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J.F. KOSOLAPOV

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КОСОЛАПОВ Ю.Ф.

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Викладаються окремі розділи дуже розгалуженої теорії рядів (числові ряди з додатними та довільними дійсними членами, найпростіші ознаки збіжності рядів з додатними членами, абсолютна й умовна збіжності, поняття про функціональний ряд та область його збіжності, степеневі ряди, їх властивості та застосування, ряди Фур"є). Докладно розглядаються приклади розв'язання типових задач. Вміщено англо-українсько-російський термінологічний словник до всіх розділів. Дано завдання для самостійного розв'язання.

Значну допомогу в створенні посібника надали автору студенти факультету економіки і менеджменту ДонНТУ Константинова А., Хорунжая О., Маринова К., Мамічева В., Місіньова О. (впорядкування студентських лекційних конспектів та створення їх електронних версій, редагування англомовного тексту, робота над термінологічним словником). Суттєвий внесок в написання посібника внесла старший викладач Слов''янського педагогічного університету Косолапова Н. В. (підготовка ілюстративного матеріалу, робота над термінологією).

Автор висловлює щиру подяку всім своїм помічникам.

Для студентів і викладачів технічних вузів.

УКЛАДАЧ: Косолапов Ю.Ф.

РЕЦЕНЗЕНТ: кандидат фізико-математичних наук, доцент Кочергін Є.В.

ВІДПОВІДАЛЬНИЙ ЗА ВИПУСК:
зав. кафедри вищої математики ДонНТУ,
доктор технічних наук, професор Улітін Г.М.

SERIES

LECTURE NO. 28. NUMERICAL SERIES

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POINT 1. CONVERGENCE AND DIVERGENCE OF NUMERICAL SERIES

Def. 1. A numerical series with terms $u_1, u_2, u_3, ..., u_n, ...$ is called an expression (a symbol)

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + ... + u_n + u_{n+1} + u_{n+2} + ... + u_{n+k} + ... = \qquad (1)$$

$$= \sum_{m=1}^{n} u_m + \sum_{m=n+1}^{\infty} u_m = S_n + \sum_{m=n+1}^{\infty} u_m$$

Def. 2. The numerical u_n is called the **general term** of the series (1).

Ex. 1. Find the general term of the series

$$\frac{1}{2\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{8\cdot 13} + \frac{1}{11\cdot 17} + ... \,.$$

The first and second factors in the denominators form the arithmetical progressions with the first terms $a_1 = 2, b_1 = 5$, arithmetical ratios [differences] $d_1 = 3$, $d_2 = 4$ and the n-th terms

$$a_n = a_1 + d_1(n-1) = 2 + 3(n-1) = 3n - 1, b_n = b_1 + d_2(n-1) = 5 + 4(n-1) = 4n + 1\,.$$

Therefore the general term in question equals

$$u_n = \frac{1}{a_n b_n} = \frac{1}{(3n-1)(4n+1)}\,.$$

Def. 3. The sum of n first terms of the series (1), namely

$$S_n = \sum_{m=1}^{n} u_m = u_1 + u_2 + \ldots + u_n, \qquad (2)$$

is called its ***n*-th partial sum**.

For example, the first, second and third partial sums are equal to

$$S_1 = u_1,\ S_2 = u_1 + u_2,\ S_3 = u_1 + u_2 + u_3, \ldots$$

Def. 4. The series

$$\sum_{m=n+1}^{\infty} u_m = u_{n+1} + u_{n+2} + \ldots + u_{n+k} + \ldots \qquad (3)$$

is called the ***n*-th remainder** of the series (1).

Def. 5. If there exists the limit of the n-th partial sum of the series (1) for $n \to \infty$,

$$\exists \lim_{n\to\infty} S_n = S \neq \infty, \qquad (4)$$

the series is called **convergent** one. The number S is called the **sum** of the series, and one can write

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \ldots + u_n + \ldots = S \qquad (5)$$

and tell that the series converges to S.

