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ON Γ-CONVERGENCE OF INTEGRAL FUNCTIONALS DEFINED ON VARIOUS WEIGHTED SOBOLEV SPACES

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We consider weighted Sobolev spaces correlated with a sequence of n-dimensional domains. We prove a theorem on the choice of a subsequence Γ-convergent to an integral functional defined on a "limit" weighted Sobolev space from a sequence of integral functionals defined on the spaces indicated.

1. Introduction

The Γ-convergence is a special convergence of functionals accompanied in many important cases by the convergence of solutions of the corresponding variational problems. For functionals with common domain of definition, the notion of Γ-convergence was introduced in [1], where, for the first time, the general properties of this convergence were described and its applications to variational problems were described. Problems of Γ-convergence of integral functionals with common domain of definition were studied in works of many Italian mathematicians (see, e.g., [2–4] and the bibliography in [3, 4]). These problems were also considered by Zhikov in [5–8]. The main results of these investigations are theorems on Γ-compactness for sequences of functionals of variational calculus and the integral representation of their Γ-limits.

For functionals with different domains of definition, including integral functionals, the notion of Γ-convergence was studied, e.g., in [9–16]. In these works, the functionals were defined on nonweighted Sobolev spaces.

In the present paper, we consider weighted Sobolev spaces correlated with a sequence of n-dimensional domains and integral functionals defined on these spaces. The condition that characterizes the behavior of the Lagrangians of these functionals [see condition (8) below] contains the weight function ν and a certain, generally speaking, unbounded sequence of functions ψ_s. The main result of the present paper (Theorem 2) gives sufficient conditions for the weight ν and a function that, in a certain sense, majorizes the sequence $\{\psi_s\}$ for which there exists a subsequence of the considered sequence of integral functionals that Γ-converges to an integral functional defined on a certain "limit" weighted Sobolev space. In the proof of this theorem, we use some ideas of [8, 14, 17, 18]. Note that one of the essential elements of the proof (as, e.g., in [14]) is the use of special local characteristics of the functionals under investigation. In the nonweighted case, analogous characteristics and related conditions for the convergence of the points of minimum of the corresponding integral functionals defined on different Sobolev spaces were studied by Khruslov [18, 19] and other authors (see, e.g., [12–14, 20, 21]).

Prior to the formulation of the result concerning Γ-compactness, we consider a general theorem on conditions for the convergence of solutions of variational problems for functionals defined on different weighted Sobolev spaces. In addition to the Γ-convergence of functionals, one of these conditions is the strong correlation of the considered spaces. In general, the notion of the strong correlation of a sequence of Sobolev spaces (or, in a different terminology, the corresponding n-dimensional domains) plays an important role in problems of averaging of boundary-value and variational problems in domains of complex structure (see [18], where this notion was introduced, and [8–13, 20, 22, 23]). The strong correlation of the spaces used in the investigation of the convergence

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of solutions of boundary-value and variational problems in "variable" (e.g., strongly perforated) domains enables one to pass from a sequence of solutions each of which is contained in its "own" space to a bounded sequence in a certain common space. This is the first step to the selection of a certain limit element of the original sequence and the subsequent proof of the statement that this element is a solution of the corresponding averaged problem. Furthermore, the strong correlation of Sobolev spaces, along with other properties of a sequence of n-dimensional domains correlated with these spaces, leads to the coercivity of Γ-limit functionals or the coercivity and monotonicity of G-limit operators for the corresponding mappings defined on these spaces (for details, see, e.g., [10, 22]). The notion of the strong correlation of weighted Sobolev spaces used in the present paper was fairly thoroughly studied in [24].

It should be noted that the available results of other authors concerning the Γ-convergence of integral functionals defined on weighted Sobolev spaces and, on the whole, the averaging of variational and boundary-value problems with degeneration deal either with functionals and operators with common domain of definition (see, e.g., [17, 25, 26]) or with operators of Dirichlet problems in perforated domains [27–29]. In the latter case, e.g., the notion of the strong correlation of the sequence of corresponding weighted Sobolev spaces is not required because this correlation is realized "automatically." On the contrary, the "variable" weighted spaces considered in the present paper are aimed at variational problems of the "Neumann" type. For the investigation of the convergence of solutions of these problems, the requirement of the strong correlation of these spaces is essential.

The present paper is organized as follows: In Sec. 2, we consider the weighted Lebesgue and Sobolev spaces used in what follows. Necessary definitions and a general theorem on the convergence of solutions of variational problems for functionals defined on the considered ("variable") weighted Sobolev spaces are given in Sec. 3. The main result of the paper (a theorem on Γ-compactness for integral functions) is given in Sec. 4. Note that this result (without proof) was announced in [30].

2. Functional Spaces

Let $n \in \mathbb{N},\ n \geq 2,$ let Ω be a bounded domain in $\mathbb{R}^n$, and let $p \in (1, n)$. Assume that ν is a nonnegative function on $\Omega,\ \nu > 0$ almost everywhere in $\Omega,$ and

$$\nu \in L^1_{\mathrm{loc}}(\Omega), \qquad \left(\frac{1}{\nu}\right)^{1/(p-1)} \in L^1_{\mathrm{loc}}(\Omega). \tag{1}$$

By $L^p(\nu, \Omega)$ we denote the set of all measurable functions $u\colon \Omega \to \mathbb{R}$ such that the function $\nu|u|^p$ is summable on Ω. Let $L^p(\nu, \Omega)$ be a Banach space with the norm

$$\|u\|_{L^p(\nu,\Omega)} = \left(\int\limits_\Omega \nu|u|^p dx\right)^{1/p}.$$

By virtue of the Young inequality and the second inclusion in (1), we get $L^p(\nu, \Omega) \subset L^1_{\mathrm{loc}}(\Omega)$. Let $W^{1,p}(\nu, \Omega)$ denote the set of all functions $u \in L^p(\nu, \Omega)$ such that, for any $i \in \{1, \ldots, n\}$, there exists a generalized derivative $D_i u,\ D_i u \in L^p(\nu, \Omega)$. Let $W^{1,p}(\nu, \Omega)$ be a reflexive Banach space with the norm

$$\|u\|_{1,p,\nu} = \left(\int\limits_\Omega \nu|u|^p dx + \sum_{i=1}^n \int\limits_\Omega \nu|D_i u|^p dx\right)^{1/p}.$$

The completeness of the space $W^{1,p}(\nu,\Omega)$ is established by using the second inclusion in (1). The reflexivity of this space follows from its uniform convexity, which is proved by using the Clarkson inequalities (for these inequalities, see, e.g., [31]).

By virtue of the first inclusion in (1), we have $C_0^\infty(\Omega)\subset W^{1,p}(\nu,\Omega)$. Denote the closure of the set of functions $C_0^\infty(\Omega)$ in $W^{1,p}(\nu,\Omega)$ by $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Let $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ be a reflexive Banach space with the norm induced from the space $W^{1,p}(\nu,\Omega)$.

Further, let $\{\Omega_s\}$ be a sequence of domains in $\mathbb{R}^n$ contained in Ω.

By analogy with the spaces introduced above, we define functional spaces corresponding to the domains Ω_s.

Let $s\in\mathbb{N}$. Denote the space of all measurable functions $u\colon\Omega_s\to\mathbb{R}$ such that the function $\nu|u|^p$ is summable on Ω_s by $L^p(\nu,\Omega_s)$. Let $L^p(\nu,\Omega_s)$ be a Banach space with the norm

$$\|u\|_{L^p(\nu,\Omega_s)}=\left(\int\limits_{\Omega_s}\nu|u|^p dx\right)^{1/p}.$$

By virtue of the second inclusion in (1), we have $L^p(\nu,\Omega_s)\subset L^1_{\mathrm{loc}}(\Omega_s)$. By $W^{1,p}(\nu,\Omega_s)$ we denote the set of all functions $u\in L^p(\nu,\Omega_s)$ such that, for any $i\in\{1,\ldots,n\}$, there exists a generalized derivative D_iu, $D_iu\in L^p(\nu,\Omega_s)$. Let $W^{1,p}(\nu,\Omega_s)$ be a Banach space with the norm

$$\|u\|_{1,p,\nu,s}=\left(\int\limits_{\Omega_s}\nu|u|^p dx+\sum_{i=1}^n\int\limits_{\Omega_s}\nu|D_iu|^p dx\right)^{1/p}.$$

Denote the set of all restrictions of functions from $C_0^\infty(\Omega)$ to Ω_s by $\widetilde{C}_0^\infty(\Omega_s)$. By virtue of the first inclusion in (1), we have $\widetilde{C}_0^\infty(\Omega_s)\subset W^{1,p}(\nu,\Omega_s)$. Let $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ denote the closure of the set $\widetilde{C}_0^\infty(\Omega_s)$ in $W^{1,p}(\nu,\Omega_s)$.

