mátrix
11.Элементарное преобразование	Елементарне перетворення	Èleméntary trànsformátion
12.Минор наивысшего порядка, отличный от нуля	Мінор найвищого порядку, відмінний від нуля	Híghest órder nonzéro mínor [nonzéro mínor of the híghest órder]
13.Единичная матрица	Одинична матриця	Únit [únity, idéntity] mátrix
14.Обратное линейное преобразование неизвестных	Обернене лінійне перетворення невідомих	Ìnvérse línear trànsformátion of unknowns
15.Обратная матрица	Обернена матриця	Ìnvérse mátrix
16.Главная диагональ квадратной матрицы	Головна діагональ квадратної матриці	Príncipal [léading, máin, pósitive] diágonal of a square [quadrátic] mátrix

17.Главная матрица системы	Головна матриця системи	Príncipal [léading, sýstem] mátrix, mátrix of a sýstem
18.Линейная операция над матрицами	Лінійна операція над матрицями	Línear operátion on mátrices
19.Линейное преобразование неизвестных	Лінійне перетворення невідомих	Línear trànsformátion of ùnknówns
20.Матрица	Матриця	Mátrix (*pl* mátrices [mátrixes])
21.Матричное уравнение	Матричне рівняння	Mátrix equátion
22.Матрица размера $m \times n$	Матриця розміру $m \times n$	Mátrix of dimension [órder, extént] $m \times n$
23.Матрица линейного преобразования неизвестных	Матриця лінійного перетворення невідомих	Mátrix of a línear trànsformátion of ùnknówns; trànsformátion mátrix
24.Матрица с m строками и n столбцами	Матриця з m рядками і n стовпцями	Mátrix with m róws and n cólumns
25.Минор (первого, второго, третьего, n-го порядка) матрицы	Мінор (першого, другого, третього, n-го порядку) матриці	Minor (of the first, second, third, n-th order) [(first-, second-, third-, n-th órder) mínor] of a mátrix
26.Умножение матрицы (на число, на матрицу (слева, справа))	Множення матриці (на число, на матрицю (зліва, справа))	Mùltiplicátion of a mátrix (by a númber, by a mátrix (from the left [right]))
27.Умножить матрицу (на число, на матрицу (слева, справа))	Помножити матрицю (на число, на матрицю (зліва, справа))	Múltiply a mátrix (by a númber, by a mátrix (from the left [right]))
28.Отрицательно определенная квадратичная форма	Від"ємно визначена квадратична [квадратна] форма	Négativ définite quadrátic form
29.Положительно определённая квадратичная форма	Додатно визначена квадратична [квадратна] форма	Pósitive définite quadrátic form
30.Главный минор квадратичной формы	Головний мінор квадратичної [квадратної] форми	Príncipal mínor of a quadrátic form
31.Главный [базисный] минор	Головний [базисний] мінор	Príncipal [base] mínor
32.Произведение матрицы (на число, на матрицу (слева, справа))	Добуток матриці (на число, на матрицю (зліва, справа))	Próduct of a mátrix (by a númber, by a mátrix (from the left [right]))
33.Произведение линейных преобразований не-	Добуток лінійних перетворень невідомих	Próduct of línear trànsformátions of ùnknówns

известных		
34.Произведение матриц	Добуток матриць	Próduct of mátrices
35.Квадратичная форма	Квадратична [квадратна] форма	Quadrátic form
36.Ранг матрицы	Ранг матриці	Rank of a mátrix
37.Невырожденная [неосоая] матрица	Невироджена/неособлива матриця	Régular [nónsíngular] mátrix
38.Результат последовательного выполнения (двух) линейных преобразоваий неизвестных	Результат послідовного виконання (двох) лінійних перетворень невідомих	Resúlt of succéssive [sequential, consécutive] realization [fulfillment, accómplishment] of (two) línear trànsformátions of ùnknówns
39.Матрица-строка (1 × *n*, с *n* элементами)	Матриця-рядок (1 × *n*, з *n* елементами)	Row matrix [one-by-*n* row mátrix, síngle-row mátrix] (with *n* élements)
40.Вырожденная матрица	Вироджена/особлива матриця	Síngular mátrix
41.Квадратная матрица (первого, второго, третьего, *n*-го порядка)	Квадратна матриця (першого, другого, третього, *n*-го порядку)	Squáre [quadrátic] mátrix (of the first, second, third, *n*-th órder); (first-, second-, third-, *n*-th órder) square [quadrátic] mátrix; *n*-by-*n* mátrix
42.Матрица системы	Матриця системи	Sýstem mátrix, mátrix of a sýstem
43.Наивысший порядок миноров (отличных от нуля)	Найвищий порядок мінорів (відмінних від нуля)	The híghest órder of (the nòn-zéro/non-null, the dífferent from zéro/null) mínors
44.Транспонированная матрица	Транспонована матриця	Transpósed mátrix [transpóse of a mátrix]

MATRIX AND SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS: basic terminology ERU

1. Addítion of mátrices (of the sáme dimension [order, extént])	Сложение матриц (одного и того же размера)	Додавання матриць (одного й того ж розміру)
2. Augménted [dilated, exténded] mátrix (of a sýstem of línear equátions)	Расширенная матрица (системы линейных уравнений)	Розширена матриця (системи лінійних рівнянь)
3. Báse/básic/príncipal mínor	Базисный/главный минор	Базисний/головний мінор
4. Báse/básic rо́ws/cólumns of a príncipal/léading mátrix [of the sýstem mátrix, of the mátrix of còeffícients of ùnknо́wns]	Базисные строки/столбцы главной матрицы [матрицы системы, матрицы коэффициентов при неизвестных]	Базисні рядки/стовпці головної матриці [матриці системи, матриці коефіцієнтів при невідомих]
5. Báse/básic ùnknо́wn	Базисная неизвестная	Базисна невідома
6. Còeffícient mátrix; mátrix of còeffícients (of ùnknо́wns)	Матрица коэффициентов (при неизвестных)	Матриця коэфіцієнтів (при невідомих)
7. Cólumn mátrix of ábsolute [free,cónstant] terms	Матрица-столбец свободных членов	Матриця-стовпець вільних членів
8. Cólumn mátrix of ùnknо́wns	Матрица-столбец неизвестных	Матриця-стовпець невідомих
9. Column matrix [one-by-m cólumn mátrix, single-cólumn mátrix] (with m élements)	Матрица-столбец ($m\times1$, с m элементами)	Матриця-стовпець ($m\times1$, з m елементами)
10. Detérminant of a square [quadrátic] mátrix	Определитель квадратной матрицы	Визначник квадратної матриці
11. Èleméntary trànsformátion	Элементарное преобразование	Елементарне перетворення
12. Híghest órder nonzéro mínor [nonzéro mínor of the híghest órder]	Минор наивысшего порядка, отличный от нуля	Мінор найвищого порядку, відмінний від нуля
13. Únit [únity, idéntity] mátrix	Единичная матрица	Одинична матриця
14. Ìnvérse línear trànsformátion of unknowns	Обратное линейное преобразование неизвестных	Обернене лінійне перетворення невідомих
15. Ìnvérse mátrix	Обратная матрица	Обернена матриця
16. Príncipal [léading, máin, pósitive] diágonal of a square [quadrátic] mátrix	Главная диагональ квадратной матрицы	Головна діагональ квадратної матриці

17.Príncipal [léading, sýstem] mátrix [mátrix of a sýstem]	Главная матрица системы	Головна матриця системи
18.Línear operátion on mátrices	Линейная операция над матрицами	Лінійна операція над матрицями
19.Línear trànsformátion of ùnknówns	Линейное преобразование неизвестных	Лінійне перетворення невідомих
20.Mátrix (*pl* mátrices [mátrixes])	Матрица	Матриця
21.Mátrix equátion	Матричное уравнение	Матричне рівняння
22.Mátrix of dimension [órder, extént] $m \times n$	Матрица размера $m \times n$	Матриця розміру $m \times n$
23.Mátrix of a línear trànsformátion of ùnknówns; trànsformátion mátrix	Матрица линейного преобразования неизвестных	Матриця лінійного перетворення невідомих
24.Mátrix with m róws and n cólumns	Матрица с m строками и n столбцами	Матриця з m рядками і n стовпцями
25.Minor (of the first, second, third, n-th order) [(first-, second-, third-, n-th órder) mínor] of a mátrix	Минор (первого, второго, третьего, n-го порядка) матрицы	Мінор (першого, другого, третього, n-го порядку) матриці
26.Mùltiplicátion of a mátrix (by a númber, by a mátrix (from the left [right]))	Умножение матрицы (на число, на матрицу (слева, справа))	Множення матриці (на число, на матрицю (зліва, справа))
27.Múltiply a mátrix (by a númber, by a mátrix (from the left [right]))	Умножить матрицу (на число, на матрицу (слева, справа))	Помножити матрицю (на число, на матрицю (зліва, справа))
28.Négativ définite quadrátic form	Отрицательно определенная квадратичная форма	Від"ємно визначена квадратична [квадратна] форма
29.Pósitive définite quadrátic form	Положительно определённая квадратичная форма	Додатно визначена квадратична [квадратна] форма
30.Príncipal mínor of a quadrátic form	Главный минор квадратичной формы	Головний мінор квадратичної [квадратної] форми
31.Príncipal [base] mínor	Главный [базисный] минор	Головний [базисний] мінор
32.Próduct of a mátrix (by a númber, by a mátrix (from the left [right]))	Произведение матрицы (на число, на матрицу (слева, справа))	Добуток матриці (на число, на матрицю (зліва, справа))
33.Próduct of línear tràns-	Произведение линейных	Добуток лінійних перет-

formátions of ùnknówns	преобразований неизвестных	ворень невідомих
34.Próduct of mátrices	Произведение матриц	Добуток матриць
35.Quadrátic form	Квадратичная форма	Квадратична [квадратна] форма
36.Rank of a mátrix	Ранг матрицы	Ранг матриці
37.Régular [nónsíngular] mátrix	Невырожденная [неособая] матрица	Невироджена/неособлива матриця
38.Resúlt of succéssive [sequential, consécutive] realization [fulfillment, accómplishment] of (two) línear trànsformátions of ùnknówns	Результат последовательного выполнения (двух) линейных преобразований неизвестных	Результат послідовного виконання (двох) лінійних перетворень невідомих
39.Row matrix [one-by-n row mátrix, síngle-row mátrix] (with n élements)	Матрица-строка ($1 \times n$, с n элементами)	Матриця-рядок ($1 \times n$, з n елементами)
40.Síngular mátrix	Вырожденная матрица	Вироджена/особлива матриця
41.Squáre [quadrátic] mátrix (of the first, second, third, n-th órder); (first-, second-, third-, n-th órder) square [quadrátic] mátrix; n-by-n mátrix	Квадратная матрица (первого, второго, третьего, n-го порядка)	Квадратна матриця (першого, другого, третього, n-го порядку)
42.Sýstem mátrix, mátrix of a sýstem	Матрица системы	Матриця системи
43.The híghest órder of (the nòn-zéro/non-null, the dífferent from zéro/null) mínors	Наивысший порядок миноров (отличных от нуля)	Найвищий порядок мінорів (відмінних від нуля)
44.Transpósed mátrix [transpóse of a mátrix]	Транспонированная матрица	Транспонована матриця

ANALYTIC GEOMETRY
ANALYTIC GEOMETRY ON THE PLANE

LECTURE NO. 7. A STRAIGHT LINE AND A CIRCLE

POINT 1. LINE EQUATION. CIRCLE
POINT 2. A STRAIGHT LINE
POINT 3. MUTUAL DISPOSITION OF TWO STRAIGHT LINES
POINT 4. EXAMPLES

POINT 1. LINE EQUATION. CIRCLE

Analytic geometry is a branch of mathematics where geometric problems are solved by analytical methods. In the basis of all these methods lies the conception of coordinate system (coordinate method). Foundations of analytic geometry have laid Viète[1], Fermat[2] and Descartes[3].

Let be given so-called Cartesian rectangular coordinate system on the xOy-plane, which is generated by two mutually perpendicular axes Ox, Oy with the same unit length [unit measure] (see fig. 1). We'll give at first examples of application of coordinate method to two simplest problems.

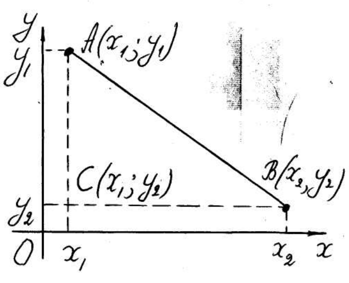


Fig. 1

Ex. 1. Find the distance between two points $A(x_1;\, y_1)$ and $B(x_2;\, y_2)$ with known coordinates (fig. 1).

By Pythagorean theorem

$$AB = \sqrt{CB^2 + CA^2}\,;\; CB = |x_2 - x_1|,\; CB^2 = |x_2 - x_1|^2 = (x_2 - x_1)^2;\; CA^2 = (y_2 - y_1)^2,$$

and therefore

[1] Viète, F. (1540 - 1603), a French mathematician
[2] Fermat, P. (1601 - 1665), a famous French mathematician
[3] Descartes, R. (1596 - 1650), a famous French mathematician and philosopher

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ . \qquad (1)$$

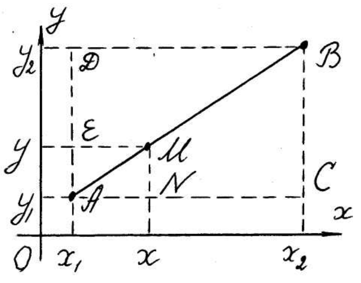


Fig. 2

Ex. 2. Find coordinates of a point $M(x;\, y)$ which divides a segment with end points $A(x_1;\, y_1)$, $B(x_2;\, y_2)$ in the given ratio λ, that is (fig. 2)

$$\frac{AM}{MB} = \lambda\ .$$

By corresponding geometrical theorem

$$\frac{AM}{MB} = \frac{AN}{NC} = \frac{x - x_1}{x_2 - x} = \lambda \Rightarrow x = \frac{x_1 + \lambda x_2}{1+\lambda};\ \frac{AM}{MB} = \frac{AE}{ED} = \frac{y - y_1}{y_2 - y} = \lambda \Rightarrow y = \frac{y_1 + \lambda y_2}{1+\lambda}.$$

Thus,

$$x = \frac{x_1 + \lambda x_2}{1+\lambda};\ y = \frac{y_1 + \lambda y_2}{1+\lambda}. \qquad (2)$$

If in particular a point $M_0(x_0;\, y_0)$ divides a segment $A(x_1;\, y_1)B(x_2;\, y_2)$ in half, then λ = 1 and the coordinates of the point $M_0(x_0;\, y_0)$ will be equal

$$x_0 = \frac{x_1 + x_2}{2};\ y_0 = \frac{y_1 + y_2}{2}$$

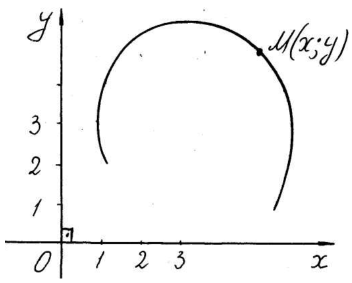


Fig. 3

Def. 1. Equation of the form

$$F(x,\, y) = 0 \qquad (3)$$

is called that of a line *L* on the xOy - plane (fig. 3) if coordinates of each point $M(x;\, y)$ of the line, and only coordinates of such points, satisfy this equation.

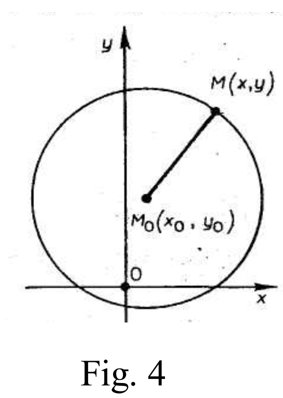


Fig. 4

A point $M(x;\, y)$ is an arbitrary [current] point of a line *L*, and its coordinates are called those current. Equation of any line must contain at least one current coordinate.

Ex. 3. Let a line *L* is determined by an equation $2x + y = 3$. A point $M(1;\, 1)$ belongs *L* because of $2 \cdot 1 + 1 = 3$ that is its coordinates satisfy the line equation, but a point $N(2;\, -3)$ doesn't belong to it for $2 \cdot 2 + (-3) = 1 \neq 3$.

Ex. 4. Write an equation of a circle (circumference) of a radius *R* and a centre $M_0(x_0; y_0)$ (fig. 4).

For an arbitrary point $M(x; y)$ of a circle we can write

$$M_0M^2 = R^2.$$

Using further the distance formula (1) we obtain the equation

$$(x - x_0)^2 + (y - y_0)^2 = R^2 \qquad (4)$$

which is called the **canonical equation** of a circle.

Ex. 5. A particular case of the equation (4)

$$x^2 + y^2 = R^2 \qquad (5)$$

is the equation of the circle of radius *R* centered at the origin. Resolving the equation with respect to *y* an then to *x* we get

a) the equation of the lower semi-circle $y = -\sqrt{R^2 - x^2}$;

b) the equation of the upper semi-circle $y = \sqrt{R^2 - x^2}$;

c) the equation of the left semi-circle $x = -\sqrt{R^2 - y^2}$;

d) the equation of the right semi-circle $x = \sqrt{R^2 - y^2}$.

Ex. 6. Let $x^2 - 4x + y^2 + 10y + 5 = 0$. Which line describes this equaton?

Completing the squares,

$$x^2 - 4x + 4 + y^2 + 10y + 25 - 4 - 25 + 5 = 0, (x - 2)^2 + (y + 5)^2 = 24,$$

we conclude that the given equation is that of a circle with the radius $R = 2\sqrt{6}$ and the centre $M(2; -5)$.

To find the **intersection point** of two lines l_1, l_2 which are represented by equations

$$l_1 : F_1(x, y) = 0, \quad l_2 : F_2(x, y) = 0$$

it's sufficient to solve a system of equations

$$\begin{cases} F_1(x, y) = 0, \\ F_2(x, y) = 0. \end{cases} \qquad (6)$$

POINT 2. A STRAIGHT LINE

Def. 2. Let α is an angle a straight line l forms with positive direction of the Ox-axis (fig. 5). Tangent of this angle is called the **slope** [the **angular coefficient**] of the straight line and is denoted by

$$k = tan\,\alpha\,. \tag{7}$$

Equation of a straight line with angular coefficient [slope intercept (form of the) equation of a straight line].

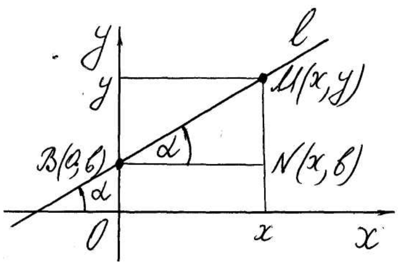


Fig. 5

If $B(0;b)$ is the intersection point of a straight line l with the Oy-axis, then the equation of the line l is

$$y = kx + b\,. \tag{8}$$

■Let $M(x;\,y)$ be an arbitrary point of a straight line, then (see fig. 5)

$$k = tan\,\alpha = \frac{NM}{BN} = \frac{y-b}{x} \Rightarrow kx = y - b,\, y = kx + b\,.$$

Ex. 7. Equation of a straight line passing through a point $B(0;-7)\in Oy$ under the angle $\alpha = \pi/3$ to the Ox-axis is

$$y = \sqrt{3}x - 7\,\left(k = tan\,\pi/3 = \sqrt{3},\, b = -7\right).$$

Ex. 8. Equation of the Ox-axis is

$$y = 0\,(k = b = 0).$$

Equation of a straight line which is perpendicular to the Ox-axis and passes through a given point

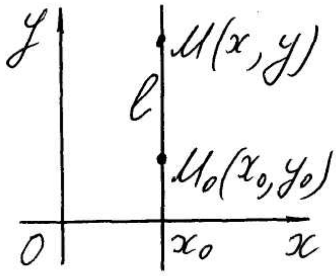


Fig. 6

Let a straight line l is perpendicular to the Ox-axis and passes through a point $M_0(x_0;\,y_0)$ (fig. 6) (the slope of such the straight line doesn't exist). Abscissa x of an arbitrary point $M\,(x;\,y)$ of line always equals x_0, therefore the line equation is

$$x = x_0 \tag{9}$$

Ex. 9. Equation of the *Oy*-axis is

$$x = 0\,(x_0 = 0).$$

General equation of a straight line

The equations (8), (9) are particular cases of an equation of the next form:

$$ax + by + c = 0.$$

■ $y = kx + b \Rightarrow kx + (-1)y + b = 0;\ x = x_0 \Rightarrow 1 \cdot x + 0 \cdot y + (-x_0) = 0$ ■

Inversely an equation

$$ax + by + c = 0 \quad (a^2 + b^2 \neq 0) \tag{10}$$

in which at least one of coefficients *a*, *b* isn't zero (otherwise $a^2 + b^2 = 0$), is the equation of a straight line.

■1) Let at first $b \neq 0$. The equation (10) yields

$$by = -ax - c,\ y = -a/b \cdot x - c/b,$$

that is the equation of a straight line with angular coefficient

$$k = -\frac{b}{a}. \tag{11}$$

2) If now $b = 0$, then (by $a^2 + b^2 \neq 0$) $a \neq 0$, and the equation (10) takes on the form $ax + c = 0$, whence it follow the equation

$$x = -c/a$$

which is that of a straight line perpendicular to the *Ox*-axis.■

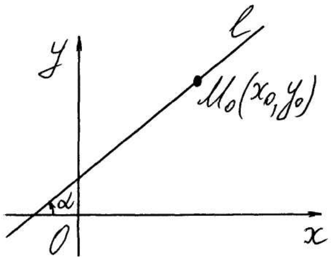
One can say that there is **one-to-one correspondence** between the set of all straight lines on the *xOy*-plane and the set of all equations of the form (10).

Equation (10) (with additional condition $a^2 + b^2 \neq 0$) is called the general equation of a straight line on the *xOy*-plane.

Ex. 10. It follows from the general equation of a straight line $5x + 6y = 30$ that its slope equals $k = -5/6$, because of $6y = 30 - 5x,\ y = -5/6\,x + 5$.

Equation of a straight line passing through a given point in a given direction

Let a straight line *l* passes through a point $M_0(x_0, y_0)$ and has a given slope $k = tan\,\alpha$ (fig. 7). For any point $M(x; y)$ of the line the equation (8) holds.



Coordinates of the given point $M_0(x_0, y_0)$ satisfy this equation, namely $y_0 = kx_0 + b$. If we subtract termwise [term by term, term-by-term] this equality from the equation (8), we'll get

$y - y_0 = (kx + b) - (kx_0 + b)$, $y - y_0 = kx - kx_0$,

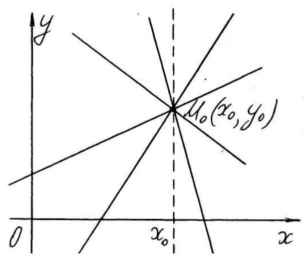
Fig. 7

and therefore the equation in question is the next one:

$$y - y_0 = k(x - x_0). \qquad (12)$$

Ex. 11. Equation of a straight line which passes through a point $M_0(2; -3)$ and forms an angle $3/4\pi$ with positive direction of the *Ox*-axis is

$$y - (-3) = tan\,3/4\,\pi \cdot (x - 2),\ y + 3 = -(x - 2) \text{ or finally } x + y + 1 = 0$$

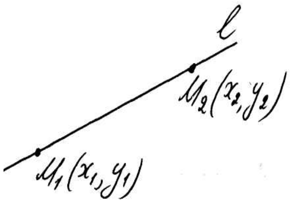


Def. 3. A set of all straight lines, which pass through a point $M_0(x_0, y_0)$ (fig. 8) is called a pencil [a bunch, a bundle] of straight lines with the centre $M_0(x_0, y_0)$.

Equation (12) gives all straight lines of the pencil excluding the line $x = x_0$ which is perpendicular to the *Ox*-axis and hasn't a slope. By this reason this equation often is called the equation of a pencil of straight lined with the centre $M_0(x_0, y_0)$.

Fig. 8

Equation of a straight line passing throught two given points [two póint (form of the) equátion of a straight line].



Let it's necessary to write the equation of a straight line passing through two given points $M_1(x_1; y_1)$, $M_2(x_2; y_2)$ (fig. 9). Using the equation (12) for the first point $M_1(x_1; y_1)$ we at first ha-

Fig. 9 ve $y-y_1=k(x-x_1)$, and it's needed to determine the slope of the straight line M_1M_2. But the coordinates of the point $M_2(x_2;\, y_2)$ must satisfy the equation, hence

$$y_2-y_1=k(x_2-x_1)$$

and therefore the slope in question is

$$k=\frac{y_2-y_1}{x_2-x_1}. \qquad (13)$$

Substituting the found value of the slope we get

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad (14)$$

or (after dividing both sides by $y-y_1$)

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \qquad (15)$$

Ex. 12. Find the centre of gravity of a triangle with vertices $A(3;-4)$, $B(5;6)$, $C(7;-8)$.

The centre of gravity of a triangle is the intersection point of its medians. Coordinates of the midpoint *D* of the side *BC* of the triangle are

$$x_D=\frac{x_B+x_C}{2}=\frac{5+7}{2}=6;\ y_D=\frac{y_B+y_C}{2}=\frac{6+(-8)}{2}=-1;\ D(6;-1),$$

and the equation of the median *AD*, by virtue of the equation (15), is

$$\frac{x-x_A}{x_D-x_A}=\frac{y-y_A}{y_D-y_A},\ \frac{x-3}{6-3}=\frac{y-(-4)}{-1-(-4)},\ \frac{x-3}{3}=\frac{y+4}{3},\ x-y-7=0.$$

By the same way we find the midpoint $E(5;-6)$ of the side *AC* and the equation of the median *BE*

$$\frac{x-x_B}{x_E-x_B}=\frac{y-y_B}{y_E-y_B},\ \frac{x-5}{5-5}=\frac{y-6}{-6-(-6)},\ \frac{x-5}{0}=\frac{y-6}{-12},\ x-5=0.$$

To find the intersection point *M* of the medians *AD* and *BE*, we solve the system of equations

$$\begin{cases} x-y-7=0, \\ x-5=0, \end{cases}$$

whence $x=5,\ y=-2$. The centre of gravity of a triangle is the point $M(5;-2)$.

Ex. 13. Set up an approximate equation of the bisector of an inner angle at a vertex A of a triangle with vertices $A(-1;5), B(-2;-2), C(5;5)$.

Solution of the problem consists in three steps.

1. The sought bisector intersects the opposite side BC of the triangle in certain point D. It divides BC in the ratio $BD:DC$ which is equal to $\lambda = AB:AC$. By the formula (1)

$$AB=\sqrt{(x_A-x_B)^2+(y_A-y_B)^2}=\sqrt{(-1-(-2))^2+(5-(-2))^2}=\sqrt{50},\ AC=6,$$

$$\lambda=\frac{AB}{AC}=\frac{\sqrt{50}}{6}\approx 1.2.$$

2. On the base of the formula we find the coordinates of the point D, namely

$$x_D=\frac{x_B+\lambda x_C}{1+\lambda}\approx\frac{-2+1.2\cdot 5}{1+1.2}\approx 1.8,\ y_D=\frac{y_B+\lambda y_C}{1+\lambda}\approx\frac{-2+1.2\cdot 5}{1+1.2}\approx 1.8;\ D(1.8;1.8).$$

3. Finally we write the approximate equation of the bisector AD as that of the straight line passing through two given points $A(-1;5), D(1.8;1.8)$; by (15)

$$\frac{x-x_A}{x_D-x_A}=\frac{y-y_A}{y_D-y_A},\ \frac{x-(-1)}{1.8-(-1)}=\frac{y-5}{1.8-5},\ \frac{x+1}{2.8}=\frac{y-5}{-3.2},\ \begin{array}{c}3.2x+2.8y-10.8=0,\\ 8x+7y-27=0.\end{array}$$

Equation of a straight line in segments [two-intercept (form of the) equation of a straight line]

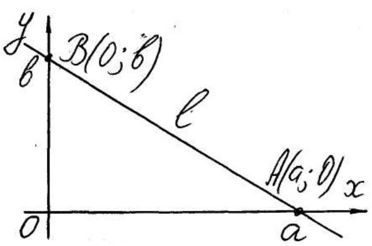


Fig. 10

Let a straight line l cuts segments OA, OB from the axes Ox, Oy correspondingly, and $A(a;0)\in Ox, B(0;b)\in Oy$ (fig. 10).

We use the equation (15) for two given points A and B of the straight line and obtain

$$\frac{x-a}{0-a}=\frac{y-0}{b-0},\ \frac{x}{-a}+\frac{-a}{-a}=\frac{y}{b},\ -\frac{x}{a}+1=\frac{y}{b},$$

$$\frac{x}{a}+\frac{y}{b}=1. \qquad (16)$$

Ex. 14. Find the area of a triangle bounded by a straight line $3x+5y+30=0$ and the coordinate axes.

Let's reduce the equation of the straight line to two-intercept form (16),

$$3x+5y=-30, \frac{3}{-30}x+\frac{5}{-30}y=1, \frac{x}{-10}+\frac{y}{-6}=1 \quad (a=-10, b=-6).$$

The sought area equals

$$S=\frac{1}{2}|a||b|=\frac{1}{2}\cdot 10\cdot 6=30 \text{ (units area).}$$

POINT 3. MUTUAL DISPOSITION OF TWO STRAIGHT LINES

Angle between two straight lines

Let be given two straight lines l_1, l_2 with slopes $k_1=\tan\alpha_1, k_2=\tan\alpha_2$ (fig. 11), and it's required to find the angle φ between them. We see from the figure 11 that

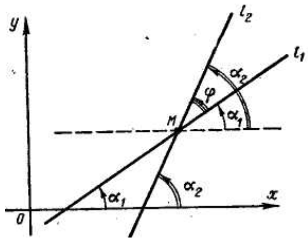


Fig. 11

$$\varphi=\alpha_2-\alpha_1,$$

whence it follows that

$$\tan\varphi=\tan(\alpha_2-\alpha_1)=\frac{\tan\alpha_2-\tan\alpha_1}{1+\tan\alpha_1\tan\alpha_2}=\frac{k_2-k_1}{1+k_1k_2}.$$

Thus the angle φ between two straight lines is given by the next formula:

$$\tan\varphi=\frac{k_2-k_1}{1+k_1k_2}. \qquad (17)$$

Ex. 15. Find the angle between two straight line represented by their general equations: $(l_1): 3x-4y=12;\quad (l_2): 2x+5y=20$.

Evaluating the slopes of the straight lines we use after that the formula (17).

$$(l_1): y = \frac{3}{4}x - 3\,(k_1 = 0.75);\ (l_2): y = -\frac{2}{5}x + 4\,(k_2 = -0.4);$$
$$tan\,\varphi = \frac{(-0.4) - 0.75}{1 + (-0.4)\cdot 0.75} \approx -1.6 \Rightarrow \varphi \approx -arctan 1.6.$$

Parallelism and perpendicularity conditions of straight lines

Formula (17) permits to establish necessary and sufficient conditions for parallelism and perpendicularity of two straight lines.

Two straight lines are parallel if and only if their slopes are equal,

$$l_1 \,||\, l_2 \Leftrightarrow k_1 = k_2. \tag{18}$$

Two straight lines are perpendicular if and only if their slopes satisfy the next condition:

$$l_1 \perp l_2 \Leftrightarrow k_1 k_2 = -1. \tag{19}$$

Ex. 16. Compile an equation of a straight line passing through the vertex A of the triangle *ABC* of the Ex. 13 parallel to its side *BC*.

The slope *k* of the desired line equals $k = k_{BC}$. The formula (13) gives

$$k = k_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{5-(-2)}{5-(-2)} = \frac{7}{7} = 1,$$

an in correspondence to the formula (12) we have

$$y - y_A = k(x - x_A),\ y - 5 = 1\cdot(x - (-1)),\ y - 5 = x + 1,\ x - y + 6 = 0.$$

Ex. 17. Set up an equation of the altitude of the same triangle *ABC* dropped from the vertex *A*.

If we denote k_{alt} the slope of desired altitude, then by virtue of the perpendicularity condition

$$k_{alt}\cdot k_{BC} = -1$$

whence it follows that

$$k_{alt} = -\frac{1}{k_{BC}} = -\frac{1}{1} = -1,$$

and the equation of the altitude, by the equation (12), will be

$$y - y_A = k_{alt}(x - x_A),\ y - 5 = -1\cdot(x - (-1)),\ y - 5 = -x - 1,\ x + y - 4 = 0.$$

POINT 4. EXAMPLES

Ex. 18. Find a point Q which is symmetric to a point $P(-8;12)$ about a straight line passing through two given points $A(2;-3), B(-5;1)$ (see fig. 12). Find the distance of the point $P(-8;12)$ from the straight line *AB*.

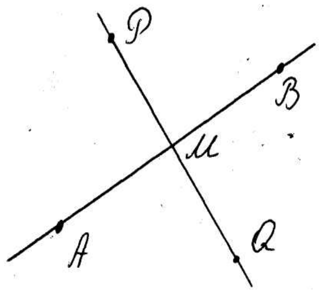


Fig. 12

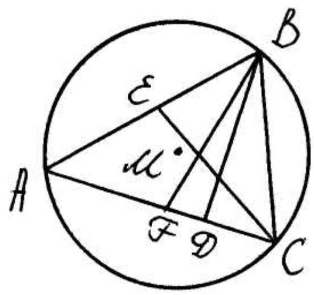


Fig. 13

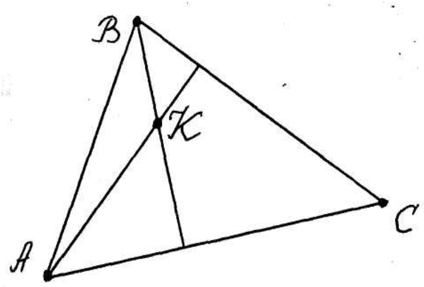


Fig. 14

1. We set up the equation of the straight line *AB* (by (15))

$$\frac{x-x_A}{x_B-x_A}=\frac{y-y_A}{y_B-y_A}, \frac{x-2}{(-5)-2}=\frac{y-(-3)}{1-(-3)}, \frac{x-2}{-7}=\frac{y+3}{4} \Rightarrow 4x+7y+13=0, k_{AB}=-\frac{4}{7}.$$

2. We write the equation of the straight line *PQ* which is perpendicular to *AB* and passes through the point $P(-8;12)$. On the base of perpendicularity condition

$k_{PQ}\cdot k_{AB}=-1, k_{PQ}=-\frac{1}{k_{AB}}=\frac{7}{4}$. Using the equation (12) we obtain

$$y-y_P=k_{PQ}(x-x_P), y-12=\frac{7}{4}(x-(-8)), 4y-48=7x+56, 7x-4y+104=0.$$

3. Let's find an intersection point *M* of the straight lines *PQ* and *AB*. Solving the corresponding system of equations

$$\begin{cases} 4x+7y+13=0, \\ 7x-4y+104=0, \end{cases}$$

we get $x=x_M=-12, y=y_M=5 \Rightarrow M(-12;5)$.

