Harmonic Balance Method and Combination Resonances in Nonlinear Systems with Polynomial Nonlinearities and Periodic Excitation

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Abstract—**In the paper the use of a complex form of the harmonic balance method for the analysis of the dynamic systems with polynomial nonlinearities is described. Two examples are considered, combination resonances are investigated, bifurcation curves are constructed.**

Keywords—Harmonic balance method, dynamic systems, combination resonance, bifurcation curves

I. Introduction

Yet 50-60 years ago harmonic balance method was one of the main methods of investigation of nonlinear systems [1, 2], but in recent decades more popular, perhaps, is the method of the Poincare maps [3].And while not denying advantages of the latter we would like to note some attractive features of the first one that slightly open the door to the global analysis of dynamic systems. And our purpose here is to draw this to the attention of the researchers. Really, the harmonic balance method gives an opportunity to reduce analysis of the differential equations to the solving of polynomial equations. Then, with use of the Newton polyhedra theory [4] we can determine the number of all solutions of the system of polynomial equations and applying after it the interval approaches [5] you may identify all solutions of polynomial equations and perform, in such a way, global analysis of dynamic system essentially. The first step in this direction is the reduction of differential equations to the systems of polynomial equations. For dynamic systems with polynomial nonlinearities this can been done successfully with use of the complex version of the harmonic balance method. Below we illustrate this fact by examples of analysis of two dynamic systems.

II. One-Masses Vibrating System

Here we consider the equation [6]

$$\frac{d^2\xi}{d\tau^2}+\mu\omega_0(1+\beta\xi+\gamma\xi^2)\frac{d\xi}{d\tau}+(1+\beta\xi+\gamma\xi^2)\xi=P\cos\eta\tau, \quad (1)$$

which in dimensionless form describes the motion of one-masses vibrating machine with force or kinematic excitation [7]. According to the harmonic balance method its solution we find in the view

$$\xi(\tau)=\sum_{n=-N}^{N} c_n e^{i n\eta\tau}, \quad (2)$$

where N is the number of harmonics taking out into consideration. The expansion (2) is connected with the trigonometric form $\xi(\tau)=\sum_{k=0}^{N} A_k\cos(k\eta\tau-\varphi_k)$ of the solution in the following way: $A_j=2\sqrt{c_j\,c_{-j}}$ and initial phase $\varphi_j=\arccos\frac{c_j+c_{-j}}{2\sqrt{c_j\,c_{-j}}}$ or $\varphi_j=-\arccos\frac{c_j+c_{-j}}{2\sqrt{c_j\,c_{-j}}}$, if $(\Im c_{-j}=0\ \wedge\ \Re c_{-j}<0)\ \vee\ \Im c_{-j}<0$, $\varphi_j\in[-\pi,\pi)$. After substituting (2) into (1), performing the obvious transformations and equating coefficients of equal powers of $e^{i\eta\tau}$ you may get the system of polynomial equations with respect to C_n

$$\begin{aligned}&(1+i\,\mu\,\omega_0\,n\frac{1}{q}\eta-n^2\frac{1}{q^2}\eta^2)\,c_n+\\&\beta\sum_{k=-N}^{N}c_k\,c_{n-k}\,(1+i\,\mu\,\omega_0\,(n-k)\frac{1}{q}\eta)+\\&\gamma\sum_{k=-N}^{N}\sum_{m=-N}^{N}c_k\,c_m\,c_{n-k-m}\,(1+i\,\mu\,\omega_0\,(n-k-m)\frac{1}{q}\eta)=\\&\begin{cases}P/2, & n=\pm q\\ 0, & n\neq\pm q\end{cases}\end{aligned} \quad (3)$$

where $n, n-k, n-k-m\in[-N,N]$ and q is a control parameter. If $q = 1$ the system (3) describes $T = 2\pi/\eta$-periodical regimes, that are basic and superharmonic motions, if $q = 2,3,\ldots$ subharmonic ones. Now consistently changing one of the parameters and solving (3) you may construct bifurcations curves. This procedure is realized with the help of the original software [8].The points of bifurcation in it are found by control the sign of the system Jacobian , stability of the regimes in the first approximation is studied with the help of the Floquet-Lyapunov theory, the finding of isolated branches of the bifurcation curves is realized by changing initial conditions in the specified part of the phase space with use of the quasi-random Sobol sequence [11]. Below in Fig.1 amplitude- and phase-frequency characteristics (AFC and

PFC) are presented for certain parameters of the model in the zones of super-, principal and sub-resonances. The unstable regimes are indicated by the dash lines. The basins of attraction (BOA) are found by the scanning method [12].