Note that if $u\in\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ and $s\in\mathbb{N}$, then $u|_{\Omega_s}\in\widetilde{W}_0^{1,p}(\nu,\Omega_s)$.

3. Main Definitions and General Theorem on Convergence of Solutions of Variational Problems

We introduce the following notation: If $s\in\mathbb{N}$, then q_s is a mapping of $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ into $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that $q_su=u|_{\Omega_s}$ for any function $u\in\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$.

Definition 1. *We say that a sequence of spaces $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ is strongly correlated with the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ if there exists a sequence of linear continuous operators $l_s\colon\widetilde{W}_0^{1,p}(\nu,\Omega_s)\to\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ such that*

$$\sup_{s\in\mathbb{N}}\|l_s\|<+\infty$$

and $q_s(l_su)=u$ almost everywhere on Ω_s for any $s\in\mathbb{N}$ and $u\in\widetilde{W}_0^{1,p}(\nu,\Omega_s)$.

Proposition 1. *Suppose that the imbedding of $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ into $L^p(\nu,\Omega)$ is compact and the sequence of spaces $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ is strongly correlated with the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Also assume that $u_s\in\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ for* *any* $s \in \mathbb{N}$, *and, furthermore, the sequence of norms* $\|u_s\|_{1,p,\nu,s}$ *is bounded. Then there exist an increasing sequence* $\{s_j\} \subset \mathbb{N}$ *and a function* $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ *such that*

$$\lim_{j\to\infty} \|u_{s_j} - q_{s_j} u\|_{L^p(\nu,\Omega_{s_j})} = 0.$$

The proof of this proposition was given in [24].

Definition 2. *Assume that, for any* $s \in \mathbb{N}$, I_s *is a functional on* $\widetilde{W}_0^{1,p}(\nu, \Omega_s)$, *and* I *is a functional on* $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$. *We say that the sequence* $\{I_s\}$ Γ*-converges to the functional* I *if the following conditions are satisfied:*

(i) for any function $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$, *there exists a sequence* $w_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$ *such that*

$$\lim_{s\to\infty} \|w_s - q_s u\|_{L^p(\nu,\Omega_s)} = 0 \qquad \text{and} \qquad \lim_{s\to\infty} I_s(w_s) = I(u);$$

(ii) for any function $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ *and any sequence* $u_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$ *such that*

$$\lim_{s\to\infty} \|u_s - q_s u\|_{L^p(\nu,\Omega_s)} = 0,$$

the following relation is true:

$$\liminf_{s\to\infty} I_s(u_s) \geq I(u).$$

Theorem 1. *Suppose that the imbedding of* $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ *into* $L^p(\nu, \Omega)$ *is compact and the sequence of spaces* $\widetilde{W}_0^{1,p}(\nu, \Omega_s)$ *is strongly correlated with the space* $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$. *Assume that, for any* $s \in \mathbb{N}$, I_s *is a functional on* $\widetilde{W}_0^{1,p}(\nu, \Omega_s)$, I *is a functional on* $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$, *and the sequence* $\{I_s\}$ Γ*-converges to the functional* I. *Also assume that the function* u_s *minimizes the functional* I_s *on* $\widetilde{W}_0^{1,p}(\nu, \Omega_s)$ *for any* $s \in \mathbb{N}$, *and, furthermore, the sequence of norms* $\|u_s\|_{1,p,\nu,s}$ *is bounded. Then there exist an increasing sequence* $\{s_j\} \subset \mathbb{N}$ *and a function* $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ *such that the function* u *minimizes the functional* I *on* $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$,

$$\lim_{j\to\infty} \|u_{s_j} - q_{s_j} u\|_{L^p(\nu,\Omega_{s_j})} = 0,$$

and

$$\lim_{j\to\infty} I_{s_j}(u_{s_j}) = I(u).$$

Proof. By virtue of Proposition 1, there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a function $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ such that

$$\lim_{j\to\infty} \|u_{s_j} - q_{s_j} u\|_{L^p(\nu,\Omega_{s_j})} = 0.$$

Then, by virtue of the Γ-convergence of the sequence $\{I_s\}$ to the functional I, we have

$$\liminf_{j\to\infty} I_{s_j}(u_{s_j}) \geq I(u). \tag{2}$$

Now let $w \in \overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Since the sequence $\{I_s\}$ Γ-converges to the functional I, there exists a sequence $w_s \in \widetilde{W}_0^{1,p}(\nu,\Omega_s)$ such that

$$\lim_{s\to\infty} I_s(w_s) = I(w).$$

Using this result and the fact that, for any $s \in \mathbb{N}$, the function u_s minimizes the functional I_s on $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$, we get

$$\limsup_{j\to\infty} I_{s_j}(u_{s_j}) \leq I(w). \tag{3}$$

It follows from (2) and (3) that the function u minimizes the functional I on $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. Furthermore, setting $w = u$ in (3), we deduce the following relation from (3) and (2):

$$\lim_{j\to\infty} I_{s_j}(u_{s_j}) = I(u).$$

The theorem is proved.

Note that, in the nonweighted case, results analogous to Theorem 1 were obtained in [10, 11, 14].

We now make several remarks on the conditions of Theorem 1. For the compactness of the imbedding of the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ into the space $L^p(\nu,\Omega)$, the following statements are true:

Proposition 2. *Suppose that* $t \geq 1/(p-1)$, $t > n/p$, $t_1 > nt/(tp-n)$, *and* $1/\nu \in L^t(\Omega)$, $\nu \in L^{t_1}(\Omega)$. *Then the imbedding of* $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ *into* $L^p(\nu,\Omega)$ *is compact.*

Proposition 3. *Suppose that the function* ν *is the restriction of a certain function of the Muckenhoupt class* A_p *to* Ω. *Then the imbedding of* $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ *into* $L^p(\nu,\Omega)$ *is compact.*

The detailed proof of these propositions was given in [24]. Note that, for weight functions satisfying conditions similar to those in Proposition 2, imbeddings of weighted Sobolev spaces into nonweighted and weighted Lebesgue spaces were considered, e.g., in [17, 32–35]. For the definition of the Muckenhoupt class A_p, see [36]. Representatives of this class are, e.g., functions of the form $w(x) = |x|^\gamma$, $x \in \mathbb{R}^n \setminus \{0\}$, where $\gamma \in (-n, n(p-1))$.

The strong correlation of the sequence of spaces $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ with the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$ takes place, e.g., in the case of a special perforated structure of the domains Ω_s and a certain behavior of the function ν in the neighborhoods of "holes" (for details, see [24]). One of the main conditions of Theorem 1 is the Γ-convergence of functionals. From the viewpoint of applications, of major interest is the investigation of the Γ-convergence of integral functionals. The Γ-convergence of these functionals can be proved and an efficient representation for the Lagrangian of the corresponding Γ-limit can be obtained, e.g., in the case of periodicity of the Lagrangians of the original functionals with respect to the space variable or in the case of periodicity of the structure of the domains Ω_s (see, e.g., [6, 8, 12] for integral functionals defined on nonweighted Sobolev spaces). In the general case, of special interest are theorems on Γ-compactness. Finally, the condition of Theorem 1 concerning the boundedness of the sequence of norms of minimizers of the functionals I_s is satisfied if, e.g., the sequence $\{I_s(0)\}$ is bounded and, for any $s \in \mathbb{N}$ and $u \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$, one has $I_s(u) \geq \Phi(\|u\|_{1,p,\nu,s})$, where $\Phi\colon [0, +\infty) \to \mathbb{R}$ and $\Phi(r) \to +\infty$ as $r \to +\infty$. For integral functionals, these requirements are satisfied if their Lagrangians satisfy the corresponding growth and coercivity conditions.