Ex. 2. Investigate for convergence the series

$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)\cdot(3n+2)} = \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \ldots + \frac{1}{(3n-1)\cdot(3n+2)} + \ldots.$$

Let's at first remark that

$$\frac{1}{(3n-1)(3n+2)} = \frac{1/3}{3n-1} - \frac{1/3}{3n+2} = \frac{1}{3}\left(\frac{1}{3n-1} - \frac{1}{3n+2}\right)$$

because of

$$\frac{1}{(3n-1)(3n+2)} = \frac{A}{3n-1} + \frac{B}{3n+2},\ 1 = A(3n+2) + B(3n-1),\ 1 = (3A+3B)n + (2A-B),$$

$$\begin{cases} 3A + 3B = 0, \\ 2A - B = 1; \end{cases} \quad \begin{cases} A + B = 0, \\ 2A - B = 1; \end{cases} \quad \Rightarrow A = \frac{1}{3},\ B = -\frac{1}{3}..$$

Assigning successively the values 1, 2, 3,… to n we'll represent the n-th partial sum of the given series as follows

$$S_n = \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \frac{1}{11\cdot 13} + \ldots + \frac{1}{(3n-1)(3n+2)} =$$

$$= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5}\right) + \frac{1}{3}\left(\frac{1}{5} - \frac{1}{8}\right) + \frac{1}{3}\left(\frac{1}{8} - \frac{1}{11}\right) + \ldots + \frac{1}{3}\left(\frac{1}{3n-4} - \frac{1}{3n-1}\right) + \frac{1}{3}\left(\frac{1}{3n-1} - \frac{1}{3n+2}\right) =$$

$$= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{5} + \frac{1}{5} - \frac{1}{8} + \frac{1}{8} - \frac{1}{11} + \ldots + \frac{1}{3n-4} - \frac{1}{3n-1} + \frac{1}{3n-1} - \frac{1}{3n+2}\right) = \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right).$$

Hence

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3n+2}\right) = \frac{1}{3}\lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{3n+2}\right) = \frac{1}{3}\left(\frac{1}{2} - \lim_{n\to\infty}\frac{1}{3n+2}\right) = \frac{1}{3}\left(\frac{1}{2} - 0\right) = \frac{1}{6} \neq \infty$$

By the definition of convergence the given series converges to the sum $S = 1/6$.

Ex. 3. Investigate for convergence the series

$$\sum_{n=1}^{\infty} \frac{14}{49n^2 - 70n - 24}.$$

By analogy with preceding example we represent the general term as the difference of two simple fractions

$$\frac{14}{49n^2 - 70n - 24} = \frac{14}{(7n+2)(7n-12)} = \frac{A}{7n-12} + \frac{B}{7n+2} = \frac{1}{7n-12} - \frac{1}{7n+2}$$

and then (taking successively $n = 1, 2, 3,\ldots$) obtain the n-th partial sum and the sum of the series

$$S_n = -\frac{1}{5} - \frac{1}{9} + \frac{1}{2} - \frac{1}{16} + \frac{1}{9} - \frac{1}{23} + \frac{1}{16} - \frac{1}{30} + \ldots + \frac{1}{7n-33} - \frac{1}{7n-19} + \frac{1}{7n-26} - \frac{1}{7n-12} +$$

$$+ \frac{1}{7n-19} - \frac{1}{7n-5} + \frac{1}{7n-12} - \frac{1}{7n+2} = -\frac{1}{5} + \frac{1}{2} - \frac{1}{7n-5} - \frac{1}{7n+2};$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(-\frac{1}{5} + \frac{1}{2} - \frac{1}{7n-5} - \frac{1}{7n+2}\right) = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}.$$

Thus the given series converges and has the sum $S = 0.3$ (it converges to 0.3).

Ex. 4. Prove yourselves that the series

$$\sum_{n=1}^{\infty} \frac{6}{9n^2 + 12n - 5}$$

converges and has the sum $S = 0.7$.

Ex. 5. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{3n+8}{n(n+1)(n+2)}.$$

Answer.

$$\frac{3n+8}{n(n+1)(n+2)} = \frac{4}{n} - \frac{5}{n+1} + \frac{1}{n+2};\ S_n = 4 - \frac{5}{2} + 2 + \frac{1}{n+1} - \frac{5}{n+1} + \frac{1}{n+2};\ S = 3.5.$$

Ex. 6. The **geometric progression**

$$a + aq + aq^2 + \ldots + aq^{n-1} + \ldots \qquad (6)$$

with the ratio q converges in the case $|q| < 1$ and has the sum

$$S = \frac{a}{1-q},$$

that is

$$\sum_{n=1}^{\infty} aq^{n-1} = a + aq + aq^2 + \ldots + aq^{n-1} + \ldots = \frac{a}{1-q},\ |q| < 1. \qquad (7)$$

Indeed, the n-th partial sum of the progression equals

$$S_n = a + aq + aq^2 + \ldots + aq^{n-1} = \frac{a(1-q^n)}{1-q} = \frac{a}{1-q} - \frac{a}{1-q} \cdot q^n$$

and has the limit $S = a/(1-q)$ for $n \to \infty$, because of $\lim_{n\to\infty} q^n = 0$ if $|q| < 1$.