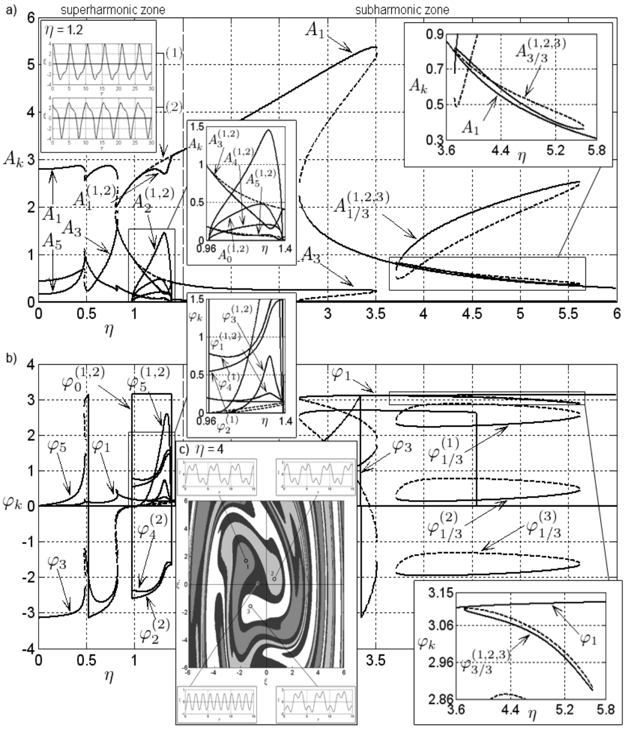

Fig. 1. AFC (a), PFC (b) and BOA (c) of (1) for $\mu\omega_0 = 0.1$, $\beta = 0$, $\gamma = 0.5$, $P = 10$, where $A_k^{(m)}$, $\varphi_k^{(m)}$ – are the amplitude and initial phase of k-th harmonic of m-th regime

of the quasi-random Sobol sequence [11]. Below in Fig.1 the amplitude- and phase-frequency characteristics (AFC and PFC) are presented for certain parameters of the model in the zones of super-, principal and sub-resonances. The unstable regimes are indicated by the dash lines. The basins of attraction (BOA) are found by the scanning method [12].

III. Two-Masses Vibrating System

Consider the principal schema of a vibrating machine (Fig.2) and dimensionless equations of its motion [12]

$$\begin{cases} \dfrac{d^2\xi_1}{d\tau^2} + b_{10}\dfrac{d\xi_1}{d\tau} + b_{11}\dfrac{d\xi}{d\tau} + b_{12}\,\xi\dfrac{d\xi}{d\tau} + b_{13}\,\xi^2\dfrac{d\xi}{d\tau} + \\ \quad + k_{10}\,\xi_1 + k_{11}\,\xi + k_{12}\,\xi^2 + k_{13}\,\xi^3 = P_1\cos\eta\tau \\ \dfrac{d^2\xi}{d\tau^2} + b_{20}\dfrac{d\xi_1}{d\tau} + b_{21}\dfrac{d\xi}{d\tau} + b_{22}\,\xi\dfrac{d\xi}{d\tau} + b_{23}\,\xi^2\dfrac{d\xi}{d\tau} + \\ \quad + k_{20}\,\xi_1 + k_{21}\,\xi + k_{22}\,\xi^2 + k_{23}\,\xi^3 = P_2\cos\eta\tau, \end{cases} \tag{4}$$

where $b_{10} = \dfrac{\mu\,k_0}{m_1\,\omega_1}$, $b_{11} = -\dfrac{\mu\,k_1'}{m_1\,\omega_1}$, $b_{12} = -\dfrac{\mu\,k_2'\,\Delta}{m_1\,\omega_1}$, $b_{13} = -\dfrac{\mu\,k_3'\,\Delta^2}{m_1\,\omega_1}$, $b_{20} = -\dfrac{\mu\,k_0}{m_1\,\omega_1}$, $b_{21} = \dfrac{\mu\,(m_1+m_2)\,k_1'}{m_1\,m_2\,\omega_1}$,