4. Theorem on Γ-Compactness for Integral Functionals

Let $b \in L^1(\Omega)$, let $b \geq 0$ in Ω, and let $\{\psi_s\}$ be a sequence of functions that satisfy the following conditions:

(i) $\psi_s \in L^1(\Omega_s)$ and $\psi_s \geq 0$ in Ω_s for any $s \in \mathbb{N}$;

(ii) for any open cube $Q \subset \mathbb{R}^n$, one has

$$\limsup_{s \to \infty} \int\limits_{Q \cap \Omega_s} \psi_s dx \leq \int\limits_{Q \cap \Omega} b dx.$$

Let $c_1, c_2 > 0$ and let $f_s\colon \Omega_s \times \mathbb{R}^n \to \mathbb{R}$, $s \in \mathbb{N}$, be a sequence of functions satisfying the following conditions:

(iii) for any $s \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$, the function $f_s(\cdot, \xi)$ is measurable on Ω_s;

(iv) for any $s \in \mathbb{N}$ and almost all $x \in \Omega_s$, the function $f_s(x, \cdot)$ is convex on $\mathbb{R}^n$;

(v) for any $s \in \mathbb{N}$, almost all $x \in \Omega_s$, and any $\xi \in \mathbb{R}^n$, one has

$$c_1\nu(x)|\xi|^p - \psi_s(x) \leq f_s(x, \xi) \leq c_2\nu(x)|\xi|^p + \psi_s(x). \tag{4}$$

By virtue of conditions (iv) and (v), the function $f_s(x, \cdot)$ is continuous on $\mathbb{R}^n$ for any $s \in \mathbb{N}$ and almost all $x \in \Omega_s$. This and condition (iii) imply that the function f_s satisfies the Carathéodory conditions for any $s \in \mathbb{N}$. Then, by virtue of condition (v), the function $f_s(x, \nabla u)$ is summable on Ω_s for any $s \in \mathbb{N}$ and $u \in W^{1,p}(\nu, \Omega_s)$.

We introduce the following notation: If $s \in \mathbb{N}$, then J_s is a functional on $\widetilde{W}_0^{1,p}(\nu, \Omega_s)$ such that, for any function $u \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$, one has

$$J_s(u) = \int\limits_{\Omega_s} f_s(x, \nabla u) dx. \tag{5}$$

Let $\mathcal{F}$ denote the set of all functions $f\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following conditions: The function $f(\cdot, \xi)$ is measurable on Ω for any $\xi \in \mathbb{R}^n$, the function $f(x, \cdot)$ is convex on $\mathbb{R}^n$ for almost all $x \in \Omega$, and $-b(x) \leq f(x, \xi) \leq c_2\nu(x)|\xi|^p + b(x)$ for almost all $x \in \Omega$ and any $\xi \in \mathbb{R}^n$.

It is easy to see that the function $f(x, \nabla u)$ is measurable on Ω for any $f \in \mathcal{F}$ and $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$.

Finally, we introduce the following definition: If $f \in \mathcal{F}$, then J^f is a functional on $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$ such that, for any function $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$, one has

$$J^f(u) = \int\limits_{\Omega} f(x, \nabla u) dx. \tag{6}$$

Theorem 2. *Suppose that there exists a sequence of nonempty open sets* $\Omega^{(k)}$ *in* $\mathbb{R}^n$ *such that the following conditions are satisfied:*

(a) $\overline{\Omega^{(k)}} \subset \Omega^{(k+1)} \subset \Omega$ *for any* $k \in \mathbb{N}$;

(b) $\lim\limits_{k\to\infty} \operatorname{meas}\left(\Omega \setminus \Omega^{(k)}\right) = 0$;

(c) *the functions* ν *and* b *are bounded on* $\Omega^{(k)}$ *for any* $k \in \mathbb{N}$.

Then there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ *and a function* $f \in \mathcal{F}$ *such that the sequence* $\{J_{s_j}\}$ Γ*-converges to the functional* J^f.

Proof. We prove the theorem in several steps. First of all, we briefly describe them. In Step 1, we introduce some local characteristics of the functionals J_s and establish their properties necessary for what follows. The next four steps of the proof contain constructions necessary for the transition from the indicated local characteristics to limit functions used for the definition of a certain function $f \in \mathcal{F}$. In Steps 6 and 7, we establish a very important limit relation for the function f. In the next four steps, we prove the Γ-convergence of a certain subsequence of the sequence $\{J_s\}$ to the functional J^f. Using the results obtained in the previous steps, we establish the corresponding properties from the definition of Γ-convergence first for functions from $C_0^\infty(\Omega)$ and then for functions from $\overset{\circ}{W}{}^{1,p}(\nu, \Omega)$.

We now pass to the exposition of these steps of the proof of the theorem.

Step 1. Let us introduce some local characteristics of the functionals J_s. We set

$$Q_t(y) = \left\{x \in \mathbb{R}^n \colon |x_i - y_i| < 1/(2t),\ i = 1, \ldots, n\right\} \quad \text{for any } y \in \mathbb{R}^n \text{ and } t \in \mathbb{N}$$

and

$$Y_t = \left\{y \in \mathbb{R}^n \colon t y_i \in \mathbb{Z},\ i = 1, \ldots, n\right\} \quad \text{for any } t \in \mathbb{N}.$$

Note that

$$\forall t \in \mathbb{N} \colon \bigcup_{y \in Y_t} \overline{Q_t(y)} = \mathbb{R}^n$$

and

$$\forall t \in \mathbb{N},\ \forall y, y' \in Y_t,\ y \neq y' \colon\ Q_t(y) \cap Q_t(y') = \varnothing.$$

Further, for any $t \in \mathbb{N}$, we set $Y_t' = \left\{y \in Y_t \colon \overline{Q_t(y)} \subset \Omega\right\}$. It is clear that there exists $t_0 \in \mathbb{N}$ such that the set Y_t' is not empty for any $t \in \mathbb{N}$, $t \geq t_0$.

For any $t \in \mathbb{N}$, $t \geq t_0$, $s \in \mathbb{N}$, and $y \in Y_t'$, we set

$$V_{t,s}(y) = \left\{ u \in \widetilde{W}_0^{1,p}(\nu, \Omega_s) \colon \int\limits_{Q_t(y) \cap \Omega_s} \nu |u|^p dx \leq t^{-n-3p} \right\}. \tag{7}$$

For any $t \in \mathbb{N},\ t \geq t_0,\ s \in \mathbb{N},\ y \in Y_t',$ and $\xi \in \mathbb{R}^n,$ we now set

$$F_{t,s}(y,\xi) = t^n \inf_{u \in V_{t,s}(y)} \int\limits_{Q_t(y)\cap\Omega_s} f_s(x, \xi + \nabla u) dx. \tag{8}$$

The numbers $F_{t,s}(y,\xi)$ are specific local characteristics of the functionals J_s.