4. To find the point *Q* in question we take into account that the found point $M(-12;5)$ is a middle [a midpoint] of a segment *PQ* and so

$$x_M=\frac{x_P+x_Q}{2} \Rightarrow x_Q=2x_M-x_P=-24-(-8)=-16, y_Q=2y_M-y_P=10-12=-2.$$

Thus the required point is $Q(-16; -2)$.

5. To find the distance of the point $P(-8; 12)$ from the straight line *AB* it's sufficient to calculate the distance of the points $P(-8; 12)$ and $M(-12; 5)$,

$$d_{P,AB} = PM = \sqrt{(x_P - x_M)^2 + (y_P - y_M)^2} = \sqrt{((-8)-(-12))^2 + (12-5)^2} =$$
$$= \sqrt{4^2 + 7^2} = \sqrt{16+49} = \sqrt{65} \approx 8.06.$$

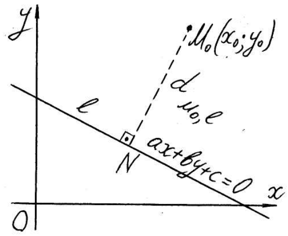


Fig. 15

Note. Distance of a point $M_0(x_0; y_0)$ from a straight line l represented by its general equation

$$ax + by + c = 0$$

(fig. 15) can be calculated with the help of the next general formula

$$d = d_{M_0,l} = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}. \qquad (20)$$

It's a particular case of more general formula what will be proved below.

Ex. 19. Let there be given vertices $A(4; -5), B(7; 1), C(-2; 6)$ of a triangle *ABC* (see fig. 13). Set up yourselves equations of the height *BD*, the median *CE* and the bisector *BF*. Compile the equation of the circumcircle [of the circumscribed circle] of the triangle.

Let a point $M(a; b)$ be the centre of the circumcircle. It means that

$$\begin{cases} MA = MB, \\ MA = MC; \end{cases} \begin{cases} MA^2 = MB^2, \\ MA^2 = MC^2; \end{cases} \begin{cases} (a-4)^2 + (b+5)^2 = (a-7)^2 + (b-1)^2, \\ (a-4)^2 + (b+5)^2 = (a+2)^2 + (b-6)^2. \end{cases}$$

We come to the system of equations in coordinates of the centre $M(a; b)$. Squaring and collecting similar terms we reduce it to the next system of linear equations

$$\begin{cases} 2a + 4b = 3, \\ 12a - 22b = 1; \end{cases} a = \frac{35}{46} \approx 0.76, b = \frac{17}{46} \approx 0.37.$$

The square of the radius of the circumcircle equals

$$R^2 = MA^2 \approx (0.76-4)^2 + (0.37+5)^2 \approx 39.33.$$

Approximate equation of the circumcircle in question

$$(x-0.76)^2+(y-0.37)^2=39.33.$$

Ex. 20. Two vertices $A(3;\,6), B(-4;\,-3)$ and the intersection point $K(4;\,1)$ of altitudes of a triangle *ABC* are given (fig. 14). Find coordinates of the vertex *C*.

Solution.

1. We find the slope k_{AK} of the altitude *AK* knowing the coordinates of the points *A*, *K* and using the formula (15):

$$k_{AK}=\frac{y_K-y_A}{x_K-x_A}=\frac{1-6}{4-3}=-5.$$

2. With the help of the perpendicularity condition $k_{AK}\cdot k_{BC}=-1$ of the straight lines AK, BC we find the slope k_{BC} of the staight line *BC*,

$$k_{BC}=-\frac{1}{k_{AK}}=\frac{1}{5}.$$

3. Now we compile the equation of the straight line *BC* using the equation (12):

$$y=y_B+\frac{1}{5}(x-x_B),\quad y=-3+\frac{1}{5}(x+4),\quad x-5y-11=0\quad (BC).$$

4. By the same way we compile the equation of the straight line AC:

$$k_{BK}=\frac{y_K-y_B}{x_K-x_B}=\frac{1+3}{4+4}=\frac{1}{2};\quad k_{AC}=-\frac{1}{k_{BK}}=-2;$$

$$y=y_A-2(x-x_A),\quad y=6-2(x-3),\quad 2x+y-12=0\quad (AC).$$

5. Finally we find the coordinates of the point *C* as the intersection point of the straight lines *BC, AC* , is we'll solve the system of equations of these lines.

$$\begin{cases}x-5y-11=0,\\ 2x+y-12=0;\end{cases}\quad \begin{cases}x-5y-11=0,\\ 11y+10=0;\end{cases}\quad y=-\frac{10}{11}\Rightarrow x=11+5x=\frac{71}{11}\Rightarrow C\left(\frac{71}{11};-\frac{10}{11}\right).$$

Answer. The point *C* in question is

$$C\left(\frac{71}{11};-\frac{10}{11}\right).$$

Ex. 21. Write equations of the tangents to a circle $x^2+y^2=R^2$ drawn [lined] from a point $M_0(x_0;\,0)$ of the *Ox*-axis ($|x_0|>R$).

We find the tangents in the form (12)

$$y = k(x - x_0).$$

At first we seek intersection points of the lines $y = k(x - x_0)$ with the circle solving a system of equations

$$\begin{cases} x^2 + y^2 = R^2, \\ y = k(x - x_0). \end{cases}$$

Substitution of the value of y in the equation of the circle gives

$$x^2 + k^2(x - x_0)^2 = R^2, x^2 + k^2x^2 - 2k^2xx_0 + k^2x_0^2 - R^2 = 0,$$
$$(1 + k^2)x^2 - 2k^2xx_0 + k^2x_0^2 - R^2 = 0.$$

To get the tangency (not intersection) of the circle and the straight lines we must equate the discriminant of the quadratic equation to zero, whence it follows that

$$k^4x_0^2 - (1 + k^2)(k^2x_0^2 - R^2) = 0,\ k^4x_0^2 - k^2x_0^2 + R^2 - k^4x_0^2 + k^2R^2 = 0, k^2(x_0^2 - R^2) = R^2,$$

$$k^2 = \frac{R^2}{x_0^2 - R^2},\ k = \pm\sqrt{\frac{R^2}{x_0^2 - R^2}}\ (|x_0| > R).$$

The sought equations of the tangents to the given circle are

$$y = \pm\sqrt{\frac{R^2}{x_0^2 - R^2}} \cdot (x - x_0).$$

Ex. 22. Compile the equation of a circle which touches [tangents] a given straight line $5x - 9y + 3 = 0$ and is centered at a point $A(-4; 7)$.

A radius of the circle equals the distance of the point $A(-4; 7)$ from the line $5x - 9y + 3 = 0$. We make use of the formula (20), and so

$$R = \frac{|5 \cdot (-4) - 9 \cdot 7 + 3|}{\sqrt{5^2 + (-9)^2}} = \frac{|-80|}{\sqrt{106}} = \frac{80}{\sqrt{106}};\ R^2 = \frac{6400}{106} = \frac{3200}{53}.$$

The equation of the circle is

$$(x - (-4))^2 + (y - 7)^2 = \frac{3200}{53},\quad (x + 4)^2 + (y - 7)^2 = \frac{3200}{53}..$$

LINE, STRAIGHT LINE, CIRCLE: basic terminology RUE

1. Абсцисса точки	Абсциса точки	Abscíssa (*pl* abscíssas, abscíssae) of a póint
2. Аналитическая геометрия (на плоскости)	Аналітична геометрія (на площині)	Ànalýtic(al) geómetry (on the pláne)
3. Взаимное [совместное] расположение двух линий	Взаємне [сумісне] розташування двох ліній	Mútual/recíprocal dìsposítion [posítion] of two lines/cúrves
4. Декартова система координат	Декартова система координат	Cartésian coórdinate sýstem [Cartésian sýstem of coórdinates]
5. Декартовы (прямоугольные) координаты	Декартові (прямокутні) координати	Cartésian (orthógonal/rectángular/grid) coórdinates
6. Деление отрезка в данном отношении	Поділ відрізка в даному відношенні	Divísion of a ségment in the given rátio
7. Деление отрезка пополам	Поділ відрізка навпіл	Divísion of a ségment in half [in halves]
8. Квадрант	Квадрант	Quádrant
9. Конец отрезка	Кінець відрізка	End [éndpóint] of a ségment
10. Координата точки	Координата точки	Coórdinate of a póint
11. Координатная ось	Координатна вісь	Áxis (*pl* áxes) of coórdinates, coórdinate áxis
12. Координатная плоскость	Координатна площина	Coórdinate pláne
13. Координатный угол	Координатний кут	Coórdinate ángle
14. Линия, кривая линия	Лінія, крива лінія	Líne, cúrved líne
15. Наклон (прямой к оси)	Нахил (прямої до осі)	Ìnclinátion, slópe (of a straight line to an áxis)
16. Начало координат	Початок координат	Órigin of coórdinates
17. Необходимое и достаточное условие (параллельности, перпендикулярности прямых)	Необхідна і достатня умова (паралельності, перпендикулярності прямих)	Nécessary and sufficient condítion (for párallelism/pérpendìculárity of straight lines)
18. Общее уравнение прямой	Загальне рівняння прямої	Géneral equátion of a straight line [géneral straight line equátion]
19. Окружность радиуса *R* с центром (в точке) *A*	Коло радіуса *R* з центром (в точці) *A*	Circúmference [circle] with a rádius *R* and with a céntre (at the póint) *A*

		[céntered at (the póint) *A*]
20.Окружность радиуса *R* с центром в начале координат (в *O*)	Коло радіуса *R* з центром в початку координат (в *O*)	Circúmference [circle] of/ with a rádius *R* céntered at the órigin of coórdinates (at *O*)
21.Окружность, касающаяся прямой	Коло, яке дотикається прямої	Circúmference [circle] tángent [touching] a straight line
22.Опустить перпендикуляр (на ось)	Опустити перпендикуляр (на вісь)	Drop a pèrpendícular (to the áxis)
23.Ордината ***b*** точки пересечения прямой с осью *Oy*	Ордината ***b*** точки перетину прямої з віссю *Oy*	*y*-ìntercépt ***b*** of a straight line
24.Ордината точки	Ордината точки	Órdinate [*y*-coórdinate] of a póint
25.Ось абсцисс, ось *Ox*	Вісь абсцисс, вісь *Ox*	Áxis (*pl* áxes) of abscíssas/abscíssae, *x*-áxis, *Ox*-áxis
26.Ось ординат, ось *Oy*	Вісь ординат, вісь *Oy*	Áxis (*pl* áxes) of órdinates, *y*-áxis, *Oy*-áxis
27.Отрезок	Відрізок	Ségment
28.Параллельная прямая	Паралельна пряма	Párallel straight line
29.Параллельность прямых	Параллельність прямих	Párallelism of straight lines
30.Перпендикуляр	Перпендикуляр	Pèrpendícular
31.Перпендикулярная прямая	Перпендикулярна пряма	Pérpendícular straight line
32.Перпендикулярность прямых	Перпендикулярність прямих	Pérpendìculárity of straight lines
33.Плоскость *Oxy*	Площина *Oxy*	*Oxy*-plane
34.Провести перпендикуляр	Провести перпендикуляр	Dráw a pèrpendícular
35.Произвольная точка	Довільна точка	Árbitrary póint
36.Проходить через данную точку (в данном направлении)	Проходити через дану точку (в заданому напрямку)	Pass thróugh a given póint (in the given diréction)
37.Проходить через две точки	Проходити через дві точки	Pass thróugh two póints
38.Прямая (линия)	Пряма (лінія)	Straight line
39.Прямая, параллельная прямой, оси	Пряма, яка є параллельною прямій, осі	Straight line párallel to a line, to an áxis
40.Прямая, перпендикулярная прямой, оси	Пряма, яка є перпендикулярною прямій, осі	Straight line pérpendícular to a straight line, to an áxis

41.Прямая, проходящая через данную точку (в данном направлении)	Пряма, яка проходить через дану точку (в заданому напрямку)	Straight line pássing through a given póint (in a given diréction)
42.Прямая, проходящая через две данные точки	Пряма, яка проходить через дві дані точки	Straight line pássing thróugh two given póints
43.Прямая, проходящая через начало координат	Пряма, яка проходить через початок координат	Straight line pássing thróugh the órigin of coórdinates
44.Прямоугольная система координат	Прямокутна система координат	Rectángular coórdinate sýstem
45.Пучок прямых (с центром *O*)	Пучок [жмуток, в"язка] прямих (з центром *O*)	Búndle [pencil] of straight lines (with the céntre *O*)
46.Разделить отрезок в данном отношении	Поділити відрізок в даному відношенні	Divíde a ségment in the given rátio
47.Разделить отрезок пополам	Поділити відрізок навпіл	Halve/bisect [divide in half/halves] (a ségment)
48.Расстояние между двумя точками	Відстань між двома точками	Dístance betwéen two póints
49.Расстояние от точки до прямой	Відстань від точки до прямої [точки від прямої]	Dístance of a póint from [betwéen a póint and] a straight line
50.Середина отрезка	Середина отрезка	Middle [mídpòint, bisécting póint] of a ségment
51.Система (декартовых прямоугольных) координат	Система (декартових прямокутних) координат	Sýstem (of Cartésian orthógonal/rectángular/grid) coórdinates, (Cartésian rectángular) coórdinate sýstem
52.Текущие координаты точки	Поточні [змінні] координати точки	Cúrrent [móving] coórdinates of a póint
53.Точка пересечения линий	Точка перетину ліній	Ìnterséction/cross póint [póint of ìnterséction] of lines/cúrves
54.Точки, не лежащие на одной прямой	Точки, які не лежать на одній прямій	Non alígned póints
55.Угловой коэффициент прямой	Кутовий коэфіцієнт прямої	Ángular còeffícient [slópe] of a straight line
56.Угловой коэффициент прямой	Кутовий коэфіцієнт прямої	Slópe [ángular còeffícient] of a straight line
57.Угол между двумя прямыми	Кут між двома прямими	Ángle (inclúded) betwéen two straight lines
58.Угол между прямой и осью	Кут між прямою і віссю	Ángle of/betwéen a straight line and an áxis
59.Угол наклона (прямой	Кут нахилу (прямої до	Slópe ángle, ángle of

к оси)	осі)	ìnclinátion (of a straight line to an áxis)
60. Удовлетворять уравнению линии	Задовольняти рівняння лінії	Sátisfy an equátion of a line/cúrve
61. Уравнение линии, кривой	Рівняння лінії, кривої	Equátion of a líne/cúrve
62. Уравнение прямой в отрезках на оси	Рівняння прямої у відрізках на осі	Equátion of a straight line in ségments, two-ìntercépt (form of the) equátion of a straight line
63. Уравнение прямой с угловым коэффициентом	Рівняння прямої з кутовим коефіцієнтом	Equátion of a stráight line [stráight line equátion] with an ángular còefficíent, slópe ìntercépt form of the equátion of a stráight line
64. Уравнение прямой, проходящей через данную точку в заданном направ-лении	Рівняння прямої, яка проходить через дану точку в заданому напрямку	Equátion of a straight line which passes through a given póint in a given diréction
65. Уравнение прямой, проходящей через две данные точки	Рівняння прямої, яка проходить через дві дані точки	Equátion of a straight line pássing thróugh two given póints, two póint form of the equátion of a straight line
66. Уравнение пучка прямых	Рівняння пучка прямих	Equátion of a bundle [pencil] of straight lines

LINE, STRAIGHT LINE, CIRCLE: basic terminology ERU

1. Abscíssa (*pl* abscíssas, abscíssae) of a póint	Абсцисса точки	Абсциса точки
2. Ànalýtic(al) geómetry (on the pláne)	Аналитическая геометрия (на плоскости)	Аналітична геометрія (на площині)
3. Ángle (inclúded) betwéen two straight lines	Угол между двумя прямыми	Кут між двома прямими
4. Ángle of/betwéen a straight line and an áxis	Угол между прямой и осью	Кут між прямою і віссю
5. Ángular còeffícient [slópe] of a straight line	Угловой коэффициент прямой	Кутовий коэфіцієнт прямої
6. Árbitrary póint	Произвольная точка	Довільна точка
7. Áxis (*pl* áxes) of abscíssas/abscíssae, *x*-áxis, *Ox*-áxis	Ось абсцисс, ось Ox	Вісь абсцисс, вісь Ox
8. Áxis (*pl* áxes) of coórdinates, coórdinate áxis	Координатная ось	Координатна вісь
9. Áxis (*pl* áxes) of órdinates, *y*-áxis, *Oy*-áxis	Ось ординат, ось *Oy*	Вісь ординат, вісь *Oy*
10.Búndle [pencil] of straight lines (with the céntre *O*)	Пучок прямых (с центром *O*)	Пучок [жмуток, в"язка] прямих (з центром *O*)
11.Cartésian (orthógonal/ rectángular/grid) coórdinates	Декартовы (прямоугольные) координаты	Декартові (прямокутні) координати
12.Cartésian coórdinate sýstem [Cartésian sýstem of coórdinates]	Декартова система координат	Декартова система координат
13.Circúmference [circle] of/with a rádius *R* céntered at the órigin of coórdinates (at *O*)	Окружность радиуса *R* с центром в начале координат (в *O*)	Коло радіуса *R* з центром в початку координат (в *O*)
14.Circúmference [circle] tángent [touching] a straight line	Окружность, касающаяся прямой	Коло, яке дотикається прямої
15.Circúmference [circle] with a rádius *R* and with a céntre (at the póint) *A* [céntered at (the póint) *A*	Окружность радиуса *R* с центром (в точке) *A*	Коло радіуса *R* з центром (в точці) *A*
16.Coórdinate ángle	Координатный угол	Координатний кут
17.Coórdinate of a póint	Координата точки	Координата точки
18.Coórdinate pláne	Координатная плоскость	Координатна площина
19.Cúrrent [móving] coór-	Текущие координаты то-	Поточні [змінні] коор-

dinates of a póint	чки	динати точки
20.Dístance betwéen two póints	Расстояние между двумя точками	Відстань між двома точками
21.Dístance of a póint from [betwéen a póint and] a straight line	Расстояние от точки до прямой	Відстань від точки до прямої [точки від прямої]
22.Divíde a ségment in the given rátio	Разделить отрезок в данном отношении	Поділити відрізок в даному відношенні
23.Divísion of a ségment in half [in halves]	Деление отрезка пополам	Поділ відрізка навпіл
24.Divísion of a ségment in the given rátio	Деление отрезка в данном отношении	Поділ відрізка в даному відношенні
25.Dráw a pèrpendícular	Провести перпендикуляр	Провести перпендикуляр
26.Drop a pèrpendícular (to the áxis)	Опустить перпендикуляр (на ось)	Опустити перпендикуляр (на вісь)
27.End [éndpóint] of a ségment	Конец отрезка	Кінець відрізка
28.Equátion of a bundle [pencil] of straight lines	Уравнение пучка прямых	Рівняння пучка прямих
29.Equátion of a líne/cúrve	Уравнение линии, кривой	Рівняння лінії, кривої
30.Equátion of a stráight line [stráight line equátion] with an ángular còefficíent, slópe ìntercépt form of the equátion of a stráight line	Уравнение прямой с угловым коэффициентом	Рівняння прямої з кутовим коефіцієнтом
31.Equátion of a straight line in ségments, two-ìntercépt (form of the) equátion of a straight line	Уравнение прямой в отрезках на оси	Рівняння прямої у відрізках на осі
32.Equátion of a straight line pássing thróugh two given póints, two póint form of the equátion of a straight line	Уравнение прямой, проходящей через две данные точки	Рівняння прямої, яка проходить через дві дані точки
33.Equátion of a straight line which passes through a given póint in a given diréction	Уравнение прямой, проходящей через данную точку в заданном направлении	Рівняння прямої, яка проходить через дану точку в заданому напрямку
34.Géneral equátion of a straight line [géneral straight line equátion]	Общее уравнение прямой	Загальне рівняння прямої
35.Halve/bisect [divide in	Разделить отрезок попо-	Поділити відрізок навпіл

half/halves] (a ségment)	лам	
36.Ìnclinátion, slópe (of a straight line to an áxis)	Наклон (прямой к оси)	Нахил (прямої до осі)
37.Ìntersection/cross póint [póint of ìnterséction] of lines/cúrves	Точка пересечения линий	Точка перетину ліній
38.Líne, cúrved líne	Линия, кривая линия	Лінія, крива лінія
39.Middle [mídpòint, bisécting póint] of a ségment	Середина отрезка	Середина отрезка
40.Mútual/recíprocal dìsposítion [posítion] of two lines/cúrves	Взаимное [совместное] расположение двух линий	Взаємне [сумісне] розташування двох ліній
41.Nécessary and sufficient condítion (for párallelism/pérpendìculárity of straight lines)	Необходимое и достаточное условие (параллельности, перпендикулярности прямых)	Необхідна і достатня умова (паралельності, перпендикулярності прямих)
42.Non alígned póints	Точки, не лежащие на одной прямой	Точки, які не лежать на одній прямій
43.Órdinate [y-coórdinate] of a póint	Ордината точки	Ордината точки
44.Órigin of coórdinates	Начало координат	Початок координат
45.Oxy-plane	Плоскость Oxy	Площина Oxy
46.Párallel straight line	Параллельная прямая	Паралельна пряма
47.Párallelism of straight lines	Параллельность прямых	Параллельність прямих
48.Pass thróugh a given póint (in the given diréction)	Проходить через данную точку (в данном направлении)	Проходити через дану точку (в заданому напрямку)
49.Pass thróugh two póints	Проходить через две точки	Проходити через дві точки
50.Pèrpendícular	Перпендикуляр	Перпендикуляр
51.Pérpendícular straight line	Перпендикулярная прямая	Перпендикулярна пряма
52.Pérpendìculárity of straight lines	Перпендикулярность прямых	Перпендикулярність прямих
53.Quádrant	Квадрант	Квадрант
54.Rectángular coórdinate sýstem	Прямоугольная система координат	Прямокутна система координат
55.Sátisfy an equátion of a line/cúrve	Удовлетворять уравнению линии	Задовольняти рівняння лінії
56.Ségment	Отрезок	Відрізок
57.Slópe [ángular còefficient] of a straight line	Угловой коэффициент прямой	Кутовий коэфіцієнт прямої

58. Slópe ángle, ángle of ìnclinátion of a straight line to an áxis	Угол наклона прямой к оси	Кут нахилу прямої до осі
59. Straight line párallel to a line, to an áxis	Прямая, параллельная прямой, оси	Пряма, яка є паралельною прямій, осі
60. Straight line pássing through a given póint (in a given diréction)	Прямая, проходящая через данную точку (в данном направлении)	Пряма, яка проходить через дану точку (в заданому напрямку)
61. Straight line pássing thróugh the órigin of coórdinates	Прямая, проходящая через начало координат	Пряма, яка проходить через початок координат
62. Straight line pássing thróugh two given póints	Прямая, проходящая через две данные точки	Пряма, яка проходить через дві дані точки
63. Straight line pérpendícular to a straight line, to an áxis	Прямая, перпендикулярная прямой, оси	Пряма, яка є перпендикулярною прямій, осі
64. Straight/right line	Прямая (линия)	Пряма (лінія)
65. Sýstem (of Cartésian orthógonal/rectángular/grid) coórdinates, (Cartésian rectángular) coórdinate sýstem	Система (декартовых прямоугольных) координат	Система (декартових прямокутних) координат
66. y-ìntercépt $\boldsymbol{b}$ of a straight line	Ордината $\boldsymbol{b}$ точки пересечения прямой с осью Oy	Ордината $\boldsymbol{b}$ точки перетину прямої з віссю Oy

LECTURE NO. 8. SECOND ORDER CURVES (CONICS)

POINT 1. SECOND ORDER CURVES (CONICS)

POINT 2. POLAR COORDINATES

POINT 3. TRANSFORMATION OF COORDINATES

POINT 4. WAYS OF REPRESENTATION OF CURVES

POINT 1. SECOND ORDER CURVES (CONICS)

As it's known an equation of the form

$$ax + by + c = 0 \quad for\ a^2 + b^2 \neq 0$$

is the general equation of a straight line. It's called **the general equation of the first order line**.

The second degree equation in two variables x and y

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0 \qquad (1)$$

is called the **general equation of the second order curve**.

Let $A = C$ (for example $A = C = 1$) and $B = 0$,

$$x^2 + y^2 + 2Dx + 2Ey + F = 0. \qquad (2)$$

After completing the squares we get

$$x^2 + 2Dx + D^2 + y^2 + 2Ey + E^2 = D^2 + E^2 - F,$$

$$(x + D)^2 + (y + E)^2 = D^2 + E^2 - F \qquad (3)$$

Equation (3) is that of a circle if $D^2 + E^2 - F > 0$. It gives only one point $(-D; -E)$ if $D^2 + E^2 - F = 0$. In the case $D^2 + E^2 - F < 0$ there isn't any line corresponding to this equation.

Ex. 1. Recognize a curve $x^2 + y^2 - 8x + 12y + 1 = 0$.

Let's complete the square,

$$x^2 - 8x + 16 + y^2 + 12y + 36 + 1 = 52 \Rightarrow (x - 4)^2 + (y + 6)^2 = 51.$$

The given equation is that of the circle of the radius $R=\sqrt{51}$ centered at the point $M_0(4;-6)$.

Circle, ellipse, hyperbola and parabola are the major cases of the second order curves. We'll introduce and study these last three curves.

The ellipse

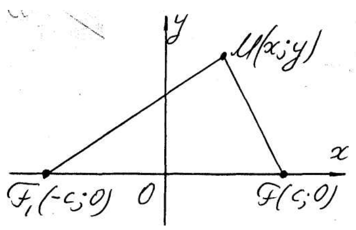


Fig. 1

Def. 1. The ellipse is called a plane curve which possesses the next property: sum of distances of any its point to two given points F_1, F_2 (foci) is constant one.

Let $M(x;\, y)$ be an arbitrary point of an ellipse, its foci lie on the *Ox*-axis such that $F_1(-c;\, 0), F_2(c;\, 0)$, and $2c$ is the distance between the foci (focal distance) (fig. 1). By definition

$$MF_1+MF_2=const=2a \quad \left(MF_1+MF_2>F_1F_2,\, 2a>2c,\, a>c\right),$$

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a. \tag{4}$$

We'll simplify the ellipse equation (4) using the same transformations as for solving irrational equations.

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2},$$

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2,$$

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2,$$

$$4a\sqrt{(x-c)^2+y^2}=4a^2-4cx,\; a\sqrt{(x-c)^2+y^2}=a^2-cx,$$

$$a^2\left(x^2-2cx+c^2+y^2\right)=a^4-2a^2cx+c^2x^2,$$

$$a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2,\; a^2x^2-c^2x^2+a^2y^2=a^4-a^2c^2,$$

$$\left(a^2-c^2\right)x^2+a^2y^2=a^2\left(a^2-c^2\right). \tag{5}$$

Let's introduce the next notation

$$a^2-c^2=b^2 \text{ or } a^2-b^2=c^2,\, a^2=b^2+c^2. \tag{6}$$

Hence

$$b^2x^2+a^2y^2=a^2b^2,$$

and after division both members by a^2b^2 we get so-called canonical equation of an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad (8)$$

An ellipse is symmetric about the coordinate axes because of if a some point $M_0(x_0;\,y_0)$ belongs to an ellipse, that is

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1,$$

then the points $M_1(x_0;-y_0),\, M_2(-x_0;\,y_0),\, M_3(-x_0;-y_0)$ also belong to it.

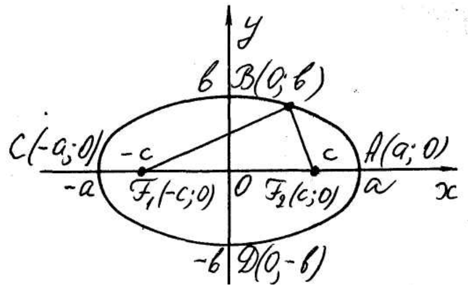


Fig. 2

Thus the coordinate axes are those of symmetry, and the origin $O(0;\,0)$ is the centre of symmetry of an ellipse (8).

Let's find vertices of an ellipse (8), namely its intersection points with the coordinate axes. Taking $y=0$, then $x=0$ in the equation (8) we obtain respectively $x=\pm a,\, y=\pm b$, and the vertices are $A(a;\,0),\, B(0;b),\, C(-a;\,0),\, D(0;-b)$ (see fig. 2).

The number *a* is called a longer [major] semiaxis [semimajor axis], *b* a shorter [minor] semiaxis [semiminor axis] of an ellipse.

Def. 2. Eccentricity of an ellipse is called the next number

$$\varepsilon=\frac{c}{a},\, 0<\varepsilon<1. \qquad (9)$$

It's the measure of flatness of an ellipse. Indeed, by (6)

$$\varepsilon=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\left(\frac{b}{a}\right)^2}.$$

Let *a* is fixed. If $b\to a$, that is the form of an ellipse tends to that of a circle, then

$$\varepsilon\to 0.$$

If, of the other hand, $b\to 0$, that is all points of an ellipse go to the segment *AC*, then

$$\varepsilon\to 1.$$

Ex. 2. Compile the canonical equation of an ellipse passing through two given points $M_1\left(4; -\sqrt{3}\right), M_2\left(2\sqrt{2}; 3\right)$.

We seek an equation of the form (8). Points $M_1\left(4; -\sqrt{3}\right), M_2\left(2\sqrt{2}; 3\right)$ must satisfy the equation, hence must be

$$\begin{cases} \dfrac{4^2}{a^2} + \dfrac{\left(-\sqrt{3}\right)^2}{b^2} = 1, \\ \dfrac{\left(2\sqrt{2}\right)^2}{a^2} + \dfrac{3^2}{b^2} = 1, \end{cases} \quad \begin{cases} \dfrac{16}{a^2} + \dfrac{3}{b^2} = 1, \\ \dfrac{8}{a^2} + \dfrac{9}{b^2} = 1. \end{cases} \quad \text{Let } \begin{matrix} \dfrac{1}{a^2} = m, \\ \dfrac{1}{b^2} = n, \end{matrix} \quad \text{then } \begin{cases} 16m + 3n = 1, \\ 8m + 9n = 1, \end{cases} \begin{matrix} m = \dfrac{1}{a^2} = \dfrac{1}{20}, \\ n = \dfrac{1}{b^2} = \dfrac{1}{15}. \end{matrix}$$

The equation in question is

$$\frac{x^2}{20} + \frac{y^2}{15} = 1.$$

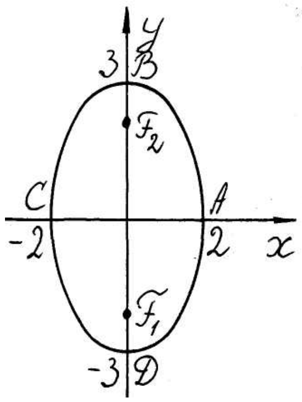


Fig. 3

Ex. 3. Prove that the equation $9x^2 + 4y^2 = 36$ is that of an ellipse. Represent it on the xOy-plane, find its vertices and foci.

Dividing both members of the equation by 36 we get

$$\frac{x^2}{4} + \frac{y^2}{9} = 1, \frac{x^2}{2^2} + \frac{y^2}{3^2} = 1,$$

that is the equation of an ellipse with $a = 2, b = 3$. But in this case $b = 3$ is the longer semiaxis, $a = 2$ is the shorter semiaxis, the formula (6) must be written in the other form $b^2 - a^2 = c^2$, $c^2 = 9 - 4 = 5, c = \sqrt{5}$, and the foci of the ellipse lie on the Oy-axis, $F_1\left(0; -\sqrt{5}\right), F_2\left(0; \sqrt{5}\right)$ (fig. 3).

The hyperbola

Def. 3. The hyperbola is called a plane curve which possesses the next property: difference of distances of any its point from two given points F_1, F_2 (foci) is constant one.

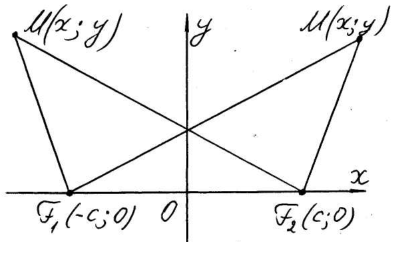


Fig. 4

Let $M(x; y)$ be an arbitrary point of a hyperbola, its foci $F_1(-c; 0), F_2(c; 0)$ lie on the Ox-axis symmetrically with respect to the origin , and $2c$ is the focal distance (fig. 4). By definition

$$MF_1 - MF_2 = const = \pm 2a \quad \left(MF_1 - MF_2 < F_1F_2,\, 2a < 2c,\, a < c\right),$$

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a. \qquad (10)$$

By analogy with the case of an ellipse

$$\sqrt{(x+c)^2 + y^2} = \pm 2a - \sqrt{(x-c)^2 + y^2},$$

$$(x+c)^2 + y^2 = 4a^2 \pm 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2,$$

$$x^2 + 2cx + c^2 + y^2 = 4a^2 \pm 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2,$$

$$\pm 4a\sqrt{(x-c)^2 + y^2} = 4a^2 - 4cx, \pm a\sqrt{(x-c)^2 + y^2} = a^2 - cx,$$

$$a^2\left(x^2 - 2cx + c^2 + y^2\right) = a^4 - 2a^2cx + c^2x^2,$$

$$a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 = a^4 - 2a^2cx + c^2x^2, a^2x^2 - c^2x^2 + a^2y^2 = a^4 - a^2c^2,$$

$$\left(a^2 - c^2\right)x^2 + a^2y^2 = a^2\left(a^2 - c^2\right). \qquad (11)$$

If one denotes

$$a^2 - c^2 = -b^2 \text{ or } c^2 = a^2 + b^2, \qquad (12)$$

then

$$-b^2x^2 + a^2y^2 = -a^2b^2,$$

and after division both members by $\left(-a^2b^2\right)$ we get the canonical equation of a hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (13)$$

A hyperbola is symmetric with respect to the coordinate axes and the origin. If we'll seek its vertices, then the equation (13) gives $x = \pm a$ for $x = 0$ and $y^2 = -b^2$.for $y = 0$. It means that a hyperbola has only two vertices $A(a; 0)$, $C(-a; 0)$, it doesn't intersect the *Oy*-axis and consists of two branches (fig. 5).

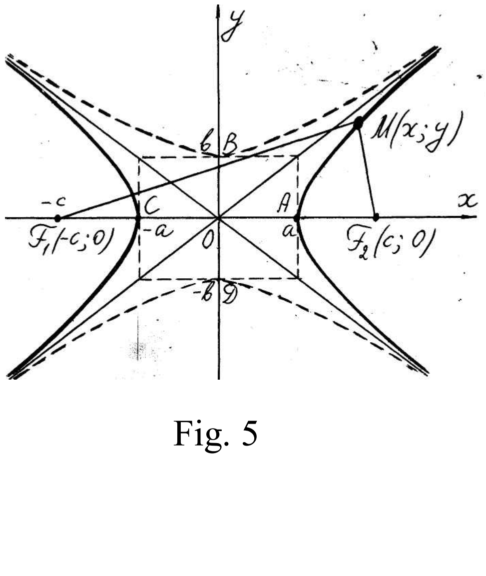


Fig. 5

The number *a* is called a real semiaxis [semi-transversal [semitransverse] axis], *b* an imaginary semiaxis [semiconjugate axis] of a hyperbola.