$b_{22} = \dfrac{\mu\,(m_1+m_2)\,k_2'\,\Delta}{m_1\,m_2\,\omega_1}$, $b_{23} = \dfrac{\mu\,(m_1+m_2)\,k_3'\,\Delta^2}{m_1\,m_2\,\omega_1}$,

$k_{10} = \dfrac{k_0}{m_1\,\omega_1^2}$, $k_{11} = -\dfrac{k_1}{m_1\,\omega_1^2}$, $k_{12} = -\dfrac{k_2\,\Delta}{m_1\,\omega_1^2}$,

$k_{13} = -\dfrac{k_3\,\Delta^2}{m_1\,\omega_1^2}$, $k_{20} = -\dfrac{k_0}{m_1\,\omega_1^2}$, $k_{21} = \dfrac{k_1\,(m_1+m_2)}{m_1\,m_2\,\omega_1^2}$,

$k_{22} = \dfrac{k_2\,(m_1+m_2)\,\Delta}{m_1\,m_2\,\omega_1^2}$, $k_{23} = \dfrac{k_3\,(m_1+m_2)\,\Delta^2}{m_1\,m_2\,\omega_1^2}$, $P_1 = \dfrac{m_0\,r}{m_1\,\Delta}\eta^2$,

$P_2 = -P_1$, $\xi_1 = x_1/\Delta$, $\xi = x/\Delta$, $x = x_2 - x_1$, x_1 – displacement of frame, x_2 – displacement of a working organ, $\Delta = 10^{-3}$ m, m_0 – unbalanced mass, m_1 – mass of a frame, m_2 – mass of a screen box, k_0 – stiffness of isolators, k_1, k_2, k_3 – parameters of elastic ties and k'_1, k'_2, k'_3 – of dissipation, r – eccentricity of an exciter, μ – coefficient of dissipation, $\eta = \omega/\omega_1$, ω – frequency of an vibroexciter, ω_1 – the first natural frequency of a vibromachine, $\tau = \omega_1 t$. The exciter is supposed to be ideal. Similarly (1), solutions of (4) are found in the form

$$\xi_1(\tau) = \sum_{n=-N}^{N} c_n^{(1)}\,e^{i\,n\,\eta\,\tau},\ \xi(\tau) = \sum_{n=-N}^{N} c_n\,e^{i\,n\,\eta\,\tau}. \tag{5}$$

After substitution (5) into (4) and equating the coefficients we get the system of polynomial equations with respect to $c_n^{(1)}$ and c_n

$$\begin{cases} (k_{10} + b_{10}\,i\,\eta\,n - \eta^2\,n^2)\,c_n^{(1)} + (k_{11} + b_{11}\,i\,\eta\,n)\,c_n + \\ \quad + \sum\limits_{j=-N}^{N} c_j\,c_{n-j}\,(k_{12} + b_{12}\,i\,\eta\,(n-j)) + \\ \quad + \sum\limits_{j=-N}^{N}\sum\limits_{m=-N}^{N} c_j\,c_m\,c_{n-j-m}\,(k_{13} + b_{13}\,i\,\eta\,(n-j-m)) = \\ \quad = \begin{cases} P_1/2, & n = \pm 1 \\ 0, & n \neq \pm 1 \end{cases} \\ (k_{20} + b_{20}\,i\,\eta\,n)\,c_n^{(1)} + (k_{21} + b_{21}\,i\,\eta\,n - \eta^2\,n^2)\,c_n + \\ \quad + \sum\limits_{j=-N}^{N} c_j\,c_{n-j}\,(k_{22} + b_{22}\,i\,\eta\,(n-j)) + \\ \quad + \sum\limits_{j=-N}^{N}\sum\limits_{m=-N}^{N} c_j\,c_m\,c_{n-j-m}\,(k_{23} + b_{23}\,i\,\eta\,(n-j-m)) = \\ \quad = \begin{cases} P_2/2, & n = \pm 1 \\ 0, & n \neq \pm 1 \end{cases}, \end{cases} \tag{6}$$

where $n, n-j, n-j-m \in [-N, N]$. Considering oscillations in the frequency zone located between the natural ones (Fig.3) and using the frequency ratio featured in literature [12] we changed initial conditions in the chosen part of the phase space and discovered pure resonances of the order 2:1, 3:1 and 1:3 with the help of the multistart method. In Figs.4, 5 and 6 there are presented the corresponding AFC and PFC of the frame and screen box

