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Classification of the Necessary Tests of Convergence in the Theory of Numerical Series

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Классификация необходимых условий сходимости в теории числовых рядов

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Класифікація необхідних умов збіжності в теорії числових рядів

The paper suggests three types of the necessary tests for the convergence of numerical series with positive terms. It is shown that these tests are consequences of the limit test of comparison with respect to the standard series: harmonic and logarithmic ones. The new tests have different capabilities and can be applied to almost any series, whose terms form a monotone sequence. The obtained necessary tests for the convergence of numerical series have more opportunities, than the accepted standard one $\lim\limits_{n\to\infty} u_n = 0$.

Keywords: limit, L'Hospital's rule, series, convergence, divergence, test of comparison of series, harmonic series, logarithmic series and geometric progression series.

В работе предложены три вида необходимых признаков сходимости числовых рядов с положительными членами. Показано, что эти признаки являются следствиями применения предельного признака сравнения к стандартным рядам: гармоническому и логарифмическому. Новые признаки сходимости имеют различные возможности и могут применяться практически к любым рядам, члены которых образуют монотонную последовательность. Полученные признаки имеют более широкие возможности, чем стандартный необходимый признак сходимости рядов $\lim\limits_{n\to\infty} u_n = 0$.

Ключевые слова: предел, правило Лопиталя, сходимость, расходимость, признак сравнения рядов, гармонический ряд, логарифмический ряд, ряд геометрической прогрессии.

У роботі запропоновані три види необхідних ознак збіжності числових рядів з додатними членами. Показано, що ці ознаки є наслідками застосування граничного ознаки порівняння зі стандартними рядами: гармонічним та логарифмічним. Нові ознаки збіжності мають різні можливості і можуть застосовуватися практично до будь-яких рядах, члени яких утворюють монотонну послідовність. Отримані ознаки мають більш широкі можливості, ніж стандартна необхідна ознака збіжності рядів $\lim\limits_{n\to\infty} u_n = 0$.

Ключові слова: границя, правило Лопіталя, збіжність, розбіжність, ознака порівняння рядів, гармонічний ряд, логарифмічний ряд, ряд геометричної прогресії.

Introduction

The necessary tests for the convergence of series play an important practical role in the evaluation of the convergence and divergence of numerical series with non-negative terms. It is believed that they are negative, because they can only define the divergence of the series. If the necessary tests are violated, we can definitely affirm the divergence of the series. If they are satisfied, the question of convergence can be finally resolved only by sufficient tests for the convergence of series [1]. First of all, these are tests of comparison with the standard series, such as the series of geometric progression, generalized harmonic series and generalized logarithmic series [2].

In the theory of the numerical series the test of comparison of series with positive terms commonly used in two forms - in the finite form and in the limit form [3]. In the first case, the two series are compared. The first series is

$$\sum_{n=1}^{\infty} u_n,\ u_n \geq 0 \tag{1}$$

and the second series is $\sum_{n=1}^{\infty} v_n,\ v_n \geq 0$. If there is a number $M > 0$ such that, starting from a certain number N, the inequality $u_n \leq M \cdot v_n$ takes place and the series $\sum_{n=1}^{\infty} v_n$ converges, then the series $\sum_{n=1}^{\infty} u_n$ converges too. But if the series $\sum_{n=1}^{\infty} u_n$ diverges, then the series $\sum_{n=1}^{\infty} v_n$ diverges also. [1]

In the limit test of comparison the expression $\lim_{n\to\infty} u_n / v_n$ is considered. If the series $\sum_{n=1}^{\infty} v_n,\ v_n > 0$ converges, and the limit value is equal to $C < \infty$ or $C = 0$, then the series $\sum_{n=1}^{\infty} u_n$ converges also. If the series $\sum_{n=1}^{\infty} v_n$ diverges, and the size of the limit is equal to $C \neq 0$ or $C = \infty$, then the series $\sum_{n=1}^{\infty} u_n$ diverges also [3], [4].