Straight lines $x = a, x = -a, y = b, y = -b$ generate so-called major rectangle of a hyperbola, and its diagonals

$$y = \pm\frac{b}{a}x \qquad (14)$$

are called asymptotes of a hyperbola. Sense of the term asymptote consists in following. If a point of a hyperbola recédes into infinity, it approaches one of its asympto-tes.

■Let for example a point $M(x; y)$ moves into infinity and is situated in the first quadrant. We express y in terms of x from the equation (13),

$$\frac{x^2}{a^2} - 1 = \frac{y^2}{b^2}, \frac{y^2}{b^2} = \frac{x^2 - a^2}{a^2}, y^2 = \frac{b^2}{a^2}\left(x^2 - a^2\right), y = \frac{b}{a}\sqrt{x^2 - a^2} \ (\text{for } y > 0)$$

and then see that for $x \to +\infty$ a difference

$$\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2} = \frac{b}{a}\left(x - \sqrt{x^2 - a^2}\right) = \frac{b}{a}\frac{\left(x - \sqrt{x^2 - a^2}\right)\left(x + \sqrt{x^2 - a^2}\right)}{x + \sqrt{x^2 - a^2}} = \frac{b}{a}\cdot\frac{a^2}{x + \sqrt{x^2 - a^2}}$$

tends to zero.■

Def. 4. Eccentricity of a hyperbola is defined by the same formula as that of an ellipse, namely:

$$\varepsilon = \frac{c}{a}, \ \varepsilon > 1. \qquad (15)$$

Remark. A curve with the equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \qquad (16)$$

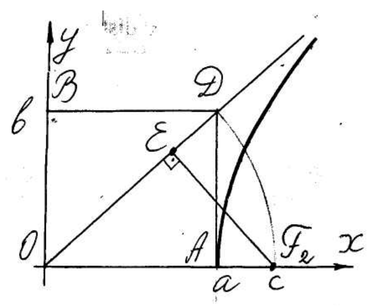


Fig. 6

is also a hyperbola (see the dotted [dashed-line, broken] line on the fig. 5), it is called a conjugate hyperbola for that defined by the equation (13).

Ex. 4. Find the distance of the focus $F_2(c; 0)$ of the hyperbola (13) from its asymptotes (14).

Let's find for example the distance F_2E from the asymptote $y = \frac{b}{a}x$ (fig. 6).

The right triangles $0F_2E, OAD$ are equal because of equal hypotenuses

$$OF_2 = c, OD = \sqrt{a^2 + b^2} = c$$

and common acute angle AOD. Therefore $F_2E = AD = b$.

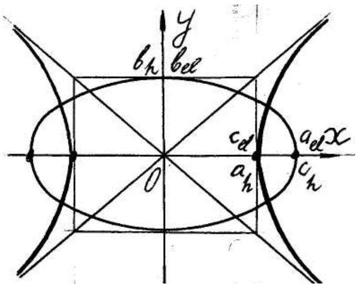


Fig. 7

Ex. 5. Write the canonical equation of an ellipse the vertices of which are located at the foci of the hyperbola

$$\frac{x^2}{5} - \frac{y^2}{4} = 1,$$

and the foci at its vertices (fig. 7).

Let's write the canonical equations of the hyperbola and an ellipse in the next form

$$\frac{x^2}{a_h^2} - \frac{y^2}{b_h^2} = 1, \qquad \begin{array}{c} a_h = \sqrt{5}, \\ b_h = 2, \\ c_h = \sqrt{a_h^2 + b_h^2} = 3; \end{array} \qquad \frac{x^2}{a_{el}^2} + \frac{y^2}{b_{el}^2} = 1.$$

We see from the fig. 7 and formulas (6), (12) that

$$a_{el} = c_h = 3, c_{el} = a_h = \sqrt{5}, b_{el} = \sqrt{a_{el}^2 - c_{el}^2} = \sqrt{c_h^2 - a_h^2} = b_h = 2,$$

and therefore the required equation of the ellipse is

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

The parabola

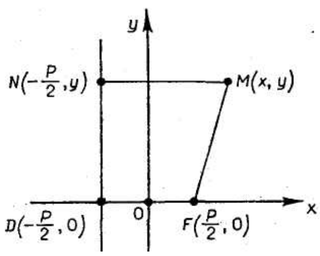


Fig. 8

Def. 5. The parabola is called a plane curve every point of which is equidistant from a given point F (focus) and a given straight line (directrix).

We choose the coordinate axes such that the focus is the point $F\left(\frac{p}{2}; 0\right)$ and the directrix is the straight line $x = -\frac{p}{2}$ (see fig. 8). Here p is some positive number which is equal to the distance between the focus and directrix and is called the parameter of a parabola. For arbitrary point $M(x; y)$ of a parabola

$$MF = MN, \text{ where } N\left(-\frac{p}{2}; y\right), \text{ or } \sqrt{\left(x-\frac{p}{2}\right)^2 + y^2} = x + \frac{p}{2}.$$

Squaring and reducing similar terms we get

$$\left(x-\frac{p}{2}\right)^2 + y^2 = \left(x+\frac{p}{2}\right)^2, x^2 - px + \frac{p^2}{4} + y^2 = x^2 + px + \frac{p^2}{4},$$

$$y^2 = 2px. \qquad (17)$$

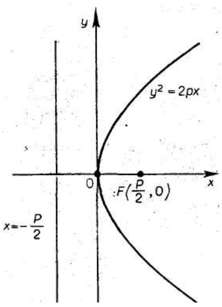


Fig. 9

Equation (17) is the canonical equation of a parabola (fig. 9). Parabola is symmetric with respect to the *Ox*-axis and the origin is its vertex.

Choosing the focus $F\left(-\frac{p}{2}; 0\right)$ and the directrix $x = \frac{p}{2}$ we obtain the canonical equation of a parabola of the form

$$y^2 = -2px. \qquad (18)$$

Ex. 6. Find the parameter, the focus and the directrix of a parabola $y = -6x^2$.

Answer. $y = -2 \cdot 3x^2,\ p = 3,\ F\left(0; -\frac{3}{2}\right),\ y = \frac{3}{2}$.

Ex. 7. Write the canonical equations of parabolas for which foci are situated on the *Ox*-axis and the parameter equals the distance of the focus of the hyperbola

$$\frac{x^2}{5} - \frac{y^2}{4} = 1$$

to its asymptote.

On the base of Ex. 3 $p = b = 2$, hence $y^2 = \pm 2px,\ y^2 = \pm 4x.$

POINT 2. POLAR COORDINATES

A position of a point *M* on the plane is completely determined by its polar coordinates with the next elements: a pole, a polar axis, a polar radius and a polar angle.

Let be a point O and a ray $O\rho$ which are called the pole and the polar axis respectively (fig. 10). The distance $\rho = OM$ of a point M from the pole is called a polar radius of the point, and an angle φ between the polar radius and the polar axis is cal-led a polar angle of the point M. Now we can represent this point with its polar coordinates, namely $M(\rho; \varphi)$ (fig. 10).

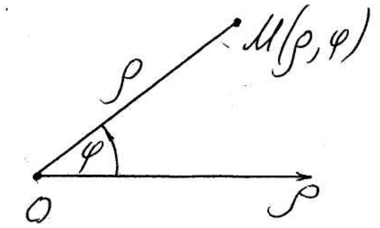


Fig. 10

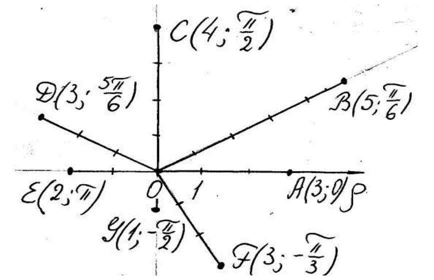


Fig. 11

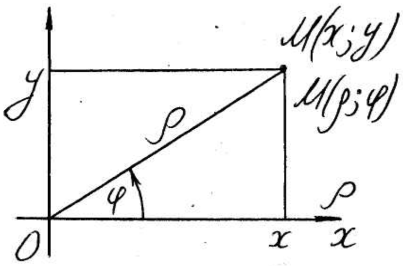


Fig. 12

Let be a point O and a ray $O\rho$ which are called the pole and the polar axis respectively (fig. 10). The distance $\rho = OM$ of a point M from the pole is called a polar radius of the point, and an angle φ between the polar radius and the polar axis is cal-led a polar angle of the point M. Now we can represent this point with its polar coordinates, namely $M(\rho; \varphi)$ (fig. 10).

To construct a point $M_0(\rho_0; \varphi_0)$ by its polar coordinates ρ_0, φ_0 it's necessary to draw a ray from the pole at the angle $\varphi = \varphi_0$ to the polar axis and to plot a polar radius [radial distance] $OM_0 = \rho_0$ of the point on this ray.

Ex. 8. Construct points by their polar coordinates
$A(3; 0)$, $B(5; \pi/6)$, $C(4; \pi/2)$, $D(3; 5\pi/6)$, $E(2; \pi)$, $F(3; -\pi/3)$, $G(1; -\pi/2)$ (fig. 11).

Change [transition] from Cartesian rectangular coordinates to those polar and vice versa

Let's coincide the pole and the origin of Cartesian rectangular coordinates and the polar axis with positive semiaxis of the Ox-axis (fig. 12). If x, y and ρ, φ be Cartesian and polar coordinates of a point M respectively $(M(x; y), M(\rho, \varphi))$, then

$$\left.\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \end{aligned}\right\} \qquad (19)$$

(transition formulas from Cartesian to polar coordinates) and

$$\left.\begin{aligned} x^2 + y^2 &= \rho^2 \\ tg\varphi &= \frac{y}{x} \end{aligned}\right\} \qquad (20)$$

(transition formulas from polar to Cartesian coordinates).

Equations of some lines in polar coordinates.

1. Ray starting from the pole under the angle φ_0 to the polar axis (fig. 13).

For any point $M(\rho,\varphi)$ of the ray we have

$$\varphi = \varphi_0. \qquad (21)$$

2. Circle with the radius R centered at the pole (fig. 14).

For any point $M(\rho,\varphi)$ of the circle we have

$$\rho = R. \qquad (22)$$

3. Circle $\rho = 2a\cos\varphi$ (fig. 15).

φ	0	$\pm\pi/6$	$\pm\pi/3$	$\pm\pi/2$
$\cos\varphi$	1	$\sqrt{3}/2$	$1/2$	0
ρ	$2a$	$a\sqrt{3}$	a	0
Point	A	B_1, B_2	C_1, C_2	O

Assigning the values 0, $\pm\pi/6$, $\pm\pi/3$, $\pm\pi/2$ to φ we find corresponding values of ρ and corresponding points of a curve. Then we join them by a smooth line.

Let's write an equation of the line in

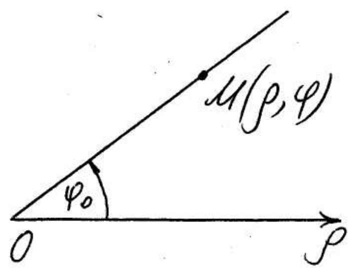


Fig. 13

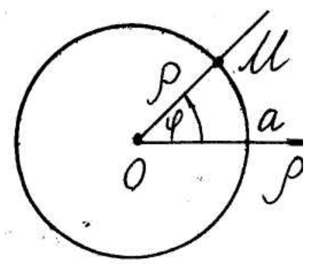


Fig. 14

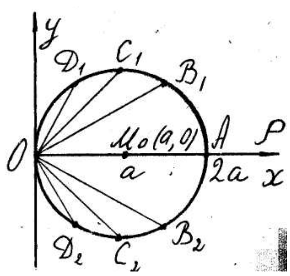


Fig. 15

Cartesian coordinates. For this purpose we multiply both its members by ρ,

$$\rho^2 = 2a\rho\cos\varphi,$$

and take into account the formulas (19), (20),

$$x^2 + y^2 = 2ax,\ x^2 - 2ax + y^2 = 0,\ x^2 - 2ax + a^2 + y^2 = a^2,\ (x-a)^2 + y^2 = a^2.$$

We obtain the equation of a circle of a radius a with the centre $M_0(a;\,0)$.

4. Cardioid $\rho = a(1+\cos\varphi)$. With the help of the table we plot some points of

	0^0	$\pm 60^0$	$\pm 90^0$	$\pm 120^0$	$\pm 180^0$
φ	0	$\pm\pi/3$	$\pm\pi/2$	$\pm 2\pi/3$	$\pm\pi$
$\cos\varphi$	1	½	0	-1/2	-1
$1+\cos\varphi$	2	3/2	1	½	0
ρ	$2a$	$3/2\ a$	a	$1/2\ a$	0
Point	A	B_1, B_2	C_1, C_2	D_1, D_2	O

the cardioid and then the whole line (see fig. 16).

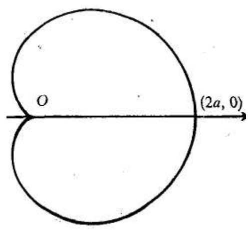


Fig. 16

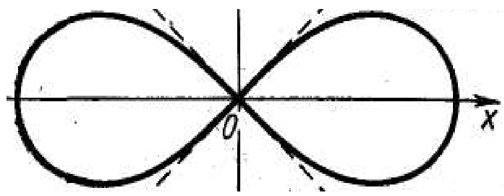


Fig. 17

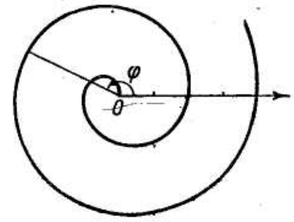


Fig. 18

5. Bernoulli[1] lemniscate

$$\left(x^2 + y^2\right)^2 = 4a^2\left(x^2 - y^2\right).$$

The line is symmetric about *Ox, Oy*-axes, so we'll study it in the first quadrant. It is evident that $x^2 - y^2 \ge 0,\ x^2 \ge y^2,\ y \le x$. Passing to polar coordinates we obtain

$$\left(\rho^2\right)^2 = 4a^2\left((\rho\cos\varphi)^2 - (\rho\sin\varphi)^2\right),\ \rho^2 = 4a^2\left(\cos^2\varphi - \sin^2\varphi\right),\ \rho^2 = 4a^2\cos 2\varphi,$$

$$\rho = 2a\sqrt{\cos 2\varphi}.$$

We assign the values 0, $\pi/12$, $\pi/8$, $\pi/4$ to φ, compile the next table and plot the lemniscate (fig. 17).

6. Construct yourselves Archimedean[1] spiral $\rho = a\varphi$ (fig. 18).

---

[1] Bernoulli, Jacob (1654 - 1705), the famous Swiss mathematician

7. Conics in polar coordinates. If we place the pole at the left focus of an elliple, at the right focus of a hyperbola, at the focus of a parabola respectively and

φ°	0^0	15^0	$22{,}5^0$	45^0
φ	0	$\pi/12$	$\pi/8$	$\pi/4$
2φ	0	$\pi/6$	$\pi/4$	$\pi/2$
$\cos 2\varphi$	1	0.9	0.7	0
$\sqrt{\cos 2\varphi}$	1	0.94	0.8	0
ρ	$2a$	$1.9\,a$	$1.6\,a$	0
Point	A	B	C	O

direct the polar axis from the left to the right focus of the ellipse and the hyperbola and from the vertex to the focus of the parabola (see fig. 19, 20, 21) then all three curves will have the same polar equation.

■Let, at first, $M(\rho,\varphi)$ is arbitrary point of the ellipse. Then

$$F_1M=\rho,\ F_2M=2a-\rho,$$

and by virtue of cosine theorem

$$F_2M^2=F_1M^2+F_1F_2^2-2F_1M\cdot F_1F_2\cos\varphi,\ (2a-\rho)^2=\rho^2+(2c)^2-4c\rho\cos\varphi,$$

$$4a^2-4a\rho+\rho^2=\rho^2+4c^2-4c\rho\cos\varphi,\ 4\left(a^2-c^2\right)=4a\rho\left(1-\frac{c}{a}\cos\varphi\right),$$

$$b^2=a\rho(1-\varepsilon\cos\varphi),\ \rho=\frac{b^2}{a(1-\varepsilon\cos\varphi)}.$$

Let's introduce the next value (so-called parameter of the ellipse)

$$p=\frac{b^2}{a}.$$

The polar equation of the ellipse is

$$\rho=\frac{p}{1-\varepsilon\cos\varphi},$$

where the eccentricity ε satisfies the inequality $0<\varepsilon<1$.

[1] Archimedes ($\approx$287 B.C. - $\approx$212 B.C.), an ancient Greek mathematician, physicist, and mechanic

For the hyperbola we deduce by the same way the same equation but for $\varepsilon > 1$ (do it yourselves).

In the end we have for the parabola (fig. 21)

$$\rho = OM = MP = \frac{p}{2} + AN = \frac{p}{2} + (AO - NO) = \frac{p}{2} + \left(\frac{p}{2} - \rho\, cos(\pi - \varphi)\right) = p + \rho\, cos\,\varphi,$$

$$\rho = \frac{p}{1 - cos\,\varphi},$$

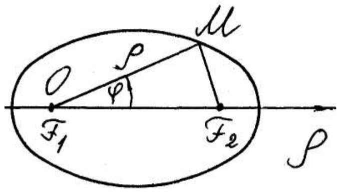


Fig. 19

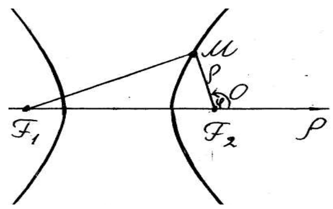


Fig. 20

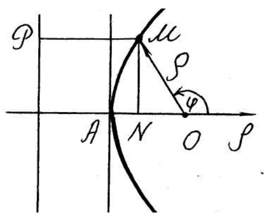


Fig. 21

that is the same equation with $\varepsilon = 1$.■

POINT 3. TRANSFORMATION OF COORDINATES

General equation of the second order curve (1) can be reduced to the canonical form by transformation of coordinates.

There exist two types of transformations: 1) translation [parallel displacement] of the coordinate axes when a new origin O' is placed in a point with coordinates x_0, y_0 in an old coordinate system and new axes $O'x', O'y'$ are parallel to old axes *Ox*, *Oy* respectively (fig. 22); 2) rotation of coordinate axes through an angle α about the origin of coordinates *O* with appearance of new coordinate axes Ox', Oy' (fig. 23).

Let *x*, *y* be coordinates of a point *M* in the *xOy* coordinate system (old coordinates) and x', y' coordinates of the same point *M* in the new coordinate system (new coordinates). It's necessary to set up the relation between old and new coordinates.

In the case of the translation of coordinate axes such the relation is obvious (see fig. 22):

$$x = x_0 + x',$$
$$y = y_0 + y'. \qquad (23)$$

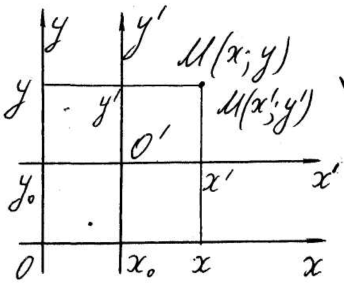


Fig. 22

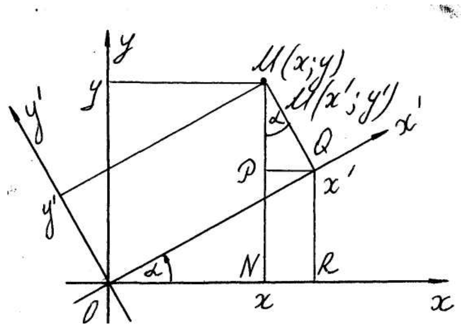


Fig. 23

In the case of the rotation of coordinate axes we do as follows (fig. 23)

$$x = ON = OR - NR = OR - PQ = OQ\cos\alpha - QM\sin\alpha = x'\cos\alpha - y'\sin\alpha;$$
$$y = NM = NP + PM = RQ + PM = OQ\sin\alpha + QM\cos\alpha = x'\sin\alpha + y'\cos\alpha.$$

Thus,

$$x = x'\cos\alpha - y'\sin\alpha;$$
$$y = x'\sin\alpha + y'\cos\alpha. \qquad (24)$$

Formulas (24) represent a linear transformation of unknowns (see Lecture 5, Point 4) with the matrix

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \text{ and so we can write } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

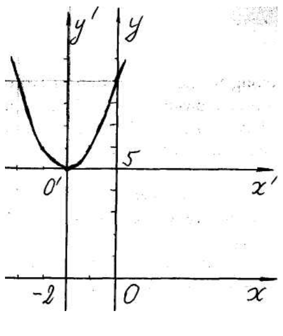


Fig. 24

Ex. 9. Recognize a line $y = x^2 + 4x + 9$.

Let's fulfill the translation of coordinate axes (23) with so far unknown coordinates x_0, y_0 of a new origin O'. Substituting x and y by $x_0 + x', y_0 + y'$ we'll have (verify!)

$$y' = x'^2 + (2x_0 + 4)x' + (x_0^2 + 4x_0 + 9 - y_0).$$

Let's choose values x_0, y_0 such that to annulate the coefficient of x' and the free term,

$$2x_0 + 4 = 0,\ x_0^2 + 4x_0 + 9 - y_0 = 0 \text{ whence it follows that } x_0 = -2,\ y_0 = 5.$$

Therefore the translation in question is

$$x = -2 + x',$$
$$y = 5 + y',$$

and the given line has the equation of usual parabola

$$y' = x'^2$$

in the new coordinate system (fig. 24).

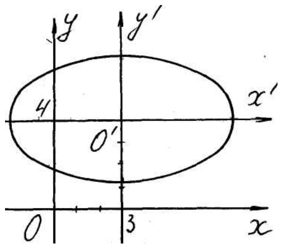


Fig. 25

Ex. 10. Putting $x-3=x', y-4=y'$ in the equation

$$\frac{(x-3)^2}{25}+\frac{(y-4)^2}{9}=1,$$

that is fulfilling the translation of coordinate axes

$$x=3+x', y=4+y',$$

we recognize the ellipse which has the canonical equation

$$\frac{x'^2}{25}+\frac{y'^2}{9}=1$$

in $x'O'y'$ coordinate system (fig. 25).

Ex. 11. Let's take the well known from the school equation of the hyperbola

$$xy = 2a^2$$

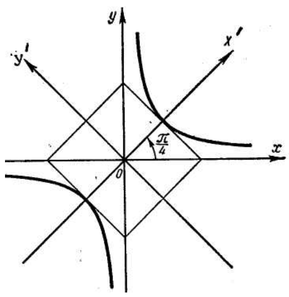


Fig. 26

and fulfill a rotation (24) of coordinate axes through an angle α so far unknown

$$(x' cos\alpha - y' sin\alpha)(x' sin\alpha + y' cos\alpha) = 2a^2,$$

$$x'^2 sin\alpha\, cos\alpha + (cos^2\alpha - sin^2\alpha)x'y' - y'^2 sin\alpha\, cos\alpha = 2a^2.$$

We choose the rotation angle α to annulate the expression in the parentheses, namely $\alpha = \pi/4$. Hence

$$x'^2 sin\frac{\pi}{4}cos\frac{\pi}{4} - y'^2 sin\frac{\pi}{4}cos\frac{\pi}{4} = 2a^2, x'^2 - y'^2 = 4a^2, \frac{x'^2}{(2a)^2} - \frac{y'^2}{(2a)^2} = 1.$$

In $x'Oy'$ coordinate system our hyperbola has the canonical equation. The *Ox, Oy* axes are its asymptotes.

POINT 4. WAYS OF REPRESENTATION OF CURVES

A curve can be represented:

1. In Cartesian coordinates:

a) by explicit equation which is resolved with respect to y or x (for example a parabola $y = ax^2 + bx + c,\ x = \frac{1}{2p}y^2$, a circle $y = \sqrt{2Rx - x^2},\ x = -\sqrt{2Ry - y^2}$);

b) by implicit equation which isn't resolved with respect to y or x (for example all the canonical equations of a circle, an ellipse, a hyperbola, a parabola);

c) parametrically, by equations (parametric equations) of the form

$$x = \varphi(t),\ y = \phi(t)$$

where t is some auxiliary variable, so-called parameter.

2. In polar coordinates (see Point 2 of this Lecture).

We'll give some examples of parametrically represented curves.

Ex. 12. Parametric equations of a circle $x^2 + y^2 = r^2$ of a radius r centered at the origin (see fig. 27) are

$$x = r\cos t,\ y = r\sin t, \tag{25}$$

where t is an angle between a radius OM ($M(x;\ y)$ is an arbitrary point of the circle) and the Ox-axis.

Ex. 13. Parametric equations of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are

$$x = a\cos t,\ y = b\sin t. \tag{26}$$

■Let $a > b$. We consider two circles of radii a, b centered at the origin $O(0;\ 0)$ (fig. 28). For an arbitrary point $M(x;\ y)$ on an ellipse (for the sake of simplicity we take it in the first quadrant)

$$x = OP = OB\cos t = a\cos t,\ y = PM = QC = OC\sin t = b\sin t.■$$

Test yourselves that the vertices $A(a;\ 0), B(0;b), C(-a;\ 0), D(0;\ -b)$ of an ellipse correspond to the values $0, \pi/2, \pi, 3/2\,\pi$ of the parameter t.

Ex. 14. An ástroid is the trájectory of a point of a circle of a radius r rotating along the inner side of a circle of a radius $a = 4r$ (fig. 30). The parametric equations of an astroid are

$$x = a\cos^3 t,\ y = a\sin^3 t. \qquad (27)$$

t	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
x	a	$0.65a$	$0.35a$	$0.13a$	0
y	0	$0.13a$	$0.35a$	$0.65a$	a
Point	A	B	C	D	E

It has also an implicit equation, namely

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}. \qquad (28)$$

On the fig. 29 we show plotting of the first part of the astroid with the help the table

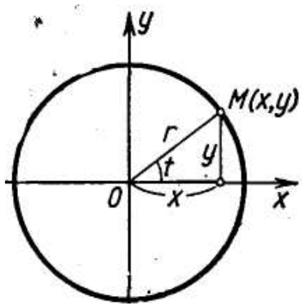


Fig. 27

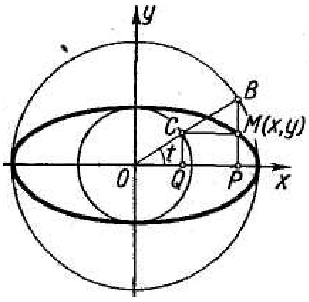


Fig. 28

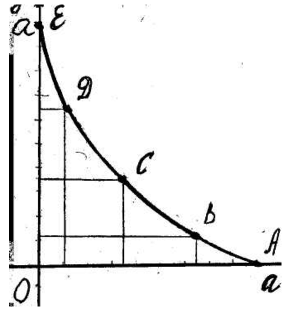


Fig. 29

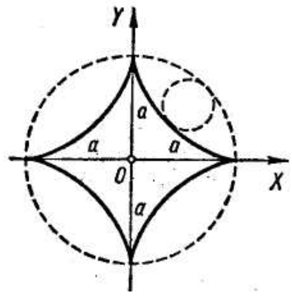


Fig. 30

Ex. 15. A cýcloid is the trajectory of a point of a circle rotating along a straight line without slide [sliding] (fig. 31). If a radius of a circle equals a, then the parametric equations of a cycloid are

$$x = a(t - \sin t),\ y = a(1 - \cos t) \qquad (30)$$

where t is the rotation angle of a radius MC of a rotating circle.

■We see from fig. 31 that for an arbitrary point $M(x; y)$ of a cycloid

$$x = OP = OB - PB = at - MK = at - a\sin t = a(t - \sin t),$$
$$y = PM = BK = BC - KC = a - a\cos t = a(1 - \cos t).$$

Ex. 16. Parametric equations of an evolvent [involute] of a circle of a radius a centered at the origin (fig. 32) are

$$x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t) \qquad (31)$$

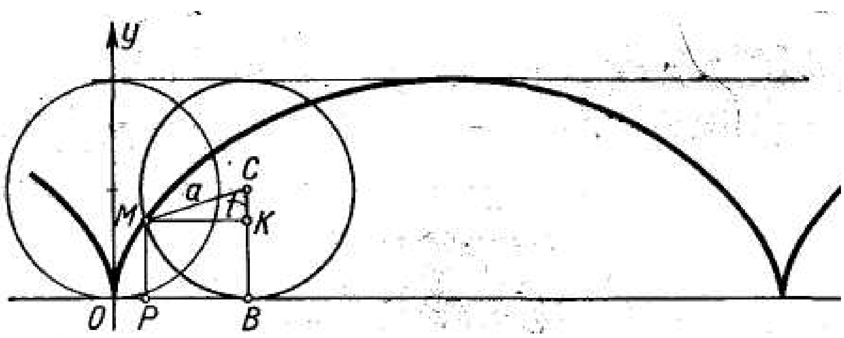


Fig. 31

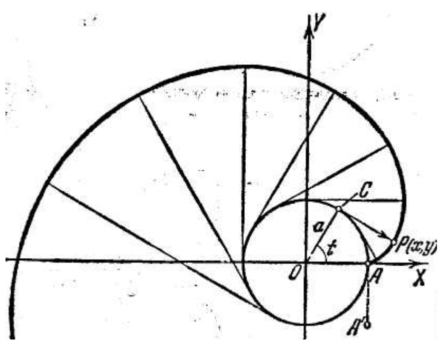


Fig. 32

SECOND ORDER CURVES (CONICS): main terminology RUE

1. Асимптота гиперболы	Асимптота гіперболи	Ásymptote of a hypérbola
2. Большая полуось эллипса	Велика піввісь еліпса	Longer [májor] semiáxis [semimájor áxis] of an ellípse
3. Вершина	Вершина	Vértex (*pl* vértices)
4. Ветвь гиперболы (левая, правая, нижняя, верхняя)	Вітка гіперболи (ліва, права, нижня, верхня)	Branch of a hypérbola (left(-hand), right(-hand), lówer, úpper)
5. Вещественная [поперечная, фокальная, главная] полуось гиперболы	Дійсна [поперечна, фокальна, головна] піввісь гіперболи	Réal sèmiáxis [sèmitransvérsal [sèmitránsverse] áxis] of a hypérbola
6. Гипербола	Гіпербола	Hypérbola (*pl* hypérbolae, hypérbolas)
7. Диагональ основного прямоугольника	Діагональ основного прямокутника	Diágonale of the májor [máin, príncipal] réctàngle
8. Директриса	Директриса	Diréctrix (*pl* dìrectríces)
9. Каноническое уравнение кривой второго порядка	Канонічне рівняння кривої другого порядку	Canónical equátion of a quadrátic [quádric, sécond-degrée/órder] cúrve
10.Коническое сечение	Конічний переріз	Cónic séction/cúrve; cónic
11.Кривая второго порядка	Крива другого порядку	Quadrátic [quádric, sécond degrée/órder] cúrve
12.Луч, исходящий из полюса (под углом φ к полярной оси)	Промінь, який виходить з полюса (під кутом φ) до полярної осі	Ray cóming/starting from the póle at the ángle φ to the pólar áxis
13.Малая полуось эллипса	Мала піввісь еліпса	Minor semiáxis [semimínor áxis] of an ellípse
14.Мера сплюснутости эллипса	Міра сплющеності еліпса	Méasure of flátness [of fláttining] of an ellípse
15.Мнимая [не поперечная, не фокальная] полуось гиперболы	Уявна [не поперечна, не фокальна] піввісь гіперболи	Imaginary semiáxis [sémicónjugate áxis] of a hyperbola
16.Новая координата	Нова координата	New coórdinate
17.Новая система координат	Нова система координат	New coórdinate sýstem
18.Новое начало (координат)	Новий початок (координат)	New órigin

19.Окружность радиуса R с центром (в точке) A	Коло радіуса R з центром (в точці) A	Circúmference [circle] with a rádius R and a céntre (at the póint) A [céntered at (the póint) A
20.Оптическое [фокальное, геометрическое] свойство	Оптична [фокальна, геометрична] властивість	Óptic(al) [reflection, fócal, geométric(al)] próperty
21.Основной прямоугольник	Основний прямокутник	Májor [máin, príncipal] réctàngle
22.Ось симметрии	Вісь симетрії	Áxis (*pl* áxes) of symmetry, sýmmetry axis
23.Отложить отрезок на *чём-л.*	Відкласти відрізок на *чомусь*	Draw/plot a ségment on *smth*
24.Парабола	Парабола	Parábola
25.Параллельный перенос координатных осей	Паралельний перенос координатних осей	Translátion of coórdinate áxes
26.Параметр [фокальный параметр] параболы	Параметр [фокальний параметр] параболи	Parameter [fócal parámeter] of a parábola
27.Поворот координатных осей (на угол α около начала координат)	Поворот координатних осей (на кут α навколо початку координат)	Rotátion of coórdinate áxes (through an ángle α abóut the órigin of coórdinates)
28.Полуокружность (левая, правая, нижняя, вер-хняя)	Півколо (ліве, праве, нижнє, верхнє)	Sémicìrcle (left(-hand), right(-hand), lówer, úpper)
29.Полюс	Полюс	Póle
30.Полярная ось	Полярна вісь	Pólar áxis
31.Полярная система координат	Полярна система координат	Pólar coórdinate sýstem, sýstem of pólar coórdinates
32.Полярное уравнение	Полярне рівняння	Pólar equátion
33.Полярные координаты	Полярні координати	Pólar coórdinates
34.Полярный радиус точки	Полярний радіус точки	Pólar radius of a póint
35.Полярный угол точки	Полярний кут точки	Pólar ángle of a póint
36.Преобразование координат	Перетворення координат	Trànsformátion of coórdinates
37.Приближаться к асимптоте	Наближатися до асимптоти	Appróach the ásymptote
38.Приведение общего уравнения кривой	Зведення загального рівняння кривої другого по-	Redúction of the géneral equátion of a sécond-órder

второго порядка к каноническому виду	рядку до канонічного вигляду	curve to the canónical form
39.Провести луч из полюса под углом φ к полярной оси	Провести промінь з полюса під кутом φ до полярної осі	Dráw a ray from the póle at the ángle φ to the pólar áxis
40.Равнобочная гипербола	Рівнобічна гіпербола	Équiláteral hypérbola
41.Равноотстоящий от *чего*	Рівновіддалений від *чогось*	Èquidístant from *smth*
42.Связь между полярными и декартовыми прямоугольными координатами	Зв"язок між полярними і декартовими прямокутними координатами	Relátion(ship) betwéen pólar and Cartésian orthógonal/rectángular/grid coórdinates
43.Связь между старыми и новыми координатами	Зв"язок між старими і новими координатами	Relátion(ship) betwéen new and old coórdinates
44.Совместить (полюс с началом координат, полярную ось с осью абсцисс)	Сумістити (полюс з початком координат, полярну вісь з віссю абсцис)	Sùperpóse (the póle and the órigin of coórdinates, the pólar áxis and that of abscíssas/abscíssae)
45.Совпадать (о точках, началах координат и др.)	Збігатися (про точки, початки координат тощо)	Còincíde (abóut points, the órigins of coórdinates)
46.Сопряжённая гипербола (для данной гиперболы)	Спряжена гіпербола (для даної гіперболи)	Cónjugate hypérbola of the given hypérbola
47.Сплюснутость эллипса	Сплющеність еліпса	Flátness/fláttining of an ellípse
48.Старая координата	Стара координата	Old coórdinate
49.Старая система координат	Стара система координат	Old coórdinate sýstem
50.Старое начало	Старий початок	Old órigin
51.Удаляться в бесконечность	Віддалятися в нескінченність	Recéde [retíre, móve] to/ínto infínity
52.Упрощение общего уравнения	Спрощення загального рівняння	Sìmplificátion of the géneral equátion
53.Фокальный радиус	Фокальний радіус	Fócal ráduis (*pl* rádii)
54.Фокус	Фокус	Fócus (*pl* fóci)
55.Фокусное расстояние	Фокусна відстань	Fócal distance, dístance betwéen the fóci
56.Центр симметрии	Центр симетрії	Céntre of sýmmetry, sýmmetry céntre
57.Эксцентриситет	Ексцентриситет	Èccentrícity
58.Эллипс	Еліпс	Ellípse