(The unstable regimes are not marked here). Computations are fulfilled

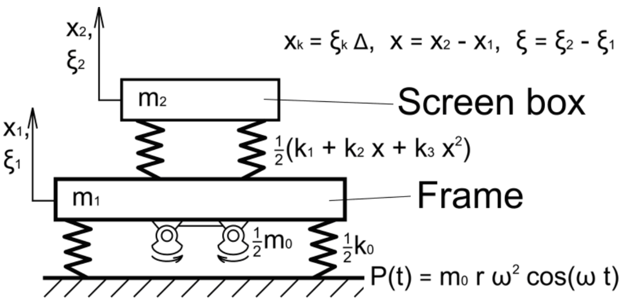

Fig. 2. Principal schema of the vibrating machine

for five harmonic components in (5), i.e. N = 5, for the vibrating machine having the following values of its main parameters: m_1 = 650 kg, m_2 = 550 kg, $k_0 = 0.12 \cdot 10^6$ N/m, $k_1 = 5.5 \cdot 10^6$ N/m, m_0 = 50 kg, r = 0.088 m, μ = 0.0008 sec. It was supposed here $k_1' = k_1$, $k_2' = k_3' = 0$.

Without dwelling on the results we only mention that the analysis of PFC is quite useful here and helps to find out certain peculiarities of the regimes. For example in Fig.4 the presence of opposite regimes is explained when the superharmonic resonance 2:1 is excited because of the even harmonic phase difference on π radians.

IV. Conclusion

Of course on the way of the harmonic balance method use for the global analysis of dynamic systems there is a number of unresolved problems. Here we mention only one of them. The matter is that for the motions of dynamic systems there correspond such solutions of the polynomial systems (4) and (7) for which $c_n = \overline{c}_{-n}$, but the Bernstein's theorem [13] gives the total number of its solutions. The following example shows how it is important. For three harmonics taken out in consideration in expansions (2) and the value η = 4.0 the number of all solutions of (3), determined with the help of the program MixedVol [14], which realizes the computation of mixed volumes of the Newton polyhedra, equals 414, while the number of the 'complex conjugate' solutions, determined by the multistart method, proved equal to 6 only.