Ex. 7. Test for convergence the series

$$\sum_{n=1}^{\infty} \frac{2^n - 3^n + 5^n}{30^n}$$

on the base of the definition of convergence.

Dividing termwise we represent the n-th partial sum of the series as follows

$$S_n = \sum_{k=1}^{n} \frac{2^k - 3^k + 5^k}{30^k} = \sum_{k=1}^{n} \left(\left(\frac{1}{15}\right)^k - \left(\frac{1}{10}\right)^k + \left(\frac{1}{6}\right)^k \right) = \sum_{k=1}^{n} \left(\frac{1}{15}\right)^k - \sum_{k=1}^{n} \left(\frac{1}{10}\right)^k + \sum_{k=1}^{n} \left(\frac{1}{6}\right)^k.$$

We obtain three geometric progressions with the first terms

$$a_1 = \frac{1}{15},\ a_2 = \frac{1}{10},\ a_3 = \frac{1}{6}$$

and the ratios

$$q_1 = \frac{1}{15},\ q_2 = \frac{1}{10},\ q_3 = \frac{1}{6}.$$

Hence the n-th partial sum is

$$S_n = \frac{1/15 - (1/15)^n}{1 - 1/15} - \frac{1/10 - (1/10)^n}{1 - 1/10} + \frac{1/6 - (1/6)^n}{1 - 1/6},$$

and the sum of the series equals

$$S = \lim_{n\to\infty} S_n = \frac{1/15}{1-1/15} - \frac{1/10}{1-1/10} + \frac{1/6}{1-1/6} = \frac{1}{14} - \frac{1}{9} + \frac{1}{5} = \frac{101}{630} \approx 0.16.$$

Def. 6. If

$$\lim_{n\to\infty} S_n = \infty$$

or the limit

$$\lim_{n\to\infty} S_n$$

doesn't exist, the series is called **divergent** one. One can say that the series diverges.

Ex. 8. The arithmetic progression

$$1 + 2 + 3 + 4 + \ldots + n + \ldots$$

diverges, because of its n-th partial sum equals

$$S_n = 1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$$

and has the infinite limit for $n \to \infty$.

Ex. 9. The geometric progression (6) diverges for $|q| \ge 1$ and $a \ne 0$.

■ a) If $|q| > 1$ then $|q^n| \to \infty$ as $n \to \infty$, and the limit of the n-th partial sum, $\lim\limits_{n\to\infty} S_n$, is infinite or doesn't exist.

b) If $q = 1$, the progression takes on the form $a + a + a + \ldots + a + \ldots$, has the n-th partial sum $S_n = an$ whose limit equals $+\infty$ for $a > 0$ and $-\infty$ for $a < 0$.

c) If $q = -1$, the progression has the form

$$a + (-a) + a + (-a) + a + (-a) + a + (-a) + \ldots = a - a + a - a + a - a + a - a + \ldots,$$

its n-th partial sum equals 0 for even n and a for odd n. Therefore the limit $\lim\limits_{n\to\infty} S_n$ doesn't exist.

Thus in all three cases a), b), c) the progression diverges.■

Ex. 10. **Harmonic series**

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots+\frac{1}{n^p}+\ldots \qquad (8)$$

converges for $p>1$ and diverges for $p\le 1$.

We'll prove this fact later.

For example the series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}+\ldots,\quad 1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\ldots+\frac{1}{n\sqrt{n}}+\ldots$$

converge ($p=2>1$, $p=3/2>1$ respectively), and the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots,\quad 1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+\frac{1}{\sqrt[3]{4}}+\ldots+\frac{1}{\sqrt[3]{n}}+\ldots$$

diverge ($p=1$, $p=1/3<1$ respectively).