SECOND ORDER CURVES (CONICS): main terminology ERU

1. Appróach the ásymptote	Приближаться к асимп-тоте	Наближатися до асимптоти
2. Ásymptote of a hypérbola	Асимптота гиперболы	Асимптота гіперболи
3. Áxis (*pl* áxes) of symmetry, sýmmetry axis	Ось симметрии	Вісь симетрії
4. Branch of a hypérbola (left(-hand), right(-hand), lówer, úpper)	Ветвь гиперболы (левая, правая, нижняя, верхняя)	Вітка гіперболи (ліва, права, нижня, верхня)
5. Canónical equátion of a quadrátic [quádric, sécond-degrée/órder] cúrve	Каноническое уравнение кривой второго порядка	Канонічне рівняння кривої другого порядку
6. Céntre of sýmmetry, sýmmetry céntre	Центр симметрии	Центр симетрії
7. Circúmference [circle] with a rádius R and a céntre (at the póint) A [céntered at (the póint) A	Окружность радиуса R с центром (в точке) A	Коло радіуса R з цент-ром (в точці) A
8. Còincíde (abóut points, the órigins of coórdinates)	Совпадать (о точках, началах координат и др.)	Збігатися (про точки, початки координат тощо)
9. Cónic séction/cúrve; cónic	Коническое сечение	Конічний переріз
10.Cónjugate hypérbola of the given hypérbola	Сопряжённая гипербола (для данной гиперболы)	Спряжена гіпербола (для даної гіперболи)
11.Diágonale of the májor [máin, príncipal] réctàngle	Диагональ основного прямоугольника	Діагональ основного прямокутника
12.Diréctrix (*pl* dìrectrí-ces)	Директриса	Директриса
13.Dráw a ray from the póle at the ángle φ to the pólar áxis	Провести луч из полюса под углом φ к полярной оси	Провести промінь з полюса під кутом φ до полярної осі
14.Draw/plot a ségment on *smth*	Отложить отрезок на *чём-л.*	Відкласти відрізок на *чо-мусь*
15.Èccentrícity	Эксцентриситет	Ексцентриситет
16.Ellípse	Эллипс	Еліпс
17.Èquidístant from *smth*	Равноотстоящий от *чего*	Рівновіддалений від *чогось*
18.Équiláteral hypérbola	Равнобочная гипербола	Рівнобічна гіпербола
19.Flátness/fláttining of an ellípse	Сплюснутость эллипса	Сплющеність еліпса

20.Fócal distance, dístance betwéen the fóci	Фокусное расстояние	Фокусна відстань
21.Fócal ráduis (*pl* rádii)	Фокальный радиус	Фокальний радіус
22.Fócus (*pl* fóci)	Фокус	Фокус
23.Hypérbola (*pl* hypérbolae, hypérbolas)	Гипербола	Гіпербола
24.Imaginary semiáxis [sémicónjugate áxis] of a hyperbola	Мнимая [не поперечная, не фокальная] полуось гиперболы	Уявна [не поперечна, не фокальна] піввісь гіперболи
25.Longer [májor] semiáxis [semimájor áxis] of an ellípse	Большая полуось эллипса	Велика піввісь еліпса
26.Májor [máin, príncipal] réctàngle	Основной прямоугольник	Основний прямокутник
27.Méasure of flátness/ fláttining of an ellípse	Мера сплюснутости эллипса	Міра сплющеності еліпса
28.Minor semiáxis [semimínor áxis] of an ellípse	Малая полуось эллипса	Мала піввісь еліпса
29.New coórdinate	Новая координата	Нова координата
30.New coórdinate sýstem	Новая система координат	Нова система координат
31.New órigin	Новое начало (координат)	Новий початок (координат)
32.Old coórdinate	Старая координата	Стара координата
33.Old coórdinate sýstem	Старая система координат	Стара система координат
34.Old órigin	Старое начало	Старий початок
35.Óptic(al) [reflection, fócal, geométric(al)] próperty	Оптическое [фокальное, геометрическое] свойство	Оптична [фокальна, геометрична] властивість
36.Parábola	Парабола	Парабола
37.Parameter [fócal parámeter] of a parábola	Параметр [фокальный параметр] параболы	Параметр [фокальний параметр] параболи
38.Pólar ángle of a póint	Полярный угол точки	Полярний кут точки
39.Pólar áxis	Полярная ось	Полярна вісь
40.Pólar coórdinate sýstem, sýstem of pólar coórdinates	Полярная система координат	Полярна система координат
41.Pólar coórdinates	Полярные координаты	Полярні координати
42.Pólar equátion	Полярное уравнение	Полярне рівняння
43.Pólar radius of a póint	Полярный радиус точки	Полярний радіус точки
44.Póle	Полюс	Полюс
45.Quadrátic [quádric, sécond degrée/órder] cúrve	Кривая второго порядка	Крива другого порядку

46.Ray cóming/starting from the póle at the ángle φ to the pólar áxis	Луч, исходящий из полюса (под углом φ к полярной оси)	Промінь, який виходить з полюса (під кутом φ) до полярної осі
47.Réal sèmiáxis [sèmitransvérsal [sèmitránsver-se] áxis] of a hypérbola	Вещественная [попереч-ная, фокальная, главная] полуось гиперболы	Дійсна [поперечна, фокальна, головна] піввісь гіперболи
48.Recéde [retíre, móve] to/ínto infínity	Удаляться в бесконеч-ность	Віддалятися в нескінчен-ність
49.Redúction of the géneral equátion of a sécond-órder curve to the canónical form	Приведение общего уравнения кривой второго порядка к каноническому виду	Зведення загального рів-няння кривої другого по-рядку до канонічного ви-гляду
50.Relátion(ship) betwéen new and old coórdinates	Связь между старыми и новыми координатами	Зв"язок між старими і новими координатами
51.Relátion(ship) betwéen pólar and Cartésian orthó-gonal/rectángular/grid co-órdinates	Связь между полярными и декартовыми прямоугольными координатами	Зв"язок між полярними і декартовими прямокутними координатами
52.Rotátion of coórdinate áxes (through an ángle α abóut the órigin of coórdinates)	Поворот координатных осей на угол α около начала координат	Поворот координатних осей на кут α навколо початку координат
53.Sémicìrcle (left(-hand) right(-hand), lówer, úpper)	Полуокружность (левая, правая, нижняя, верхняя)	Півколо (ліве, праве, нижнє, верхнє)
54.Sìmplificátion of the géneral equátion	Упрощение общего уравнения	Спрощення загального рівняння
55.Sùperpóse (the póle and the órigin of coórdina-tes, the pólar áxis and that of abscíssas/abscíssae)	Совместить (полюс с на-чалом координат, поляр-ную ось с осью абсцисс)	Сумістити (полюс з початком координат, полярну вісь з віссю абсцис)
56.Trànsformátion of coórdinates	Преобразование координат	Перетворення координат
57.Translátion of coórdinate áxes	Параллельный перенос координатных осей	Паралельний перенос координатних осей
58.Vértex (*pl* vértices)	Вершина	Вершина

SPACE ANALYTIC GEOMETRY

LECTURE NO. 9. VECTORS

POINT 1. VECTORS AND LINEAR OPERATIONS ON THEM

There are scalar variables (**scalars**) which are completely defined only by a number value (number of some things, human weight and height, atmospheric pressure and so on) and vector variables (**vectors**) completely defined nor only by a number value but also by a direction (force, speed, acceleration and so on).

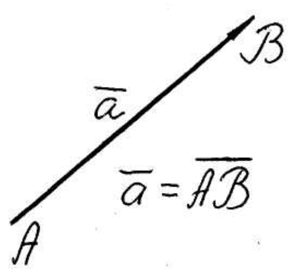


Fig. 1

Def. 1. A vector $\bar{a} = \overline{AB}$ is called a directed segment (see fig. 1). The point A is its origin, B its extremity, and a number $|\bar{a}| = |\overline{AB}|$ its length (or modulus).

Def. 2. A vector with unit length is called that unit [unitary] or normalized.

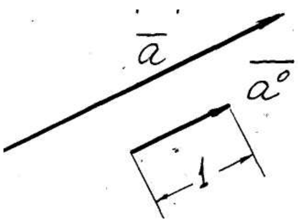


Fig. 2

Def. 3. A vector with zero length is called that zero. Its origin and extremity coincide

Def. 4. A unit vector of the same direction as a vector $\bar{a}$ (fig. 2) is called the unit vector of this vector and is denoted by $\overline{a^\circ}$.

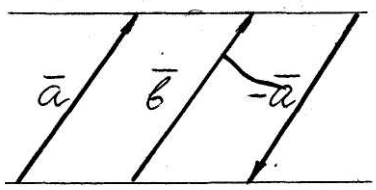


Fig. 3

Def. 5. Vectors of the same modulus and direction are called equal ($\overline{a}$ and $\overline{b}$ on fig. 3).

Def. 6. Vectors $\overline{a}$ and $-\overline{a}$ of the same modulus and opposite directions are called those opposite (fig. 3).

Def. 7. Vectors are called those collinear if they lie on one straight line or are parallel to the same straight line (fig. 4).

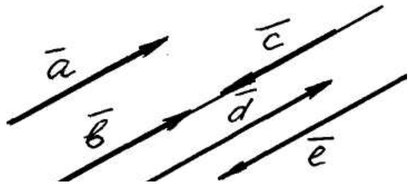


Fig. 4

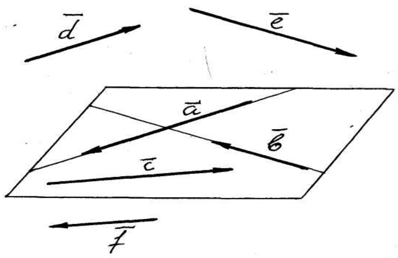


Fig. 5

To notice that two vectors have the same direction there is a symbol ↑↑ ($\overline{a}$ ↑↑ $\overline{b}$ on fig. 4), for vectors of opposite directions there is a symbol ↑↓ ($\overline{a}$ ↑↓ $\overline{c}$ on fig. 4).

Def. 8. Vectors are called those coplanar [complanar] if they lie in one plane or are parallel to the same plane (see fig. 5).

Linear operations on vectors

There are two linear operations on vectors namely addition [adding, composition] of vectors and multiplication of a vector by a number (by a scalar).

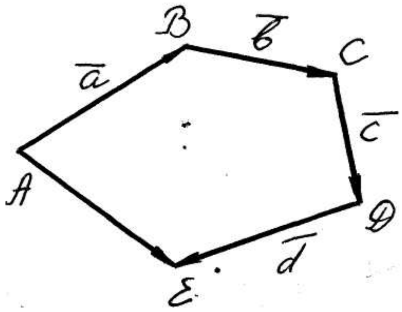


Fig. 6

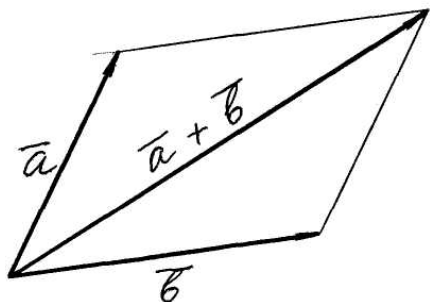


Fig. 7

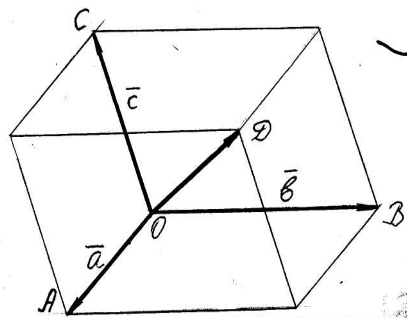


Fig. 8

Def. 9. **Sum of vectors** is defined by some rules: **polygon rule** for arbitrary number of vectors ($\overline{AE} = \overline{a} + \overline{b} + \overline{c} + \overline{d}$, fig. 6), **parallelogram rule** for two vectors (fig. 7), **parallelepiped rule** for three vectors ($\overline{OD} = \overline{a} + \overline{b} + \overline{c}$, fig. 8).

Def. 10. Product $k\overline{a}$ of a vector $\overline{a}$ by a number k is called the vector of the length $|k||\overline{a}|$ which has the direction of the vector $\overline{a}$ if $k > 0$ and opposite direction otherwise,

$$\overline{b} = k\overline{a} \text{ if a) } |\overline{b}| = |k||\overline{a}|\text{; b) } \overline{b} \uparrow\uparrow \overline{a} \text{ for } k > 0\text{; c) } \overline{b} \uparrow\downarrow \overline{a} \text{ for } k < 0.$$

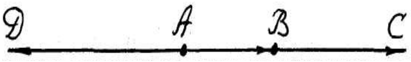


Fig. 9

Ex. 1. $\overline{AC} = 2.5\overline{AB}, \overline{AD} = -2\overline{AB}$ on the fig. 9.

Theorem 1. Any vector equals the product of its length and its unit vector

$$\overline{a} = |\overline{a}|\overline{a^\circ}. \qquad (1)$$

Theorem 2. Two vectors $\overline{a}, \overline{b}$ are collinear if and only if they differ by certain scalar factor, that is

$$\overline{a} \parallel \overline{b} \Leftrightarrow \exists k : \overline{b} = k\overline{a}\ . \qquad (2)$$

■If $\overline{a} \parallel \overline{b}$ and vectors have the same direction ($\overline{a} \uparrow\uparrow \overline{b}$), then

$$\overline{a^\circ} = \overline{b^\circ} \Rightarrow \overline{b} = |\overline{b}|\overline{b^\circ} = |\overline{b}|\overline{a^\circ} = |\overline{b}|\frac{\overline{a}}{|\overline{a}|} = \frac{|\overline{b}|}{|\overline{a}|}\overline{a},\ k = \frac{|\overline{b}|}{|\overline{a}|}.$$

If $\overline{a}, \overline{b}$ have opposite directions ($\overline{a} \uparrow\downarrow \overline{b}$), then

$$\overline{b^\circ} = -\overline{a^\circ} \Rightarrow \overline{b} = |\overline{b}|\overline{b^\circ} = -|\overline{b}|\overline{a^\circ} = -|\overline{b}|\frac{\overline{a}}{|\overline{a}|} = -\frac{|\overline{b}|}{|\overline{a}|}\overline{a},\ k = -\frac{|\overline{b}|}{|\overline{a}|}.■$$

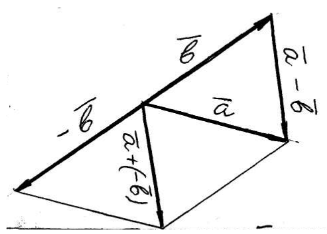


Fig. 10

Def. 11. A vector $\overline{c}$ is called the **difference** of vectors $\overline{a}$ and $\overline{b}$ ($\overline{c} = \overline{a} - \overline{b}$) if the sum of vectors $\overline{b}$ and $\overline{c}$ equals the vector $\overline{a}$ (fig. 10).

It's seen from the fig. 9 that

$$\overline{a} - \overline{b} = \overline{a} + (-1)\overline{b} = \overline{a} + (-\overline{b}).$$

Properties of linear operations on vectors

I. Properties of addition

1) $\overline{a} + \overline{b} = \overline{b} + \overline{a}$ (commutativity, additive commutation).

2) $\overline{a} + (\overline{b} + \overline{c}) = (\overline{a} + \overline{b}) + \overline{c}$ (associativity, additive association).

3) $\overline{a} + \overline{0} = \overline{a}$ for any vector $\overline{a}$.

4) $\overline{a}+\left(-\overline{a}\right)=\overline{0}$ for any vector $\overline{a}$.

II. Properties of multiplication of a vector by a number

5) $1\cdot\overline{a}=\overline{a}$ for any vector $\overline{a}$.

6) $\left(-1\right)\cdot\overline{a}=-\overline{a}$ for any vector $\overline{a}$.

7) $0\cdot\overline{a}=\overline{0}$ for any vector $\overline{a}$.

8) $k\cdot\overline{0}=\overline{0}$ for any number k

9) $\left(kl\right)\overline{a}=k\left(l\overline{a}\right)=l\left(k\overline{a}\right)$ for any numbers k, l and vector $\overline{a}$.

III. Common properties of addition and multiplication by a number

10) $\left(k+l\right)\overline{a}=k\overline{a}+l\overline{a}$ for any numbers k, l and vector $\overline{a}$.

11) $k\left(\overline{a}+\overline{b}\right)=k\overline{a}+k\overline{b}$ for any number k and vectors $\overline{a},\overline{b}$.

POINT 2. PROJECTION OF A VECTOR ON AN AXIS

Def. 12. Let there be given some axis ***u*** with unit vector $\overline{u^0}$ and a vector $\overline{AB}$. Let $AA_1\perp u$ and $BB_1\perp u$ (fig. 11). The vector $\overline{A_1B_1}$ is called the **component** of the vector $\overline{AB}$ along the axis ***u*** [***u***-component of the vector $\overline{AB}$].

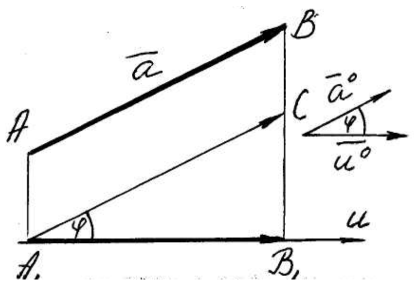


Fig. 11

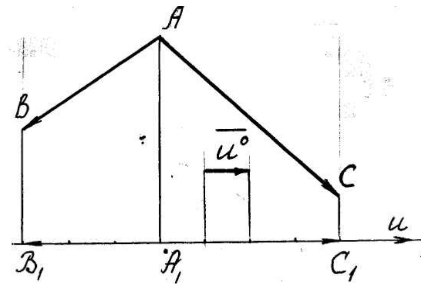


Fig. 12

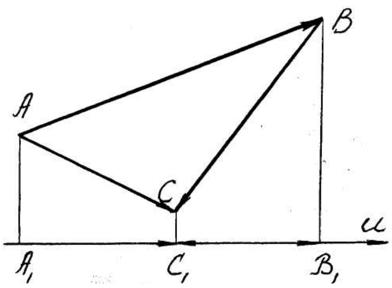


Fig. 13

Def. 13. Projection (orthogonal projection, normal component, rectangular component) of the vector $\overline{AB}$ on the axis ***u*** is called the modulus $\left|\overline{A_1B_1}\right|$ of its component $\overline{A_1B_1}$ if the latter has the same direction as the unit vector $\overline{u^0}$ of the axis and opposite number (-$\left|\overline{A_1B_1}\right|$) otherwise,

$$Pr_u \overline{AB} = \begin{cases} \left|\overline{A_1B_1}\right| & \text{if } \overline{A_1B_1} \uparrow\uparrow \overline{u^0}, \\ -\left|\overline{A_1B_1}\right| & \text{if } \overline{A_1B_1} \uparrow\downarrow \overline{u^0}. \end{cases} \qquad (3)$$

Ex. 2. For vectors $\overline{AC}, \overline{AB}$ represented on fig. 12

$$Pr_u \overline{AC} = \left|\overline{A_1C_1}\right| = 4,\ Pr_u \overline{AB} = -\left|\overline{A_1B_1}\right| = -3.$$

Def. 14. An angle between a vector and an axis, between two vectors is called the angle between their unit vectors.

Theorem 3. Projection of the vector on the axis equals the product of the length of the vector and the cosine of the angle between the vector and the axis,

$$Pr_u \overline{AB} = \left|\overline{AB}\right| cos\,\varphi,\ \varphi = \left(\overline{AB}, \hat{} u\right) \qquad (4)$$

■Let for simplicity $\overline{A_1B_1} \uparrow\uparrow \overline{u^\circ}$ (fig. 11). We draw $A_1C \| AB$ whence it follows that

$$Pr_u \overline{AB} = \left|\overline{A_1B_1}\right| = A_1C\,cos\,\varphi = \left|\overline{A_1C}\right| cos\,\varphi = \left|\overline{AB}\right| cos\,\varphi.$$

The case of $\overline{A_1B_1} \uparrow\downarrow \overline{u^\circ}$ consider yourselves.■

Theorem 4 (relationship between the projection of a vector on the axis and the component of this vector along the axis)**.**

$$\overline{A_1B_1} = \mathrm{Pr}_u \overline{AB} \cdot \overline{u^0} \qquad (5)$$

Properties of projections

1. Projection of a sum of two vectors equals the sum of their projections,

$$\mathrm{Pr}_u\left(\overline{a} + \overline{b}\right) = \mathrm{Pr}_u \overline{a} + \mathrm{Pr}_u \overline{b}.$$

■Let for example $\overline{AC} = \overline{AB} + \overline{BC}$ (fig. 13). Then

$$Pr_u\left(\overline{AB} + \overline{BC}\right) = Pr_u \overline{AC} = \left|\overline{A_1C_1}\right| = \left|\overline{A_1B_1}\right| - \left|\overline{B_1C_1}\right| = \left|\overline{A_1B_1}\right| + \left(-\left|\overline{B_1C_1}\right|\right) =$$

$$= \mathrm{Pr}_u \overline{AB} + \mathrm{Pr}_u \overline{BC}$$ ■

2. Scalar factor *k* can be taken outside the projection sign,

$$\mathrm{Pr}_u\left(k \cdot \overline{a}\right) = k \cdot \mathrm{Pr}_u \overline{a}$$

POINT 3. BASES. DECOMPOSITION OF A VECTOR WITH RESPECT TO A BASE

Arbitrary bases

Def 15. A base in the plane is called any ordered pair $\overline{e_1}, \overline{e_2}$ of non-collinear vectors with common origin *O* (fig. 14).

Def 16. A base in the space is called any ordered triplet $\overline{e_1}, \overline{e_2}, \overline{e_3}$ of non-co-planar vectors with common origin O (fig. 15).

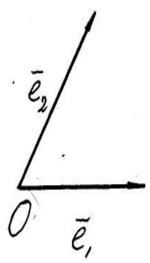


Fig. 14

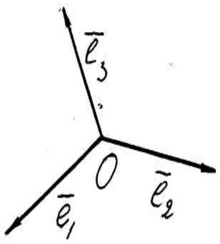


Fig. 15

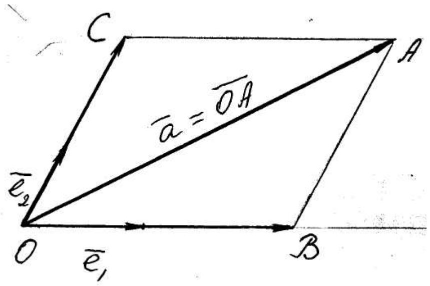


Fig. 16

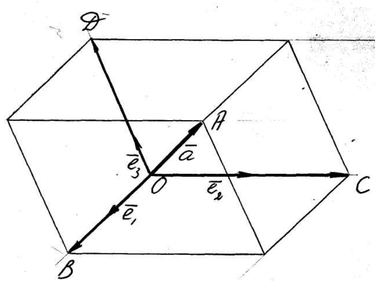


Fig. 17

Theorem 5 (decomposition of a vector with respect to a base). Every vector $\overline{a}$ can be uniquely represented in the next form

$$\overline{a} = a_1\overline{e_1} + a_2\overline{e_2} \text{ (in the plane)} \quad (6)$$

$$\overline{a} = a_1\overline{e_1} + a_2\overline{e_2} + a_3\overline{e_3} \text{ (in the space)} \quad (7)$$

where a_1, a_2 (correspondingly a_1, a_2, a_3) are some numbers (which are called the **coordinates of the vector** $\overline{a}$ in the given base).

■Proving for the base $\overline{e_1}, \overline{e_2}$ on the plane. Let a vector in question $\overline{a} = \overline{OA}$ (see fig. 16). We draw $AB \parallel \overline{e_2}$, $AC \parallel \overline{e_1}$ to meet the straight lines of the vectors $\overline{e_2}, \overline{e_1}$ at the points *B*, *C* correspondingly. $\overline{OB} \parallel \overline{e_1}$, $\overline{OC} \parallel \overline{e_2}$, so by the theorem 2 there are two numbers a_1, a_2 such that $\overline{OB} = a_1 \cdot \overline{e_1}$, $\overline{OC} = a_2 \cdot \overline{e_2}$. Therefore $\overline{a} = \overline{OA} = \overline{OB} + \overline{OC} = a_1 \cdot \overline{e_1} + a_2 \cdot \overline{e_2}$ and the equality (6) is proved.

If it were another decomposition $\overline{a} = a_1'\overline{e_1} + a_2'\overline{e_2}$ of the vector $\overline{a} = \overline{OA}$ we would have

$a_1\overline{e_1} + a_2\overline{e_2} = a_1'\overline{e_1} + a_2'\overline{e_2}, (a_1 - a_1')\overline{e_1} + (a_2 - a_2')\overline{e_2} = \overline{0}, a_1 - a_1' = 0, a_2 - a_2' = 0, a_1 = a_1', a_2 = a_2'$ by virtue of non-collinearity of the vectors $\overline{e_1}, \overline{e_2}$.■

By analogous way we can prove this theorem for the space (see fig. 17).

Corollary. Two vectors are equal if and only if their correspondent coordinates are equal.

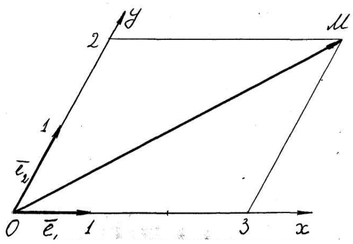


Fig. 18

A base $\overline{e_1}, \overline{e_2}$ on the plane ($\overline{e_1}, \overline{e_2}, \overline{e_3}$ in the space) generates the coordinate system O*xy* on the plane (O*xyz* in the space) with units of scale [of measurement, of measuring] $|\overline{e_1}|, |\overline{e_2}|$ ($|\overline{e_1}|, |\overline{e_2}|, |\overline{e_3}|$) on every axis O*x*, O*y* (O*x*, O*y*, O*z*) respectively.

Ex. 3. A point *M* in the coordinate system represented by fig. 18 has coordinates 3 and 2, *M* (3; 2). The same coordinates has the vector $\overline{OM}$, because of

$$\overline{OM} = 3\overline{e_1} + 2\overline{e_2} .$$

Def 17. A vector $\overline{OA}$ is called a radius vector of a point *A*.

Theorem 6. Coordinates of any point coincide with coordinates of its radius vector.

■We have seen validity of the theorem in Ex. 3. In general case the proving is similar.■

Very important case of a base is that Cartesian.

Cartesian (orthonormal, orthonormalized] base

Def. 18. Cartesian base (orthonormal [orthonormalized] base) in the plane (in the space) is called the base which is made up by two (three) unit mutually perpendicular [orthogonal] vectors $\overline{i}, \overline{j}$ ($\overline{i}, \overline{j}, \overline{k}$),

$$|\overline{i}| = |\overline{j}| = |\overline{k}| = 1, \ \overline{i} \perp \overline{j}, \overline{i} \perp \overline{k}, \overline{j} \perp \overline{k} .$$

Cartesian base generates so-called Cartesian rectangular [orthogonal, grid] coordinate system *Oxy* on the plane (*Oxyz* in the spáce). It consists of two (three) mutually perpendicular (orthogonal) axes *Ox*, *Oy*, *Oz* with the same unit of scale [of measurement, of measuring] $|\bar{i}| = |\bar{j}| = |\bar{k}| = 1$.

Theorem 7. Coordinates of a vector $\bar{a}$ in Cartesian base $\bar{i}, \bar{j}$ ($\bar{i}, \bar{j}, \bar{k}$) are its projections on coordinate axes,

$$a_1 = Pr_{Ox}\,\bar{a},\ a_2 = Pr_{Oy}\,\bar{a}\ \ (a_1 = Pr_{Ox}\,\bar{a},\ a_2 = Pr_{Oy}\,\bar{a},\ a_3 = Pr_{Oz}\,\bar{a}),$$

and so the decomposition of a vector with respect to this base has the next form

$$\bar{a} = a_x \cdot \bar{i} + a_y \cdot \bar{j},\ \text{ where }\ a_x = Pr_{Ox}\,\bar{a},\ a_y = Pr_{Oy}\,\bar{a} \qquad (8)$$

on the plane and

$$\bar{a} = a_x \cdot \bar{i} + a_y \cdot \bar{j} + a_z \cdot \bar{k}\ \text{ where }\ a_x = Pr_{Ox}\,\bar{a},\ a_y = Pr_{Oy}\,\bar{a},\ a_z = Pr_{Oz}\,\bar{a} \qquad (9)$$

in the space.

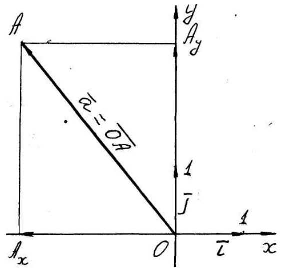


Fig. 19

■ (for the plane and the case $\bar{a} = \overline{OA}$, fig. 19). $\bar{a} = \overline{OA} = \overline{OA_x} + \overline{OA_y}$, where $\overline{OA_x}, \overline{OA_y}$ be the components of the vector $\bar{a} = \overline{OA}$ along the axes Ox and Oy correspondingly. By the formula (5)

$$\overline{OB} = Pr_{Ox}\,\bar{a} \cdot \bar{i} = a_x \cdot \bar{i},\ \overline{OC} = Pr_{Oy}\,\bar{a} \cdot \bar{j} = a_y \cdot \bar{j},$$

and therefore

$$\bar{a} = \overline{OA} = a_x \bar{i} + a_y \bar{j},\ a_x = Pr_{Ox}\,\bar{a},\ a_y = Pr_{Oy}\,\bar{a}.■$$

POINT 4. COROLLARIES

1. To linear operations on vectors there correspond the same operations on their coordinates, i.e. (id est *лат*) if

$$\bar{a} = a_1\bar{e}_1 + a_2\bar{e}_2 + a_3\bar{e}_3,\ \bar{b} = b_1\bar{e}_1 + b_2\bar{e}_2 + b_3\bar{e}_3$$

then

$$\bar{a} + \bar{b} = (a_1 + b_1)\bar{e}_1 + (a_2 + b_2)\bar{e}_2 + (a_3 + b_3)\bar{e}_3,\ k \cdot \bar{a} = (ka_1)\bar{e}_1 + (ka_2)\bar{e}_2 + (ka_3)\bar{e}_3.$$

■By virtue of corresponding properties of linear operations on vectors

$$\bar{a} + \bar{b} = (a_1\bar{e}_1 + a_2\bar{e}_2 + a_3\bar{e}_3) + (b_1\bar{e}_1 + b_2\bar{e}_2 + b_3\bar{e}_3) = (a_1 + b_1)\bar{e}_1 + (a_2 + b_2)\bar{e}_2 + (a_3 + b_3)\bar{e}_3,$$

$$k\overline{a} = k\left(a_1\overline{e_1} + a_2\overline{e_2} + a_3\overline{e_3}\right) = (ka_1)\overline{e_1} + (ka_2)\overline{e_2} + (ka_3)\overline{e_3}. \blacksquare$$

Corollary. As to [as for, as concerns, concerning] linear operations we identify a vector and a row of its coordinates, and we'll write as usually

$$\overline{a} = (a_1, a_2, a_3)$$

instead

$$\overline{a} = a_1\overline{e_1} + a_2\overline{e_2} + a_3\overline{e_3}.$$

Ex. 4. Decompose a vector $\overline{m} = (0,6,3)$ with respect to the next three vectors $\overline{a} = (4,1,3)$, $\overline{b} = (-1,4,2), \overline{c} = (2,-1,4)$. Do the vectors $\overline{a}, \overline{b}, \overline{c}$ constitute a base of the space?

Solution. We must find three numbers x, y, z to have

$$\overline{m} = x \cdot \overline{a} + y \cdot \overline{b} + z \cdot \overline{c}.$$

But

$$\begin{aligned} x \cdot \overline{a} + y \cdot \overline{b} + z \cdot \overline{c} &= x \cdot (4,1,3) + y \cdot (-1,4,2) + z \cdot (2,-1,4) = \\ &= (4x - y + 2z, x + 4y - z, 3x + 2y + 4z); \overline{m} = (0,6,3). \end{aligned}$$

Equating corresponding coordinates of the vectors $x \cdot \overline{a} + y \cdot \overline{b} + z \cdot \overline{c}$ and $\overline{m}$ we get a system of linear equations in x, y, z

$$\begin{cases} 4x - y + 2z = 0, \\ x + 4y - z = 6, \\ 3x + 2y + 4z = 3. \end{cases}$$

The principal determinant of the system $\Delta = 59 \neq 0$, and so it possesses unique solution $(27/59, 78/59, -15/59)$. Therefore the decomposition in question is

$$\overline{m} = \frac{27}{59} \cdot \overline{a} + \frac{78}{59} \cdot \overline{b} - \frac{15}{59} \cdot \overline{c}.$$

The same problem can be solved for arbitrary spatial vector, because of the solving leads to a system with the same left sides and the principal determinant $\Delta =$ $= 59 \neq 0$. Therefore any vector of the space can be decomposed with respect to the vectors $\overline{a}, \overline{b}, \overline{c}$, and so they constitute a base in the space.

2. Necessary and sufficient condition for collinearity of two vectors is proportionality of their corresponding coordinates.

■ Let vectors $\overline{a}=(a_1,a_2,a_3), \overline{b}=(b_1,b_2,b_3)$ be collinear. By the theorem 2 there exists a number k such that $\overline{b}=k\cdot\overline{a}$, $(b_1,b_2,b_3)=(k\cdot a_1, k\cdot a_2, k\cdot a_3)$, from which $b_1=k\cdot a_1, b_2=k\cdot a_2, b_3=k\cdot a_3$ and $b_1:a_1=b_2:a_2=b_3:a_3=k$.

