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Harmonic balance method and global analysis of dynamical systems

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Belovodskiy V.N., Smirnov A.N. "Harmonic balance method and global analysis of dynamical systems". *One of the main problems in the global analysis of oscillatory systems is the finding of all its periodic motions for given correlations of the parameters and priori considerations give reason to believe that for systems of differential equations with polynomial nonlinearity the use of the harmonic balance method (HBM) for this purpose seems to be very attractive. Indeed, the HBM enables us to reduce the finding of stationary motions of such systems to the solving of systems of polynomial equations, the number of solutions of which, presumably, can be set using the theory of Newton polyhedra. And, then, with the help of the interval approaches or methods of continuation, which are currently being developed within the framework of tropical geometry, you can determine the whole set of solutions of polynomial equations and thus, the entire range of motions of the dynamical system.*

In this paper, this hypothesis is being tested for the differential equation with cubic nonlinearity and harmonic exciting force. We consider two versions of HBM, – trigonometric one and complex exponential form. On their basis for the differential equation with cubic nonlinearity the construction of polynomial equations is fulfilled and in accordance with the theorem of Bernstein, attempt to estimate the number of solutions of the obtained system has been undertaken. Then, with use of interval bisection method solutions of the system of polynomial equations in a given part of phase space are determined, comparative evaluation of the complexity of the considered versions of HBM is conducted, advantages and disadvantages of the described approach are marked.

Keywords: harmonic balance method, dynamical system, global analysis, Duffing equation, Newton polyhedron, interval bisection method.

Introduction

The following considerations are the impetus of this research.

On the one hand it is obvious, that the successful performance of global analysis of dynamical systems is based on the finding of all solutions of the corresponding differential systems of equations. And traditionally it is being done with use of multistart methods by trying a certain volume of initial conditions. But, unfortunately, even for great number of initial points such procedure is not exhaustive. On the other hand the HBM gives an opportunity to reduce the finding of solutions of differential equations to the solving of systems of polynomial ones, for which there is the Bernstein's theorem describing the number of solutions through the mixed volumes of Newton polytopes [1]. And based on this, the following scheme of the implementation of the global analysis of dynamical systems seems to be very attractive. At first, by preserving the sufficient number of terms in the Fourier decompositions, we reduce the original system of differential equations to a system of polynomial ones. Then, using the theory of Newton polyhedra we determine the number of solutions of the resulting system. And, further, applying, one or the other method of the global search, we are guaranteed to determine the entire range of solutions and, thus, theoretically evidential global analysis of dynamical systems is carried out.

It should be noted that at present, perhaps, the methods of continuation and various modifications of the multistart method are dominant in the global analysis of dynamical systems [2, 3], though the first of them are still in the development stage, and the latter is not exhaustive. Along with this under computational mathematics during the last decades there is being developed an interval analysis, which is increasingly beginning to use to the global analysis of equations and systems. And some authors believe that namely interval approaches are able to provide reliable analysis of nonlinear systems [4].

Below these hypotheses are being tested on the example of the Duffing equation in the area of the principal resonance.

The model under consideration

So, here we consider the Diffing equation

$$\ddot{x} + b\dot{x} + x + \gamma x^3 = P\cos\omega t \qquad (1)$$

and using HBM find its solutions in the area of the principal resonance. There are two main versions of HBM, – complex and trigonometric. In the first case, the solution is sought in the form

$$x(t) = \sum_{n=-N}^{N} c_n e^{in\eta t}, \quad (2)$$

where N is the number of harmonics taken into account, in the second, – in the form

$$x = \sum_{n=0}^{N} (A_n \cos n\omega t + B_n \sin n\omega t) \quad (3)$$

or $$x = \sum_{n=0}^{N} Am_n \cos(n\omega t - \varphi_j).$$

The connection between the coefficients of these expansions are described by the following relationships: $Am_n = \sqrt{A_n^2 + B_n^2} = 2\sqrt{c_n c_{-n}}$,

$\varphi_n = \arccos \dfrac{c_n + c_{-n}}{2\sqrt{c_n c_{-n}}}$ or $\varphi_n = -\arccos \dfrac{c_n + c_{-n}}{2\sqrt{c_n c_{-n}}}$,

if $(\Im c_{-n} = 0 \ \wedge \ \Re c_{-n} < 0) \ \vee \ \Im c_{-n} < 0$,

$\varphi_n \in [-\pi, \pi)$, and $c_n = \dfrac{A_n - iB_n}{2}$, $c_{-n} = \dfrac{A_n + iB_n}{2}$.

