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CLASSIFICATION OF SECOND ORDER CURVES ON THE STATIONARY POINTS OF THE CURVES

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The idea of the method consists in finding stationary points of the curve of the second order of general form. Then the analysis of factors of the quadratic form of general view (1) is done. The coefficients of the quadratic form vary in such a way that the stationary points of the curve are placed on the axes of a Cartesian coordinate system. Consider the general equation of the second degree with two unknowns x and y:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0, \qquad (1)$$

where $a_{11}, a_{12}, \ldots, a_{33}$ are factors of the equation, which are specified by the real numbers.

Calculate the derivative y'_x of the quadratic form (1), assuming that y is a function of x:

$$y' = -\frac{a_{11}x + a_{12}y + a_{13}}{a_{12}x + a_{22}y + a_{23}}. \qquad (2)$$

The stationary points of the function $y(x)$ can be found from the equation $y' = 0$:

$$a_{11}x + a_{12}y + a_{13} = 0. \qquad (3)$$

This is the general equation of a straight line with the angular factor $k = -a_{11} / a_{12}$. It crosses the y-axis at the point $b = -a_{13} / a_{12}$.

The straight line (3) forms a set of stationary points of the second order curve (1). At these points the tangent to the second order curves should be located horizontally, i.e. parallel to the abscissa. Similarly, we obtain the equation of the vertical straight line, every point of which is the tangent to the second order curves, where the derivatives are equal to infinity

$$a_{12}x + a_{22}y + a_{23} = 0. \qquad (4)$$

Hence, analysis of the equations of second order curves can be done either by the equation (3) or equation (4).

Ellipse, hyperbola and the imaginary ellipse.

The easiest choice of the location of the line (3) is a coincidence with the y-axis (or x-axis). In this case stationary points lay on the y-axis and in the equation (3) it is necessary to consider $a_{12} = a_{13} = 0$ and the equation (1) takes the form

$$a_{11}x^2 + a_{22}y^2 + 2a_{23}y + a_{33} = 0. \qquad (5)$$

Geometrically this means the rotation of the coordinate system ($a_{12} = 0$) and the parallel transport along the x-axis ($a_{13} = 0$).

In the equation (5) at $a_{22} \neq 0$ always it is possible to allocate a full square on a variable y and to write down this equation in the form

$$a_{11}x^2 + a_{22}y'^2 + a'_{33} = 0, \qquad (6)$$

where $y' = y - y_o$, $y_o = a_{23} / a_{22}$, $a'_{33} = a_{22}y_o^2 + a_{33}$.

Geometrically parameter y_o means the parallel transport along the y-axis of the coordinate system. According to the signs of the factors, we obtain the equations for the three geometric forms – an ellipse, a hyperbola and the equation of an imaginary ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1. \qquad (7)$$

In terms of the classification of second-order curves for the stationary points we take in equation (5) $a_{23} = 0$. It means the symmetry of the stationary points of these curves with respect to the x-axis.

Degenerate cases of second-order curves.

Particular cases of equation (6) are obtained when some factors of this equation are equal to zero. If $a'_{33} = 0$, then according to the signs of the factors, we have two geometric images – or a pair of skew lines, or a pair of imaginary skew lines

$$a_{11}x^2 - a_{22}y'^2 = 0, \quad a_{11}x^2 + a_{22}y'^2 = 0.$$

Two other special cases of the equation (6) can be obtained at $a_{22} = 0$. Then we have a pair of parallel straight lines and a pair of imaginary straight lines $x^2 = |a''_{33}|$, $x^2 = -|a''_{33}|$, where $|a''_{33}| = a'_{33} / a_{11}$.

Finally, when $a'_{33} = a_{22} = 0$ we have a pair of matching straight lines $x^2 = 0$.

Parabola.

The case of the parabola is obtained from the condition $y' \neq 0$ for different values of x and y. Then in the equation (3) should be taken $a_{11} = a_{12} = 0,\ a_{13} \neq 0$. Get the equation

$$a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0. \qquad (8)$$

If to allocate a full square on y, we get the equation $a_{22}y'^2 + 2a_{13}x + a'_{33} = 0$. *In the last equation, we define new* coordinates $x' = x - x_o,\ x_o = -a'_{33} / 2a_{13}$ and obtain the canonical equation of a parabola

$$y^2 = 2px,\ \ p = -a_{13} / a_{22}. \qquad (9)$$

The equilateral hyperbola.

In the equation of a hyperbola (7) it is believed $a = b$, that is equivalent to $a_{11} = -a_{22}$ in equation (6). Apply the formula for the difference of the squares and get

$$a_{11}(x - y)(x + y) + a'_{33} = 0 \qquad (10)$$

We designate $p = -a'_{33} / a_{11}$ and make the change of variables $X = x - y,\ Y = x + y$. Then equation (10) takes the familiar form

$$XY = p.$$

Table 1 – Classification of second-order curves for the stationary points of these curves

The equation of the stationary points of the curve (1) $y' = 0$ *is equivalent to the equation of a straight line* $a_{12}x + a_{22}y + a_{23} = 0$.	
Particular cases of the equation $a_{11}x + a_{12}y + a_{13} = 0$	Position of stationary points on the plane, and their equation
$a_{12} = a_{13} = 0$	Stationary points are located on the y-axis $a_{11}x^2 + a_{22}y^2 + 2a_{23}y + a_{33} = 0$
$a_{12} = a_{13} = 0,$ $a_{23} = 0$	Stationary points are located on the y-axis symmetrically with respect to the x-axis $a_{11}x^2 + a_{22}y^2 + a_{33} = 0$ Ellipse, Hyperbola.
Stationary points are not present $y' \neq 0$	
a_{1} 0	$a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0.$ A parabola with vertex at some point in the plane
a_{1} 0, a_{1} 0	$a_{22}y^2 + 2a_{13}x = 0.$ A parabola with vertex at the coordinate origin

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