Let inversely $b_1:a_1=b_2:a_2=b_3:a_3=k$. Then

$$b_1=ka_1, b_2=ka_2, b_3=ka_3, \overline{b}=(ka_1,ka_2,ka_3)=k(a_1,a_2,a_3)=k\overline{a},$$

and by virtue of the same theorem $\overline{a}\|\overline{b}$.■

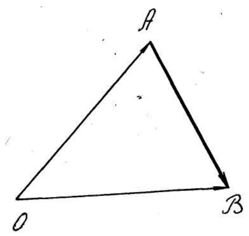


Fig. 20

3. To find a vector $\overline{AB}$ knowing coordinates of its origin $A(x_1,y_1,z_1)$ and end point $B(x_2,y_2,z_2)$ it's necessary to subtract the coordinates of the origin from corresponding coordinates of the end point (fig. 20),

$$\overline{AB}=(x_2-x_1, y_2-y_1, z_2-z_1). \quad (10)$$

■The fig. 20 shows that

$$\overline{AB}=\overline{OB}-\overline{OA}.$$

The vectors $\overline{OA}, \overline{OB}$ are the radius vectors of the points A, B. On the base of the theorem 6

$$\overline{OA}=(x_1;y_1;z_1), \overline{OB}=(x_2;y_2;z_2),$$

hence

$$\overline{AB}=\overline{OB}-\overline{OA}=(x_2;y_2;z_2)-(x_1;y_1;z_1)=(x_2-x_1, y_2-y_1, z_2-z_1).■$$

Additional corollaries as to vectors represented in Cartesian base

4. Length [modulus, absolute value] of a vector equals the root of the sum of squares [the root-sum squares, the root of sum-of-squares] of its coordinates in Cartesian base,

$$|\overline{a}|=|a_x\overline{i}+a_y\overline{j}|=\sqrt{a_x^2+a_y^2} \text{ on the plane,} \quad (11)$$

$$|\bar{a}| = |a_x \bar{i} + a_y \bar{j} + a_z \bar{k}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \text{ in the space.} \qquad (12)$$

■ Let for example a vector $\bar{a}$ on the plane is one at the origin ($\bar{a} = \overline{OA}$, fig.19). Its length equals the length of the diagonal of the rectangle which dimensions are equal to moduli of the projections a_x, a_y of the vector on the coordinate axes. By analogy the length of a space vector $\bar{a}$ (see fig. 21) equals the length of the diagonal of the rectangular parallelepiped with dimensions $|a_x|, |a_y|, |a_z|$■

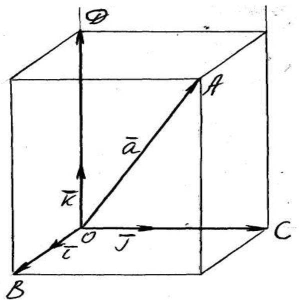


Fig. 21

Def 19. Direction cosines of a vector $\bar{a}$ are called the cosines of the angles α, β, γ which the vector forms with the coordinate axes *Ox*, *Oy*, *Oz* respectively (fig. 22).

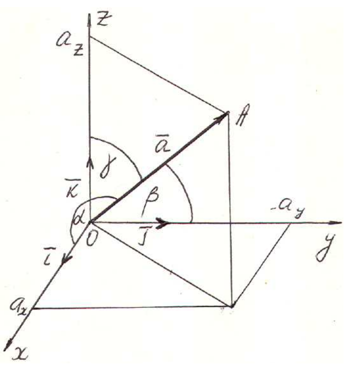


Fig. 22

5. Direction cosines of a vector $\bar{a}$ are given by formulae

$$\cos\alpha = \frac{a_x}{|\bar{a}|}, \cos\beta = \frac{a_y}{|\bar{a}|}, \cos\gamma = \frac{a_z}{|\bar{a}|}, \qquad (13)$$

because by virtue of the formula (4)

$$a_x = Pr_{Ox}\,\bar{a} = |\bar{a}|\cos\alpha,\ a_y = Pr_{Oy}\,\bar{a} = |\bar{a}|\cos\beta,\ a_z = Pr_{Oz}\,\bar{a} = |\bar{a}|\cos\gamma .$$

6. Sum of squares of direction cosines equals 1,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \qquad (14)$$

■On the base of formulas (13), (12)

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_x^2 + a_y^2 + a_z^2}{|\bar{a}|^2} = \frac{|\bar{a}|^2}{|\bar{a}|^2} = 1.■$$

Corollary. The unit vector of a vector $\bar{a}$ which is determined in Cartesian base is

$\overline{a^0} = \cos\alpha \cdot \bar{i} + \cos\beta \cdot \bar{j}$ (or simply $\overline{a^0} = (\cos\alpha, \cos\beta)$) on the plane,

$\overline{a^0} = \cos\alpha \cdot \bar{i} + \cos\beta \cdot \bar{j} + \cos\gamma \cdot \bar{k}$ (or simply $\overline{a^0} = (\cos\alpha, \cos\beta, \cos\gamma)$) in the space.

7. Distance between two space points $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$, by virtue of the

formulas (10), (12), equals the square root of the sum of squares [to the root-sum square, to the root of sum-of-squares] of differences of their coordinates

$$AB = |AB| = |\overline{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\,. \qquad (15)$$

For two points $A(x_1, y_1), B(x_2, y_2)$ of the plane

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,. \qquad (16)$$

In conclusion we'll enumerate three main ways [methods, modes] of represent-ting [representation, definition, determination] of a vector: a) by its coordinates in so-me base; b) by coordinates of its origin and end point; c) in Cartesian base – by its length [modulus] and direction cosines.

Ex. 5. A vector is given in Cartesian base by its origin $A(1; 0; -2)$ and end point $B(3; -4; 4)$. Find the vector $\overline{AB}$, its modulus, unit vector, direction cosines, distance between the points *A* and *B*. Then find a vector $\overline{x}$ of opposite direction ha-ving the length 12.

Solution. The formula (10) yields

$$\overline{AB} = (3-1; -4-0; 4-(-2)) = (2; -4; 6); \quad \overline{AB} = (2; -4; 6).$$

By the formulae (15), (13), (1)

$$|\overline{AB}| = \sqrt{2^2 + (-4)^2 + 6^2} = \sqrt{56} = |AB|;\ cos\,\alpha = \frac{2}{\sqrt{56}},\ cos\,\beta = -\frac{4}{\sqrt{56}},\ cos\,\gamma = \frac{6}{\sqrt{56}},$$

$$\overline{AB^0} = \frac{\overline{AB}}{|\overline{AB}|} = (cos\,\alpha; cos\,\beta; cos\,\gamma) = \left(\frac{2}{\sqrt{56}}, -\frac{4}{\sqrt{56}}, \frac{6}{\sqrt{56}}\right) = \frac{1}{\sqrt{56}}(2, -4, 6),$$

$$\overline{x^0} = -\overline{AB^0} = \left(-\frac{2}{\sqrt{56}}, \frac{4}{\sqrt{56}}, -\frac{6}{\sqrt{56}}\right),$$

$$\overline{x} = |\overline{x}| \cdot \overline{x^0} = 12 \cdot \overline{x^0} = 12 \cdot \left(-\frac{2}{\sqrt{56}}, \frac{4}{\sqrt{56}}, -\frac{6}{\sqrt{56}}\right) = \left(-\frac{24}{\sqrt{56}}, \frac{48}{\sqrt{56}}, -\frac{72}{\sqrt{56}}\right).$$

POINT 5. DIVISION OF A SEGMENT IN A GIVEN RATIO

Def. 11. One says that a point *M* divides a segment *AB* in a given ratio λ if an equality

$$\overline{AM} = \lambda \cdot \overline{MB} \qquad (17)$$

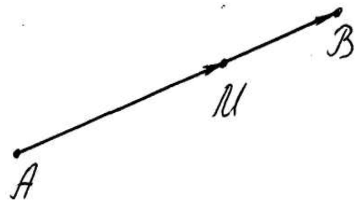


Fig. 23

holds (fig. 23).

For space segment $A(x_1, y_1, z_1)B(x_2, y_2, z_2)$ coordinates of a point $M(x, y, z)$, dividing the segment in the ratio λ, are given by the formulas

$$x = \frac{x_1 + \lambda \cdot x_2}{1+\lambda},\ y = \frac{y_1 + \lambda \cdot y_2}{1+\lambda},\ z = \frac{z_1 + \lambda \cdot z_2}{1+\lambda} \qquad (18)$$

$$\blacksquare\ \overline{AM} = (x - x_1, y - y_1, z - z_1),\ \overline{MB} = (x_2 - x, y_2 - y, z_2 - z),,$$

$$\lambda \cdot \overline{MB} = (\lambda(x_2 - x), \lambda(y_2 - y), \lambda(z_2 - z)).$$

Equating corresponding coordinates of the vectors $\overline{AM}, \lambda \cdot \overline{MB}$ we get successively

$$x - x_1 = \lambda(x_2 - x),\ y - y_1 = \lambda(y_2 - y),\ z - z_1 = \lambda(z_2 - z),$$
$$x(1+\lambda) = x_1 + \lambda x_2,\ y(1+\lambda) = y_1 + \lambda y_2,\ z(1+\lambda) = z_1 + \lambda z_2,$$

$$x = \frac{x_1 + \lambda \cdot x_2}{1+\lambda},\ y = \frac{y_1 + \lambda \cdot y_2}{1+\lambda},\ z = \frac{z_1 + \lambda \cdot z_2}{1+\lambda}\ .\blacksquare$$

Remark. If a point M lies between the points A and B (and therefore $\lambda > 0$) one can write more simply, namely $|AM|/|MB| = \lambda$, instead (17).

If a segment AB is divided in half by a point $M(x, y, z)$, we have $\lambda = 1$, and so

$$x = \frac{x_1 + x_2}{2},\ y = \frac{y_1 + y_2}{2},\ z = \frac{z_1 + z_2}{2}\ . \qquad (19)$$

Ex. 6. A point $C(2; -1; 6)$ divides in halves a segment AB with known end point $A(-3; 4; 5)$. Find coordinates of the point B.

Solution. Applying the formulae (19) we have

$$x_C = \frac{x_A + x_B}{2},\ y_C = \frac{y_A + y_B}{2},\ z_C = \frac{z_A + z_B}{2};\ x_B = 2x_C - x_A,\ y_B = 2y_C - y_A,\ z_B = 2z_C - z_A,$$

whence it follows that

$$x_B = 2 \cdot 2 - (-3) = 7,\ y_B = 2 \cdot (-1) - 4 = -6,\ z_B = 2 \cdot 6 - 5 = 7; \quad B(7; -6; 7).$$

Ex. 7. Find a median BK and a bisector AL of a triangle with known vertices $A(-4; 5; 3), B(3; -2; 4), C(2; -4; 6)$ (fig. 24).

By the formulas (19) and (15)

$$x_K = \frac{x_A + x_C}{2} = -1,\ y_K = \frac{y_A + y_C}{2} = 0.5,\ z_K = \frac{z_A + z_C}{2} = 1.5,$$

$$BK = \sqrt{(x_K - x_B)^2 + (y_K - y_B)^2 + (z_K - z_B)^2} = \sqrt{4^2 + 2.5^2 + 2.5^2} = \sqrt{28.50} \approx 5.3.$$

By (15)

$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} = \sqrt{7^2 + 7^2 + 1^2} = \sqrt{99},$$

$$AC = \sqrt{(x_C - x_A)^2 + (y_C - y_A)^2 + (z_C - z_A)^2} = \sqrt{6^2 + 9^2 + 3^2} = \sqrt{126}.$$

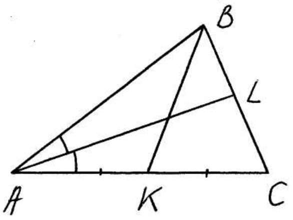


Fig. 24

Hence

$$BL : LC = AB : AC = \sqrt{99/126} \approx 0.89.$$

Now by the formulas (18)

$$x_L = \frac{x_B + \lambda x_C}{1+\lambda} = \frac{3 + 0.89 \cdot 2}{1.89} \approx 2.53,\ y_L = \frac{y_B + \lambda y_C}{1+\lambda} = \frac{-2 + 0.89 \cdot (-4)}{1.89} \approx -2.94,$$

$$z_L = \frac{z_B + \lambda z_C}{1+\lambda} = \frac{4 + 0.89 \cdot 6}{1.89} \approx 4.94,$$

and then by (15)

$$AL = \sqrt{(x_L - x_A)^2 + (y_L - y_A)^2 + (z_L - z_A)^2} = \sqrt{6.53^2 + 7.94^2 + 1.94^2} = \sqrt{109.44} \approx 10.5.$$

Answer. The median $BK \approx 5.3$, the bisector $AL \approx 10.5$.

LECTURE NO. 10. PRODUCTS OF VECTORS

POINT 1. SCALAR PRODUCT

POINT 2. VECTOR PRODUCT

POINT 3. n-DIMENSION VECTORS

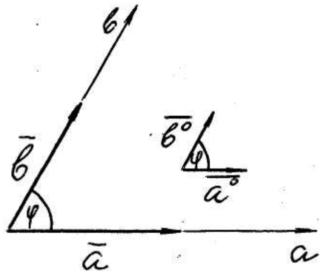
POINT 1. SCALAR PRODUCT

Def. 1. A scalar [a dot, an Euclidean, an inner, an interior, an internal] product of two vectors $\overline{a}, \overline{b}$ is called a number which equals the product of the lengths of these vectors and the cosine of an angle between them,

$$\overline{a}\overline{b} \equiv \overline{a} \cdot \overline{b} \equiv \left(\overline{a}, \overline{b}\right) = \left|\overline{a}\right|\left|\overline{b}\right| \cos\varphi,\ \varphi = \left(\overline{a}\,\hat{,}\,\overline{b}\right) = \left(\overline{a^\circ}\,\hat{,}\,\overline{b^\circ}\right). \qquad (1)$$

In accordance with the formula (4) of the preceding lecture we can write

$$\overline{a}\overline{b} = \left|\overline{a}\right| Pr_a \overline{b} = \left|\overline{b}\right| Pr_b \overline{a} \quad or \quad \overline{a}\overline{b} = \left|\overline{a}\right| Pr_{\overline{a}} \overline{b} = \left|\overline{b}\right| Pr_{\overline{b}} \overline{a}, \qquad (2)$$



where *a* (corr. *b*) is the axis determined by the vector $\overline{a}$ (corr. by $\overline{b}$), briefly the axis of the vector $\overline{a}$ (corr. $\overline{b}$) (see fig. 1). Thus, a scalar product of two vectors equals a product of the length of one vector and the projection of the other on the axis of the first one (or simply on the first one).

Fig. 1

Ex. 1. Let the lengths of vectors $\overline{a}, \overline{b}$ equal $\left|\overline{a}\right| = 5, \left|\overline{b}\right| = 8$ and the angle between them is $\varphi = \pi/3$. On the base of the definition 1 and the formula (2)

$$\overline{a}\overline{b} = \left|\overline{a}\right|\left|\overline{b}\right| \cos\frac{\pi}{3} = 5 \cdot 8 \cdot \frac{1}{2} = 20,\ Pr_b\, \overline{a} = \frac{\overline{a}\overline{b}}{\left|\overline{b}\right|} = \frac{20}{8} = 2.5,\ Pr_a\, \overline{b} = \frac{\overline{a}\overline{b}}{\left|\overline{a}\right|} = \frac{20}{5} = 4.$$

Ex. 2. For the vectors of Cartesian (orthonormal) base $\overline{i}, \overline{j}, \overline{k}$

$$\overline{i} \cdot \overline{j} = \overline{i} \cdot \overline{k} = \overline{k} \cdot \overline{j} = 0,$$

because of $cos\left(\overline{i}\,\hat{,}\,\overline{j}\right) = cos\left(\overline{i}\,\hat{,}\,\overline{k}\right) = cos\left(\overline{j}\,\hat{,}\,\overline{k}\right) = cos(\pi/2) = 0$.

Properties of a scalar product

1. $\overline{a}\overline{b}=\overline{b}\overline{a}$ (commutativity).

■The property is obvious from the definition 1 (see the formula (1)).■

2. $(\overline{a}+\overline{b})\overline{c}=\overline{a}\overline{c}+\overline{b}\overline{c}$ (distributivity).

■By virtue of the formula (2) and the corresponding property of projections

$(\overline{a}+\overline{b})\overline{c}=|\overline{c}|\,Pr_c(\overline{a}+\overline{b})=|\overline{c}|(Pr_c\,\overline{a}+Pr_c\,\overline{b})=|\overline{c}|\,Pr_c\,\overline{a}+|\overline{c}|\,Pr_c\,\overline{b}=\overline{a}\overline{c}+\overline{b}\overline{c}$.■

3. For any number (scalar) k

$$k(\overline{a}\cdot\overline{b})=(k\overline{a})\cdot\overline{b}=\overline{a}\cdot(k\overline{b}).$$

4. Scalar square $\overline{a}^2=\overline{a}\cdot\overline{a}$ of a vector equals the square of its length,

$$\overline{a}^2=\overline{a}\cdot\overline{a}=|\overline{a}|^2. \qquad (3)$$

■$\overline{a}^2=\overline{a}\cdot\overline{a}=|\overline{a}||\overline{a}|cos(\overline{a}\,\hat{,}\,\overline{a})=|\overline{a}|^2\,cos\,0=|\overline{a}|^2$.■

5. Two non-zero vectors are orthogonal if and only if their scalar product is equal to zero.

■a) If $\overline{a}\perp\overline{b}$, then $\varphi=(\overline{a}\,\hat{,}\,\overline{b})=(\overline{a^\circ}\,\hat{,}\,\overline{b^\circ})=\pi/2\ or\ \varphi=3\pi/2\Rightarrow cos\,\varphi=0$; hence $\overline{a}\cdot\overline{b}=0$.

b) If $\overline{a}\cdot\overline{b}=0$ and $\overline{a}\neq\overline{0},\overline{b}\neq\overline{0}$, then by (1) $cos\,\varphi=cos(\overline{a}\,\hat{,}\,\overline{b})=0$, and $\overline{a}\perp\overline{b}$.■

Ex. 3. Lengths of vectors $\overline{a},\overline{b}$ are equal to $|\overline{a}|=5,|\overline{b}|=8$, the angle between them $\frac{2\pi}{3}$. Find the length of the vector $3\overline{a}-2\overline{b}$.

On the base of the properties 4, 2, 3 of a scalar product

$$|3\overline{a}-2\overline{b}|^2=(3\overline{a}-2\overline{b})^2=(3\overline{a}-2\overline{b})(3\overline{a}-2\overline{b})=9\overline{a}^2+4\overline{b}^2-12\overline{a}\overline{b}=$$

$$=9|\overline{a}|^2+4|\overline{b}|^2-12|\overline{a}||\overline{b}|\cos(\overline{a}\,\hat{,}\,\overline{b})=9\cdot25+4\cdot64-12\cdot5\cdot8\cdot\cos\frac{2\pi}{3}=481-480\cdot\left(-\frac{1}{2}\right)=721,$$

$$|3\overline{a}-2\overline{b}|=\sqrt{721}\approx26.9.$$

Expression of a scalar product of vectors in terms of their coordinates in Cartesian (orthonormal) base

Let vectors $\bar{a}, \bar{b}$ be decomposed with respect to Cartesian (orthonormal) base $\bar{i}, \bar{j}, \bar{k}$, that is

$$\bar{a} = a_x \bar{i} + a_y \bar{j} + a_z \bar{k},\ \bar{b} = b_x \bar{i} + b_y \bar{j} + b_z \bar{k} \text{ or briefly } \bar{a} = (a_x, a_y, a_z), \bar{b} = (b_x, b_y, b_z).$$

In this case a scalar product of the vectors equals the sum of products of their corresponding coordinates,

$$\bar{a} \cdot \bar{b} = a_x b_x + a_y b_y + a_z b_z . \tag{4}$$

■By virtue of the properties 2, 3, 4 of a scalar product

$$\bar{a}\bar{b} = (a_x \bar{i} + a_y \bar{j} + a_z \bar{k}) \cdot (b_x \bar{i} + b_y \bar{j} + b_z \bar{k}) = (a_x b_x)(\bar{i} \cdot \bar{i}) + (a_x b_y)(\bar{i} \cdot \bar{j}) + (a_x b_z)(\bar{i} \cdot \bar{k}) +$$

$$(a_y b_x)(\bar{j} \cdot \bar{i}) + (a_y b_y)(\bar{j} \cdot \bar{j}) + (a_y b_z)(\bar{j} \cdot \bar{k}) + (a_z b_x)(\bar{k} \cdot \bar{i}) + (a_z b_y)(\bar{k} \cdot \bar{j}) + (a_z b_z)(\bar{k} \cdot \bar{k}) =$$

$$= (a_x b_x)\bar{i}^2 + (a_x b_y) \cdot 0 + (a_x b_z) \cdot 0 + (a_y b_x) \cdot 0 + (a_y b_y)(\bar{j}^2) + (a_y b_z) \cdot 0 +$$

$$(a_z b_x) \cdot 0 + (a_z b_y) \cdot 0 + (a_z b_z)\bar{k}^2 = (a_x b_x) \cdot 1 + (a_y b_y) \cdot 1 + (a_z b_z) \cdot 1 = a_x b_x + a_y b_y + a_z b_z . \blacksquare$$

Some applications of a scalar product

1. Finding an angle between two vectors. By virtue of the formula (1)

$$cos\left(\bar{a} \,\hat{,}\, \bar{b}\right) = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}||\bar{b}|} \tag{5}$$

2. Finding a projection of one vector onto (an axis of) the other vector. On the base of the formula (2)

$$Pr_a \bar{b} = Pr_{\bar{a}} \bar{b} = \frac{\bar{a} \cdot \bar{b}}{|\bar{a}|} \tag{6}$$

3. Scalar product in economics. Let a shop realizes daily

a_1 items of the first kind with a unit price p_1,

a_2 items of the second kind with a unit price p_2,

a_3 items of the third kind with a unit price p_3.

A vector

$$\overline{a} = (a_1, a_2, a_3)$$

represents the set of products, and a vector

$$\overline{p} = (p_1, p_2, p_3)$$

is that of price. Then the daily proceeds of sales *S* of the shop is the next scalar product

$$S = a_1 p_1 + a_2 p_2 + a_3 p_3 = \overline{a} \cdot \overline{p}.$$

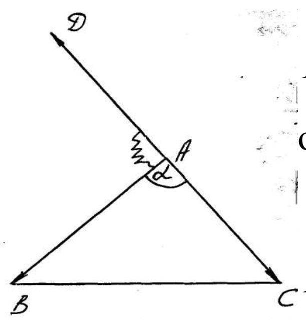


Fig. 2

Ex. 4. Whether a triangle with given vertices $A(2; 3; 1)$, $B(4; 7; -1)$, $C(1; 2; 0)$ (fig. 2) is the right one? Find its inner and outer angles at the vertex *A*.

Solution. The sides of the triangle *ABC* are equal to

$$AB = \sqrt{2^2 + 4^2 + (-2)^2} = \sqrt{24},\ AC = \sqrt{(-1)^2 + (-1)^2 + (-1)^2} = \sqrt{3},$$

$$BC = \sqrt{(-3)^2 + (-5)^2 + 1^2} = \sqrt{36}.$$

$$BC^2 = 36;\ AB^2 + AC^2 = 27;\ BC^2 \neq AB^2 + AC^2,$$

and so the triangle isn't that right. To find angles in question we introduce vectors

$$\overline{AB} = (2, 4, -2),\ \overline{AC} = (-1, -1, -1),\ \overline{AD} = -\overline{AC} = (1, 1, 1).$$

Using the formula (5) we get

$$\cos\alpha = \cos BAC = \frac{\overline{AB} \cdot \overline{AC}}{\left|\overline{AB}\right|\left|\overline{AC}\right|} = \frac{2\cdot(-1) + 4\cdot(-1) + (-2)\cdot(-1)}{AB \cdot AC} = \frac{-4}{\sqrt{24}\cdot\sqrt{3}} = -\frac{\sqrt{2}}{3} \approx -0.47;$$

$$\cos BAD = \frac{\overline{AB} \cdot \overline{AD}}{\left|\overline{AB}\right|\left|\overline{AD}\right|} = \frac{2\cdot 1 + 4\cdot 1 + (-2)\cdot 1}{AB \cdot AC} = \frac{4}{\sqrt{24}\cdot\sqrt{3}} = \frac{\sqrt{2}}{3} \approx 0.47;$$

$$\alpha = \angle BAC \approx \arccos(-0.47) \approx 118°, \quad \angle BAD \approx \arccos 0.47 \approx 62°.$$

Ex. 5. Find a vector of the length 14 if it's perpendicular to two given vectors $\overline{a} = (3, 2, 2)$, $\overline{b} = (18, -22, -5)$ and forms an acute angle with the *Oz*-axis.

Let we must find a vector $\overline{x} = (x, y, z)$. By conditions ($\overline{x} \perp \overline{a}, \overline{x} \perp \overline{b}, \left|\overline{x}\right| = 14,$ an angle $\left(\overline{x}, \hat{\ } Oz\right)$ is acute one) we get

$$\bar{a}\cdot\bar{x}=3x+2y+2z=0,\ \bar{b}\cdot\bar{x}=18x-22y-5z=0,\ \left|\bar{x}\right|^2=x^2+y^2+z^2=196,\ z>0,$$

hence we have to solve the next system of equations

$$\begin{cases} 3x+2y+2z=0, \\ 18x-22y-5z=0, \\ x^2+y^2+z^2=196,\ z>0. \end{cases}$$

From two first equations we can express x and y through z, namely

$$\begin{cases} 3x+2y=-2z, \\ 18x-22y=5z, \end{cases} \Delta=\begin{vmatrix} 3 & 2 \\ 18 & -22 \end{vmatrix}=-102,\ \Delta_1=\begin{vmatrix} -2z & 2 \\ 5z & -22 \end{vmatrix}=34z,\ \Delta_2=\begin{vmatrix} 3 & -2z \\ 18 & 5z \end{vmatrix}=51z,$$

and by Cramer rule

$$x=\frac{\Delta_1}{\Delta}=\frac{34z}{-102}=-\frac{1}{3}z,\ y=\frac{\Delta_1}{\Delta}=\frac{51z}{-102}=-\frac{1}{2}z.$$

Now the third equation gives

$$\left(-\frac{1}{3}z\right)^2+\left(-\frac{1}{2}z\right)^2+z^2=196,\ \frac{1}{9}z^2+\frac{1}{4}z^2+z^2=196,\ z^2=144,\ z=12>0,$$

and so $x=-1/3z=-4,\ y=-1/2,\ z=-6$.

Answer: $\bar{x}=(-4,-6,12)$.

Ex. 6. Find a vector perpendicular to given vectors $\bar{a}=(4,-3,1),\bar{b}=(1,4,-2)$ if its projection on (the axis of) a vector $\bar{c}=(2,5,-6)$ equals $\frac{3}{\sqrt{65}}$.

Let, by analogy with the preceding example, $\bar{x}=(x,y,z)$. Then in accordance with the property 5 and the formula (6)

$$\begin{cases} \bar{a}\cdot\bar{x}=0, \\ \bar{b}\cdot\bar{x}=0, \\ \dfrac{\bar{c}\cdot\bar{x}}{\left|\bar{c}\right|}=\dfrac{3}{\sqrt{65}}; \end{cases} \begin{cases} 4x-3y+z=0, \\ x+4y-2z=0, \\ \dfrac{2x+5y-6z}{\sqrt{65}}=\dfrac{3}{\sqrt{65}}; \end{cases} \begin{cases} 4x-3y+z=0, \\ x+4y-2z=0, \\ 2x+5y-6z=3; \end{cases} \begin{matrix} x=-6/65, \\ y=-27/65, \\ z=-57/65. \end{matrix}$$

Answer. $\bar{x}=\left(-\frac{6}{65},-\frac{27}{65},-\frac{57}{65}\right)$.

POINT 2. VECTOR PRODUCT

Def. 2. An ordered triple of three non-coplanar vectors $\bar{a},\bar{b},\bar{c}$ with common origin is called the right-handed triple of vectors if a rotation of the first vector $\bar{a}$ to the second vector $\bar{b}$ by the shortest route is seen from the end point of the third vector $\bar{c}$ as performed anticlockwise (fig. 3).

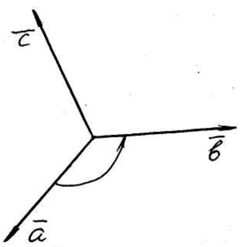


Fig. 3

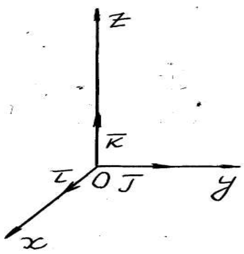


Fig. 4

Henceforth we'll consider only the right-handed Cartesian (orthonormal) base $\bar{i},\bar{j},\bar{k}$ and the right-handed Cartesian coordinate system generated by this base (see fig. 4).

It's evident that for the right-handed triple $\bar{i},\bar{j},\bar{k}$ the other triples $\bar{j},\bar{k},\bar{i}$ and $\bar{k},\bar{i},\bar{j}$ are also those right-handed.

Def. 3. A vector [a cross, an exterior, an outer] product $\bar{a}\times\bar{b}\equiv\left[\bar{a},\bar{b}\right]$ of vectors $\bar{a},\bar{b}$ is called a vector which possesses the next three properties:

a) it is perpendicular to the vectors $\bar{a},\bar{b}$;

b) its length equals the product of the lengths of the vectors $\bar{a},\bar{b}$ and the sine of an angle between them:

$$\left|\bar{a}\times\bar{b}\right|=\left|\bar{a}\right|\left|\bar{b}\right|\sin\varphi,\ \varphi=\left(\bar{a}\hat{;}\bar{b}\right);$$

c) the vectors $\bar{a},\bar{b}$ and $\bar{a}\times\bar{b}$ form the right-handed triple of vectors.

Ex. 7. The table of pairwise multiplication of the basic vectors $\bar{i},\bar{j},\bar{k}$

$$\bar{i}\times\bar{j}=\bar{k},\ \bar{j}\times\bar{k}=\bar{i},\ \bar{k}\times\bar{i}=\bar{j},\ \bar{j}\times\bar{i}=-\bar{k},\ \bar{k}\times\bar{j}=-\bar{i},\ \bar{i}\times\bar{k}=-\bar{j}. \qquad (7)$$

■Let's prove the equalities $\bar{i}\times\bar{j}=\bar{k},\ \bar{j}\times\bar{i}=-\bar{k}$. Prove the other equalities yourselves.

a) $\left(\bar{i}\times\bar{j}\right)\perp\bar{i},\left(\bar{i}\times\bar{j}\right)\perp\bar{j},\left(\bar{j}\times\bar{i}\right)\perp\bar{i},\left(\bar{j}\times\bar{i}\right)\perp\bar{j}$, so $\left(\bar{i}\times\bar{j}\right)\parallel\bar{k},\left(\bar{j}\times\bar{i}\right)\parallel\bar{k}$;

b) $\left|\bar{i}\times\bar{j}\right|=\left|\bar{i}\right|\left|\bar{j}\right|\sin\pi/2=1,\left|\bar{j}\times\bar{i}\right|=\left|\bar{j}\right|\left|\bar{i}\right|\sin\pi/2=1$, so $\bar{i}\times\bar{j}=\bar{k}\ or\ \bar{i}\times\bar{j}=-\bar{k}$, $\bar{j}\times\bar{i}=\bar{k}\ or\ \bar{j}\times\bar{i}=-\bar{k}$;

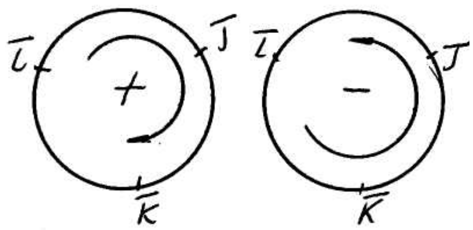


Fig. 5

c) the vectors $\bar{i}, \bar{j}, \bar{k}$ and $\bar{j}, \bar{i}, -\bar{k}$ form the right-handed triples, and therefore $\bar{i}\times\bar{j}=\bar{k},\ \bar{j}\times\bar{i}=-\bar{k}$.■

To remember the "multiplication table" (7) obtained in Ex. 7 we'll introduce the next scheme (fig. 5).

Properties of a vector product

1. $\bar{a}\times\bar{b}=-\bar{b}\times\bar{a}$ (anticommutativity).

■Vectors $\bar{a}\times\bar{b}$ and $\bar{b}\times\bar{a}$ have the same length and the opposite directions.■

2. The length of a vector product numerical equals the area of the parallelogram constructed on these vectors as the sides.

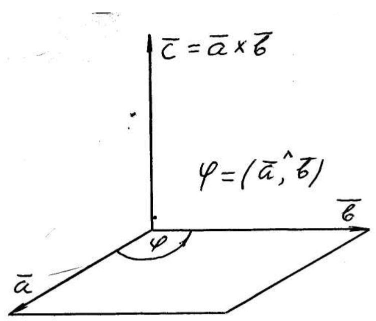


Fig. 6

■The area of such the parallelogram equals

$$S=|\bar{a}||\bar{b}|\sin(\bar{a}\,\hat{,}\,\bar{b})=|\bar{a}\times\bar{b}|$$ ■

On the base of this property and definition of a vector product we can represent this product with the help of a fig. 6.

3. $(\bar{a}+\bar{b})\times\bar{c}=\bar{a}\times\bar{c}+\bar{b}\times\bar{c},\ \bar{a}\times(\bar{b}+\bar{c})=\bar{a}\times\bar{b}+\bar{a}\times\bar{c}$ (distributivity).

4. $k(\bar{a}\times\bar{b})=(k\bar{a})\times\bar{b}=\bar{a}\times(k\bar{b})$ for any number (scalar) *k*.

5. Two non-zero vectors are collinear if and only if their vector product equals zero vector.

■a) If $\bar{a}\parallel\bar{b}$ then $(\bar{a}\,\hat{,}\,\bar{b})=0$ *or* $(\bar{a}\,\hat{,}\,\bar{b})=\pi,\ \sin(\bar{a}\,\hat{,}\,\bar{b})=0,\ |\bar{a}\times\bar{b}|=0,\ \bar{a}\times\bar{b}=\bar{0}$;

b) if now $\bar{a}\times\bar{b}=\bar{0}$, then $|\bar{a}\times\bar{b}|=0,\ \sin(\bar{a}\,\hat{,}\,\bar{b})=0$ and therefore $\bar{a}\parallel\bar{b}$.■

Expression of a vector product in terms of coordinates of vectors in Cartesian (orthonormal) base

Let vectors $\bar{a}, \bar{b}$ be decomposed with respect to Cartesian (orthonormal) base

$\bar{i}, \bar{j}, \bar{k}$, that is

$$\bar{a} = a_x \bar{i} + a_y \bar{j} + a_z \bar{k}, \bar{b} = b_x \bar{i} + b_y \bar{j} + b_z \bar{k} \text{ or simply } \bar{a} = (a_x, a_y, a_z), \bar{b} = (b_x, b_y, b_z).$$

In this case their vector product equals the third order determinant

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}, \qquad (6)$$

the first row of which consists of the basic vectors $\bar{i}, \bar{j}, \bar{k}$, the second row contains coordinates of the first vector and the third row contains those of the second vector.