Theorem 1. The **necessary** (but **not sufficient**) condition for convergence of the series (1) is the next:

$$\lim_{n\to\infty} u_n = 0. \qquad (9)$$

Theorem 1 means that if a series (1) converges, then the limit of its general term u_n, as $n\to\infty$, must be equal to zero.

■Let the series (1) converges to $S\ne\infty$. It means that

$$\exists\lim_{n\to\infty} S_n = S,\quad \exists\lim_{n\to\infty} S_{n-1} = S.$$

But

$$u_n = S_n - S_{n-1},$$

and so

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(S_n - S_{n-1}\right) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$■

Ex. 8. The series

$$\text{a) } \sum_{n=1}^{\infty} \frac{3n-2}{4n+7}, \text{ b) } \sum_{n=1}^{\infty} \frac{e^n}{3n+2}$$

diverge, for

$$\text{a) } \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{3n-2}{4n+7} = \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{n(3-2/n)}{n(4+7/n)} = \lim_{n\to\infty} \frac{3-2/n}{4+7/n} = \frac{3-0}{4+0} = \frac{3}{4},$$

$$\text{b) } \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{e^n}{3n+2} = \left(\frac{\infty}{\infty}\right) = \left| \lim_{x\to+\infty} \frac{e^x}{3x+2} = \lim_{x\to+\infty} \frac{(e^x)'}{(3x+2)'} = \lim_{x\to+\infty} \frac{e^x}{3} = +\infty \right| = +\infty,$$

and the necessary condition for convergence of the series isn't fulfilled.

Ex. 11. The necessary condition for convergence is fulfilled for the next two se-ries

$$\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}, \quad \sum_{n=1}^{\infty} \frac{n}{n^2+1},$$

but one can say nothing as to their convergence or divergence. Later we'll prove that the first series converges and the second diverges.

Theorem 2. If the series (1) converges, then for any n its n-th remainder (3) converges. If for some n the n-th remainder (3) of the series (1) converges, then the series by itself converges.

■We'll prove the first part of the theorem. Let the series (1) converges to S and σ_k is the k-th partial sum of the remainder (3),

$$\sigma_k = u_{n+1} + u_{n+2} + \ldots + u_{n+k}.$$

It is obvious that

$$\sigma_k = S_{n+k} - S_n,$$

hence there exists the limit

$$\lim_{k\to\infty} \sigma_k = \lim_{k\to\infty} (S_{n+k} - S_n) = \lim_{k\to\infty} S_{n+k} - S_n = S - S_n \neq \infty.$$

It means that the remainder (3) converges for each n.■

The meaning [essentiality, substance] of the theorem 2 consists in follows: convergence or divergence of a series doesn't change if one adds to it or rejects from it finite number of terms.

Corollary 1. Let's denote by R_n the sum of the *n*-th remainder of a convergent series. From the proving of the theorem 2 we'll obtain

$$R_n = \lim_{k\to\infty} \sigma_k = S - S_n$$

And therefore

$$S = S_n + R_n. \qquad (10)$$

The formula (10) represents the sum *S* of a convergent series as the sum of its *n*-th partial sum S_n and the sum R_n of corresponding *n*-th remainder.

Corollary 2. The sum R_n of the *n*-th remainder of a convergent series goes to zero for $n \to \infty$,

$$\lim_{n\to\infty} R_n = 0 \qquad (11)$$

■It follows from the formula (10) that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}(S - S_n) = S - \lim_{n\to\infty} S_n = S - S = 0 \;■$$

Corollary 3. For large *n* the sum *S* of a convergent series approximately equals

$$S \approx S_n \qquad (12)$$

with absolute error

$$\alpha = |R_n| \qquad (13)$$

which can be done however small for sufficiently large values of *n*.

In practice it isn't necessary to investigate a series for convergence with the help of the definition 5, that is by seeking the limit of the *n*-th partial sum. Often it is sufficient to ascertain its convergence or divergence from the other considerations [arguments, reasons] and in the case of convergence to find the sum of the series approximately.

There are many tests for convergence or divergence of series. We'll begin from stating without proof the next theorem.