The series of comparison is usually taken from the three standard series: the harmonic one with the general term $v_n = 1/n$, the generalized harmonic one with the general term $v_n = 1/n^{\alpha}$ and the series of geometric progression with the general term $v_n = q^n$.

1 The Necessary Test for the Convergence of Series in the Classical Approach

Theorem. If the series (1) $\sum_{n=1}^{\infty} u_n,\ u_n \geq 0$ converges, then the limit of the general term of the series is zero

$$\lim_{n\to\infty} u_n = 0. \tag{2}$$

Proof. If the series (1) converges, then there is a finite limit of the sequence of partial sums $s_k = \sum_{n=1}^{k} u_n$, where $\lim_{k\to\infty} s_k = s$. The same limit is the limit of the sequence of partial sums $\{s_{k-1}\}$, where $\lim_{k\to\infty} s_{k-1} = s$. Then we have $\lim_{n\to\infty} u_n = \lim_{n\to\infty}(s_n - s_{n-1}) = s - s = 0$.

Consequence. The series $\sum_{n=1}^{\infty} u_n,\ u_n \geq 0$ diverges at $\lim_{n\to\infty} u_n \neq 0$.

Remark. The condition $\lim_{n\to\infty} u_n = 0$ is not the sufficient condition for the convergence of series (1), so it is usually used to establish the divergence of the series.

Examples.

1. The series $\sum_{n=1}^{\infty}\frac{n-1}{2n+1}$ diverges, since $\lim_{n\to\infty}\frac{n-1}{n+1}=\frac{1}{2}\neq 0$.

2. The series $\sum_{n=1}^{\infty}\frac{n-1}{(2n+1)^2}$ is given. The necessary test (2) of convergence is performed, since $\lim_{n\to\infty}\frac{n-1}{(2n+1)^2}=0$, but the conclusion about the convergence of the series cannot be done. It is necessary to use any sufficient test of convergence.

3. The series $\sum_{n=1}^{\infty}\frac{n-1}{(2n+1)^2\ln(n+1)}$ is given. The necessary test (2) of convergence is performed, since $\lim_{n\to\infty}\frac{n-1}{(2n+1)^2\ln(n+1)}=0$. As in the previous example, the conclusion about the convergence of the series cannot be done.

2 The Necessary Test for the Convergence of Series Compared with Divergent Series

1. The necessary test of convergence of the series in comparison with the harmonic series. Let us write the limit test of comparison of any arbitrary series (1) with respect to the harmonic series $\sum_{n=1}^{\infty}v_n$, $,v_n=1/n$: $\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{u_n}{1/n}=\lim_{n\to\infty}n\cdot u_n$.

Consider all the possible values of this limit.

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}n\cdot u_n=\left\{\begin{matrix}\infty\\ C\neq 0\\ 0\end{matrix}\right\} \qquad (3)$$

It is known that the harmonic series diverges. If this limit is equal to ∞ or $C\neq 0$, then the series (1) diverges. If the limit is equal to 0, then the series (1) either converges or diverges. So the test of comparison of series in the limit form works. Let us change the statement of the problem. Now assume that the series (1) converges. Then the first two values of the limit (3) ∞ or $C\neq 0$ are not possible, because they disagree to the assumption that the series (1) converges. So remains only the third option, which is equal to zero. This case should be understood as the necessary test for convergence of the series (1).

$$\lim_{n\to\infty}n\cdot u_n=0. \qquad (4)$$

Thus, it has been established from the limit comparison test with respect to the harmonic series, that the new necessary test for the convergence of the series in the form (4) is much "stronger" than the standard one in the form (2).

Examples.

2. The series $\sum_{n=1}^{\infty}\frac{n-1}{(2n+1)^2}$ is given. The necessary test for the convergence of the series in the form (4) is not performed, since $\lim_{n\to\infty}n\cdot\frac{n-1}{(2n+1)^2}=\frac{1}{4}$. Hence the series diverges. Above in Example 2, when the necessary test (2) was applied to the same series, the conclusion of convergence of the series was impossible.