In each of these approaches solving of the differential equation (1), ultimately, reduces to the solving of a polynomial system of equations, however, in the first case, – to the system with complex coefficients, in the second case, – to the system with real ones. Consider these options in more detail.

Comparison and selection of the form of the HBM

Here, we restrict ourselves by the analysis of the harmonic solutions of the equation (1), i.e. in expansions (2), (3) and suppose that N = 1 and $A_0 = B_0 = C_0 = 0$. Substitute, now, each of these expansions into equation (1), perform algebraic transformations and after comparison of the respective harmonics we obtain a system of equations to determine the coefficients:

– in the case (2), –

$$\begin{cases} (1-\omega^2 + ib\omega)c_1 + 3\gamma c_1^2 c_{-1} = \frac{1}{2}P \\ 3\gamma c_1 c_{-1}^2 + (1-\omega^2 - ib\omega)c_{-1} = \frac{1}{2}P \end{cases} \quad (4)$$

– in the case (3), –

$$\begin{cases} (1-\omega^2)A_1 + b\omega B_1 + \frac{3}{4}\gamma A_1(A_1^2 + B_1^2) = P \\ -b\omega A_1 + (1-\omega^2)B_1 + \frac{3}{4}\gamma B_1(A_1^2 + B_1^2) = 0 \end{cases}. \quad (5)$$

Note that the solutions of the equation (1) correspond to a self-adjoint solutions of the system (4), i.e., those for which $c_{-1} = \overline{c_1}$. Given this fact, along with (4) we also consider the equation

$$(1-\omega^2 + ib\omega)c_1 + 3\gamma c_1^2 \overline{c_1} = \frac{1}{2}P, \quad (6)$$

which is a shortened version of this system which doesn't contain, unlike it, the extra solutions. To assess the comparative computational complexity of the considered forms HBM the construction of amplitude-frequency characteristics (AFC) for system (1) (Fig. 1) was carried out in Matlab, version 8.5 (R2015a), by solving the systems (4), (5) and (6) for the parameters of equation (1): $b = 0.1$, $\gamma = 0.5$, $P = 1$ and $\omega = [0.01, 3.0]$, with step $\Delta\omega = 0.01$. When solving these systems as the initial conditions for the next value of ω the set c_1, c_{-1} (or A_1, B_1) has been taken, respectively, which was obtained for its previous value. Computational experiments were conducted with a computer having processor Intel Pentium Dual Core 2.2 GHz and memory 4GB. The results are presented in Table 1 and they demonstrate that the least time consuming corresponds to a trigonometric version of HBM, despite a few more number of calls to the functions describing the system of equations, compared with the complex version of HBM.

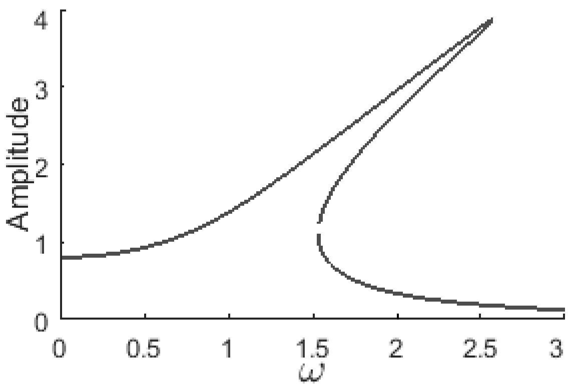

Figure 1. – AFC for the equation (1)

Estimation of the number of solutions of the system

Taking in mind the described experiments, further we restrict ourselves by the trigonometric version of HBM and consider the system (5).