By virtue of condition (v), for any $t \in \mathbb{N},\ t \geq t_0,\ s \in \mathbb{N},\ y \in Y_t',$ and $\xi \in \mathbb{R}^n$ we have

$$-t^n \int\limits_{Q_t(y)\cap\Omega_s} \psi_s dx \leq F_{t,s}(y,\xi) \leq c_2 |\xi|^p t^n \int\limits_{Q_t(y)\cap\Omega_s} \nu dx + t^n \int\limits_{Q_t(y)\cap\Omega_s} \psi_s dx. \tag{9}$$

Moreover, the following assertions are true:

$(*_1)$ if $t \in \mathbb{N},\ t \geq t_0,\ s \in \mathbb{N},\ y \in Y_t',\ \xi, \xi' \in \mathbb{R}^n,$ and $\tau \in [0,1],$ then

$$F_{t,s}\big(y, (1-\tau)\xi + \tau\xi'\big) \leq (1-\tau) F_{t,s}(y,\xi) + \tau F_{t,s}(y,\xi');$$

$(*_2)$ if $t \in \mathbb{N},\ t \geq t_0,\ s \in \mathbb{N},\ y \in Y_t',$ and $\xi, \xi' \in \mathbb{R}^n,$ then

$$\big|F_{t,s}(y,\xi) - F_{t,s}(y,\xi')\big|$$

$$\leq 2^p c_2 \left(1 + |\xi| + |\xi'|\right)^{p-1} |\xi - \xi'| t^n \int_{Q_t(y)\cap\Omega_s} \nu dx + 2|\xi - \xi'| t^n \int_{Q_t(y)\cap\Omega_s} \psi_s dx.$$

Assertion $(*_1)$ is a corollary of condition (iv). It is proved by analogy with the proof of Lemma 1 in [14]. Assertion $(*_2)$ follows from relation (9) and assertion $(*_1)$.

Step 2. Using condition (ii), estimate (9), and assertion $(*_2)$, we establish that there exist an increasing sequence $\{s_j\} \subset \mathbb{N}$ and a sequence of functions $\Phi_t \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that, for any $t \in \mathbb{N},\ t \geq t_0,\ y \in Y_t',$ and $\xi \in \mathbb{R}^n,$ one has

$$\lim_{j \to \infty} F_{t,s_j}(y,\xi) = \Phi_t(y,\xi). \tag{10}$$

By virtue of condition (ii), estimate (9), and relation (10), for any $t \in \mathbb{N},\ t \geq t_0,\ y \in Y_t',$ and $\xi \in \mathbb{R}^n$ we get

$$-t^n \int\limits_{Q_t(y)} b dx \leq \Phi_t(y,\xi) \leq c_2 |\xi|^p t^n \int\limits_{Q_t(y)} \nu dx + t^n \int\limits_{Q_t(y)} b dx. \tag{11}$$

Furthermore, it follows from assertion $(*_1)$ and relation (10) that, for any $t \in \mathbb{N},\ t \geq t_0,\ y \in Y_t',\ \xi, \xi' \in \mathbb{R}^n,$ and $\tau \in [0,1],$ one has

$$\Phi_t(y, (1-\tau)\xi + \tau\xi') \leq (1-\tau)\Phi_t(y,\xi) + \tau\Phi_t(y,\xi'). \tag{12}$$

Step 3. For any $t \in \mathbb{N}$ and $y \in \Omega$ such that $\overline{Q_t(y)} \subset \Omega,$ let $\chi_{t,y} \colon \Omega \to \mathbb{R}$ be the characteristic function of the set $Q_t(y)$.

For any $k, t \in \mathbb{N}$, we set $Y_{k,t} = \{y \in Y_t \colon Q_t(y) \subset \Omega^{(k)}\}$.

We introduce the following definition: If $k, t \in \mathbb{N}$ and $Y_{k,t} \neq \varnothing$, then $H_t^{(k)}$ is a function on $\Omega \times \mathbb{R}^n$ such that, for any pair $(x, \xi) \in \Omega \times \mathbb{R}^n$, one has

$$H_t^{(k)}(x, \xi) = \sum_{y \in Y_{k,t}} \chi_{t,y}(x) \Phi_t(y, \xi),$$

and if $k, t \in \mathbb{N}$ and $Y_{k,t} = \varnothing$, then $H_t^{(k)}$ is a function on $\Omega \times \mathbb{R}^n$ such that, for any pair $(x, \xi) \in \Omega \times \mathbb{R}^n$, one has

$$H_t^{(k)}(x, \xi) = 0.$$

Further, for any $k \in \mathbb{N}$, we set

$$n_k = \sup_{x \in \Omega^{(k)}} \nu(x) \qquad \text{and} \qquad m_k = \sup_{x \in \Omega^{(k)}} b(x).$$

By virtue of condition (c), we have $n_k, m_k \in [0, +\infty)$ for any $k \in \mathbb{N}$.

It is easy to see that, for any $k, t \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$, the function $H_t^{(k)}(\cdot, \xi)$ is measurable on Ω. Furthermore, by virtue of estimate (11), for any $k \in \mathbb{N}$, $t \in \mathbb{N}$, $t \geq t_0$, $x \in \Omega$, and $\xi \in \mathbb{R}^n$ we have

$$-m_k \leq H_t^{(k)}(x, \xi) \leq c_2 |\xi|^p n_k + m_k. \tag{13}$$

This and relation (12) yield the following inequality for any $k \in \mathbb{N}$, $t \in \mathbb{N}$, $t \geq t_0$, $x \in \Omega$, and $\xi, \xi' \in \mathbb{R}^n$:

$$\left| H_t^{(k)}(x, \xi) - H_t^{(k)}(x, \xi') \right| \leq 2^p c_2 n_k (1 + |\xi| + |\xi'|)^{p-1} |\xi - \xi'| + 2 m_k |\xi - \xi'|. \tag{14}$$

Step 4. Let $\mathbb{Q}^n$ denote the set of all elements of $\mathbb{R}^n$ with rational coordinates. Using estimate (13), we establish that there exist an increasing sequence $\{t_i\} \subset \mathbb{N}$ and functions $h_\xi^{(k)} \in L^2(\Omega)$, $k \in \mathbb{N}$, $\xi \in \mathbb{Q}^n$, such that, for any $k \in \mathbb{N}$ and $\xi \in \mathbb{Q}^n$, one has

$$H_{t_i}^{(k)}(\cdot, \xi) \to h_\xi^{(k)} \quad \text{weakly in } L^2(\Omega). \tag{15}$$

Let E denote the intersection of the sets of Lebesgue points of the functions ν, b, and $h_\xi^{(k)}$, $k \in \mathbb{N}$, $\xi \in \mathbb{Q}^n$. Note that $\operatorname{meas} E = \operatorname{meas} \Omega$.

For any $k \in \mathbb{N}$, $\xi \in \mathbb{Q}^n$, and $z \in E$, we have

$$-b(z) \leq h_\xi^{(k)}(z) \leq c_2 \nu(z) |\xi|^p + b(z). \tag{16}$$

Indeed, let $k \in \mathbb{N}$, $\xi \in \mathbb{Q}^n$, and $z \in E$. We fix $\tau_0 \in \mathbb{N}$ such that $\overline{Q_{\tau_0}(z)} \subset \Omega$. Now assume that $\tau \in \mathbb{N}$ and $\tau > \tau_0$. By virtue of (15), we get

$$\lim_{i \to \infty} \int\limits_{Q_\tau(z)} H_{t_i}^{(k)}(\cdot, \xi) dx = \int\limits_{Q_\tau(z)} h_\xi^{(k)} dx. \tag{17}$$

Assume that $t \in \mathbb{N}$ and $t \geq \max\{t_0, 2\tau^2\}$. Let $Y_{k,t} \neq \varnothing$. We set $Y = \{y \in Y_{k,t} \colon Q_t(y) \cap Q_\tau(z) \neq \varnothing\}$ and assume that $Y \neq \varnothing$. By the definition of the function $H_t^{(k)}$, we have

$$\int\limits_{Q_\tau(z)} H_t^{(k)}(\cdot,\xi)dx = \sum_{y\in Y} \int\limits_{Q_t(y)\cap Q_\tau(z)} H_t^{(k)}(\cdot,\xi)dx = \sum_{y\in Y} \Phi_t(y,\xi)\,\mathrm{meas}\,\big[Q_t(y)\cap Q_\tau(z)\big].$$

Using estimate (11) and the fact that $Q_t(y) \subset Q_{\tau-1}(z)$ for any $y \in Y$, we obtain

$$-\int\limits_{Q_{\tau-1}(z)} bdx \leq \int\limits_{Q_\tau(z)} H_t^{(k)}(\cdot,\xi)dx \leq c_2|\xi|^p \int\limits_{Q_{\tau-1}(z)} \nu dx + \int\limits_{Q_{\tau-1}(z)} bdx. \tag{18}$$

It is easy to see that this inequality is also true if $Y = \varnothing$, as well as in the case where $Y_{k,t} = \varnothing$. By virtue of (17) and (18), we get

$$-\tau^n \int\limits_{Q_{\tau-1}(z)} bdx \leq \tau^n \int\limits_{Q_\tau(z)} h_\xi^{(k)} dx \leq c_2|\xi|^p\tau^n \int\limits_{Q_{\tau-1}(z)} \nu dx + \tau^n \int\limits_{Q_{\tau-1}(z)} bdx.$$

Passing to the limit as $\tau \to \infty$, we obtain inequality (16).