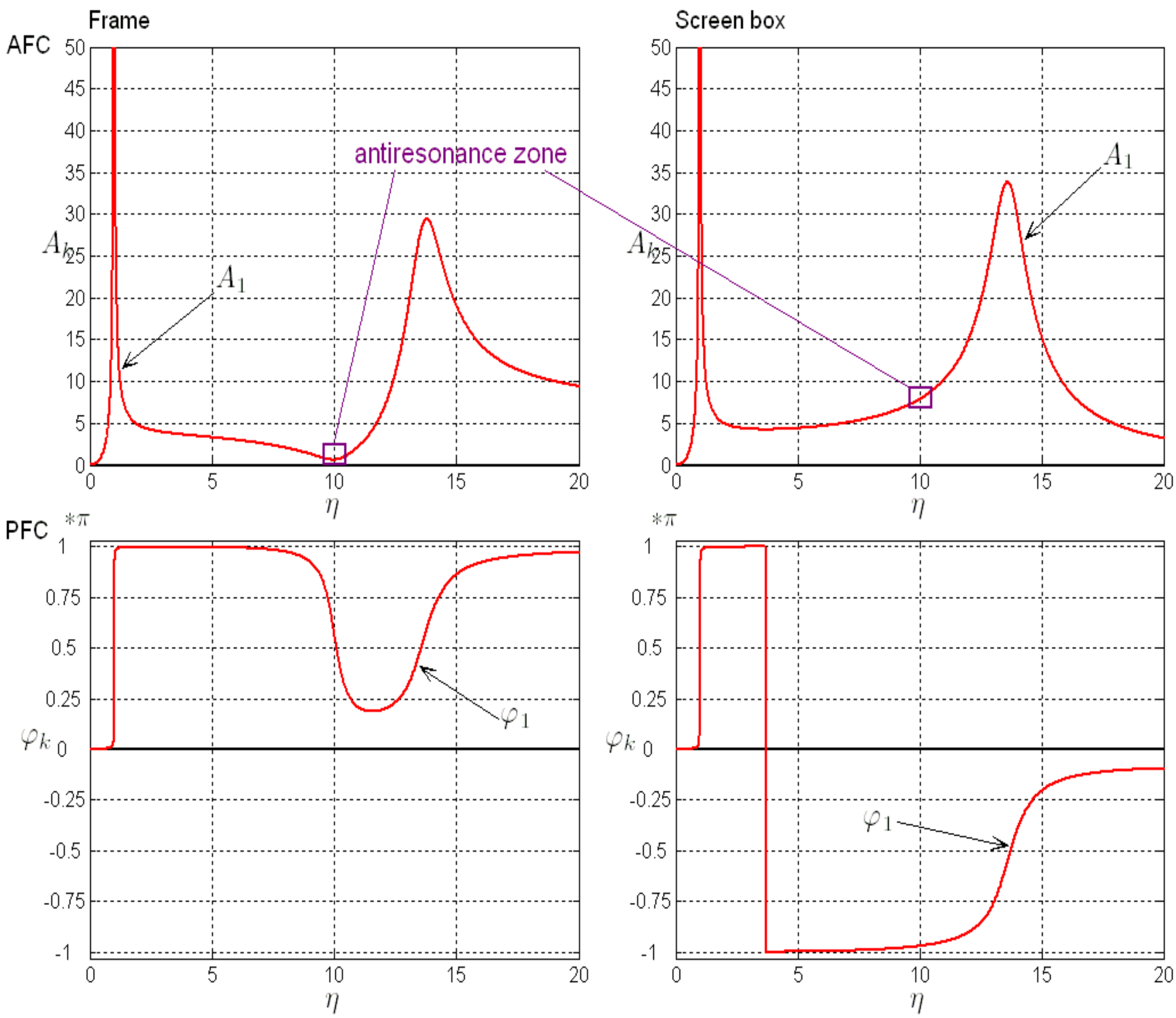

Fig. 3. AFC and PHC of linear system

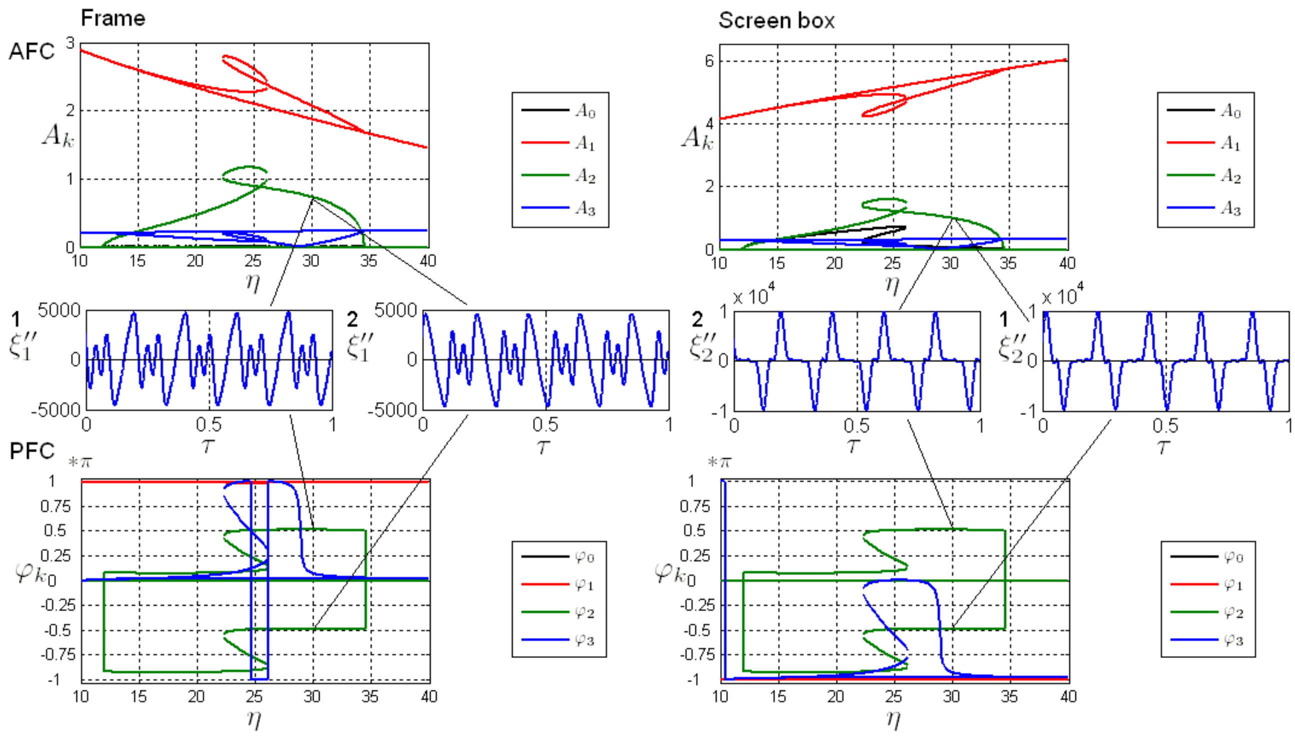

Fig. 4. Superharmonic