Note that the estimation of the total number of solutions of the nonlinear system in a given area or in a given interval, in itself, is quite interesting and important. For algebraic equations such instruments are known. They are, in particular, and the Fundamental Theorem of Algebra, and Shturm's Theorem, and Budan's and Fourier's procedures, etc., which allow you to set the number of real and complex roots and implement their separation [5]. For systems of polynomial equations the range of such possibilities is more narrow, although there are also some tools for estimating the number of solutions and they are based on the theory of Newton polytopes [1]. Thus, in particular, according to Bernstein's theorem, the number of non-trivial solutions of the system with generic coefficients is equal to the mixed volume of such polyhedra constructed for polynomial equations.

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Table 1. – The results of the run-time

	Trigonometric version of HBM, system (5)	Complex version of HBM, system (4)	Shortened version of HBM, system (6)
Total work time, (s)	5.877	6.338	13.997
Average time of one solving of system of equations, (s)	0.0107	0.0110	0.2495
Average time of one calling of system of equations, (s)	$0.4738{\cdot}10^{-4}$	$0.7276{\cdot}10^{-4}$	$0.5586{\cdot}10^{-4}$
Total number of calls to the function describing system of equations	6775	6473	22379

We return to system (5). The computation of the mixed volume of Newton polytopes, carried out with use of the program MixedVol [7], gave 9 solutions. Although from the theory of vibrating systems it is known that the Duffing system in the area of the primary resonance has no more than three regimes, – two stable and one unstable. Taking in mind that system (5) has no trivial solution perform the following transformation. Namely, to the first equation multiplied by ($-B_1$) we add the second one multiplied by A_1 and as the result we obtain

$$\begin{cases} A_1^2 + B_1^2 = PB_1 / b\omega \\ -b\omega A_1 + (1-\omega^2)B_1 + \dfrac{3}{4}\gamma B_1 (A_1^2 + B_1^2) = 0 \end{cases}. \quad (7)$$

The mixed volume of the Newton polyhedra for the system (7) is already six. Finally, if we substitute the ratio from the first equation into the second of system (7), we get

$$\begin{cases} A_1^2 + B_1^2 = PB_1 / b\omega \\ -b\omega A_1 + (1-\omega^2)B_1 + \dfrac{3}{4}\gamma PB_1^2 / b\omega = 0 \end{cases}. \quad (8)$$

And for this system the mixed volume of Newton polyhedra is already four.

Thus, for equivalent, essentially, systems (5), (7) and (8) the number of solutions determined with help of the Bernstein theorem turns out different. What's the deal?

To clarify the situation let us turn to the simple examples.

Example 1. Consider

$$\begin{cases} x^3 + xy^2 - 2 = 0 \\ x^2 y + y^3 - 2 = 0 \end{cases}.$$

Program MixedVol gives for this system nine solutions. However, subtracting the second equation from the first one and considering, further, two cases

$$\begin{cases} x - y = 0 \\ x^2 y + y^3 - 2 = 0 \end{cases} \text{ and } \begin{cases} x^2 + y^2 = 0 \\ y(x^2 + y^2) - 2 = 0 \end{cases},$$

i.e., essentially, performing identical transformations, we'll come to a conclusion that the original system has only three solutions.

Example 2. Consider

$$\begin{cases} x^2 - y^2 + 1 = 0 \\ x^2 + y^2 + 1 = 0 \end{cases}.$$

Adding the second equation to the first one we have

$$\begin{cases} x^2 + 1 = 0 \\ x^2 + y^2 + 1 = 0 \end{cases} \Rightarrow \begin{cases} x^2 + 1 = 0 \\ y^2 = 0 \end{cases} \Rightarrow$$

$$\begin{cases} x_{1,2} = \pm i \\ y_{1,2} = 0 \end{cases} \Rightarrow (\pm i, 0) \ u \ (\pm i, 0),$$

i.e. two pairs of multiple solutions, the mixed volume is equal to 4.

At last,

Example 3. The system

$$\begin{cases} x^2 + y^2 - 1 = 0 \\ x^2 + y^2 - 1 = 0 \end{cases}$$

has an infinite number of solutions, while the mixed volume of Newton polytopes equals 4.