By analogy, using (15), (11), and (12), we establish that the following relation holds for any $k \in \mathbb{N}$, $\xi, \xi' \in \mathbb{Q}^n$, and $z \in E$:

$$\left|h_\xi^{(k)}(z) - h_{\xi'}^{(k)}(z)\right| \leq 2^p c_2 \nu(z)(1 + |\xi| + |\xi'|)^{p-1}|\xi - \xi'| + 2b(z)|\xi - \xi'|. \tag{19}$$

Step 5. By virtue of (19), for any $k \in \mathbb{N}$, $\xi \in \mathbb{R}^n$, and $x \in E$ there exists a number $\tilde{h}_\xi^{(k)}(x)$ such that the fact that $\{\xi^{(l)}\} \subset \mathbb{Q}^n$ and $\xi^{(l)} \to \xi$ in $\mathbb{R}^n$ implies that $h_{\xi^{(l)}}^{(k)}(x) \to \tilde{h}_\xi^{(k)}(x)$.

For any $k \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$, we introduce a function $g_\xi^{(k)}$ on Ω as follows: $g_\xi^{(k)}(x) = \tilde{h}_\xi^{(k)}(x)$ if $x \in E$, and $g_\xi^{(k)}(x) = 0$ if $x \in \Omega \setminus E$.

It is clear that if $k \in \mathbb{N}$, $\xi \in \mathbb{R}^n$, $x \in E$, $\{\xi^{(l)}\} \subset \mathbb{Q}^n$, and $\xi^{(l)} \to \xi$ in $\mathbb{R}^n$, then

$$h_{\xi^{(l)}}^{(k)}(x) \to g_\xi^{(k)}(x). \tag{20}$$

Further, let $\widetilde{\Omega}$ be the union of all sets $\Omega^{(k)}$, let $\chi\colon \Omega \to \mathbb{R}$ be the characteristic function of the set $\widetilde{\Omega}$, and let $\bar{k}$ be the mapping of Ω into $\mathbb{N}$ such that $\bar{k}(x) = \min\{k \in \mathbb{N} \colon x \in \Omega^{(k)}\}$ if $x \in \widetilde{\Omega}$, and $\bar{k}(x) = 1$ if $x \in \Omega \setminus \widetilde{\Omega}$.

Now let f be the function on $\Omega \times \mathbb{R}^n$ such that $f(x,\xi) = \chi(x) g_\xi^{(\bar{k}(x))}(x)$ for any pair $(x,\xi) \in \Omega \times \mathbb{R}^n$. Using (20), one can easily verify that, for any $\xi \in \mathbb{R}^n$, the function $f(\cdot,\xi)$ is measurable on Ω. Moreover, by virtue of relation (20) and estimates (16) and (19), for any $x \in \Omega$ and $\xi, \xi' \in \mathbb{R}^n$ we have

$$-b(x) \leq f(x,\xi) \leq c_2\nu(x)|\xi|^p + b(x), \tag{21}$$

$$\big|f(x,\xi) - f(x,\xi')\big| \leq 2^p c_2 \nu(x)(1 + |\xi| + |\xi'|)^{p-1}|\xi - \xi'| + 2b(x)|\xi - \xi'|. \tag{22}$$

By virtue of inequality (22), the function $f(x,\cdot)$ is continuous on $\mathbb{R}^n$ for any $x \in \Omega$. Thus, the function f satisfies the Carathéodory conditions. Also note that, by virtue of (12), (15), and (20), the function $f(x,\cdot)$ is convex on $\mathbb{R}^n$ for any $x \in \Omega$. It is now clear that $f \in \mathcal{F}$.

Step 6. Let $k \in \mathbb{N}$, $\eta \in \mathbb{Q}^n$, $\varphi \in L^\infty(\Omega)$, $m \in \mathbb{N}$, $m \le k$, and $z \in \Omega^{(m)} \cap E$. We fix $\tau \in \mathbb{N}$, $\tau > 1$, such that $Q_{\tau-1}(z) \subset \Omega^{(m)}$. Let $t \in \mathbb{N}$ and $t \ge 2\tau^2$. We have $Y_{m,t} \ne \varnothing$. Furthermore, by virtue of condition (a) of the theorem, we get $\Omega^{(m)} \subset \Omega^{(k)}$ and, hence, $Y_{m,t} \subset Y_{k,t}$. This implies that $H_t^{(k)}(\cdot,\eta) = H_t^{(m)}(\cdot,\eta)$ on $Q_\tau(z)$. Then, using (15), we establish that $h_\eta^{(k)}(z) = h_\eta^{(m)}(z)$. This and relation (20) imply that $h_\eta^{(k)}(x) = f(x,\eta)$ for any $x \in \Omega^{(k)} \cap E$. Then, using (15), we establish that the integrals of the functions $H_{t_i}^{(k)}(\cdot,\eta)\varphi$ over $\Omega^{(k)}$ converge as $i \to \infty$ to the integral of the function $f(\cdot,\eta)\varphi$ over $\Omega^{(k)}$.

Using this result and relations (14) and (22), we establish that, for any $k \in \mathbb{N}$, $\xi \in \mathbb{R}^n$, and $\varphi \in L^\infty(\Omega)$, one has

$$\lim_{i\to\infty} \int\limits_{\Omega^{(k)}} H_{t_i}^{(k)}(\cdot,\xi)\varphi dx = \int\limits_{\Omega^{(k)}} f(\cdot,\xi)\varphi dx. \tag{23}$$

Step 7. We introduce the following notation: If $k, t \in \mathbb{N}$ and $Y_{k,t} \ne \varnothing$, then

$$E_{k,t} = \bigcup_{y \in Y_{k,t}} Q_t(y);$$

if $u \in C_0^\infty(\Omega)$, $k, t \in \mathbb{N}$, and $Y_{k,t} \ne \varnothing$, then

$$\lambda_t^{(k)}(u) = \sum_{y \in Y_{k,t}} \Phi_t(y, \nabla u(y)) t^{-n};$$

and if $u \in C_0^\infty(\Omega)$, $k, t \in \mathbb{N}$, and $Y_{k,t} = \varnothing$, then $\lambda_t^{(k)}(u) = 0$.

We show that, for any $u \in C_0^\infty(\Omega)$ and $k \in \mathbb{N}$, one has

$$\lim_{i\to\infty} \lambda_{t_i}^{(k)}(u) = \int\limits_{\Omega^{(k)}} f(x, \nabla u) dx. \tag{24}$$