■Using at first the properties 3, 4 of a vector product and then the table (5) of pairwise multiplication of the basic vectors $\bar{i}, \bar{j}, \bar{k}$, we get

$$\bar{a} \times \bar{b} = (a_x \bar{i} + a_y \bar{j} + a_z \bar{k}) \times (b_x \bar{i} + b_y \bar{j} + b_z \bar{k}) =$$

$$= (a_x b_x)(\bar{i} \times \bar{i}) + (a_x b_y)(\bar{i} \times \bar{j}) + (a_x b_z)(\bar{i} \times \bar{k}) + (a_y b_x)(\bar{j} \times \bar{i}) + (a_y b_y)(\bar{j} \times \bar{j}) + (a_y b_z)(\bar{j} \times \bar{k}) +$$

$$= (a_z b_x)(\bar{k} \times \bar{i}) + (a_z b_y)(\bar{k} \times \bar{j}) + (a_z b_z)(\bar{k} \times \bar{k}) = (a_x b_x)\bar{0} + (a_x b_y)\bar{k} - (a_x b_z)\bar{j} -$$

$$-(a_y b_x)\bar{k} + (a_y b_y)\bar{0} + (a_y b_z)\bar{i} + (a_z b_x)\bar{j} - (a_z b_y)\bar{i} + (a_z b_z)\bar{0} =$$

$$= (a_y b_z - a_z b_y)\bar{i} - (a_x b_z - a_z b_x)\bar{j} + (a_x b_y - a_y b_x)\bar{k} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}\bar{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix}\bar{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\bar{k} =$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \blacksquare$$

Ex. 8. Find the area and the altitude *BK* drawn from a vertex *B* of a triangle with vertices *A*(1; 3; 5), *B*(2; -4; 6), *C*(0; -2; 3) (fig. 7).

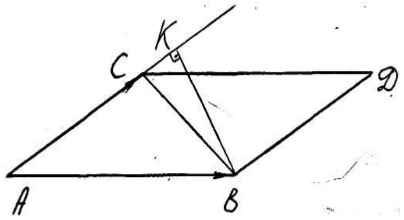


Fig. 7

Solution. Let's introduce two vectors $\overline{AB} = (1, -7, 1)$ and $\overline{AC} = (-1, -5, -2)$. The area of the triangle *ABC* equals one half of the area of the parallelogram *ABDC* constructed on the vectors $\overline{AB}, \overline{AC}$ as the sides. This latter area on the base of the property 2 of a vector product equals

$$S_{ABDC}=\left|\overline{AB}\times\overline{AC}\right|,$$

and so

$$S_{\Delta ABC}=\frac{1}{2}\left|\overline{AB}\times\overline{AC}\right|.$$

To find the altitude *BK* we take into account that

$$S_{\Delta ABC}=\frac{1}{2}\left|\overline{AB}\times\overline{AC}\right|=\frac{1}{2}\left|\overline{AC}\right|\cdot BK,\ BK=\frac{\left|\overline{AB}\times\overline{AC}\right|}{\left|\overline{AC}\right|}.$$

By the formula (6)

$$\overline{AB}\times\overline{AC}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & -7 & 1\\ -1 & -5 & -2\end{vmatrix}=19\bar{i}+\bar{j}-12\bar{k},$$

$$\left|\overline{AB}\times\overline{AC}\right|=\sqrt{19^2+1^2+(-12)^2}=\sqrt{506}\approx 22.49.$$

Therefore

$$S_{\Delta ABC}=\frac{1}{2}\sqrt{506}\approx 11.25,\ BK=\frac{\sqrt{506}}{\sqrt{(-1)^2+(-5)^2+(-2)^2}}=\sqrt{\frac{506}{30}}\approx 4.11.$$

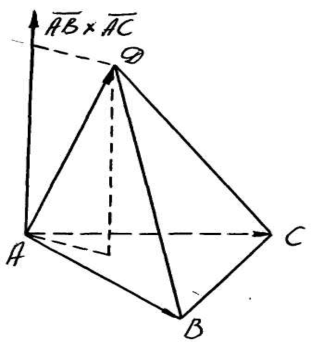


Fig. 8

Ex. 9. Find the volume of a triangular pyramid with given vertices $A(1;3;5)$, $B(2;-4;6)$, $C(0;-2;3)$, $D(-2;1;5)$ (fig. 8).

$$V_{ABCD}=\frac{1}{3}S_{ABC}H=\frac{1}{3}\cdot\frac{1}{2}\left|\overline{AB}\times\overline{AC}\right|H,\ H=\left|Pr_{\overline{AB}\times\overline{AC}}\overline{AD}\right|=$$
$$=\frac{\left|\left(\overline{AB}\times\overline{AC}\right)\cdot\overline{AD}\right|}{\left|\overline{AB}\times\overline{AC}\right|},\ V_{ABCD}=\frac{1}{6}\left|\left(\overline{AB}\times\overline{AC}\right)\cdot\overline{AD}\right|.$$

From the preceding example we know that

$$\overline{AB}=(1,-7,1),\ \overline{AC}=(-1,-5,-2),\ \overline{AB}\times\overline{AC}=(19,1,-12).$$

Hence

$$\left(\overline{AB}\times\overline{AC}\right)\cdot\overline{AD}=(19,1,-12)\cdot(-3,-2,0)=19\cdot(-3)+1\cdot(-2)+(-12)\cdot 0=-59,$$
$$\left|\left(\overline{AB}\times\overline{AC}\right)\cdot\overline{AD}\right|=59,\ V_{ABCD}=\frac{59}{6}\approx 9.8.$$

Ex. 10. With the help of a vector product solve the problem which we've solved in Ex. 5 (find a vector of the length 14 if it's perpendicular to two given vectors

$\overline{a}=(3,2,2), \overline{b}=(18,-22,-5)$ and forms an acute angle with the *Oz*-axis).

Solution. A sought vector $\overline{x}$, being perpendicular to the vectors $\overline{a}, \overline{b}$, is collinear to their vector product

$$\overline{a}\times\overline{b}=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 3 & 2 & 2\\ 18 & -22 & -5\end{vmatrix}=34\overline{i}+51\overline{j}-102\overline{k}=17\left(2\overline{i}+3\overline{j}-6\overline{k}\right)$$

or to the vector

$$\overline{m}=(2,3,-6).$$

This latter forms an obtuse angle with the *Oz*-axis, hence the unite vector of the vector $\overline{x}$ equals

$$\overline{x^{\circ}}=-\overline{m^{\circ}}=-\frac{\overline{m}}{\left|\overline{m}\right|}=-\frac{(2,3,-6)}{\sqrt{2^2+3^2+(-6)^2}}=\frac{(-2,-3,6)}{\sqrt{49}}=\left(-\frac{2}{7},-\frac{3}{7},\frac{6}{7}\right).$$

Finally we get

$$\overline{x}=14\overline{x^{\circ}}=14\left(-\frac{2}{7},-\frac{3}{7},\frac{6}{7}\right)=(-4,-6,12).$$

Ex. 11. Solve the problem of Ex. 6 with the help of a vector product (find a vector which is perpendicular to given vectors $\overline{a}=(4,-3,1), \overline{b}=(1,4,-2)$ if its projection on (the axis of) a vector $\overline{c}=(2,5,-6)$ equals $\frac{3}{\sqrt{65}}$).

Solution. By analogy on the preceding example the unknown vector $\overline{x}$ must be collinear to the vector product

$$\overline{a}\times\overline{b}=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 4 & -3 & 1\\ 1 & 4 & -2\end{vmatrix}=2\overline{i}+9\overline{j}+19\overline{k}$$

and so must have the next form

$$\overline{x}=k\left(\overline{a}\times\overline{b}\right)=k(2,9,19)=(2k,9k,19k)$$

where k is some unknown number. By the condition

$$Pr_{\bar{c}}\,\bar{x}=\frac{\bar{x}\cdot\bar{c}}{|\bar{c}|}=\frac{2k\cdot 2+9k\cdot 5+19k\cdot(-6)}{\sqrt{2^2+5^2+(-6)^2}}=\frac{-65k}{\sqrt{65}};\ \frac{-65k}{\sqrt{65}}=\frac{3}{\sqrt{65}},\ -65k=3,\ k=-\frac{3}{65},$$

therefore

$$\bar{x}=-\frac{3}{65}(2,9,19)=\left(-\frac{6}{65},-\frac{27}{65},-\frac{57}{65}\right).$$

POINT 3. n-DIMENSIONAL VECTORS

Def. 4. n-dimensional vector is called an ordered set of n numbers.

There are n-dimensional vectors-rows

$$\bar{a}=(a_1,a_2,...,a_n)$$

and n-dimensional vectors-columns

$$\bar{a}=\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}$$

Ex. 12. A matrix of dimension $m\times n$

$$\begin{pmatrix}a_{11}&a_{12}&\dots&a_{1n}\\a_{21}&a_{22}&\dots&a_{2n}\\....&....&....&.....\\a_{m1}&a_{m2}&\dots&a_{mn}\end{pmatrix}$$

is the set of m n-dimensional vectors-rows and n m-dimensional vectors-columns

$$\begin{aligned}\overline{a_1}&=(a_{11},a_{12},...,a_{1n}),\\\overline{a_2}&=(a_{21},a_{22},...,a_{2n}),\\&..............................\\\overline{a_m}&=(a_{m1},a_{m2},...,a_{mn});\end{aligned}\qquad \overline{b_1}=\begin{pmatrix}a_{11}\\a_{21}\\\vdots\\a_{m1}\end{pmatrix},\overline{b_2}=\begin{pmatrix}a_{12}\\a_{22}\\\vdots\\a_{m2}\end{pmatrix},...,\overline{b_n}=\begin{pmatrix}a_{1n}\\a_{2n}\\\vdots\\a_{mn}\end{pmatrix}.$$

It's possible to consider such a matrix as mn-dimensional vector.

Ex. 13. One can treat solutions

$$(x_{i1},x_{i2},...,x_{in})$$

of a system of m linear algebraic equations in n unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m, \end{cases}$$

as n-dimensional vectors-rows. The system by itself can be written in the vector form. Indeed, let's introduce the next m-dimensional vectors-columns:

$$\overline{a_1} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \overline{a_1} = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \overline{a_1} = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}, \overline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

The system passes in a vector equation

$$x_1\overline{a_1} + x_2\overline{a_2} + \ldots + x_n\overline{a_n} = \overline{b}.$$

Linear operations on n-dimensional vectors

Let

$$\overline{a} = (a_1, a_2, \ldots, a_n), \quad \overline{b} = (b_1, b_2, \ldots, b_n).$$

Then by definition a vector

$$k\overline{a} = (ka_1, ka_2, \ldots, ka_n)$$

is the product of a vector $\overline{a}$ by a number (scalar) k, and a vector

$$\overline{a} + \overline{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

is the sum of vectors $\overline{a}$ and $\overline{b}$.

Linear dependence and independence of n-dimensional vectors

Def. 5. A system of n-dimensional vectors

$$\overline{a_1}, \overline{a_2}, \ldots, \overline{a_l} \qquad (7)$$

is called the linearly dependent system if there is non-trivial set of numbers

$$k_1, k_2, \dots, k_l, \quad k_1^2 + k_2^2 + \dots + k_l^2 \neq 0 \text{ (at least one of } k_i, i = \overline{1, l} \text{ isn't zero)} \qquad (8)$$

such that the next equality

$$k_1 \overline{a_1} + k_2 \overline{a_2} + \dots + k_l \, \overline{a_l} = \overline{0} \qquad (9)$$

holds, where $\overline{0} = (0, 0, \dots, 0)$ is zero vector.

Def. 6. If the equality (9) holds only for trivial set of numbers (8), that is only if

$$k_1 = k_2 = \dots = k_l = 0, \qquad (10)$$

then a system of vectors (7) is called that linearly independent.

Ex. 14. A system of n-dimensional vectors

$$\overline{e_1} = (1, 0, 0, \dots, 0), \overline{e_2} = (0, 1, 0, \dots, 0), \overline{e_3} = (0, 0, 1, \dots, 0), \dots, \overline{e_3} = (0, 0, 0, \dots, 1) \qquad (11)$$

is linearly independent one. Every n-dimensional vector can be decomposed with respect to this vector system.

■ $k_1 \overline{e_1} + k_2 \overline{e_2} + k_3 \overline{e_3} + \dots + k_n \overline{e_n} = (k_1, k_2, k_3, \dots, k_n) = \overline{O} = (0, 0, 0, \dots, 0)$

in unique case, namely only if $k_1 = k_2 = \dots = k_l = 0$ which means linear independence of the system (11).

For any n-dimensional vector $\overline{a} = (a_1, a_2, \dots, a_n)$

$$\overline{a} = (a_1, a_2, \dots, a_n) = a_1 \overline{e_1} + a_2 \overline{e_2} + \dots + a_n \overline{e_n} . \blacksquare \qquad (12)$$

Def. 7. An express of the form

$$\lambda_1 \overline{a_1} + \lambda_2 \overline{a_2} + \dots + \lambda_p \overline{a_p},$$

that is a sum of products of vectors $\overline{a_1}, \overline{a_2}, \dots, \overline{a_p}$ and some numbers $\lambda_1, \lambda_2, \dots, \lambda_p$ is called a linear combination of the vectors $\overline{a_1}, \overline{a_2}, \dots, \overline{a_p}$.

The equality (12) means that a vector $\overline{a} = (a_1, a_2, \dots, a_n)$ can be represented as a linear combination of the vectors (11).

Theorem 1. If a vector system (7) contains zero vector or two equal vectors or linearly dependent subsystem, then it is linearly dependent one.

■a) For the system $\overline{0}, \overline{a_2}, \dots, \overline{a_l}$ (the first vector is zero one) we can write for example

$$1\cdot\overline{0}+0\cdot\overline{a_2}+\ldots+0\cdot\overline{a_l}=\overline{0}$$

with non-trivial set of numbers $1, 0, 0,\ldots, 0$.

b) For the system $\overline{a_1}, \overline{a_1}, \overline{a_3},\ldots, \overline{a_l}$ (the first two vectors are equal) we can write for example

$$1\cdot\overline{a_1}+(-1)\cdot\overline{a_1}+0\cdot\overline{a_3}+\ldots+0\cdot\overline{a_l}=\overline{0}$$

with non-trivial set of numbers $1, -1, 0,\ldots, 0$.

c) Let some subsystem $\overline{a_1}, \overline{a_2},\ldots, \overline{a_m}$ $(m<l)$ of the system (7) is linearly dependent, that is

$$k_1\overline{a_1}+k_2\overline{a_2}+\ldots+k_m\,\overline{a_m}=\overline{0} \text{ and } k_1^2+k_2^2+\ldots+k_m^2\neq 0.$$

In this case we can take non-trivial set of numbers

$$k_1, k_2,\ldots, k_m, 0, 0,\ldots, 0 \quad k_1^2+k_2^2+\ldots+k_m^2+0^2+0^2+\ldots+0^2\neq 0$$

to get

$$k_1\overline{a_1}+k_2\overline{a_2}+\ldots+k_m\,\overline{a_m}+0\cdot\overline{a_{m+1}}+0\cdot\overline{a_{m+2}}+\ldots+0\cdot\overline{a_l}=\overline{0},$$

and the system (7) is linearly dependent.■

Theorem 2. A system of vectors (7) is that linearly dependent if and only if at least one of its vectors can be represented as a linear combination of the other vectors of the system.

■a) Let the system (7) is linearly dependent one and $k_1\neq 0$ in the equality (9). Then we get from (9)

$$k_1\overline{a_1}=-k_2\overline{a_2}-k_3\overline{a_3}-\ldots-k_l\,\overline{a_l}, \overline{a_1}=\lambda_2\overline{a_2}+\lambda_3\overline{a_3}+\ldots+\lambda_l\overline{a_l},$$

where $\lambda_2=-\frac{k_2}{k_1}, \lambda_3=-\frac{k_3}{k_1},\ldots, \lambda_l=-\frac{k_l}{k_1}$, and the vector $\overline{a_1}$ is a linear combination of the vectors $\overline{a_2}, \overline{a_3},\ldots, \overline{a_l}$.

b) Let inversely the vector $\overline{a_1}$ of the system (7) is a linear combination of the other vectors, namely there are some numbers $\lambda_2, \lambda_3,\ldots, \lambda_l$ such that

$$\overline{a_1}=\lambda_2\overline{a_2}+\lambda_3\overline{a_3}+\ldots+\lambda_l\overline{a_l}.$$

In this case we can write

$$(-1)\overline{a_1} + \lambda_2\overline{a_2} + \lambda_3\overline{a_3} + \ldots + \lambda_l\overline{a_l} = \overline{0},$$

that is we get the equality of the type (9) with non-trivial set of numbers

$$k_1 = -1 \neq 0, k_2 = \lambda_2, \ldots, k_l = \lambda_l. ■$$

Rank of a system of vectors

Def. 8. The rank of a system of vectors is called the greatest number of linearly independent vectors of this system.

Theorem 2. The rank of the set of all *n*-dimensional vectors equals *n*.

■a) On the one hand there is *n* linearly independent vectors of this set, namely the vectors (11).

b) On the other hand every *n* +1, *n* + 2, … vectors of the set of all *n*-dimensional vectors are linearly dependent.

Let's prove this statement for particular case of four three-dimensional vectors

$$\overline{a_1} = (a_{11}, a_{12}, a_{13}), \overline{a_2} = (a_{21}, a_{22}, a_{23}), \overline{a_3} = (a_{31}, a_{32}, a_{33}), \overline{a_4} = (a_{41}, a_{42}, a_{43}).$$

We'll try to prove existence of non-trivial set of numbers k_1, k_2, k_3, k_4 for which the equality

$$k_1\overline{a_1} + k_2\overline{a_2} + k_3\overline{a_3} + k_4\overline{a_4} = \overline{0} \qquad (13)$$

holds. Representing it in extended form we have

$$k_1(a_{11}, a_{12}, a_{13}) + k_2(a_{21}, a_{22}, a_{23}) + k_3(a_{31}, a_{32}, a_{33}) + k_4(a_{41}, a_{42}, a_{43}) = (0, 0, 0),$$

$$(k_1a_{11}, k_1a_{12}, k_1a_{13}) + (k_2a_{21}, k_2a_{22}, k_2a_{23}) + (k_3a_{31}, k_3a_{32}, k_3a_{33}) + (k_4a_{41}, k_4a_{42}, k_4a_{43}) = (0,0,0)$$

$$(k_1a_{11} + k_2a_{21} + k_3a_{31} + k_4a_{41}, k_1a_{12} + k_2a_{22} + k_3a_{32} + k_4a_{42}, k_1a_{13} + k_2a_{23} + k_3a_{33} + k_4a_{43}) = \\ = (0, 0, 0),$$

$$\begin{cases} k_1a_{11} + k_2a_{21} + k_3a_{31} + k_4a_{41} = 0, \\ k_1a_{12} + k_2a_{22} + k_3a_{32} + k_4a_{42} = 0, \\ k_1a_{13} + k_2a_{23} + k_3a_{33} + k_4a_{43} = 0. \end{cases} \qquad (14)$$

The problem in question is reduced to that of existing non-trivial solutions of the system (14) of linear homogeneous equations in k_1, k_2, k_3, k_4. But such the solutions exist because of the number 4 of unknowns is greater than the number 3 of equations.■

Theorem 3. For any matrix the rank of the system of its rows equals the rank of the system of its columns and equals the rank of the matrix.

Theorem 4. The rank of a system of non-zero solutions of a system of *m* linear homogeneous algebraic equations in *n* unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = 0, \end{cases}$$

equals $n - r$, where r is the rank of the system matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}.$$

Euclidean spaces

Def. 9. The set of all *n*-dimensional vectors is called **Euclidean space** if in it it's defined the scalar product of any two vectors $\bar{a} = (a_1, a_2, \ldots, a_n), \bar{b} = (b_1, b_2, \ldots, b_n)$, namely

$$\bar{a}\bar{b} \equiv \bar{a} \cdot \bar{b} \equiv (\bar{a}, \bar{b}) = a_1b_1 + a_2b_2 + \ldots + a_nb_n. \qquad (15)$$

For $\bar{a} = \bar{b}$ the formula gives so-called length [modulus, norm] of a vector $\bar{a}$, that is

$$|\bar{a}| = \sqrt{\bar{a}^2} = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}. \qquad (16)$$

On the base of definition (15) one can directly prove the next properties of the scalar product:

1. $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$.

2. $\left(\overline{a}+\overline{b}\right)\cdot\overline{c}=\overline{a}\cdot\overline{c}+\overline{b}\cdot\overline{c}$.

3. $k\left(\overline{a}\cdot\overline{b}\right)=\left(k\overline{a}\right)\cdot\overline{b}=\overline{a}\cdot\left(k\overline{b}\right)$.

4. $\left|\overline{a}\right|\ge 0, \left|\overline{a}\right|=0$ only if $\overline{a}=\overline{0}$.

For two arbitrary vectors $\overline{a}, \overline{b}$ of Euclidean space the next Cauchy[1] – Bunyakowsky[2] inequality

$$\left|\overline{a}\cdot\overline{b}\right|\le\left|\overline{a}\right|\left|\overline{b}\right| \qquad (17)$$

holds.

■For any real number k

$$\left(k\overline{a}+\overline{b}\right)^2\ge 0, k^2\overline{a}^2+2k\left(\overline{a}\cdot\overline{b}\right)+\overline{b}^2\ge 0, k^2\left|\overline{a}\right|^2+2k\left(\overline{a}\cdot\overline{b}\right)+\left|\overline{b}\right|^2\ge 0.$$

The latter quadratic trinomial in k must have non-positive discriminant to be non-negative for any k, whence it follows that

$$\left(\overline{a}\cdot\overline{b}\right)^2-\left|\overline{a}\right|^2\left|\overline{b}\right|^2\le 0, \left(\overline{a}\cdot\overline{b}\right)^2\le\left|\overline{a}\right|^2\left|\overline{b}\right|^2, \left|\overline{a}\cdot\overline{b}\right|\le\left|\overline{a}\right|\left|\overline{b}\right|.$$ ■

Corollary. For any two vectors $\overline{a}, \overline{b}$

$$\left|\overline{a}+\overline{b}\right|\le\left|\overline{a}\right|+\left|\overline{b}\right|. \qquad (18)$$

■Using the formula (16), properties 2,3 and inequality (17) we'll have

$$\left|\overline{a}+\overline{b}\right|^2=\left(\overline{a}+\overline{b}\right)^2=\overline{a}^2+2\overline{a}\overline{b}+\overline{b}^2=\left|\overline{a}\right|^2+2\overline{a}\overline{b}+\left|\overline{b}\right|^2\le\left|\overline{a}\right|^2+2\left|\overline{a}\right|\left|\overline{b}\right|+\left|\overline{b}\right|^2=\left(\left|\overline{a}\right|+\left|\overline{b}\right|\right)^2$$

whence it follows the inequality (18).■

An angle between two vectors $\overline{a}, \overline{b}$ of Euclidean space can be defined by the next formula:

$$cos\left(\overline{a}\,\hat{,}\,\overline{b}\right)=\frac{\overline{a}\cdot\overline{b}}{\left|\overline{a}\right|\left|\overline{b}\right|}. \qquad (19)$$

On the base of Cauchy – Bunyakowsky inequality (17) we have

$$\left|\cos\left(\overline{a}\,\hat{,}\,\overline{b}\right)\right|\le 1. \qquad (20)$$

[1] Cauchy, A.L. (1780 - 1859), a famous French mathematician
[2] Bunyakowsky, V.J. (1804 - 1889), a Ukranian mathematician

VECTORS: basic terminology RUE

1. Базис (на плоскости, в пространстве)	Базис (на площині, в просторі)	Bàse/básis (on the pláne, in the spáce)
2. Быть заданным: а) координатами начала и конца; б) длиной и направляющими косинусами; в) координатами/ проекциями	Бути заданим: а) координатами початку й кінця; б) довжиною і напрямними косинусами; в) координатами/проекціями	Be defíned/detérmined/ spécified: a) by coórdinates of the órigin and end point; b) by length and diréction cósines; c) by coórdinates/projéctions
3. Быть параллельными одной и той же прямой /плоскости (о векторах)	Бути паралельними одній і тій же прямій/пло-щині (про вектори)	Be párallel to the sáme (straight) líne [to the sáme plane] (about véctors)
4. Быть численно равным	Бути чисельно рівним	Be numérically équal to…
5. Вектор	Вектор	Véctor
6. Векторная величина	Векторна величина	Véctor/vectórial quántity
7. Векторное произведение двух векторов	Векторний добуток двох векторів	Véctor [cross, extérior, óuter] próduct of two véctors
8. Взаимно перпендикулярные (ортогональные) векторы, оси	Взаємно перпендикулярні (ортогональні) вектори, осі	Mútually pèrpendícular (orthógonal) véctors, áxes
9. Выразить произведение векторов через коор-динаты сомножителей	Виразити добуток векторів через координати співмножників	Expréss a próduct of véctors by [in terms of, through] coórdinates of fáctors
10.Вычитание векторов	Віднімання векторів	Subtráction of véctors
11.Геометрический смысл	Геометричний сенс	Geométric(al) méaning/ sense
12.Двумерный вектор	Двовимірний вектор	Two-diménsional véctor
13.Декартов базис/репер	Декартів базис/репер	Cartésian base [básis], fráme
14.Декартовы координаты вектора	Декартові координати вектора	Cartésian coórdinates of a véctor
15.Декартовы прямоугольные координаты вектора	Декартові прямокутні координати вектора	Cartésian orthógonal/rectángular coórdinates of a véctor
16.Деление отрезка в	Поділ відрізка в даному	Divísion of a ségment in

данном отношении	відношенні	the given rátio
17.Длина вектора	Довжина вектора	Length of a véctor
18.Иметь одно и то же начало, направление	Мати один і той же початок, напрямок	Have the same órigin, diréction
19.Квадратный корень из суммы квадратов	Квадратний корінь з суми квадратів	Squáre róot of the sum of squáres [róot-sum squáre, róot of sum-of-squáres]
20.Коллинеарность векторов	Колінеарність векторів	Collìneárity of véctors
21.Коллинеарные векторы	Колінеарні вектори	Collínear véctors
22.Компланарность векторов	Компланарність векторів	Cò(m)planárity of véctors
23.Компланарные векторы	Компланарні вектори	Có(m)planar véctors
24.Конец [конечная точка] вектора	Кінець [кінцева точка] вектора	Extrémity [end, end/términal póint, términus] of a véctor
25.Координата вектора в данном базисе	Координати вектора в даному базисі	Coórdinate of a véctor in a given báse/básis
26.Лежать в одной (и той же) плоскости	Лежати в одній (і тій же) площині	Lie [be situáted] in the same plane
27.Лежать на одной (и той же) прямой	Лежати на одній (і тій же) прямій	Lie [be, be situáted/fóund] on the same straight line
28.Линейная (не)зависимость векторов	Лінійна (не)залежність векторів	Línear (ìn)depéndence of véctors
29.Линейная комбинация векторов	Лінійна комбінація векторів	Línear còmbinátion of véctors
30.Линейная операция	Лінійна операція	Línear òperátion
31.Линейно (не)зависимые векторы	Лінійно (не)залежні вектори	Línearly (ìn)depéndent véctors
32.Направленный отрезок	Напрямлений відрізок	Diréсted ségment
33.Направляющий косинус	Напрямний косинус	Diréction cósine
34.Начало [начальная точка, точка приложения] вектора	Початок [початкова точка, точка прикладення] вектора	Órigin [inítial póint, póint of àpplicátion] of a véctor
35.Необходимое и достаточное условие (коллинеарности двух векторов, компланарности трёх векторов)	Необхідна і достатня умова (колінеарності двох векторів, компланарності трьох векторів)	Nécessary and suffícient condítion (for collìneárity of two véctors, for còplanárity of three véctors)

36.Нормированный вектор	Нормований вектор	Nórmalized/stánderdized véctor
37.Нулевой вектор	Нульовий вектор	Zéro/nill/null véctor
38.Определяться величиной [числом, числовым значением] и направлением	Визначатися величиною [числом, числовим значенням] і напрямком	Be detérmined/defíned by a mágnitude [númber, númber válue] and by a diréction
39.Определяться только величиной [числом, числовым значением]	Визначатися величиною [числом, числовим значенням]	Be detérmined/defíned ónly by a mágnitude [númber, númber válue]
40.Орт вектора	Орт вектора	Únit/únitary/diréction/ méasuring/básis véctor
41.Ортогональная проекция	Ортогональна проекція	Orthógonal projéction
42.Ортонормированный базис	Ортонормований базис	Órthonórmal base/básis
43.Отнять, вычесть вектор ***a*** из вектора ***b***	Відняти вектор ***a*** и вектора ***b***	Subtráct the véctor ***a*** from the véctor ***b***
44.Пара векторов (упорядоченная)	Пара векторів (впорядкована)	Páir (órdered páir) of véctors
45.Параллелепипед, построенный на векторах ***a***, ***b***, ***c*** как на сторонах [образованный векторами]	Паралелепіпед, побудований на векторах ***a***, ***b***, ***c*** як на сторонах [утворений векторами]	Pàrallelépiped constructed on the véctors ***a***, ***b***, ***c*** as the sídes [fórmed/detérmi-ned by the véctors ***a***, ***b***, ***c***; with sídes ***a***, ***b***, ***c***]
46.Параллелограмм, построенный на векторах ***a***, ***b*** как на сторонах [образованный векторами]	Паралелограм, побудований на векторах ***a***, ***b*** як на сторонах [утворе-ний векторами]	Pàrallélogram constructed on the véctors ***a***, ***b*** as the sídes [fórmed/detérmined by the véctors ***a***, ***b*** ; with sídes ***a***, ***b***]
47.Переместительное (коммутативное) свойство	Переставна (комутативна) властивість	Commútative [còmmutátive] próperty
48.Перпендикулярный (ортогональный) вектор	Перпендикулярний (ортогональний) вектор	Pèrpendícular (orthógonal) véctor
49.Попарно ортогональные векторы	Парами ортогональні вектори	Páirwìse orthógonal véctors
50.Правая тройка векторов	Права трійка векторів	Right-hánded tríple(t) of véctors
51.Правило (закон) многоугольника, параллелограмма, параллелепипеда, треугольника	Правило (закон) многокутника, паралелогра-ма, паралелепіпеда, трикутника	Rúle [láw]: pólygone [polygonal] rúle, pàrallélogram rúle, pàrallelépiped rúle, tríangle rúle

52.Правило правой руки	Правило правої руки	Right-hand rule
53.Привести векторы к общему началу	Звести вектори до спільного початку	Redúce/lead/bring véctors to [locáte véctors at] a cómmon órigine
54.Проекция (вектора на ось)	Проекція (вектора на вісь)	Projéction (of a véctor on(to) an áxis)
55.Произведение вектора на число (скаляр)	Добуток вектора на число (скаляр)	Próduct of a véctor by a númber/scálar
56.Происходить [производиться, осуществляться, совершаться] против часовой стрелки	Відбуватися [виконуватися] проти годинникової стрілки	Take pláce [occúr, be perfórmed] ánticlóckwise/ cóunterclóckwise
57.Противоположно направленные векторы	Протилежно напрямлені вектори	Véctors of ópposite diréc-tions
58.Противоположный вектор	Протилежний вектор	Ópposite véctor
59.Прямоугольный параллелепипед	Прямокутний паралеле-піпед	Rectángular pàrallelépiped
60.Равные векторы	Рівні вектори	Équal véctors
61.Радиус-вектор точки	Радіус-вектор точки	Rádius véctor of a póint
62.Разложение вектора по базису	Розвинення вектора за базисом	Décomposìtion of a véctor with respéct to a báse/bá-sis
63.Разложить вектор по базису	Розвинути вектор за базисом	Dècompóse/expánd a véc-tor with respéct to a báse/ básis
64.Разность векторов	Різниця векторів	Dífference of véctors
65.Распределительное (дистрибутивное) свой-ство	Розподільча (дистрибутивна) властивість	Distríbutive próperty
66.Свободный вектор	Вільний вектор	Free [nónlócalized] véctor
67.Связанный вектор	Зв'язаний вектор	Bóund(ed) [fíxed, lócalized] véctor
68.Система векторов	Система векторів	Véctor sýstem
69.Скаляр (число)	Скаляр (число)	Scálar (númber)
70.Скалярная величина	Скалярна величина	Scálar quántity
71.Скалярное произведение двух векторов	Скалярний добуток двох векторів	Scálar [dot, Éuclidean, ínner, intérior, índernal] próduct of two véctors
72.Скалярный квадрат вектора	Скалярний квадрат вектора	Scálar squáre of a véctor
73.Скалярный	Скалярний множник	Scálar fáctor

множитель		
74. Скользящий (вдоль прямой) вектор	Ковзний (вздовж прямої) вектор	Glíding [slíding, lócalized on a line, nónlócalized] véctor
75. Сложение векторов	Додавання векторів	Addítion of véctors
76. Сложить векторы	Додати вектори	Add véctors
77. Сонаправленные векторы	Вектори одного й того ж напрямку	Véctors of the same diréction
78. Составляющая вектора на/по/вдоль оси	Складова вектора на/по/вздовж осі	Compónent of a véctor along/on an áxis
79. Сочетательное (ассоциативное) свойство	Сполучна (асоциативна) властивість	Assóciàtive próperty
80. Способ задания вектора	Спосіб задання вектора	Way/méthod/mode of rèpresentátion/detèrminátion of a véctor
81. Сумма векторов	Сума векторів	Sum of véctors
82. Сумма произведений соответствующих координат	Сума добутків відповідних координат	Sum of próducts of corresponding coórdinates
83. Трёхмерный вектор	Тривимірний вектор	Thrée-diménsional véctor
84. Тройка векторов (упорядоченная)	Трійка векторів (впорядкована)	Tríple(t) (órdered tríple(t)) of véctors
85. Угол между вектором и осью	Кут між вектором і віссю	Ángle betwéen a véctor and an áxis
86. Угол между двумя векторами	Кут між двома векторами	Ángle betwéen two véctors
87. Умножить вектор на число (скаляр)	Помножити вектор на число (скаляр)	Múltiply a véctor by a númber/scálar
88. Упорядоченная пара (неколлинеарных) векторов (приведённых к общему началу)	Впорядкована пара (неколінеарних) векторів (зведених до спільного початку)	Órdered páir of (non-collinear) véctors (locáted at a common órigin)
89. Упорядоченная пара чисел	Впорядкована пара чисел	Órdered páir of numbers
90. Упорядоченная тройка (некомпланарных) векторов (приведённых к общему началу)	Впорядкована трійка (некомпланарних) векторів (зведених до спільного початку)	Órdered tríple(t) of (nòncó(m)planar) véctors (locáted at a common origin)
91. Упорядоченная тройка чисел	Впорядкована трійка чисел	Órdered tríple(t) of numbers
92. Упорядоченный набор *n* чисел	Впорядкована множина *n* чисел	Arránged set of *n* númbers, órdered *n*-túple of númbers