Theorem 3 (Cauchy necessary and sufficient condition for convergence of a numerical series)**.** A numerical series (1) converges if and only if for any however small positive number ε there exists a number *N* such that for any greater number *n* and for arbitrary number *m* the inequality

$$\left|S_{n+m} - S_n\right| < \varepsilon$$

holds. Symbolically

(Series (1) converges) $\Leftrightarrow \left(\forall\varepsilon > 0, \exists N, \forall n, \forall m : \left\{n > N \Rightarrow \left|S_{n+m} - S_n\right| < \varepsilon\right\}\right)$. (14)

Theorem 4 (termwise linear operations on series). Let be given two series convergent to S and T respectively,

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots = \sum_{n=1}^{\infty} u_n = S, \quad v_1 + v_2 + v_3 + \ldots + v_n + \ldots = \sum_{n=1}^{\infty} v_n = T .$$

In this case for any number k

$$ku_1 + ku_2 + \ldots + ku_n + \ldots = \sum_{n=1}^{\infty}(ku_n) = k\sum_{n=1}^{\infty} u_n = k(u_1 + u_2 + \ldots + u_n + \ldots) = kS \qquad (15)$$

(taking a constant factor k out of a convergent series),

$$(u_1 \pm v_1) + (u_2 \pm v_2) + \ldots + (u_n \pm v_n) + \ldots = \sum_{n=1}^{\infty}(u_n \pm v_n) = \sum_{n=1}^{\infty} u_n \pm \sum_{n=1}^{\infty} v_n =$$
$$= (u_1 + u_2 + \ldots + u_n + \ldots) \pm (v_1 + v_2 + \ldots + v_n + \ldots) = S \pm T \qquad (16)$$

(termwise addition or subtraction of two convergent series), and for any numbers k and l

$$(ku_1 + lv_1) + (ku_2 + lv_2) + \ldots + (ku_n + lv_n) + \ldots = \sum_{n=1}^{\infty}(ku_n + lv_n) = k\sum_{n=1}^{\infty} u_n + l\sum_{n=1}^{\infty} v_n =$$
$$= (ku_1 + ku_2 + \ldots + ku_n + \ldots) + (lv_1 + lv_2 + \ldots + lv_n + \ldots) = k\sum_{n=1}^{\infty} u_n + l\sum_{n=1}^{\infty} v_n = kS + lT \qquad (17)$$

(termwise linear combination of two convergent series, a corollary of the formulas (15), (16)).

■Validity of the formula (15) follows from the equality which connects the n-th partial sums σ_n, S_n of the series

$$\sum_{n=1}^{\infty}(ku_n), \quad \sum_{n=1}^{\infty} u_n .$$

Namely,

$$\sigma_n = ku_1 + ku_2 + ku_3 + \ldots + ku_n = k(u_1 + u_2 + u_3 + \ldots + u_n) = kS_n$$

and therefore for the sums σ, S of the series we get

$$\sigma=\sum_{n=1}^{\infty}(ku_n)=\lim_{n\to\infty}\sigma_n=\lim_{n\to\infty}kS_n=k\lim_{n\to\infty}S_n=k\sum_{n=1}^{\infty}u_n=kS\,.\blacksquare$$

The formulas (16), (17) prove yourselves.

Ex. 12. The sum of the series

$$\sum_{n=1}^{\infty}\frac{2^n-3^n+5^n}{30^n}$$

(see Ex. 7) can be easy calculated with the help of the theorem 4 and the formula (7). Indeed,

$$\sum_{n=1}^{\infty}\frac{2^n-3^n+5^n}{30^n}=\sum_{n=1}^{\infty}\left(\left(\frac{1}{15}\right)^n-\left(\frac{1}{10}\right)^n+\left(\frac{1}{6}\right)^n\right)=\sum_{n=1}^{\infty}\left(\frac{1}{15}\right)^n-\sum_{n=1}^{\infty}\left(\frac{1}{10}\right)^n+\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^n=$$

$$=\frac{1/15}{1-1/15}-\frac{1/10}{1-1/10}+\frac{1/6}{1-1/6}=\frac{1}{14}-\frac{1}{9}+\frac{1}{5}=\frac{101}{630}\approx 0.16\,.$$

POINT 2. SUFFICIENT CONVERGENCE TESTS FOR NUMERICAL SERIES WITH POSITIVE TERMS

Cauchy theorem (theorem 3) on the necessary and sufficient condition for convergence of a numerical series is of great theoretical importance but of hard practical applications. We as usually deal with some simple sufficient conditions (sufficient tests) for convergence. At first we'll study numerical series with positive terms [positive term series] .