Indeed, let $u \in C_0^\infty(\Omega)$ and $k \in \mathbb{N}$. We set

$$\mu = \sup_{x \in \Omega} |\nabla u(x)|$$

and fix an arbitrary $\varepsilon \in (0,1)$. It is obvious that there exists $\delta \in (0,\varepsilon)$ such that, for any $x', x'' \in \Omega$ that satisfy the inequality $|x' - x''| \le \delta$, one has $|\nabla u(x') - \nabla u(x'')| \le \varepsilon$. Furthermore, it is easy to verify that there exists $\tau \in \mathbb{N}$ such that $\tau > 2^{n+2} n \delta^{-1}$, $Y_{k,\tau} \ne \varnothing$, and $\operatorname{meas}(\Omega^{(k)} \setminus E_{k,\tau}) \le \delta \operatorname{meas} \Omega$. We set

$$G_\tau = \bigcup_{z \in Y_{k,\tau}} [Q_{\tau-1}(z) \setminus Q_{\tau+1}(z)].$$

Since $2^{n+2} n/\tau < \delta$, we have $\operatorname{meas} G_\tau \le \delta \operatorname{meas} \Omega$. We fix $t \in \mathbb{N}$, $t \ge \max\{t_0, 2\tau(\tau+1)\}$. It is easy to see that $Y_{k,t} \ne \varnothing$. For any $z \in Y_{k,\tau}$, we set $X(z) = \{y \in Y_{k,t} \colon Q_t(y) \subset Q_\tau(z)\}$. Now assume that, for any $z \in Y_{k,\tau}$, we have

$$R(z) = Q_\tau(z) \setminus \bigcup_{y\in X(z)} Q_t(y).$$

Since all points of the set $\bigcup_{z\in Y_{k,\tau}} R(z)$ belong to G_τ, we have

$$\operatorname{meas}\left(\bigcup_{z\in Y_{k,\tau}} R(z)\right) \le \delta \operatorname{meas}\Omega. \tag{25}$$

We set

$$X = Y_{k,t} \setminus \bigcup_{z\in Y_{k,\tau}} X(z).$$

Assume that $X \neq \varnothing$. We have

$$\bigcup_{y\in X} Q_t(y) \subset (\Omega^{(k)} \setminus E_{k,\tau}) \cup G_\tau.$$

Using this result and the estimates for the measures of the sets $\Omega^{(k)} \setminus E_{k,\tau}$ and G_τ presented above, we get

$$\operatorname{meas}\left(\bigcup_{y\in X} Q_t(y)\right) \le 2\delta \operatorname{meas}\Omega. \tag{26}$$

Further, we have

$$\lambda_t^{(k)}(u) = \sum_{z\in Y_{k,\tau}} \int\limits_{Q_\tau(z)} H_t^{(k)}(\cdot, \nabla u(z))dx - \sum_{z\in Y_{k,\tau}} \int\limits_{R(z)} H_t^{(k)}(\cdot, \nabla u(z))dx$$

$$+ \sum_{z\in Y_{k,\tau}} \sum_{y\in X(z)} \int\limits_{Q_t(y)} \Big\{H_t^{(k)}(\cdot, \nabla u(y)) - H_t^{(k)}(\cdot, \nabla u(z))\Big\}dx$$

$$+ \sum_{y\in X} \int_{Q_t(y)} H_t^{(k)}(\cdot, \nabla u(y))dx,$$

$$\int_{\Omega^{(k)}} f(x, \nabla u)dx = \sum_{z\in Y_{k,\tau}} \int\limits_{Q_\tau(z)} f(\cdot, \nabla u(z))dx$$

$$+ \sum_{z\in Y_{k,\tau}} \int\limits_{Q_\tau(z)} \{f(x, \nabla u) - f(\cdot, \nabla u(z))\}dx + \int\limits_{\Omega^{(k)}\setminus E_{k,\tau}} f(x, \nabla u)dx.$$

Using these equalities, relations (13), (14), (21), (22), (25), and (26), the inequality $n/\tau < \delta,$ an estimate for the measure of the set $\Omega^{(k)} \setminus E_{k,\tau},$ and properties of the number $\delta,$ we obtain

$$\left| \lambda_t^{(k)}(u) - \int\limits_{\Omega^{(k)}} f(x, \nabla u) dx \right|$$

$$\leq \sum_{z \in Y_{k,\tau}} \left| \int\limits_{Q_\tau(z)} H_t^{(k)}(\cdot, \nabla u(z)) dx - \int\limits_{Q_\tau(z)} f(\cdot, \nabla u(z)) dx \right| + 8^p(1+\mu)^p(1+c_2)(n_k+m_k)\varepsilon \operatorname{meas}\Omega. \quad (27)$$

It is clear that this relation is also true in the case where $X = \varnothing.$ Using (27) and (23), we obtain (24).

We now pass to the proof of the Γ-convergence of the sequence $\{J_{s_j}\}$ to the functional $J^f.$ Let $c_i,\ i = 3, 4, \ldots,$ denote positive constants that depend only on $n,\ p,\ \operatorname{meas}\Omega,\ c_1,\ c_2,$ and $\|b\|_{L^1(\Omega)}.$

Step 8. Let $u \in C_0^\infty(\Omega).$ For any $s \in \mathbb{N},$ we have $u_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$ and

$$\lim_{s\to\infty} \|u_s - q_s u\|_{L^p(\nu,\Omega_s)} = 0. \quad (28)$$

Let us show that

$$\liminf_{j\to\infty} J_{s_j}(u_{s_j}) \geq J^f(u). \quad (29)$$

Let a be a limit point of the sequence $\{J_{s_j}(u_{s_j})\}.$ By virtue of conditions (ii) and (v), we have $a \in (-\infty, +\infty].$ If $a = +\infty,$ then we obviously have $a \geq J^f(u).$ Now let $a \neq +\infty.$ It is clear that there exists an increasing sequence $\{r_l\} \subset \{s_j\}$ such that

$$J_{r_l}(u_{r_l}) \to a. \quad (30)$$

Taking into account that $a \in \mathbb{R}$ and using conditions (ii) and (v), we establish that there exists a constant $c \geq 1$ such that, for any $l \in \mathbb{N},$ we have

$$\int\limits_{\Omega_{r_l}} \nu |\nabla u_{r_l}|^p dx \leq c. \quad (31)$$

Further, let $\varepsilon \in (0,1).$ By virtue of the absolute continuity of the Lebesgue integral, there exists $\varepsilon_1 \in (0, \varepsilon)$ such that, for any measurable set $G \subset \Omega,\ \operatorname{meas} G \leq \varepsilon_1,$ we have

$$\int\limits_G b\, dx \leq \varepsilon.$$

Moreover, by virtue of condition (b) of the theorem, there exists $k \in \mathbb{N}$ such that

$$\operatorname{meas}(\Omega \setminus \Omega^{(k)}) \leq \frac{\varepsilon_1}{2}, \qquad \left| \int\limits_{\Omega\setminus\Omega^{(k)}} f(x, \nabla u) dx \right| \leq \varepsilon. \quad (32)$$

Taking the first inequality in (32) into account, we establish that there exists $t' \in \mathbb{N}$ such that, for any $t \in \mathbb{N}$, $t \geq t'$, the set $Y_{k,t}$ is nonempty and

$$\operatorname{meas}(\Omega \setminus E_{k,t}) \leq \varepsilon_1. \tag{33}$$

It is also obvious that there exists $\delta > 0$ such that, for any $x', x'' \in \Omega$ that satisfy the inequality $|x' - x''| \leq \delta$, we have $|\nabla u(x') - \nabla u(x'')| \leq \varepsilon n_k^{-1/p}$.

We fix $t \in \mathbb{N}$, $t \geq \max\{t_0, t', n/\delta\}$. By virtue of condition (ii), inequality (33), and properties of the number ε_1, we get

$$\limsup_{s\to\infty} \int\limits_{\Omega_s \setminus E_{k,t}} \psi_s dx \leq \int\limits_{\Omega \setminus E_{k,t}} b dx \leq \varepsilon. \tag{34}$$

For any $s \in \mathbb{N}$, we set $v_s = u_s - q_s u$. By virtue of (28), there exists $s' \in \mathbb{N}$ such that, for any $s \in \mathbb{N}$, $s \geq s'$, and $y \in Y_{k,t}$, we have $v_s \in V_{t,s}(y)$. Then, for any $s \in \mathbb{N}$, $s \geq s'$, the following inequality is true:

$$\sum_{y \in Y_{k,t}} F_{t,s}(y, \nabla u(y)) t^{-n} \leq \sum_{y \in Y_{k,t}} \int\limits_{Q_t(y) \cap \Omega_s} f_s(x, \nabla u(y) + \nabla v_s) dx. \tag{35}$$