resonance 2:1, $k_{13} / k_{11} = 1$

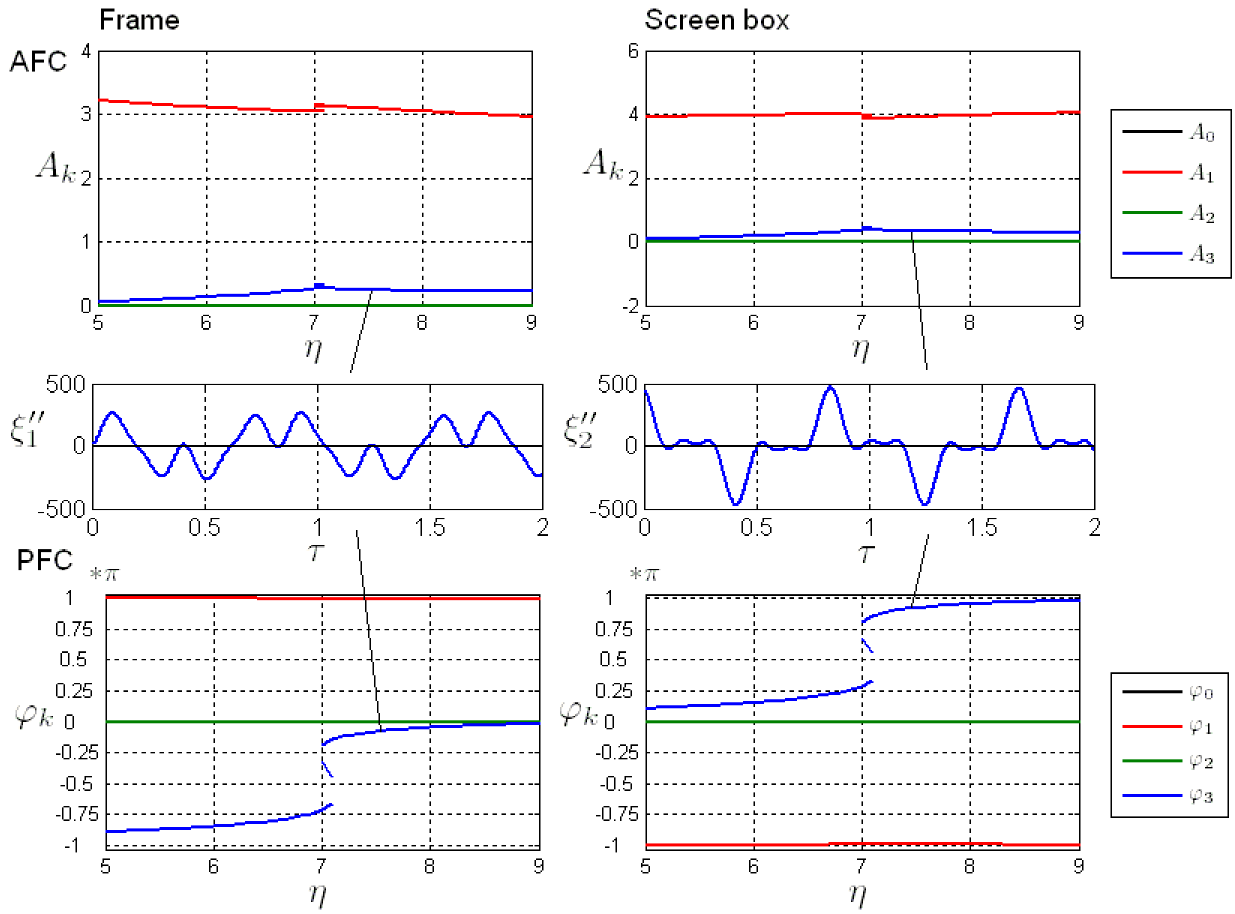

Fig. 5. Superharmonic resonance 3:1, $k_{13} / k_{11} = 1$