Thus, summing up the performed analysis, the following can be noted. The theory of Newton polytopes is applicable to nondegenerate systems, does not account the relations between the equations of the system and., because of this, it gives only an upper estimate for the number of solutions. For these reasons, the results obtained with its use cannot serve as a control in the formation of a plurality of the solutions of the given system.

Interval solving of system

It seems that the interval method of bisection can become to be a method that allows to find all solutions of the system. This method essentially generalizes the dichotomy method for the case of a system of equations and, conceptually, its approach is as follows. Consider a system of two equations

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \quad (9)$$

and suppose you want to find all solutions of this system in a rectangular area $D = X \times Y$, where $X = [a,\ b]$, $Y = [c,\ d]$, $a \le b, c \le d$. At the first step the region *D*, along one of the measurements, is divided equally into two parts and in each part the smallest and the largest values of the functions *f* and g are determined. If in some of it the found values have equal signs, i.e. $\min f \cdot \max f > 0$ or $\min g \cdot \max g > 0$, then in this part, this function takes values of the same sign and doesn't become zero. Consequently, in this part of the rectangle D system (9) has no solution, then this part is discarded and the remaining part is divided in half again. The procedure of successive division continues until each of the sides of the remaining parties becomes less certain δ, where $\delta / 2$ is the given accuracy of the approximate solution. Then, the midpoints of the remaining sections are accepted as solutions of the system of the equations.

Another variant of this approach can be implemented using interval computations. Namely, instead of the original functions $f(x,y)$, $g(x,y)$ there are considered interval functions $\mathbf{f}(X,Y)$, $\mathbf{g}(X,Y)$ which can be obtained by replacing of the original variables in the given function by the intervals X, Y and the coefficients in it are presented by "point" intervals. The functions thus defined are called the natural interval extensions of the original functions. Operations over intervals in it are carried out according to the rules of interval arithmetic [6] and as a result an interval estimation of the values of the given functions is obtained. According to the fundamental theorem of interval arithmetic the range of values of the initial function is contained in its interval estimation that is $ran(f(x,y)) \subseteq \mathbf{f}(X,Y)$. Further actions are similar to those described above. Namely, the interval estimates $\mathbf{f}(X,Y)$, $\mathbf{g}(X,Y)$ are determined. If each of them contains zero, i.e. $0 \in \mathbf{f}(X,Y)$ and $0 \in \mathbf{g}(X,Y)$, then the system (9) can have a solution in the domain $X \times Y$ and by dividing one of the intervals in half the domain is divided into two parts. In each of the obtained parts the new interval estimates are determined, the part, in which, at least, one of the interval estimates doesn't contain zero, is discarded and the procedure continues as long as the size of the remaining part or parts containing zero becomes less than the preassigned small δ. If desired, the obtained solutions can be clarified further by one of the usual methods for solving of nonlinear equations.

It should be noted that this procedure becomes less laborious and more certain, if $ran(f(x,y)) = \mathbf{f}(X,Y)$ exactly. And for some classes of functions [4, 6], in particular, for polynomials, presented in the form

$$p(x, a^{(0)}, ..., a^{(m)}) =$$
$$(...((a^{(m)}x + a^{(m-1)})^{n_{m-1}} + a^{(m-2)}) + ... + a^{(1)})^{n_1} + a^{(0)},$$

in which exponentiation is computed according to the rule $X^k = [\min_{x \in X} x^k, \max_{x \in X} x^k]$ this requirement is fulfilled. In order to bring equations of the system (8) to this kind it is sufficient to select there the perfect squares and present them in the form

$$\begin{cases} a_1 A_1 + c_1 (B_1 + \frac{b_1}{2c_1})^2 - \frac{b_1^2}{4c_1} = 0 \\ A_1^2 + (B_1 + \frac{d_1}{2})^2 - \frac{d_1^2}{4} = 0 \end{cases}, \quad (10)$$

where $a_1 = -b\omega$, $b_1 = 1 - \omega^2$, $c_1 = \frac{3\gamma P}{4b\omega}$, $d_1 = -\frac{P}{b\omega}$.