We fix $s \in \mathbb{N}$, $s \geq s'$. Let $y \in Y_{k,t}$. Using conditions (iv) and (v) and properties of the number δ, we obtain the following relation for almost all $x \in Q_t(y) \cap \Omega_s$:

$$\begin{aligned} f_s(x, \nabla u(y) + \nabla v_s(x)) &= f_s(x, (1-\varepsilon)\nabla u_s(x) + \varepsilon(\nabla u_s(x) + \varepsilon^{-1}(\nabla u(y) - \nabla u(x)))) \\ &\leq (1-\varepsilon) f_s(x, \nabla u_s(x)) + \varepsilon f_s(x, \nabla u_s(x) + \varepsilon^{-1}(\nabla u(y) - \nabla u(x))) \\ &\leq f_s(x, \nabla u_s(x)) + 2^p \varepsilon c_2 \nu(x) |\nabla u_s(x)|^p + 2^p \varepsilon c_2 + 2\varepsilon \psi_s(x). \end{aligned}$$

With regard for condition (v) and inequality (35), for any $s \in \mathbb{N}$, $s \geq s'$, we get

$$\begin{aligned} &\sum_{y \in Y_{k,t}} F_{t,s}(y, \nabla u(y)) t^{-n} \\ &\qquad \leq J_s(u_s) + 2^p \varepsilon c_2 \int\limits_{\Omega_s} \nu |\nabla u_s|^p dx + 2\varepsilon \int\limits_{\Omega_s} \psi_s dx + \int\limits_{\Omega_s \setminus E_{k,t}} \psi_s dx + 2^p \varepsilon c_2 \operatorname{meas} \Omega. \end{aligned}$$

Using this result, condition (ii), and relations (10), (30), (31), and (34), we establish that $\lambda_t^{(k)}(u) \leq a + \varepsilon c c_3$. Using (24) and the second inequality in (32), we conclude that $a \geq J^f(u)$. Therefore, inequality (29) is true.

Step 9. Let $u \in \overset{\circ}{W}{}^{1,p}(\nu, \Omega)$. For any $s \in \mathbb{N}$, the inclusion $u_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$ and equality (28) are true. Let us verify inequality (29).

Let $\{u^{(l)}\}$ be a sequence of functions from $C_0^\infty(\Omega)$ such that

$$\lim_{l\to\infty} \|u^{(l)} - u\|_{1,p,\nu} = 0. \tag{36}$$

For any $l, s \in \mathbb{N}$, we set $u_s^{(l)} = u_s + q_s(u^{(l)} - u)$. By virtue of relation (28) and the result obtained in the previous step, for any $l \in \mathbb{N}$ we get

$$\liminf_{j\to\infty} J_{s_j}(u_{s_j}^{(l)}) \geq J^f(u^{(l)}). \tag{37}$$

Furthermore, using conditions (iv) and (v), for any $l, s \in \mathbb{N}$ we obtain

$$J_s(u_s^{(l)}) \leq J_s(u_s) + c_4 \left(1 + \int\limits_{\Omega_s} \psi_s dx + \int\limits_{\Omega_s} \nu|\nabla u_s|^p dx + \|u^{(l)} - u\|_{1,p,\nu}^p\right) \|u^{(l)} - u\|_{1,p,\nu}.$$

Using conditions (ii) and (v), relations (36) and (37), and the continuity of the functional J^f, we establish that if a is a finite limit point of the sequence $\{J_{s_j}(u_{s_j})\}$, then $a \geq J^f(u)$, which proves inequality (29).

Step 10. Let $u \in C_0^\infty(\Omega)$. We show that there exists a sequence $w_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$ such that

$$\lim_{s\to\infty} \|w_s - q_s u\|_{L^p(\nu,\Omega_s)} = 0, \qquad \limsup_{j\to\infty} J_{s_j}(w_{s_j}) \leq J^f(u). \tag{38}$$

Let $\varepsilon \in (0,1)$. Using the corresponding result from Step 8, condition (ii), and relation (24), we establish that there exist numbers $k \in \mathbb{N}$, $\delta > 0$, and $t \in \mathbb{N}$ such that

$$\forall x', x'' \in \Omega, \quad |x' - x''| \leq \delta\colon \quad \big|\nabla u(x') - \nabla u(x'')\big| \leq \varepsilon n_k^{-1/p}, \tag{39}$$

$$t \geq \max\{t_0, 1/\varepsilon, n/\delta\}, \qquad Y_{k,t} \neq \varnothing, \tag{40}$$

$$\lambda_t^{(k)}(u) \leq J^f(u) + 2\varepsilon, \tag{41}$$

$$\int\limits_{\Omega\setminus E_{k,t}} \nu|\nabla u|^p dx \leq \varepsilon, \qquad \limsup_{s\to\infty} \int\limits_{\Omega_s\setminus E_{k,t}} \psi_s dx \leq \varepsilon, \tag{42}$$

$$\begin{gathered} \sum_{y\in Y_{k,t}} \int\limits_{Q_t(y)\setminus Q_{t+1}(y)} \nu|\nabla u|^p dx \leq \varepsilon, \\ \limsup_{s\to\infty} \sum_{y\in Y_{k,t}} \int\limits_{[Q_t(y)\setminus Q_{t+1}(y)]\cap\Omega_s} \psi_s dx \leq \varepsilon. \end{gathered} \tag{43}$$

Further, assume that, for any $y \in Y_{k,t}$, φ_y is a function from $C_0^\infty(\Omega)$ such that $0 \leq \varphi_y \leq 1$ on Ω, $\varphi_y = 1$ in $Q_{t+1}(y)$, $\varphi_y = 0$ on $\Omega \setminus Q_t(y)$, and $|\nabla\varphi_y| \leq c_0 t^2$ on Ω $(c_0 > 0$ depends only on $n)$, and, for any $y \in Y_{k,t}$ and $s \in \mathbb{N}$, $w_{y,s}$ is a function from $V_{t,s}(y)$ such that

$$\int\limits_{Q_t(y)\cap\Omega_s} f_s(x, \nabla u(y) + \nabla w_{y,s}) dx \leq F_{t,s}(y, \nabla u(y)) t^{-n} + \varepsilon t^{-n}. \tag{44}$$

Note that, by virtue of condition (v) and relations (39), (40), and (44), for any $y \in Y_{k,t}$ and $s \in \mathbb{N}$ we have

$$c_1 \int\limits_{Q_t(y)\cap\Omega_s} \nu|\nabla u(y) + \nabla w_{y,s}|^p dx \le 2^p c_2 \int\limits_{Q_t(y)} \nu|\nabla u|^p dx + 2 \int\limits_{Q_t(y)\cap\Omega_s} \psi_s dx + (2^p c_2 + 1)t^{-n}. \tag{45}$$

For any $s \in \mathbb{N}$, we now set

$$w_s = q_s u + \sum_{y\in Y_{k,t}} w_{y,s}\varphi_y.$$

For any $s \in \mathbb{N}$, we have $w_s \in \widetilde{W}_0^{1,p}(\nu, \Omega_s)$. Moreover, by virtue of the inclusions $w_{y,s} \in V_{t,s}(y)$ $(y \in Y_{k,t}$, $s \in \mathbb{N})$ and the inequality $t \ge 1/\varepsilon$, for any $s \in \mathbb{N}$ we get

$$\|w_s - q_s u\|_{L^p(\nu,\Omega_s)} \le \varepsilon(\operatorname{meas}\Omega)^{1/p}. \tag{46}$$

It is also clear that, for any $s \in \mathbb{N}$, we have

$$J_s(w_s) = \int\limits_{\Omega_s\setminus E_{k,t}} f_s(x, \nabla u)dx + \sum_{y\in Y_{k,t}} \int\limits_{Q_t(y)\cap\Omega_s} f_s(x, \nabla w_s)dx. \tag{47}$$

By virtue of condition (v) and inequalities (42), we obtain

$$\limsup_{s\to\infty} \int\limits_{\Omega_s\setminus E_{k,t}} f_s(x, \nabla u)dx \le (c_2 + 2)\varepsilon. \tag{48}$$