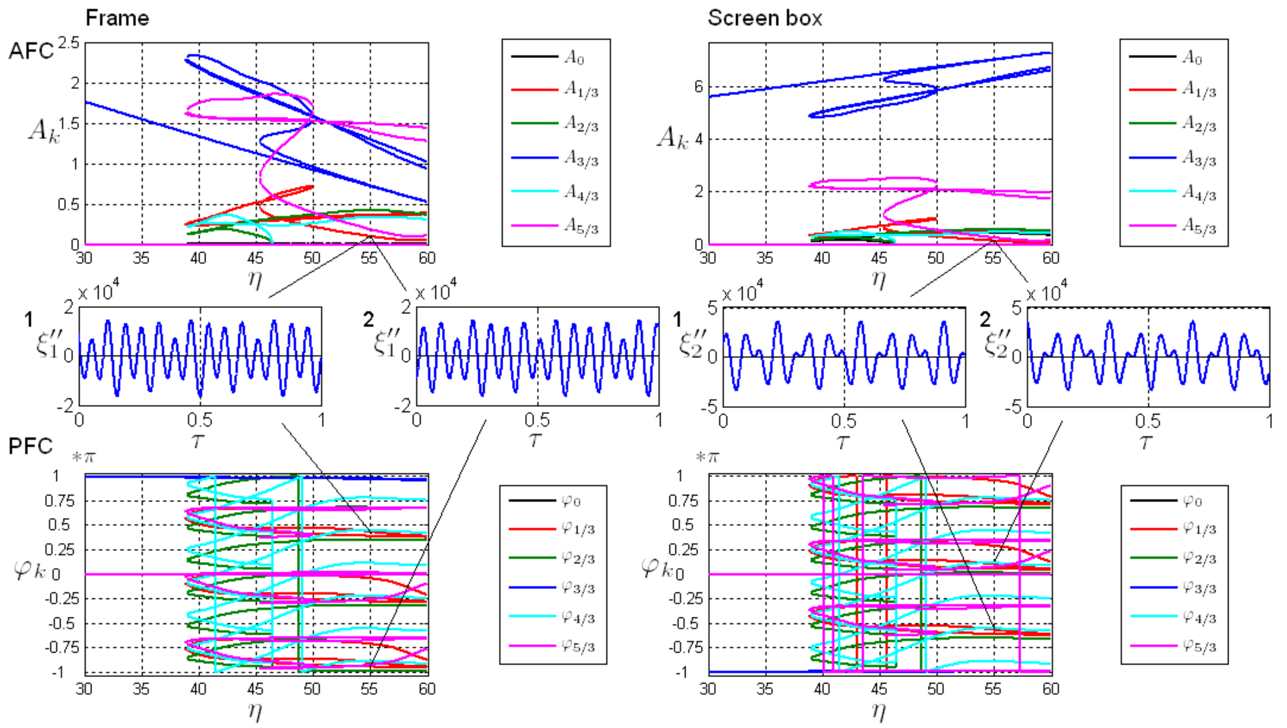

Fig. 6. Subharmonic resonance 1:3, k_{13} / k_{11} = 1

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