The described methodology has been implemented by us in Matlab and the subsequent computational experiments were performed for the parameters of the equation $b = 0.1$, $\gamma = 0.5$, $P = 1$ and $\omega = 2$. Figure 1 shows that the system at this point has three solutions

Figure 2 illustrates the procedure of successive division. In fairness, we note that for the purpose of clarifying the existence of a root in the area under the condition

$$\begin{cases} 0 \in \mathbf{f}(X_n, Y_m) \\ 0 \in \mathbf{g}(X_n, Y_m) \end{cases}$$

an additional test procedure was added to the program, which contains criteria for the check whether the point of intersection of curves $f(x,y) = 0$, $g(x,y) = 0$ lies in this plot. The matter is that the interval estimates contain zero in those cases also when the curves corresponding to the equations are contained in the parallelepiped, but do not intersect. This feature leads to the preservation of false parallelepipeds of the next generations and, perhaps, even to their accumulation. In order to avoid such situations an additional criterion was included into the algorithm, which performs the verification of the intersection of the curves in the considered part of the parallelepiped by replacing sections of the curves by the segments [8].

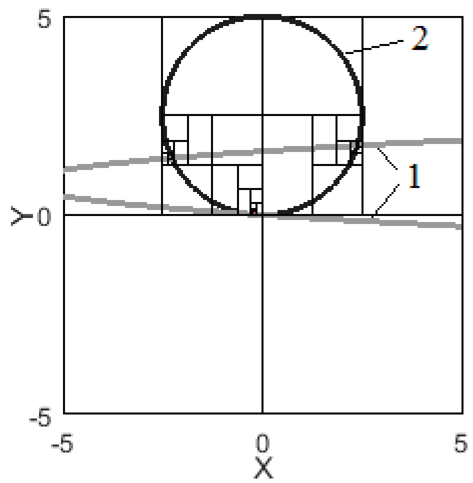

Figure 2. – Illustration of use of bisection method to the system (10): 1 – curve of the first equation of (10); 2 – curve of the second equation of (10)

In addition, given that the system (5) has no trivial solutions, in order to exclude the search of the zero solution when implementing interval method, the zero point has been removed from the originally specified area of search (Figure 3) and the solving of the system (9) was carried out, which had been adapted to perform interval computations. The solving of the "trigonometric" system (5) was carried out in Matlab with the help of the function fsolve. The final results are as follows.

Interval bisection method. For the value of $\delta = 0.001$ the elapsed time was $T = 2.206s$ and there was found the next set of interval solutions:

– $A_1 = [-0.3367; -0.3361]$, $B_1 = [0.0226; 0.0233]$;
– $A_2 = [-2.2604; -2.2597]$, $B_2 = [1.4315; 1.4321]$;
– $A_3 = [2.3831; 2.3837]$, $B_3 = [1.7453; 1.7459]$.

When $\delta = 0.01$ then time to find solutions amounted to $2.132s$, when $\delta = 0.1$, – $1.822s$.

Procedure fsolve: Volume of initial conditions in the domain $D = \{(A_1, B_1) : -5 \le A_1, B_1 \le 5\}$ (Figure 2) was fixed and amounted to 25 sample points, elapsed time $T = 0.874s$, the found solutions were: the first one, – $A_1 = -0.3366$ and $B_1 = 0.0228$, the second one, – $A_2 = -2.2602$ and $B_2 = 1.4316$, the third one, – $A_3 = 2.3835$ and $B_3 = 1.7456$.

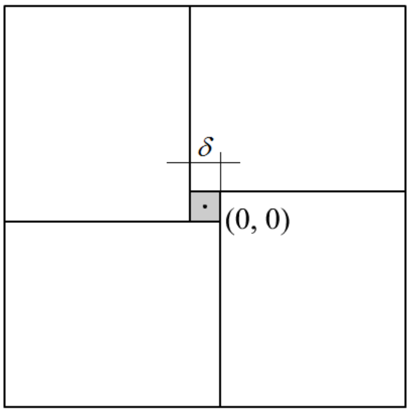

Figure 3. – Exclusion of the zero-point

Thus, interval methodology of calculations for the considered example turned out to be quite workable, although noticeably slower, compared with the multistart method. However, its undoubted advantages include a comprehensive conclusion on the number of roots, whereas when iterating the initial conditions there are no such certainty.