Using conditions (iv) and (v), inequalities (39) and (40), and properties of the functions φ_y, $y \in Y_{k,t}$, we establish that if $s \in \mathbb{N}$, $y \in Y_{k,t}$, and $\operatorname{meas}(Q_t(y) \cap \Omega_s) > 0$, then the following relation holds for almost all $x \in Q_t(y) \cap \Omega_s$:

$$\begin{aligned} f_s(x, \nabla w_s(x)) &\le (1-\varepsilon) f_s(x, \nabla u(y) + \varphi_y(x)\nabla w_{y,s}(x)) \\ &\quad + \varepsilon f_s(x, \nabla u(y) + \varphi_y(x)\nabla w_{y,s}(x) + \varepsilon^{-1}(\nabla u(x) - \nabla u(y) + w_{y,s}(x)\nabla\varphi_y(x))) \\ &\le f_s(x, \nabla u(y) + \nabla w_{y,s}(x)) + 4^{p+1}c_2(1-\varphi_y(x))\nu(x)|\nabla u(x)|^p \\ &\quad + 2(1-\varphi_y(x))\psi_s(x) + 2\varepsilon\psi_s(x) \\ &\quad + 4^p c_2\varepsilon\nu(x)|\nabla u(y) + \nabla w_{y,s}(x)|^p + 4^p c_0^p t^{3p} c_2 \varepsilon\nu(x)|w_{y,s}(x)|^p + 8^{p+1}c_2\varepsilon. \end{aligned}$$

Using properties of the functions φ_y, $y \in Y_{k,t}$, the inclusions $w_{y,s} \in V_{t,s}(y)$ $(y \in Y_{k,t},\ s \in \mathbb{N})$, condition (ii), and relations (10) and (43)–(45), we obtain

$$\limsup_{j\to\infty} \sum_{y\in Y_{k,t}} \int\limits_{Q_t(y)\cap\Omega_{s_j}} f_{s_j}(x, \nabla w_{s_j})dx \le \lambda_t^{(k)}(u) + c_5\varepsilon \int\limits_{\Omega} \nu|\nabla u|^p dx + c_6\varepsilon.$$

Using this inequality and relations (41), (47), and (48), we establish that

$$\limsup_{j\to\infty} J_{s_j}(w_{s_j}) \le J^f(u) + c_7\varepsilon\left(1+\int\limits_\Omega \nu|\nabla u|^p dx\right). \tag{49}$$

Using (46) and (49), we conclude that if $l\in\mathbb{N}$, then there exist a sequence $w_s^{(l)}\in\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ and a number $j^{(l)}\in\mathbb{N}$ such that

$$\forall s\in\mathbb{N}:\quad \|w_s^{(l)}-q_s u\|_{L^p(\nu,\Omega_s)}\le l^{-1}, \tag{50}$$

$$\forall j\in\mathbb{N},\quad j\ge j^{(l)}:\quad J_{s_j}(w_{s_j}^{(l)})\le J^f(u)+l^{-1}. \tag{51}$$

For any $l\in\mathbb{N}$, we set

$$s^{(l)} = l + \max_{1\le r\le l} s_{j^{(r)}}.$$

It is obvious that $\{s^{(l)}\}$ is an increasing sequence. Now assume that the sequence $\{w_s\}$ is such that $w_s=w_s^{(1)}$ for $s\le s^{(1)}$, and $w_s=w_s^{(l)}$ for $s^{(l)}<s\le s^{(l+1)}$, $l=1,2,\ldots$. Then, for any $s\in\mathbb{N}$, we have $w_s\in\widetilde{W}_0^{1,p}(\nu,\Omega_s)$. Moreover, using (50) and (51), we establish that relations (38) are true.

Step 11. Let $u\in\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$. It is obvious that there exists a sequence $\{u^{(l)}\}\subset C_0^\infty(\Omega)$ such that, for any $l\in\mathbb{N}$, we have

$$\|u^{(l)}-u\|_{1,p,\nu}\le 1/(2l) \qquad\text{and}\qquad J^f(u^{(l)})\le J^f(u)+1/(2l).$$

Using these inequalities and the result obtained in the previous step of the proof, we conclude that if $l\in\mathbb{N}$, then there exist a sequence $v_s^{(l)}\in\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ and numbers $s_1^{(l)}, j_1^{(l)}\in\mathbb{N}$ such that

$$\forall s\in\mathbb{N},\quad s\ge s_1^{(l)}:\quad \|v_s^{(l)}-q_s u\|_{L^p(\nu,\Omega_s)}\le l^{-1}, \tag{52}$$

$$\forall j\in\mathbb{N},\quad j\ge j_1^{(l)}:\quad J_{s_j}(v_{s_j}^{(l)})\le J^f(u)+l^{-1}. \tag{53}$$

For any $l\in\mathbb{N}$, we set

$$\bar{s}^{(l)} = l+\max_{1\le r\le l} s_1^{(r)} + \max_{1\le r\le l} s_{j_1^{(r)}}.$$

It is clear that $\{\bar{s}^{(l)}\}$ is an increasing sequence. Now let $\{v_s\}$ be the sequence such that $v_s=v_s^{(1)}$ if $s\le\bar{s}^{(1)}$, and $v_s=v_s^{(l)}$ if $\bar{s}^{(l)}<s\le\bar{s}^{(l+1)}$, $l=1,2,\ldots$. Then, for any $s\in\mathbb{N}$, we have $v_s\in\widetilde{W}_0^{1,p}(\nu,\Omega_s)$. Furthermore, by virtue of (52) and (53), we get

$$\lim_{s\to\infty}\|v_s-q_s u\|_{L^p(\nu,\Omega_s)}=0 \qquad\text{and}\qquad \limsup_{j\to\infty} J_{s_j}(v_{s_j})\le J^f(u).$$

Using this result and the result obtained in Step 9, we conclude that the sequence $\{J_{s_j}\}$ Γ-converges to the functional J^f.

The theorem is proved.

In conclusion, we make several remarks. First of all, note that conditions (a)–(c) of Theorem 2 are satisfied if, e.g., the functions ν and b are bounded in Ω or continuous in Ω except for a closed set of measure zero. Moreover, under the conditions of Theorem 2 and certain additional assumptions, including the so-called regular strong correlation of the sequence of spaces $\widetilde{W}_0^{1,p}(\nu,\Omega_s)$ with the space $\overset{\circ}{W}{}^{1,p}(\nu,\Omega)$, the Lagrangian f of the Γ-limit functional for the sequence of functionals $\{J_{s_j}\}$ is coercive, i.e., for a certain constant $c'>0$, almost all $x\in\Omega$, and any $\xi\in\mathbb{R}^n$, one has $f(x,\xi)\geq c'\nu(x)|\xi|^p-b(x)$. This result was obtained in [37]. In turn, the indicated additional conditions are satisfied in the case of a weight function ν of the form $\nu(x)=|x|^\gamma$, $x\in\Omega\setminus\{0\}$, $\gamma\in(-n,n(p-1))$, and domains Ω_s of special perforated structure (for details, see [24]). Also note that, on the basis of results on Γ-convergence or Γ-compactness for the functionals J_s, one can obtain analogous results for functionals of the form $I_s=J_s+G_s$, where

$$G_s(u)=\int\limits_{\Omega_s} g(x,u)dx, \quad u\in\widetilde{W}_0^{1,p}(\nu,\Omega_s);$$

furthermore, under proper growth and coercivity conditions for the function g $\big($e.g., if $g(x,\eta)=a_0\nu(x)|\eta|^p-g_0(x)\eta$, $a_0>0$, and $g_0(1/\nu)^{1/p}\in L^{p/(p-1)}(\Omega)\big)$, the sequence of the norms of minimizers of the functionals I_s is bounded (we emphasize this fact in connection with the arguments presented at the end of Sec. 3). Finally, note that if ψ is a nonnegative function from $L^1(Q_1(0))$ and, for any $s\in\mathbb{N}$, ψ_s is a nonnegative function on Ω_s such that $\psi_s(x)=\psi(s(x-z))$ for $x\in Q_s(z)\cap\Omega_s$ and $z\in Y_s$, then conditions (i) and (ii) are satisfied, and, furthermore, the function b takes a constant value on Ω equal to the integral of the function ψ over the cube $Q_1(0)$.

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