93. Эн-мерный вектор | *n*-вимірний вектор | *n*-diménsional véctor

VECTORS: basic terminology ERU

1. Add véctors	Сложить векторы	Додати вектори
2. Addítion of véctors	Сложение векторов	Додавання векторів
3. Ángle betwéen a véctor and an áxis	Угол между вектором и осью	Кут між вектором і віссю
4. Ángle betwéen two véctors	Угол между двумя век-торами	Кут між двома векторами
5. Arránged set of *n* númbers, órdered *n*-túple of númbers	Упорядоченный набор *n* чисел	Впорядкована множина *n* чисел
6. Assóciàtive próperty	Сочетательное (ассоциативное) свойство	Сполучна (асоциативна) властивість
7. Bàse/básis (on the pláne, in the spáce)	Базис (на плоскости, в пространстве)	Базис (на площині, в просторі)
8. Be defíned/detérmined/spéci-fied: a) by coórdinates of órigin and end point; b) by length and diréction cósines; c) by coórdinates/projéctions	Быть заданным: а) координатами начала и конца; б) длиной и направляющими косинусами; в) координа-тами/проекциями	Бути заданим: а) координатами початку й кінця; б) довжиною і напрямними косинусами; в) координатами/проекціями
9. Be detérmined/defíned by a mágnitude [númber, númber vá-lue] and by a diréction	Определяться величи-ной [числом, числовым значением] и направле-нием	Визначатися величиною [числом, числовим значенням] і напрямком
10.Be detérmined/defíned ónly by a mágnitude [númber, númber válue]	Определяться только величиной [числом, чи-словым значением]	Визначатися величиною [числом, числовим значенням]
11.Be numérically équal to…	Быть численно равным	Бути чисельно рівним
12.Be párallel to the sáme (straight) líne [to the sáme pla-ne] (about véctors)	Быть параллельными одной и той же прямой /плоскости (о векторах)	Бути паралельними одній і тій же прямій/ площині (про вектори)
13.Bóund(ed) [fíxed, lócalized] véctor	Связанный вектор	Зв’язаний вектор
14.Cartésian base [básis], fráme	Декартов базис/репер	Декартів базис/репер

15.Cartésian coórdinates of a véctor	Декартовы координаты вектора	Декартові координати вектора
16.Cartésian orthógonal/rectángular coórdinates of a véctor	Декартовы прямоуго-льные координаты вектора	Декартові прямокутні координати вектора
17.Cóplanar [cómplanar] véctors	Компланарные векторы	Компланарні вектори
18.Còplanárity [cómplanárity] of véctors	Компланарность векто-ров	Компланарність векторів
19.Collínear véctors	Коллинеарные векторы	Колінеарні вектори
20.Collìneárity of véctors	Коллинеарность векто-ров	Колінеарність векторів
21.Commútative [còmmutátive] próperty	Переместительное (коммутативное) свойство	Переставна (комутативна) властивість
22.Compónent of a véctor along an áxis	Составляющая вектора на/по/вдоль оси	Складова вектора на/по/вздовж осі
23.Coórdinate of a véctor in a given báse/básis	Координата вектора в данном базисе	Координати вектора в даному базисі
24.Dècompóse/expánd a véctor with respéct to a báse/básis	Разложить вектор по базису	Розвинути вектор за базисом
25.Décomposìtion of a véctor with respéct to a báse/básis	Разложение вектора по базису	Розвинення вектора за базисом
26.Dífference of véctors	Разность векторов	Різниця векторів
27.Dirécted ségment	Направленный отрезок	Напрямлений відрізок
28.Diréction cósine	Направляющий косинус	Напрямний косинус
29.Distríbutive próperty	Распределительное (дистрибутивное) свой-ство	Розподільча (дистрибутивна) властивість
30.Divísion of a ségment in the given rátio	Деление отрезка в дан-ном отношении	Поділ відрізка в даному відношенні
31.Équal véctors	Равные векторы	Рівні вектори
32.Expréss a próduct of véctors by [in terms of, through] coórdinates of fáctors	Выразить произведение векторов через координаты сомножителей	Виразити добуток векторів через координати співмножників
33.Extrémity [end, end/términal póint, términus] of a véctor	Конец [конечная точка] вектора	Кінець [кінцева точка] вектора

34.Free [nónlócalized] véctor	Свободный вектор	Вільний вектор
35.Geométric(al) méaning/sense	Геометрический смысл	Геометричний сенс
36.Glíding [slíding, lócalized on a line, nónlócalized] véctor	Скользящий (вдоль прямой) вектор	Ковзний (вздовж прямої) вектор
37.Have the same órigin, diréc-tion	Иметь одно и то же начало, направление	Мати один і той же початок, напрямок
38.Length of a véctor	Длина вектора	Довжина вектора
39.Lie [be situáted] in the same plane	Лежать в одной (и той же) плоскости	Лежати в одній (і тій же) площині
40.Lie [be, be situáted/fóund] on the same straight line	Лежать на одной (и той же) прямой	Лежати на одній (і тій же) прямій
41.Línear (ìn)depéndence of véctors	Линейная (не)зависи-мость векторов	Лінійна (не)залежність векторів
42.Línear còmbinátion of véc-tors	Линейная комбинация векторов	Лінійна комбінація векторів
43.Línear òperátion	Линейная операция	Лінійна операція
44.Línearly (ìn)depéndent véc-tors	Линейно (не)зависи-мые векторы	Лінійно (не)залежні вектори
45.Múltiply a véctor by a núm-ber/scálar	Умножить вектор на число (скаляр)	Помножити вектор на число (скаляр)
46.Mútually pèrpendícular (or-thógonal) véctors, áxes	Взаимно перпендикулярные (ортогональные) векторы, оси	Взаємно перпендикулярні (ортогональні) вектори, осі
47.*n*-diménsional véctor	Эн-мерный вектор	*n*-вимірний вектор
48.Nécessary and suffícient con-dítion (for collìneárity of two véctors, for còplanárity of three véctors)	Необходимое и доста-точное условие (колли-неарности двух векторов, компланарности трёх векторов)	Необхідна і достатня умова (колінеарності двох векторів, компланарності трьох векторів)
49.Nórmalized/stánderdized véctor	Нормированный вектор	Нормований вектор
50.Ópposite véctor	Противоположный вектор	Протилежний вектор
51.Órdered páir of (non-colli-near) véctors (locáted at a common órigin)	Упорядоченная пара (неколлинеарных) векторов (приведённых к общему началу)	Впорядкована пара (неколінеарних) векторів (зведених до спільного початку)

52.Órdered páir of numbers	Упорядоченная пара чисел	Впорядкована пара чи-сел
53.Órdered tríple(t) of (nòn-có(m)planar) véctors (locáted at a common origin)	Упорядоченная тройка (некомпланарных) век-торов (приведённых к общему началу)	Впорядкована трійка (некомпланарних) векторів (зведених до спільного початку)
54.Órdered tríple(t) of numbers	Упорядоченная тройка чисел	Впорядкована трійка чисел
55.Órigin [inítial póint, póint of àpplicátion] of a véctor	Начало [начальная точ-ка, точка приложения] вектора	Початок [початкова точка, точка прикладення] вектора
56.Orthógonal projéction	Ортогональная проек-ция	Ортогональні вектори
57.Órthonórmal base/básis	Ортонормированный базис	Ортонормований базис
58.Páir (órdered páir) of véctors	Пара векторов (упоря-доченная)	Пара векторів (впоряд-кована)
59.Páirwìse orthógonal véctors	Попарно ортогональные векторы	Парами ортогональні вектори
60.Pàrallelépiped constructed on the véctors ***a***, ***b***, ***c*** as the sídes [fórmed/detérmined by the véc-tors ***a***, ***b***, ***c***; with sídes ***a***, ***b***, ***c***]	Параллелепипед, пост-роенный на векторах ***a***, ***b***, ***c*** как на сторонах [образованный векторами]	Паралелепіпед, побудований на векторах ***a***, ***b***, ***c*** як на сторонах [утворений векторами]
61.Pàrallélogram constructed on the véctors ***a***, ***b*** as the sídes [fórmed/detérmined by the véc-tors ***a***, ***b*** ; with sídes ***a***, ***b***]	Параллелограмм, пост-роенный на векторах ***a***, ***b*** как на сторонах [образованный векторами]	Паралелограм, побудований на векторах ***a***, ***b*** як на сторонах [утво-рений векторами]
62.Pèrpendícular (orthógonal) véctor	Перпендикулярный (ортогональный) вектор	Перпендикулярний (ортогональний) вектор
63.Próduct of a véctor by a númber/scálar	Произведение вектора на число (скаляр)	Добуток вектора на число (скаляр)
64.Projéction (of a véctor on(to) an áxis)	Проекция (вектора на ось)	Проекція (вектора на вісь)
65.Rádius véctor of a póint	Радиус-вектор точки	Радіус-вектор точки

66.Rectángular pàrallelépiped	Прямоугольный парал-лелепипед	Прямокутний паралелепіпед
67.Redúce/lead/bring véctors to [locáte véctors at] a cómmon órigine	Привести векторы к общему началу	Звести вектори до спільного початку
68.Right-hand rule	Правило правой руки	Правило правої руки
69.Right-hánded tríple(t) of véctors	Правая тройка векторов	Права трійка векторів
70.Rúle [láw]: pólygone [polygonal] rúle, pàrallélogram rú-le, pàrallelépiped rúle, tríangle rúle	Правило (закон) много-угольника, параллелограмма, параллелепипеда, треугольника	Правило (закон) многокутника, паралелограма, паралелепіпеда, трикутника
71.Scálar (númber)	Скаляр (число)	Скаляр (число)
72.Scálar [dot, Éuclidean, ínner, intérior, ínternal] próduct of two véctors	Скалярное произведе-ние двух векторов	Скалярний добуток двох векторів
73.Scálar fáctor	Скалярный множитель	Скалярний множник
74.Scálar quántity	Скалярная величина	Скалярна величина
75.Scálar squáre of a véctor	Скалярный квадрат вектора	Скалярний квадрат вектора
76.Squáre róot of the sum of squáres [róot-sum squáre, róot of sum-of-squáres]	Квадратный корень из суммы квадратов	Квадратний корінь з суми квадратів
77.Subtráct the véctor ***a*** from the véctor ***b***	Отнять, вычесть вектор ***a*** из вектора ***b***	Відняти вектор ***a*** и вектора ***b***
78.Subtráction of véctors	Вычитание векторов	Віднімання векторів
79.Sum of próducts of corresponding coórdinates	Сумма произведений соответствующих координат	Сума добутків відповідних координат
80.Sum of véctors	Сумма векторов	Сума векторів
81.Take pláce [occúr, be perfórmed] ánticlóckwise/cóunter-clóckwise	Происходить [произво-диться, осуществляться, совершаться] против часовой стрелки	Відбуватися [виконуватися] проти годинникової стрілки
82.Thrée-diménsional véctor	Трёхмерный вектор	Тривимірний вектор
83.Two-diménsional véctor	Двумерный вектор	Двовимірний вектор
84.Tríple(t) (órdered tríple(t)) of véctors	Тройка векторов (упо-рядоченная)	Трійка векторів (впорядкована)
85.Únit/únitary/diréction/méasu-	Орт вектора	Орт вектора

ring/básis véctor		
86. Véctor	Вектор	Вектор
87. Véctor [cross, extérior, óuter] próduct of two véctors	Векторное произведение двух векторов	Векторний добуток двох векторів
88. Véctor sýstem	Система векторов	Система векторів
89. Véctor/vectórial quántity	Векторная величина	Векторна величина
90. Véctors of ópposite diréc-tions	Противоположно нап-равленные векторы	Протилежно напрямлені вектори
91. Véctors of the same diréction	Сонаправленные векторы	Вектори одного й того ж напрямку
92. Way/méthod/mode of rèpre-sentátion/detèrminátion of a véctor	Способ задания вектора	Спосіб задання вектора
93. Zéro/nill/null véctor	Нулевой вектор	Нульовий вектор

LECTURE NO. 11. ANALYTIC GEOMETRY IN THE SPACE

POINT 1. A SURFACE. A PLANE

POINT 2. A SPACE STRAIGHT LINE

POINT 3. A PLANE AND A STRAIGHT LINE

POINT 1. A SURFACE. A PLANE

Def. 1. An equation of the form

$$F(x, y, z) = 0 \qquad (1)$$

is called the equation of a surface S if coordinates of every point of the surface (and only of such the point) satisfies it.

Equation of a sphere $S(R, M_0(x_0; y_0; z_0))$ of a radius R and the centre at a point $M_0(x_0; y_0; z_0)$ (fig. 1).

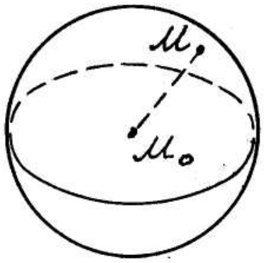


Fig. 1

For any point $M(x; y; z)$ of the sphere $S(R, M_0)$

$$M_0M^2 = R^2$$

whence it follows that

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2. \qquad (2)$$

Equation (2) is that in question.

If a sphere centered at the origin $O(0; 0; 0)$ its equation takes on the form

$$x^2 + y^2 + z^2 = R^2. \qquad (3)$$

Ex. 1. Let there be given an equation

$$x^2 + y^2 + z^2 - 6x + 8\mathrm{y} + 10z = 0.$$

Completing the squares, we'll get

$$x^2 - 6x + 9 + y^2 + 8y + 16 + z^2 + 10z + 25 = 50,$$

$$(x - 3)^2 + (y + 4)^2 + (z + 5)^2 = \left(\sqrt{50}\right)^2$$

that is the equation of a sphere $S\left(5\sqrt{2}, M_0(3; -4; -5)\right)$ of a radius $R = 5\sqrt{2}$ and a centre $M_0(3; -4; -5)$.

Equation of a plane passing through a given point perpendicularly to a given non-zero vector.

Let a plane α (fig. 2) passes through a point $M_0(x_0;\, y_0;\, z_0)$ perpendicularly to some non-zero vector $\overline{N}=(A,\, B,\, C)\neq\overline{0}$ (a **normal vector** of the plane), and it's necessary to set an equation of the plane.

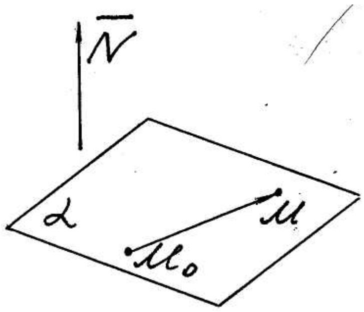


Fig. 2

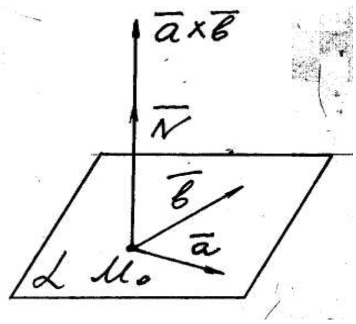


Fig. 3

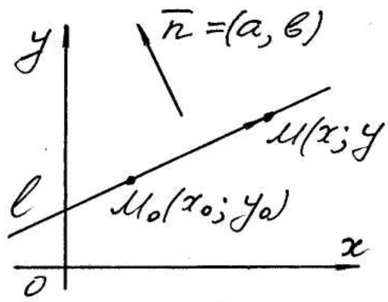


Fig. 4

For arbitrary point $M(x;\, y;\, z)$ of a plane the vectors

$$\overline{M_0M}=(x-x_0,\, y-y_0,\, z-z_0),\ \overline{N}=(A,\, B,\, C)$$

are perpendicular, and so their scalar product equals zero,

$$\overline{M_0M}\cdot\overline{N}=0,$$

whence we get the sought equation

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0,\quad A^2+B^2+C^2\neq 0. \tag{4}$$

Ex. 2. Write the equation of a plane passing through a point $M_0(-3;\, 4;\, -2)$ in parallel to two vectors $\overline{a}=(2,-3,\, 7),\ \overline{b}=(4,-3,-1)$.

If we suppose these vectors to be reduced to a common origin M_0 (fig. 3) we'll see that the normal vector of a sought plane is collinear to their vector product,

$$\overline{N}\parallel\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ 2 & -3 & 7\\ 4 & -3 & -1\end{vmatrix}=24\overline{i}+30\overline{j}+6\overline{k}=6(4\overline{i}+5\overline{j}+\overline{k});\ \overline{N}=(4,\, 5,\, 1),$$

and with the help of the equation (4) we get

$$4(x-(-3))+5(y-4)+1\cdot(z-(-2))=0,\quad 4x+5y+z-6=0.$$

Remark. Analogous problem can be solved in the *xOy*-plane: set an equation of a straight line passing through a point $M_0(x_0; y_0)$ perpendicularly to a given non-zero vector (normal vector) $\bar{n} = (a, b), a^2 + b^2 \neq 0$ (fig. 4).

For arbitrary point $M(x; y)$ of the straight line the vectors $\overline{M_0M}$ and $\bar{n}$ are perpendicular, and so their scalar product equals zero: $\overline{M_0M} \cdot \bar{n} = 0$,

$$a(x - x_0) + b(y - y_0) = 0. \qquad (5)$$

If we open the parentheses we'll get the general equation of a straight line

$$ax + by + c = 0, \quad a^2 + b^2 \neq 0. \qquad (6)$$

If $b \neq 0$ in (5), we'll get the equation of a pencil of straight lines with the centre $M_0(x_0; y_0)$ (or the equation of a straight line passing through a given point M_0 in a given direction)

$$y - y_0 = k(x - x_0), k = -\frac{a}{b}. \qquad (7)$$

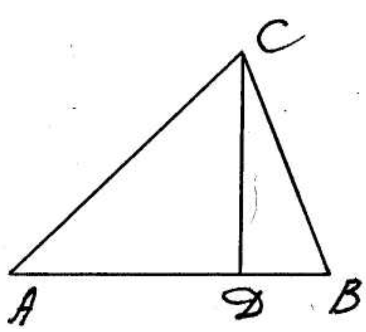


Fig. 5

Ex. 3. Compile an equation of the height drawn from the vertex *C* of a triangle *ABC*, if coordinates of its vertices are known, $A(-4; 6), B(1; -4), C(3; -4)$ (fig. 5).

A normal vector of the height

$\bar{n} \parallel \overline{AB} = (5; -10), \bar{n} = (1, -2)$, and by the equation (5)

$1 \cdot (x - 3) - 2 \cdot (y - (-4)) = 0, x - 2y - 11 = 0.$

General equation of a plane

The plane equation (4) after opening the parentheses can be written in the form

$$Ax + By + Cz + D = 0$$

where $D = -Ax_0 - By_0 - Cz_0$ and $A^2 + B^2 + C^2 \neq 0$. Inversely an equation

$$Ax + By + Cz + D = 0 \quad (A^2 + B^2 + C^2 \neq 0) \qquad (8)$$

is that of a plane.

■At least one of numbers A, B, C isn't zero. Let for example $A \neq 0$. Then we can write the equation (8) in the form

$$Ax + D + B(y - 0) + C(z - 0) = 0, \; A(x - (-D/A)) + B(y - 0) + C(z - 0) = 0$$

whence it follows that it's the equation of a plane which passes through the point $M_0(-D/A;\, 0;\, 0)$ and has the normal vector $\overline{N} = (A,\, B,\, C) \neq \overline{0}$.■

Thus there is **one-to-one correspondence** between the sets of all planes and all equations of the form (8).

Some particular cases of the general equation of a plane

1. Let $D = 0$. Equation (8) takes on the form

$$Ax + By + Cz = 0,$$

and the coordinates of the origin satisfy it. Therefore the plane **passes through the origin** $O(0;\, 0;\, 0)$.

2. Let $A = 0$ that is the equation of the plane doesn't contain x,

$$By + Cz + D = 0,$$

and the projection $Pr_{Ox}\,\overline{N} = A$ of its normal vector $\overline{N} = (0,\, B,\, C)$ equals 0. Hence the normal vector is perpendicular to the Ox-axis, and the plane is **parallel to the *Ox*-axis**.

3. If $A = 0$ and $B = 0$, the plane has the equation

$$Cz + D = 0,$$

it's parallel to the Ox-, Oy-axes and so is **parallel to the *xOy*-plane**. If in addition $D = 0$, we get the equation of the xOy-plane, namely

$$z = 0. \tag{9}$$

By analogous way (if $A = C = 0$ or $B = C = 0$) we come to the equations of the xOz-, yOz-planes

$$y = 0 \; (xOz - \text{plane}), \; x = 0 \; (yOz - \text{plane}) \tag{10}$$

Equation of a plane passing through three given points

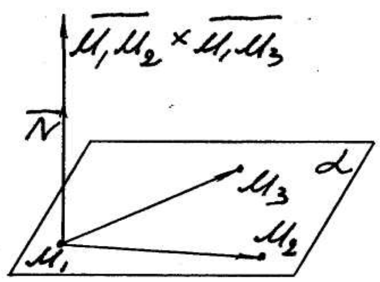


Fig. 6

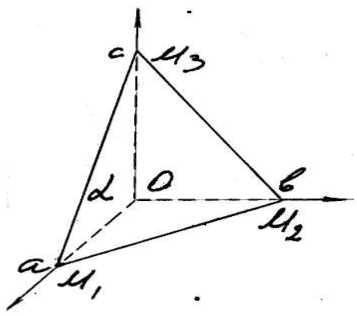


Fig. 7

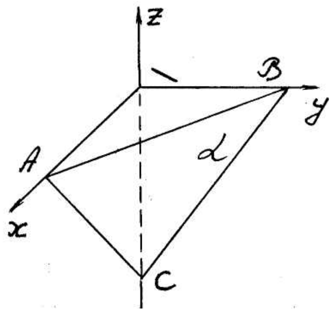


Fig. 8

If $M_1(x_1; y_1; z_1)$, $M_2(x_2; y_2; z_2)$, $M_3(x_3; y_3; z_3)$ be three known points of a plane (fig. 6), we compile its equation with the help of the equation (4) if we determine the normal vector by the condition

$$\overline{N} \parallel \overline{M_1M_2} \times \overline{M_1M_3} \qquad (12)$$

and use one of given points.

Ex. 4. Write the equation of a plane passing through the points $M_1(-6; 1; -5)$, $M_2(7; -2; -1)$, $M_3(10; -7; 1)$.

We have successfully

$$\overline{M_1M_2} = (13, -3, 4), \overline{M_1M_3} = (16, -8, 6), \ \overline{M_1M_2} \times \overline{M_1M_3} = (14, -14, -56),$$

$$\overline{N} \parallel \overline{M_1M_2} \times \overline{M_1M_3} = 14 \cdot (1, -1, -4), \overline{N} = (1, -1, -4).$$

Taking into account for example the point $M_1(-6; 1; -5)$ and making use of the equation (4) we obtain

$$1 \cdot (x - (-6)) + (-1) \cdot (y - 1) + (-4) \cdot (z - (-5)) = 0, (x + 6) - (y - 1) - 4 \cdot (z + 5) = 0;$$

opening the parentheses leads to the general equation of the plane in question

$$x - y - 4z - 13 = 0.$$

Equation of a plane in segments

Let a plane α cuts out segments a, b, c on the coordinate axes (fig. 7). Its equation in this case is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \qquad (13)$$

■The plane α passes through three points $M_1(a;\,0;\,0), M_2(0;b;\,0), M_3(0;0;c)$, and we can use above-stated theory. Namely, by (12)

$$\overline{M_1M_2}=(-a,b,0),\ \overline{M_1M_3}=(-a,0,c), \overline{M_1M_2}\times\overline{M_1M_3}=(bc,ac,ab),$$

and then by (4)

$bc(x-a)+ac(y-0)+ab(z-0)=0,\ bcx+acy+abz=abc,$ dividing by abc we get

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$■

Ex. 5. Evaluate the volume of a triangular pyramid bounded by a plane

$$6x-7y+5z+210=0$$

and the coordinate planes (fig. 7).

At first we rearrange the plane equation in the form (13),

$$6x-7y+5z=-210, -\frac{6}{210}x+\frac{7}{210}y-\frac{5}{210}z=1, -\frac{x}{35}+\frac{y}{30}-\frac{z}{42}=1,$$

the plane intercepts segments OM_1, OM_2, OM_3 ($a=-35, b=30, c=-42$) on the coordinate axes. Hence, the volume of the pyramid equals

$$V=\frac{1}{3}S_{OM_1M_2}\cdot|OM_3|=\frac{1}{3}|a|\cdot|b|\cdot|c|=\frac{1}{3}|-35|\cdot|30|\cdot|-42|=14700\ \textit{cubic units.}$$

Ex. 6. Find intersection points of the plane $3x+5y-2z-60=0$ with the coordinate axes and its intersection lines with the coordinate planes. Represent the plane.

a) Intersection points with the coordinate axes

Intersection point with the	We put	We get
Ox-axis	$y=z=0$	$x=20,\quad A(20;\,0;\,0)$
Oy-axis	$x=z=0$	$y=12,\quad B(0;\,15;\,0)$
Oz-axis	$x=y=0$	$z=30,\quad C(0;\,0;\,30)$

b) Intersection lines with the coordinate planes

Intersection lines with the We put We get

xOy-plane	$z = 0$	$AB: 3x + 5y - 60 = 0$
xOz-plane	$y = 0$	$AC: 3x - 2z - 60 = 0$
yOz-plane	$z = 0$	$BC: 5y - 2z - 60 = 0$

The plane is represented on fig. 8

Distance from a point to a plane

Let be given a plane α with an equation

$$Ax + By + Cz + D = 0$$

and a point $M_0(x_0; y_0; z_0)$. Its distance from the plane α is defined by a formula

$$d = d_{M_0,\alpha} = \frac{|Ax_0 + By_0 + Cz_0 + D|}{|\overline{N}|} = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} \qquad (14)$$

correspondingly to which one must substitute x, y, z by the coordinates of the point in the equation of the plane, find the modulus of the result and then divide it by the length of the normal vector of the plane.

■Let l is a straight line passing through the point $M_0(x_0; y_0; z_0)$ perpendicularly to the plane α, and $M_1(x_1; y_1; z_1)$ be an intersection point of l with α (fig. 9). Coordinates of this point satisfy the plane equation,

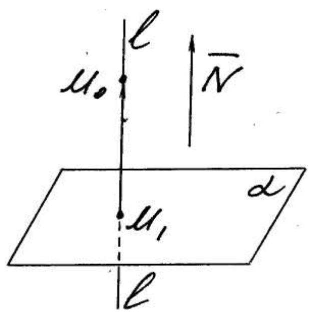


Fig. 9

$$Ax_1 + By_1 + Cz_1 + D = 0 \Rightarrow Ax_1 + By_1 + Cz_1 = -D.$$

The distance in question equals the length of the vector $\overline{M_1M_0}$, but we don't know its coordinates. We'll find the distance as the modulus of the projection of this vector on the normal vector of the plane. So

$$d = d_{M_0,\alpha} = \left|\text{Pr}_{\overline{N}}\, \overline{M_1M_0}\right| = \left|\frac{\overline{N} \cdot \overline{M_1M_0}}{|\overline{N}|}\right| =$$

$$= \frac{|(A, B, C)\cdot(x_0 - x_1, y_0 - y_1, z_0 - z_1)|}{|\overline{N}|} = \frac{|A(x_0 - x_1) + B(y_0 - y_1) + C(z_0 - z_1)|}{|\overline{N}|} =$$

$$= \frac{\left|Ax_0 + By_0 + Cz_0 - \left(Ax_1 + By_1 + Cz_1\right)\right|}{\left|\overline{N}\right|} = \frac{\left|Ax_0 + By_0 + Cz_0 - \left(-D\right)\right|}{\left|\overline{N}\right|} =$$

$$= \frac{\left|Ax_0 + By_0 + Cz_0 + D\right|}{\left|\overline{N}\right|}. \blacksquare$$

Ex. 7. Compile an equation of a sphere centered at a point $M_0(2;\, 4;\, -3)$ and touching a plane $-3x - 4y + 5z + 6 = 0$.

The radius of the sphere equals the distance of the point from the plane. By the formula (14) we have

$$R = d = d_{M_0,\alpha} = \frac{\left|-3 \cdot 2 - 4 \cdot 4 + 5 \cdot (-3) + 6\right|}{\sqrt{(-3)^2 + (-4)^2 + 5^2}} = \frac{|-31|}{\sqrt{50}} = \frac{31}{5\sqrt{2}},\ R^2 = \frac{961}{50},$$

and by virtue (2) the equation of the sphere is

$$(x-2)^2 + (y-4)^2 + (z+3)^2 = \frac{961}{50}.$$

Ex. 8. Find the volume of a triangular pyramid with vertices $M_1(-6;\, 1;\, -5)$, $M_2(7;\, -2;\, -1)$, $M_3(10;\, -7;\, 1)$, $M_4(3;\, -2;\, -6)$ (fig. 10).

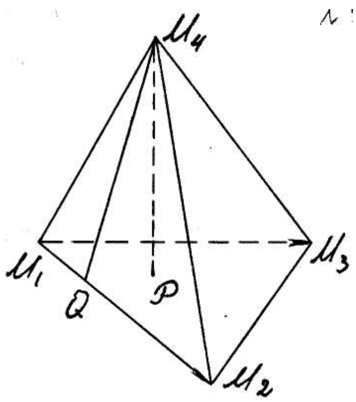


Fig. 10

If $H = M_4P$ be the height of the pyramid, then its volume equals

$$V = \frac{1}{3} \cdot S_{M_1M_2M_3} \cdot H.$$

But

$$S_{M_1M_2M_3} = \frac{1}{2} \cdot \left|\overline{M_1M_2} \times \overline{M_1M_3}\right|,$$

where

$$\overline{M_1M_2} = (13, -3, 4),\ \overline{M_1M_3} = (16, -8, 6),\ \overline{M_1M_2} \times \overline{M_1M_3} = (14, -14, -56)$$

(see Ex. 4), $\left|\overline{M_1M_2} \times \overline{M_1M_3}\right| = \sqrt{14^2 + (-14)^2 + (-56)^2} = 42\sqrt{2},\ S_{M_1M_2M_3} = 21\sqrt{2}$,

and the height $H = M_4P$ equals the distance of the point M_4 from the plane passing through the points M_1, M_2, M_3. Equation of the latter was found in Ex. 4, namely

$$x - y - 4z - 13 = 0,$$

hence

$$H = M_4P = \frac{|3-(-2)-4\cdot(-6)-13|}{\sqrt{1^2+(-1)^2+4^2}} = \frac{16}{\sqrt{18}} = \frac{8\sqrt{2}}{3}.$$

Finally,

$$V = \frac{1}{3}\cdot 21\sqrt{2}\cdot\frac{8\sqrt{2}}{3} = \frac{112}{3} \approx 37.33 \ \textit{cubic units.}$$

Angle between two planes, conditions for parallelism or perpendicularity

Let be given two planes α_1, α_2 by their general equations

$$\begin{aligned}(\alpha_1): A_1x + B_1y + C_1z + D_1 = 0,\ \overline{N_1} = (A_1, B_1, C_1);\\ (\alpha_2): A_2x + B_2y + C_2z + D_2 = 0,\ \overline{N_2} = (A_2, B_2, C_2).\end{aligned} \tag{15}$$

An angle between the planes α_1, α_2 is called the angle between their normal vectors,

$$\left(\alpha_1 \ \hat{,}\ \alpha_2\right) = \left(\overline{N_1} \ \hat{,}\ \overline{N_2}\right),$$

and therefore

$$cos\left(\alpha_1 \ \hat{,}\ \alpha_2\right) = cos\left(\overline{N_1} \ \hat{,}\ \overline{N_2}\right) = \frac{\overline{N_1}\cdot\overline{N_2}}{\left|\overline{N_1}\right|\left|\overline{N_2}\right|} = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}. \tag{16}$$

Two planes are **parallel** if and only if their normal vectors are collinear, namely

$$\alpha_1 \parallel \alpha_2 \Leftrightarrow \left(\overline{N_1} \parallel \overline{N_2},\quad \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}\right) \tag{17}$$

Two planes are **perpendicular** if and only if their normal vectors are perpendicular, that is

$$\alpha_1 \perp \alpha_2 \Leftrightarrow \left(\overline{N_1} \perp \overline{N_2},\quad \overline{N_1}\cdot\overline{N_2} = 0,\quad A_1A_2 + B_1B_2 + C_1C_2 = 0\right) \tag{18}$$

Ex. 9. Write an equation of a plane if it passes through a point $A(-4; 3; 2)$ perpendicularly to two planes $x - 2y + 3z - 5 = 0, x + 4y - z + 3 = 0$.

Normal vectors of the given planes $\overline{N_1} = (1, -2, 3), \overline{N_2} = (1, 4, -1)$. A normal vector $\overline{N}$ of a sought plane is perpendicular to the vectors $\overline{N_1}, \overline{N_2}$ and so is collinear to their vector product,

$$\overline{N} \parallel \overline{N_1} \times \overline{N_2} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & -2 & 3 \\ 1 & 4 & -1 \end{vmatrix} = -10\bar{i} + 4\bar{j} + 6\bar{k},\ \overline{N} = (5, -2, -3);$$

by virtue of the equation (4) the required equation is

$$5(x-(-4)) - 2(y-3) - 3(z-2) = 0, \quad 5x - 2y - 3z + 32 = 0.$$

Ex. 10. Find the values of parameters m, n for which two planes

$$3x + my - 5z + 8 = 0,\ 7x + 9y + nz - 1 = 0$$

are parallel.

Normal vectors of the given planes are $\overline{N_1} = (3, m, -5), \overline{N_2} = (7, 9, n)$, and by parallelism condition (17) of two planes one must have

$$\frac{3}{7} = \frac{m}{9} = \frac{-5}{n},$$

whence it follows that

$$\frac{3}{7} = \frac{m}{9}, \frac{3}{7} = \frac{-5}{n} \Rightarrow 7m = 27, 3n = -35, m = \frac{27}{7}, n = \frac{-35}{3}.$$

Problem of intersection of three planes

Let we find intersection points of three planes, therefore we study the system of equations of these planes

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, & \overline{N_1} = (A_1, B_1, C_1), \\ A_2x + B_2y + C_2z + D_2 = 0, & \overline{N_2} = (A_2, B_2, C_2), \\ A_3x + B_3y + C_3z + D_3 = 0, & \overline{N_3} = (A_3, B_3, C_3). \end{cases} \qquad (19)$$

System and dilated [extended] matrices of the system (19)

$$A = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{pmatrix}$$

Case 1. $RankA = Rank\widetilde{A} = 3$.

The principal determinant of the system $\Delta = |A| \neq 0$ distinct from zero, the system possesses unique solution, and all the planes intersect in one point.

Case 2. $RankA = 2, Rank\widetilde{A} = 3$.

System (19) is non-compatible one. By the fact $RankA = 2$ the matrix A has at least one non-zero second order minor. In this case some two planes are intersecting, and the third one is parallel to their intersection line. If for example

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \neq 0,$$

we have $A_1B_2 - A_2B_1 \neq 0, A_1/A_2 \neq B_1/B_2$, the normal vectors $\overline{N_1}, \overline{N_2}$ are non-collinear, and two first planes intersect.

Case 3. $RankA = Rank\widetilde{A} = 2$.

System (19) has infinitely many solutions, all planes have common intersect line.

Case 4. $RankA = 1, Rank\widetilde{A} = 2$.

System (19) hasn't solutions. All second order minors of the matrix A equal zero, hence the normal vectors $\overline{N_1}, \overline{N_2}, \overline{N_3}$ of the planes are pairwise collinear, the planes are parallel, but at least two of them don't coincide.

Case 5. $RankA = Rank\widetilde{A} = 1$.

All three planes coincide.