Conclusion

In the paper there was illustrated the possibility of using the interval approach to the analysis of simple dynamical systems on the example of Duffing equation in the zone of the principal resonance. The results are quite encouraging and indicate the desirability of further research in this direction and subsequent transition to the analysis of combinational resonances of the dynamical systems with one and several degrees of freedom. And according to their results, we hope, the role and place of the interval approach to the problems of global analysis can be established more specifically.

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Беловодский В.Н., Смирнов А.Н. «Метод гармонического баланса и глобальный анализ динамических систем». *Одной из основных проблем при глобальном анализе колебательных систем является задача нахождения всех ее периодических режимов при данных соотношениях параметров и априорные соображения дают основания предполагать, что для систем дифференциальных уравнений с полиномиальной нелинейностью использование метода гармонического баланса (МГБ) для этих целей представляется весьма привлекательным. Действительно, МГБ позволяет свести поиск стационарных движений таких систем к решению систем полиномиальных уравнений, число решений которых, предположительно, может быть установлено с использованием теории многогранников Ньютона. А, далее, с помощью интервальных подходов или методов гомотопии, которые в настоящее время развиваются в рамках тропической геометрии, определить все множество решений полиномиальных уравнений и, тем самым, весь спектр движений динамической системы.*

В данной работе эта гипотеза апробируется на дифференциальном уравнении с кубической нелинейностью и гармонической вынуждающей силой. Рассматриваются две версии метода гармонического баланса, – тригонометрическая и комплексно-показательная. На их основе для дифференциального уравнения с кубической нелинейностью проводится построение полиномиальных уравнений и, в соответствии с теоремой Бернштейна, предпринимается попытка оценить число решений полученных систем. Затем, с использованием интервального метода бисекции определяются решения системы полиномиальных уравнений в заданной части фазового пространства, проводятся оценки сравнительной трудоемкости рассматриваемых версий гармонического баланса, отмечаются достоинства и недостатки описываемого подхода.

Ключевые слова: метод гармонического баланса, динамическая система, глобальный анализ, уравнение Дуффинга, многогранник Ньютона, интервальный метод бисекции.

Бєловодський В.М., Смирнов О.М. «Метод гармонійного балансу та глобальний аналіз динамічних систем». *Однією із основних проблем при глобальному аналізі коливальних систем є задача знаходження всіх їхніх періодичних режимів при даних співвідношеннях параметрів і апріорні міркування дають підстави припускати, що для систем диференційних рівнянь з поліноміальною нелінійністю використання методу гармонійного балансу (МГБ) для цих цілей представляється вельми привабливим. Дійсно, МГБ дозволяє звести пошук стаціонарних рухів таких систем до рішення систем поліноміальних рівнянь, число рішень яких, імовірно, може бути встановлено з використанням теорії багатогранників Ньютона. А далі, за допомогою інтервальних підходів або методів гомотопії, які нині розвиваються в рамках тропічної геометрії, визначити всю множину рішень поліноміальних рівнянь, і тим самим, весь спектр рухів динамічної системи.*

В цій роботі ця гіпотеза апробується на диференційному рівнянні із кубічною нелінійністю та гармонічною змушуючою силою. Розглядаються дві версії методу гармонійного балансу, – тригонометрична та комплексно-показова. На їхньої основі для диференційного рівняння з кубічною нелінійністю проводиться побудова поліноміальних рівнянь і, відповідно до теореми Бернштейна, робиться спроба оцінити число рішень отриманих систем. Далі, з використанням інтервального методу бісекцій знаходяться рішення системи поліноміальних рівнянь в заданій частині фазового простору, наводяться оцінки порівняльної трудомісткості розглянутих версій гармонійного балансу, відзначаються переваги та недоліки підходу, який описується.

Ключові слова: метод гармонійного балансу, динамічна система, глобальний аналіз, рівняння Дуффінга, багатогранник Ньютона, інтервальний метод бісекції.

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