3. The series $\sum_{n=1}^{\infty}\frac{n-1}{(2n+1)^2\ln(n+1)}$ is given. The necessary test for the convergence of the series in the form (4) is performed, since $\lim_{n\to\infty} n\frac{n-1}{(2n+1)^2\ln(n+1)}=0$, but the conclusion of the convergence of this series cannot be done.

Remark. The necessary test (4) works only for series, whose terms form a monotone sequence, i.e. $u_n > u_{n+1}$. Otherwise the limit (4) may not exist. For example, in the series $\sum_{n=1}^{\infty} u_n$ with the general term $u_n=\begin{cases}1/m^2 & at\ n=m^2\\ 0 & at\ n\neq m^2\end{cases}$ the terms u_n do not form a monotone sequence. For this series the limit (4) does not exist.

4. The necessary test of convergence of the series in comparison with the logarithmic series. Let us write the limit test of comparison of any arbitrary series (1) with respect to the logarithmic series $\sum_{n=1}^{\infty} v_n$, $,v_n = 1/n\ln n$:

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{u_n}{1/n\ln n}=\lim_{n\to\infty} n\ln n\cdot u_n. \qquad (5)$$

Repeat the same reasoning as for the harmonic series. It is known that this logarithmic series diverges. Suppose that the series (1) converges. The values ∞ or $C\neq 0$ of the limit (5) are not possible, since they contradict to the assumption of convergence of the series (1). Remains only one option - the limit (5) is zero. The latter case is to be understood as the necessary test for the convergence of the series (1):

$$\lim_{n\to\infty} n\cdot\ln n\cdot u_n = 0. \qquad (6)$$

Thus, from the limit comparison test of series was established the second new necessary test for the convergence of series in the form (6), which is "stronger", than test (4).

Example.

5. The series $\sum_{n=1}^{\infty}\frac{n-1}{(2n+1)^2\ln(n+1)}$ is given. The necessary test for the convergence of the series in the form (6) is not performed, since $\lim_{n\to\infty} n\ln n\frac{n-1}{(2n+1)^2\ln(n+1)}=\frac{1}{4}$. Hence the series diverges. Above, in Examples 2 and 5, the necessary tests (2) and (4) were applied, but the conclusion of the convergence of series could not be done.

3 The Other Representations of Necessary Tests for Convergence Of Series

Let us designate the series (1) thus $\sum_{n=1}^{\infty}\frac{1}{w_n}$, $w_n = 1/u_n$. Then the first necessary test for the convergence of the series (2) will be:

$$\lim_{n\to\infty}\frac{1}{w_n}=0\ ;\ \lim_{n\to\infty} w_n=\infty. \qquad (7)$$

So the second necessary test for convergence of series (4) will be $\lim_{n\to\infty}\frac{n}{w_n}=0$. From the necessary test (2) it follows $w_n \to \infty$. Therefore, there is indeterminacy $\{\infty/\infty\}$, which will be disclosed by L'Hospital's rule $\lim\limits_{n\to\infty}\frac{n}{w_n}=\lim\limits_{n\to\infty}\frac{n'}{w_n'}=\lim\limits_{n\to\infty}\frac{1}{w_n'}$. As a result, we came to the modification of the second necessary test for convergence of series in the form:

$$\lim_{n\to\infty}\frac{1}{w_n'}=0 \text{ или } \lim_{n\to\infty} w_n'=\infty. \quad (8)$$

The third necessary test for convergence of series (4) will be $\lim\limits_{n\to\infty}\frac{n\ln n}{w_n}=0$. From the necessary test (2) it follows $w_n \to \infty$. Then there is indeterminacy $\{\infty/\infty\}$, which can be disclosed by L'Hospital's rule $\lim\limits_{n\to\infty}\frac{n\ln n}{w_n}=\lim\limits_{n\to\infty} n\lim\limits_{n\to\infty}\frac{(\ln n)'}{w_n'}=\lim\limits_{n\to\infty} n\lim\limits_{n\to\infty}\frac{1}{nw_n'}=\lim\limits_{n\to\infty}\frac{1}{w_n'}=0$. As a result it is obtained that the third modified necessary test for convergence of series coincides with the second modified necessary test for convergence of series (8). However, this third necessary test for convergence of series can be expressed via the second derivative:

$$\lim_{n\to\infty}\frac{n\ln n}{w_n}=\lim_{n\to\infty}\ln n\lim_{n\to\infty}\frac{n'}{w_n'}=\lim_{n\to\infty}\ln n\lim_{n\to\infty}\frac{1}{w_n'}=\lim_{n\to\infty}\frac{\ln n}{w_n'}=\lim_{n\to\infty}\frac{(\ln n)'}{w_n''}=\lim_{n\to\infty}\frac{1}{nw_n''}.$$

As a result we came to the modification of the third necessary test for convergence of series in the form: $\lim\limits_{n\to\infty}\frac{1}{nw_n''}=0$ or $\lim\limits_{n\to\infty} nw_n''=\infty$.

The obtained results can be tabulated (Table).

Table 1 – Classification of the necessary conditions for convergence of series with positive terms

A comparison series $\sum_{n=1}^{\infty} u_n$	Necessary test	Other forms of necessary test $\sum_{n=1}^{\infty} 1/w_n$	Necessary test opportunities
$\sum_{n=1}^{\infty} 1$	$\lim\limits_{n\to\infty} u_n=0$	$\lim\limits_{n\to\infty} w_n=\infty$	light
Harmonic series $\sum_{n=1}^{\infty} 1/n$	$\lim\limits_{n\to\infty} nu_n=0$	$\lim\limits_{n\to\infty}\frac{1}{w_n'}=0$ или $\lim\limits_{n\to\infty} w_n'=\infty$	medium
Logarithmic series $\sum_{n=1}^{\infty} 1/n\ln n$	$\lim\limits_{n\to\infty} n\ln n\cdot u_n=0$	$\lim\limits_{n\to\infty}\frac{1}{w_n'}=0$ или $\lim\limits_{n\to\infty}\frac{1}{nw_n''}=0$	strong

In Table 1 the notions light, medium, strong are relative because intermediate variants and the expansion of the table to the strongest notation are possible. For example, the logarithmic series of the second order creates stronger necessary test for the convergence of series $\lim\limits_{n\to\infty} n\cdot\ln n\cdot(\ln\ln n)\cdot u_n=0$, than $\lim\limits_{n\to\infty} n\cdot\ln n\cdot u_n=0$.

Summarizing the results of this paper, it is possible to formulate the theorem on the necessary condition for the convergence of numerical series with positive terms.

Theorem. If the series $\sum_{n=1}^{\infty} u_n$, $u_n\ge 0$, $u_n>u_{n+1}$ converges, and $\sum_{n=1}^{\infty} v_n$, $v_n>0$ is an arbitrary divergent series, then the condition $\lim_{n\to\infty}\frac{u_n}{v_n}=0$ is the necessary condition for the convergence of the initial series $\sum_{n=1}^{\infty}u_n$.

If the general term of the series v_n to take equal to $1, 1/n$ or $1/n\ln n$ it will give us the necessary conditions for the convergence of the series presented in the table (Table 1).

Findings

This paper describes three types of the necessary tests of convergence of numerical series with positive terms. Thus

1. It is shown that these tests are the consequences of the application of the limit test of comparison to the standard series: geometric, harmonic and logarithmic ones.
2. These new necessary tests of the convergence of numerical series have different capabilities and can be applied to almost any series with positive terms, whose terms form a monotone sequence.
3. The obtained necessary tests have more opportunities than the accepted standard necessary test of convergence of numerical series (2).
4. It is shown that the necessary test for the convergence of numerical series in the standard form $\lim_{n\to\infty}u_n=0$ can be successfully replaced by more powerful new necessary tests for the convergence of numerical series, which are the next: $\lim_{n\to\infty}nu_n=0$, $\lim_{n\to\infty}n\ln n\cdot u_n=0$, $\lim_{n\to\infty}n\cdot\ln n\cdot(\ln\ln n)\cdot u_n=0$ etc.
5. These results can be transferred to the improper integrals.

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