POINT 2. A SPACE STRAIGHT LINE

Equations of a straight line passing through a given point in parallel to a given vector

Let a spatial straight line l passes through a point $M_0(x_0; y_0; z_0)$ and is parallel

Fig. 11

to some vector $\bar{s} = (m, n, p)$ which is called the **directing vector** of the straight line (see fig. 11).

For any point $M(x; y; z)$ of the straight line the vectors $\overline{M_0M}, \bar{s}$ are collinear, and so their coordinates are proportional, whence it follows that

$$\frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p}. \qquad (20)$$

The formula (20) contains two independent equations which are called the **canonical equations of the straight line** *l*.

Let's denote by *t* the equal ratios in (20),

$$\frac{x - x_0}{m} = t, \frac{y - y_0}{n} = t, \frac{z - z_0}{p} = t.$$

We'll get

$$x - x_0 = mt, \; y - y_0 = nt, \; z - z_0 = pt,$$

$$\begin{cases} x = x_0 + mt, \\ y = y_0 + nt, \\ z = z_0 + pt. \end{cases} \qquad (21)$$

The equations (21) are called parametric equations of a straight line. Here *t* is an auxiliary variable which is called a parameter. For example the value $t = 0$ of the parameter corresponds to a point $M_0(x_0; y_0; z_0)$.

Ex. 11. Set parametric and canonical equations of a straight line which passes through a point $M_0(2; 6; -3)$ and is perpendicular to a plane $4x - 5y + z - 10 = 0$.

As the directing vector of a straight line we take the normal vector of the plane,

$$\bar{s} = \overline{N} = (4, -5, 1)$$

and with the help of the equations (21) and (20) we get

$$\begin{aligned} x &= 2 + 4t, \\ y &= 6 - 5t, \\ z &= -3 + t; \end{aligned} \qquad \frac{x - 2}{4} = \frac{y - 6}{-5} = \frac{z + 3}{1}.$$

Ex. 12. Write yourselves equations of the height M_4P of the triangular pyramid with the vertices $M_1(-6; 1; -5)$,
$M_2(7; -2; -1), M_3(10; -7; 1), M_4(3; -2; -6)$ (see Ex. 8 and fig. 10).

Equations of a straight line passing through two given points

Let a straight line passes through two points $M_1(x_1, y_1, z_1), M_2(x_2, y_2, z_2)$. If we take the vector

$$\overline{M_1M_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

as the directing vector of a straight line, we'll obtain parametric and canonical equations of a straight line in the next form

$$\begin{array}{l} x = x_1 + (x_2 - x_1)t, \\ y = y_1 + (y_2 - y_1)t, \\ z = z_1 + (z_2 - z_1)t; \end{array} \quad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \qquad (22)$$

In practice it's sometimes better to take $\bar{s} \parallel \overline{M_1M_2}$ and use the equations (20), (21).

Ex. 13. Write canonical and parametric equations of a straight line which passes through two given points $A(3; -3; 4), B(3; 1; -2)$.

Directing vector of a straight line

$$\bar{s} \parallel \overline{AB} = (0; 4; -6) \Rightarrow \bar{s} = (0, 2, -3),$$

and by the equations (20), (21)

$$\frac{x-3}{0} = \frac{y+3}{2} = \frac{z-4}{-3}; \quad \begin{cases} x = 3 + 0 \cdot t, \\ y = -3 + 2 \cdot t, \\ z = 4 - 3 \cdot t, \end{cases} \quad \begin{cases} x = 3, \\ y = -3 + 2t, \\ z = 4 - 3t. \end{cases}$$

The straight line is perpendicular to the Ox-axis.

General equations of a straight line

A straight line can be determined as an intersection line of two non-parallel planes.

Let

$$(\alpha_1): A_1x + B_1y + C_1z + D_1 = 0, \overline{N_1} = (A_1, B_1, C_1),$$
$$(\alpha_2): A_2x + B_2y + C_2z + D_2 = 0, \overline{N_2} = (A_2, B_2, C_2)$$

be two non-parallel planes that is their normal vectors $\overline{N_1}, \overline{N_2}$ are non-collinear. In this case a system of two linear equations

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \qquad (23)$$

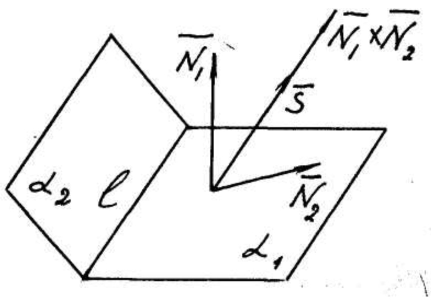


Fig. 12

represents a straight line l as the intersection line of the planes α_1, α_2 (Fig. 12). Equations (23) are called general equations of a straight line.

It's easy to pass from general equations of a straight line to those parametric and canonical. Let for example

$$\frac{A_1}{A_2} \neq \frac{B_1}{B_2}.$$

In this case we put $z = t$ in the general equations (23) and get a system of equations in x and y

$$\begin{cases} A_1x + B_1y = -C_1t - D_1, \\ A_2x + B_2y = -C_2t - D_2. \end{cases}$$

Finding x, y we get parametric equations of a straight line.

There is the other method of passage from general equations of a straight line to those parametric and canonical. Namely we can find its directing vector

$$\bar{s} \parallel \overline{N_1} \times \overline{N_2} \text{ (fig. 12)}$$

and then its point taking for example $z = 0$ in (23).

Ex. 14. Pass to parametric and canonical equations of a straight line

$$\begin{cases} 2x+3y-5z+4=0, \\ 3x-2y+z-5=0. \end{cases}$$

The first method. Putting $z=t$, we obtain a system of equations in x, y

$$\begin{cases} 2x+3y=5t-4, \\ 3x-2y=-t+5. \end{cases} \Delta=\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix}, \Delta_1=\begin{vmatrix} 5t-4 & 3 \\ -t+5 & -2 \end{vmatrix}, \Delta_2=\begin{vmatrix} 2 & 5t-4 \\ 3 & -t+5 \end{vmatrix},$$

$$\Delta=-13,\ \Delta_1=(5t-4)\cdot(-2)-3\cdot(-t+5)=-7-7t,\ \Delta_2=2\cdot(-t+5)-(5t-4)\cdot 3=22-17t$$

$$x=\frac{\Delta_1}{\Delta}=\frac{-7-7t}{-13}=\frac{7}{13}+\frac{7}{13}t,\ y=\frac{\Delta_2}{\Delta}=\frac{22-17t}{-13}=-\frac{22}{13}+\frac{17}{13}t,$$

and parametric equations of a straight line in question

$$\begin{cases} x=7/13+7/13t \\ y=-22/13+17/13t, \\ z=t. \end{cases} \qquad (\ *\ )$$

From the equations (*) we get a point $M_0\left(\frac{7}{13}; \frac{-22}{13}; 0\right)$ of a straight line and its directing vector $\bar{s}=\left(\frac{7}{13}, \frac{17}{13}, 1\right)$.

We can improve the obtained equations (*). At first we can take a directing vector in the form $\overline{s_1}=13\bar{s}=(7, 17, 13)$ and get

$$\begin{cases} x=7/13+7t \\ y=-22/13+17t, \\ z=13t. \end{cases}$$

Taking further $t=-1/13$ we find a point of the straight line with integer coordinates $x=0,\ y=-3,\ z=-1$. Finally we write the improved parametric and canonical equations of a straight line

$$\begin{cases} x=7t \\ y=-3+17t, \\ z=-1+13t; \end{cases} \quad \frac{x}{7}=\frac{y+3}{17}=\frac{z+1}{13}.$$

The second method. The directing vector of the straight line

$$\bar{s} \parallel \overline{N_1} \times \overline{N_2} = (2, 3, -5) \times (3, -2, 1) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 3 & -5 \\ 3 & -2 & 1 \end{vmatrix} = -7\bar{i} - 17\bar{j} - 13\bar{k}, \; \bar{s} = (7, 17, 13).$$

Putting $z = 0$ in the general equations of the straight line and solving a system

$$\begin{cases} 2x + 3y = -4, \\ 3x - 2y = 5, \end{cases}$$

we get a point $M_0(7/13;\, -22/13;\, 0)$ and sought equations

$$\begin{cases} x = 7/13 + 7t \\ y = -22/13 + 17t, \\ z = 13t. \end{cases} \quad \frac{x - 7/13}{7} = \frac{y + 22/13}{17} = \frac{z}{13}.$$

Angle between two straight lines, conditions for parallelism or perpendicularity

An angle between two straight lines l_1, l_2 is defined as the angle between their directing vectors,

$$\left(l_1 \,\hat{,}\, l_2\right) = \left(\overline{s_1} \,\hat{,}\, \overline{s_2}\right),$$

and therefore

$$cos\left(l_1 \,\hat{,}\, l_2\right) = cos\left(\overline{s_1} \,\hat{,}\, \overline{s_2}\right) = \frac{\overline{s_1} \cdot \overline{s_2}}{\left|\overline{s_1}\right|\left|\overline{s_2}\right|} = \frac{m_1 m_2 + n_1 n_2 + p_1 p_2}{\sqrt{m_1^2 + n_1^2 + p_1^2}\sqrt{m_2^2 + n_2^2 + p_2^2}}. \qquad (24)$$

Two straight lines are **parallel** if and only if their directing vectors are collinear, namely

$$l_1 \parallel l_2 \Leftrightarrow \left(\overline{s_1} \parallel \overline{s_2}, \quad \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{p_1}{p_2}\right). \qquad (25)$$

Two straight lines are **perpendicular** if and only if their directing vectors are perpendicular, that is

$$l_1 \perp l_2 \Leftrightarrow \left(\overline{s_1} \perp \overline{s_2}, \quad \overline{s_1} \cdot \overline{s_2} = 0, \quad m_1 m_2 + n_1 n_2 + p_1 p_2 = 0\right). \qquad (26)$$

Ex. 15. Write equations of a straight line passing through a point $A(-4;\, 3;\, -2)$ in parallel to a given straight line

$$\begin{cases} 2x-3y+z-1=0, \\ x+4y-2z+10=0. \end{cases}$$

As a directing vector of a sought straight line we can take that of the given line. This latter is collinear to the vector product of the normal vectors $\overline{N_1}=(2,-3,1)$, $\overline{N_2}=(1,4,-2)$ of the planes which determine the given line (fig. 12). Thus

$$\bar{s} \parallel \overline{N_1}\times\overline{N_2}=\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & -3 & 1 \\ 1 & 4 & -2 \end{vmatrix}=2\bar{i}+5\bar{j}+11\bar{k},\ \bar{s}=(2,5,11),$$

and parametric and canonical equations of the sought straight line are

$$\begin{cases} x=-4+2t, \\ y=3+5t, \\ z=-2+11t, \end{cases} \quad \frac{x+4}{2}=\frac{y-3}{5}=\frac{z+2}{11}.$$

Ex. 16. Prove that straight lines

$$l_1:\begin{cases} x=3+2t, \\ y=-1-3t, \\ z=2-5t, \end{cases} \quad l_2:\frac{x+2}{-3}=\frac{y-3}{1}=\frac{z-4}{-3}$$

are not parallel and lie in the same plane (fig. 13). Write the equation of this plane. Find the intersection point of the straight lines.

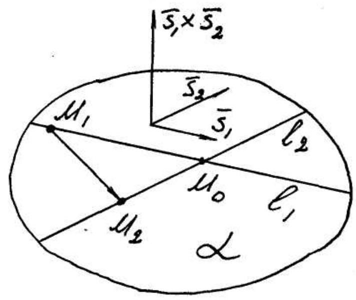


Fig. 13

Equations of the straight lines give to us the point $M_1(3;-1;2)$ and the directing vector $\overline{s_1}=(2,-3,-5)$ of the line l_1, the point $M_2(-2;3;4)$ and the directing vector $\overline{s_2}=(-3,1,-3)$ of the line l_2.

The vectors $\overline{s_1},\overline{s_2}$ aren't collinear, and so the straight lines aren't parallel.

The vector $\overline{M_1M_2}=(-5,4,2)$ and the vector product of the directing vectors

$$\overline{s_1}\times\overline{s_2}=\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & -3 & -5 \\ -3 & 1 & -3 \end{vmatrix}=14\bar{i}+21\bar{j}-7\bar{k}=7(2\bar{i}+3\bar{j}-\bar{k})$$

are perpendicular ($\overline{M_1M_2}\cdot\left(\overline{s_1}\times\overline{s_2}\right)=(-5)\cdot 14+4\cdot 21+2\cdot(-7)=0$) which means that the straight lines lie in the same plane α. Its normal vector and equation are

$$\overline{N}\parallel\overline{s_1}\times\overline{s_2},\ \overline{N}=(2,3,-1),\ 2\left(x-x_{M_1}\right)+3\left(y-y_{M_1}\right)-\left(z-z_{M_1}\right)=0,$$
$$2(x-3)+3(y+1)-(z-2)=0;\ \alpha:2x+3y-z-1=0.$$

To find the intersection point of the straight lines we write the parametric equations of the second line l_2 (with some other parameter, namely τ)

$$\begin{cases} x=-2-3\tau, \\ y=3+\tau, \\ z=4-3\tau \end{cases}$$

equate the right sides of two first equations of both straight lines

$$\begin{cases} 3+2t=-2-3\tau, \\ -1-3t=3+\tau \end{cases}$$

and solve the obtained system of equations in t and τ

$$\begin{cases} 2t+3\tau=-5, \\ 3t+\tau=-4 \end{cases}$$

whence it follows that $t=\tau=-1$. Substitution $t=-1$ in the equations of the first line or $\tau=-1$ in the parametric equations of the second one gives the coordinates of the intersection point of the lines, namely $M_0(1;\,2;\,7)$.

Ex. 17 (for individual resolution). Prove that straight lines

$$l_1:\begin{cases} x=-1+4t, \\ y=3-6t, \\ z=2t, \end{cases}\quad l_2:\frac{x-3}{2}=\frac{y+1}{-3}=\frac{z-4}{1}$$

are parallel and compile the equation of the plane in which they are situated.

POINT 3. A PLANE AND A STRAIGHT LINE

Intersection point of a straight line with a plane or a surface

Let be given a plane

$$Ax+By+Cz+D=0 \tag{27}$$

and a straight line determined by its parametric equations

$$\begin{cases} x = x_0 + mt, \\ y = y_0 + nt, \\ z = z_0 + pt. \end{cases} \qquad (28)$$

To seek eventual [possible, probable, virtual] intersection point of the straight line (28) and the plane (27) we substitute variables *x*, *y*, *z* in (27) by their values from the equations (28) obtaining an equation in *t*,

$$A(x_0 + mt) + B(y_0 + nt) + C(z_0 + pt) + D = 0,$$

$$Ax_0 + By_0 + Cz_0 + D + (Am + Bn + Cp)t = 0. \qquad (29)$$

Three cases can occur as to the equation (29).

1) $Am + Bn + Cp \neq 0$.

In this case we solve the equation (29) with respect to *t*,

$$t = t' = -\frac{Ax_0 + By_0 + Cz_0 + D}{Am + Bn + Cp};$$

substitution of *t* by t' in the equations (28) gives the coordinates of the intersection point of the straight line and the plane.

2) $Am + Bn + Cp = 0$, but $Ax_0 + By_0 + Cz_0 + D \neq 0$.

An intersection point doesn't exist in this case, the straight line is parallel to the plane, but it doesn't lie in the plane.

3) $Am + Bn + Cp = 0$ and $Ax_0 + By_0 + Cz_0 + D = 0$.

The straight line lies in the plane.

By the same way we can look up virtual intersection points of a straight line with arbitrary surface. For example in the case of a sphere of a radius *R* centered at a point $M_1(x_1; y_1; z_1)$,

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = R^2,$$

we get the next equation in *t*:

$$(x_0 + mt - x_1)^2 + (y_0 + nt - y_1)^2 + (z_0 + pt - z_1)^2 = R^2.$$

$$\left(m^2+n^2+p^2\right)t^2+2\left(m(x_0-x_1)+n(y_0-y_1)+p(z_0-z_1)\right)t+$$
$$+(x_0-x_1)^2+(y_0-y_1)^2+(z_0-z_1)^2-R^2=0.$$

It's a quadratic in t and can have two, one or no roots. Correspondingly we get two, one (case of tangency) or no intersection points.

Ex. 18. Find an intersection point of the straight line $\frac{x-2}{4}=\frac{y-6}{-5}=\frac{z+3}{1}$ and the plane $4x-5y+z-10=0$.

At first we write the parametric equations of the straight line

$$x=2+4t,$$
$$y=6-5t,$$
$$z=-3+t,$$

and then do as follows:

$$4(2+4t)-5(6-5t)+(-3+t)-10=0,\ 42t-35=0,\ t=35/42;$$
$$x=2+4\cdot 35/42=16/3,\ y=6-5\cdot 35/42=77/42,\ z=-3+35/42=-91/42.$$

Answer. The straight line intersects the plane at the point $\left(\frac{16}{3};\frac{11}{6};-\frac{91}{42}\right)$.

Angle between a straight line and a plane,
conditions for parallelism or perpendicularity

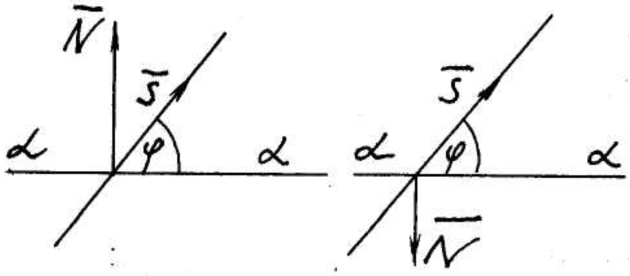


Fig. 14

As it's known the angle between a straight line l and a plane α is called the angle between the line l and its orthogonal projection onto the plane α.

Let $\bar s=(m,n,p),\ \overline{N}=(A,B,C)$ be the directing vector of the straight line l and the normal vector of the plane α correspondingly (fig. 14). One can see from the figure that the angle φ between l and α equals

$$\varphi=\left(l\ \hat{,}\ \alpha\right)=\frac{\pi}{2}-\left(\bar s\ \hat{,}\ \overline{N}\right)\text{ or }\varphi=\left(l\ \hat{,}\ \alpha\right)=\left(\bar s\ \hat{,}\ \overline{N}\right)-\frac{\pi}{2},$$

and therefore

$$\varphi = \left(l \,\hat{,}\, \alpha\right) = \pm\left(\frac{\pi}{2} - \left(\bar{s} \,\hat{,}\, \overline{N}\right)\right),\ sin\,\varphi = \pm\,sin\left(\frac{\pi}{2} - \left(\bar{s} \,\hat{,}\, \overline{N}\right)\right) = \pm\,cos\left(\bar{s} \,\hat{,}\, \overline{N}\right),$$

$$\sin\varphi = \sin\left(l \,\hat{,}\, \alpha\right) = \pm\cos\left(\bar{s} \,\hat{,}\, \overline{N}\right) = \pm\frac{\bar{s}\cdot\overline{N}}{\left|\bar{s}\right|\left|\overline{N}\right|} = \pm\frac{Am + Bn + Cp}{\sqrt{m^2+n^2+p^2}\sqrt{A^2+B^2+C^2}}. \quad (30)$$

A straight lines and a plane are **parallel** if and only if their directing and normal vectors $\bar{s}, \overline{N}$ are perpendicular, that is

$$l \parallel \alpha \Leftrightarrow \left(\bar{s} \perp \overline{N},\quad \bar{s}\cdot\overline{N} = 0,\quad Am + Bn + Cp = 0\right). \quad (31)$$

A straight lines and a plane are **perpendicular** if and only if their directing and normal vectors $\bar{s}, \overline{N}$ are collinear, namely

$$l \perp \alpha \Leftrightarrow \left(\bar{s} \parallel \overline{N},\quad \frac{m}{A} = \frac{n}{B} = \frac{p}{C}\right). \quad (32)$$

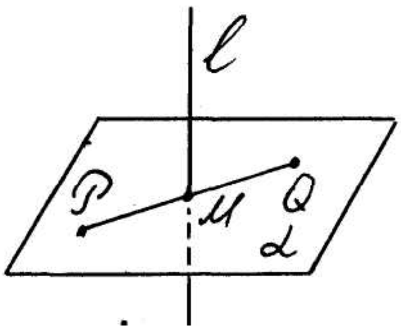


Fig. 15

Ex. 19. Find a point symmetric to a point $P(1;-1;-2)$ with respect to a straight line (fig. 15)

$$\frac{x+3}{3} = \frac{y+2}{2} = \frac{z-8}{-2}.$$

Where is the projection of the point P on the straight line and how to find the distance of the point to the line?

1) As the first step we'll write an equation of a plane α passing through the point P perpendicularly to the straight line l. As its normal vector we take the directing vector of l: $\overline{N_\alpha} = \overline{s_l} = (3, 2, -2)$, hence

$$3(x - x_P) + 2(y - y_P) - 2(z - z_P) = 0,\quad 3(x-1) + 2(y+1) - 2(z+2) = 0,$$

$$(\alpha):\ 3x + 2y - 2z - 5 = 0.$$

2) Now we find the intersection point M of the plane α and the straight line l,

$$(\alpha):\ 3x + 2y - 2z - 5 = 0,$$

$$(l): \begin{cases} x = -3 + 3t, \\ y = -2 + 2t, \\ z = 8 - 2t; \end{cases}$$

$$3(-3+3t)+2(-2+2t)-2(8-2t)-5=0,\ 17t-34=0,\ t=2,$$

$$x=x_M=-3+3\cdot 2=3,\ y=y_M=-2+2\cdot 2=2,\ z=z_M=8-2\cdot 2=4,\ M(3;\,2;\,4)$$

The point M is the projection of the point P on the straight line l. The distance between the points M and P is that of the point P from the line l.

3) In conclusion we find the sought point Q is we'll take into account that the point M divides the segment in halves, and therefore

$$x_M=\frac{x_P+x_Q}{2},\ x_Q=2x_M-x_P, \text{ and similarly } y_Q=2y_M-y_P,\ z_Q=2z_M-z_P,$$

$$x_Q=2\cdot 3-1=5,\ y_Q=2\cdot 2+1=5,\ z_Q=2\cdot 4+2=10.$$

Answer. The point in question is $Q(5;\,5;\,10)$.

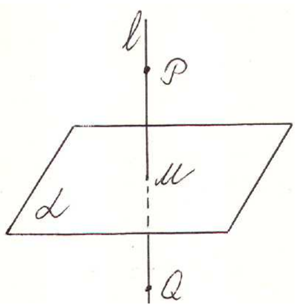


Fig. 16

Ex. 20. Find a point Q which is symmetric to a point $P(3;-4;-6)$ with respect to a plane $\alpha : x-y-4z-13=0$ (see fig. 16).

The plan of solving the problem.

1) Compile parametric equations of a straight line l passing through the point P perpendicularly to the plane α.

2) Find the intersection point M of the line l and the plane α.

3) Find the coordinates of the sought point Q taking into account that M is the centre of the segment PQ.

Realization of the plan

1) We choose the normal vector $\overline{N_\alpha}$ of the plane α as the directing vector $\overline{s_l}$ of the straight line l, $\overline{s_l}=\overline{N_\alpha}=(1,-1,-4)$, whense we get parametric equations of the line,

$$\begin{cases} x=3+t, \\ y=-4-t, \\ z=-6-4t. \end{cases}$$

2) Let's substitute the right sides of these equations in the equation of the plane α,

$$(3+t)-(-4-t)-4(-6-4t)-13=0,\quad 18t+18=0\Rightarrow t=-1.$$

Then we get the coordinates of the point M if we'll substitute the found value of t in the parametric equations of the line l,

$$x_M = 3 + (-1) = 2, \quad y_M = -4 - (-1) = -3, \quad z_M = -6 - 4(-1) = -2; \quad M(2; -3; -2).$$

3) Finally we find the coordinates of the point Q,

$$x_M = \frac{x_P + x_Q}{2},\ x_Q = 2x_M - x_P, \text{ and similarly } y_Q = 2y_M - y_P,\ z_Q = 2z_M - z_P,$$

$$x_Q = 2 \cdot 2 - 3 = 1,\ y_Q = 2 \cdot (-3) - (-4) = -2,\ z_Q = 2 \cdot (-2) - (-6) = 2 \Rightarrow Q(1; -2; 2).$$

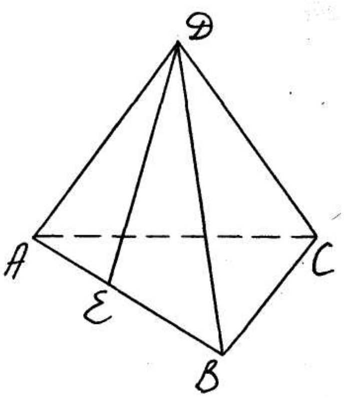


Fig. 17

Ex. 21. Compile the equations of the height DE of the lateral face ABD of the triangular pyramid with given vertices $A(2; -5; 6)$, $B(-1; 3; 4)$, $C(-3; 4; -1)$, $D(4; -1; 5)$ (see fig.17).

The plan of solving the problem.

1) Write parametric equations of the straight line AB.

2) Set an equation of a plane α passing through the point D perpendicularly to the straight line AB.

3) Find the intersection point E of the line AB and the plane α.

4) Compile the equations of the height DE in question.

Realization of the plan

1) We take the vector $\overline{AB}$ as the directing vector of the line AB, hence

$$\begin{cases} x = x_A + (x_B - x_A)t, \\ y = y_A + (y_B - y_A)t, \\ z = z_A + (z_B - z_A)t; \end{cases} \quad \begin{cases} x = 2 + ((-1) - 2)t, \\ y = -5 + (3 - (-5))t, \\ z = 6 + (4 - 6)t; \end{cases} \quad \begin{cases} x = 2 - 3t, \\ y = -5 + 8t, \\ z = 6 - 2t. \end{cases}$$

2) Now we take the vector $\overline{AB}$ as the normal vector of the plane α, whence

$$\begin{gathered} \overline{N_\alpha} \| \overline{s_{AB}} = \overline{AB} = (-3, 8, -2); \quad \overline{N_\alpha} = (3, -8, 3) \\ (\alpha): 3(x - x_D) - 8(y - y_D) + 2(z - z_D) = 0; \\ (\alpha): 3(x - 4) - 8(y - (-1)) + 2(z - 5) = 0; \\ (\alpha): 3x - 8y + 2z - 30 = 0. \end{gathered} ;$$

3) As the next step we find the intersection point E of the straight line l and the plane α,

$$3(2-3t)-8(-5+8t)+2(6-2t)-30=0;\quad -77+28t=0,\quad t=11/4;$$

$$x_E=2-3\cdot\frac{11}{4}=-\frac{25}{4};\ y_E=-5+8\cdot\frac{11}{4}=\frac{48}{4}=12;\ z_E=6-2\cdot\frac{11}{4}=-\frac{5}{2};\ E\left(-\frac{25}{4};12;-\frac{5}{2}\right)$$

4) Finally we compile the parametrical and canonical equations of the height in question:

$$\overline{s_{DE}}\parallel\overline{DE}=(-41/4;13;-15/2);\quad \overline{s_{DE}}=-4\overline{DE}=(41,-52,30);$$

$$\begin{cases} x = x_D+41t, \\ y = y_D+(-52)t, \\ z = z_D+30t; \end{cases} \begin{cases} x = 4+41t, \\ y = -1-52t, \\ z = 5+30t; \end{cases} \frac{x-4}{41}=\frac{y+1}{-52}=\frac{z-5}{30}.$$

ANALYTIC GEOMETRY IN THE SPACE: basic terminology RUE

1. Исследование общего уравнения плоскости	Дослідження загального рівняння площини	Invèstigátion of the géneral equátion of a pláne
2. Канонические уравнения прямой	Канонічні рівняння прямої	Canónical equátions of a straight líne
3. Направляющий вектор прямой	Напрямний вектор прямої	Diréction véctor of the straight line
4. Необходимое и достаточное условие параллельности [перпендикулярности] двух плоскостей, двух прямых, прямой и плоскости	Необхідна і достатня умова паралельності [перпендикулярності] двох площин, двох прямих, прямої і площини	Nécessary and suffícient condítion for párallelism [pérpendìculárity] of two planes, of two straight lines, of a straight líne and a súrface
5. Нормальный вектор плоскости	Нормальний вектор (до) площини	Nórmal véctor of/to the pláne
6. Общее уравнение плоскости	Загальне рівняння площини	Géneral equátion of a pláne
7. Общие уравнения прямой	Загальні рівняння прямої	Géneral equátions of a straight line [straight line géneral equátions]
8. Параметрические уравнения прямой	Параметричні рівняння прямої	Pàramétric equátions of a straight line
9. Переход от…к… (от общих уравнений прямой к параметрическим/ каноническим)	Перехід від … до …(від загальних рівнянь прямої до параметричних/канонічних)	Pássage from … to… (from géneral equátions of a straight line to those pàramétric/canónical)
10. Расстояние от точки до плоскости	Відстань від точки до площини	Dístance of a póint from [betwéen a póint and] a plane
11. Точка пересечения прямой и плоскости, прямой и поверхности	Точка перетину прямої і площини, прямої і поверхні	Ìnterséction/cross póint [póint of ìnterséction] of a straight líne and a plane, of a straight líne and a súrface
12. Угол между двумя плоскостями, между двумя прямыми	Кут між двома площинами, між двома прямими	Ángle (inclúded) betwéen two planes, betwéen two straight lines
13. Угол между прямой и плоскостью	Кут між прямою і площиною	Ángle (inclúded) betwéen a straight line and a pláne [ángle a straight line má-

		kes with a pláne]
14. Уравнение плоскости	Рівняння площини	Equátion of a pláne
15. Уравнение плоскости в отрезках на осях	Рівняння площини у відрізках на осях	Equátion of a pláne in ségments, threeìntercépt equátion of a pláne, (three)ìntercépt form of the equátion of a pláne
16. Уравнение плоскости, которая проходит через три данные точки	Рівняння площини, яка проходить через три дані точки	Equátion of a plane pássing thróugh three given póints; ; three-póint form of the equátion of a plane
17. Уравнение плоскости, проходящей через данную точку перпендикулярно данному вектору [нормальному вектору]	Рівняння площини, яка проходить через дану точку перпендикулярно даному вектору [нормальному вектору]	Equátion of a plane pássing thróugh a given póint pèrpendícularly [in pèrpendícular] to a given véc-tor [to a nórmal véctor]
18. Уравнения прямой	Рівняння прямої	Equátions of a straight line
19. Уравнения прямой, проходящей через данную точку параллельно данному вектору [направляющему вектору]	Рівняння прямої, яка проходить через дану точку паралельно даному вектору [напрямному вектору]	Equátions of a straight line passing thróugh a given póint in párallel to a given véctor [diréction véctor]
20. Уравнения прямой, проходящей через две данные точки	Рівняння прямої, яка проходить через дві дані точки	Equátions of a straight line pássing thróugh two given póints; two póint form of the equátions of a straight line
21. Условие параллельности двух плоскостей, двух прямых, прямой и плоскости	Умова паралельності двох площин, двох прямих, прямої і площини	Párallelism condition of two planes, of two straight lines, of a straight líne and a súrface
22. Условие перпендикулярности двух плоскостей, двух прямых, прямой и плоскости	Умова перпендикулярності двох площин, двох прямих, прямої і площини	Pérpendìculárity con-dítion of two planes, of two straight lines, of a straight líne and a plane

ANALYTIC GEOMETRY IN THE SPACE: basic terminology ERU

1. Ángle (inclúded) betwéen a straight line and a pláne [ángle a straight line mákes with a pláne]	Угол между прямой и плоскостью	Кут між прямою і площиною
2. Ángle (inclúded) betwéen two planes, betwéen two straight lines	Угол между двумя плоскостями, между двумя прямыми	Кут між двома площинами, між двома прямими
3. Canónical equátions of a straight líne	Канонические уравнения прямой	Канонічні рівняння прямої
4. Diréction véctor of the straight line	Направляющий вектор прямой	Напрямний вектор прямої
5. Dístance of a póint from [betwéen a póint and] a plane	Расстояние от точки до плоскости	Відстань від точки до площини
6. Equátion of a pláne	Уравнение плоскости	Рівняння площини
7. Equátion of a pláne in ségments, threeìntercépt equátion of a pláne, (three)ìntercépt form of the equátion of a pláne	Уравнение плоскости в отрезках на осях	Рівняння площини у відрізках на осях
8. Equátion of a plane pássing thróugh a given póint in pèrpendícular to a given véctor [to a nórmal véctor]	Уравнение плоскости, проходящей через данную точку перпендикулярно данному вектору [нормальному вектору]	Рівняння площини, яка проходить через дану точку перпендикулярно даному вектору [нормальному вектору]
9. Equátion of a plane pássing thróugh three given póints; three-póint form of the equátion of a plane	Уравнение плоскости, проходящей через три данные точки	Рівняння площини, яка проходить через три дані точки
10.Equátions of a straight line	Уравнения прямой	Рівняння прямої
11.Equátions of a straight line passing thróugh a given póint in párallel to a given véctor [diréction véctor]	Уравнения прямой, проходящей через данную точку параллельно данному вектору [направляющему вектору]	Рівняння прямої, яка проходить через дану точку паралельно даному вектору [напрямному вектору]
12.Equátions of a straight line pássing thróugh two	Уравнения прямой, проходящей через две	Рівняння прямої, яка проходить через дві дані

given póints; two póint form of the equátions of a straight line	данные точки	точки
13.Géneral equátion of a pláne	Общее уравнение плоскости	Загальне рівняння площини
14.Géneral equátions of a straight line [straight line géneral equátions]	Общие уравнения прямой	Загальні рівняння прямої
15.Ìnterséction/cross póint [póint of ìnterséction] of a straight líne and a plane, of a straight líne and a súrface	Точка пересечения прямой и плоскости, прямой и поверхности	Точка перетину прямої і площини, прямої і поверхні
16.Invèstigátion of the géneral equátion of a pláne	Исследование общего уравнения плоскости	Дослідження загального рівняння площини
17.Nécessary and suffícient condítion for párallelism [pérpendìculárity] of two planes, of two straight lines, of a straight líne and a súrface	Необходимое и достаточное условие параллельности [перпендикулярности] двух плоскостей, двух прямых, прямой и плоскости	Необхідна і достатня умова паралельності [перпендикулярності] двох площин, двох прямих, прямої і площини
18.Nórmal véctor of/to the pláne	Нормальный вектор плоскости	Нормальний вектор (до) площини
19.Párallelism condition of two planes, of two straight lines, of a straight líne and a súrface	Условие параллельности двух плоскостей, двух прямых, прямой и плоскости	Умова паралельності двох площин, двох прямих, прямої і площини
20.Pàramétric equátions of a straight line	Параметрические уравнения прямой	Параметричні рівняння прямої
21.Pássage from … to… (from géneral equátions of a straight line to those pàramétric/canónical)	Переход от…к… (от общих уравнений прямой к параметрическим/каноническим)	Перехід від … до …(від загальних рівнянь прямої до параметричних/канонічних)
22.Pérpendìculárity condítion of two planes, of two straight lines, of a straight líne and a plane	Условие перпендикулярности двух плоскостей, двух прямых, прямой и плоскости	Умова перпендикулярності двох площин, двох прямих, прямої і площини

CONTENTS