Let be given a numerical series with positive terms

$$\sum_{n=1}^{\infty}u_n=u_1+u_2+\ldots+u_n+\ldots,\ \forall n: u_n>0\,. \qquad (18)$$

Its partial sums form the increasing number [numerical] sequence

$$S_1<S_2<S_3<\ldots<S_n<\ldots, \qquad (19)$$

and on the base of the lecture No. 12 (point 3, general properties of limits of functions, property 5) we obtain the next theorem which states a very general sufficient condition for convergence.

Theorem 5. For converges of a positive term series it's sufficient the sequence of its partial sums to be bounded above.

In other words if it exists some number C such that for any n

$$S_n \le C, \tag{20}$$

then the series (18) converges.

Let we have two series with positive terms, namely (18) and a series

$$\sum_{n=1}^{\infty} v_n = v_1 + v_2 + \ldots + v_n + \ldots, \forall n : v_n > 0. \tag{21}$$

Theorem 6 (the **first comparing test** for positive term series). Let (at least for sufficient great values of n)

$$u_n \le v_n \text{ (in particular } u_n < v_n\text{)}. \tag{22}$$

1) If the series (21) converges, then the series (18) also converges.

2) If the series (18) diverges, then the series (21) also diverges.

■Let for example the series (21) converges to some number T, namely there exists the limit of its n-th partial sum σ_n,

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} (v_1 + v_2 + \ldots + v_n) = T.$$

It's obvious that

$$\sigma_n \le T.$$

By virtue of the inequality (22) (which can be supposed to be fulfil for any n) we have for the n-th partial sum S_n of the series (18)

$$S_n = u_1 + u_2 + \ldots + u_n \le u_1 + u_2 + \ldots + u_n = \sigma_n \le T,\ S_n \le T.$$

Therefore the sequence of partial sums of the series (18) is bounded above by the number T, and by the theorem 5 the series converges.■

With the help of the theory of limits one can prove the next theorem.

Theorem 7 (the **second comparing test** for series with positive terms). Let it exists the limit of the ratio of the general terms of the series (18) and (21),

$$\lim_{n\to\infty}\frac{u_n}{v_n}=k\,. \qquad (23)$$

If k is a finite positive number ($k\neq 0, k\neq\infty$), then both series (18), (21) converge or diverge simultaneously.

Remark 1. For limiting cases $k=0$ and $k=\infty$ we able to conclude as follows.

If $k=0$, then the series (18) converges in the case of convergence of (21), and (21) diverges in the case of divergence of (18).

If $k=\infty$, (18) diverges in the case of divergence of (21), and (21) converges in the case of convergence of (18).

To apply comparing tests we must possess some series with known convergence or divergence. One often uses various cases of the geometric progression (6) (see Ex. 6) and the harmonic series (8) (see Ex.10).

Ex. 13. Test for convergence a series

$$\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\dots.$$

Finding the general term of the series we can represent the latter as follows

$$\frac{1}{1\cdot 2^1}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\dots+\frac{1}{n\cdot 2^n}+\dots.$$

Then we compare it with the convergent series

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots+\frac{1}{2^n}+\dots$$

(the geometric progression (6) with the ratio $q=1/2$, $0<q<1$). The comparison gives

$$\frac{1}{n\cdot 2^n}<\frac{1}{2^n}$$

for any $n>1$. On the base of the theorem 6 (case 1)) the given series converges.

Ex. 14. Solve the same problem for the series

$$\sum_{n=2}^{\infty}\frac{\ln n}{n}=\frac{\ln 2}{2}+\frac{\ln 3}{3}+\frac{\ln 4}{4}+\dots+\frac{\ln n}{n}+\dots.$$

Observing that $\ln 2 > 1, \ln 3 > 1, \ln 3 > 1, ..., \ln n > 1, ...$ we compare the given series with the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n} + ...$$

(the harmonic series (7) with $p = 1$). The comparison gives

$$\frac{\ln n}{n} > \frac{1}{n}$$

for $n \geq 2$. By virtue of the theorem 6 (case 2)) the given series diverges.

Ex. 15. To investigate the series for convergence

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n^3 - 4n^2 + 3n + 7}}$$

we'll take for comparison the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$$

(the harmonic series (7) with $p = 3/2 > 1$) and make use of the second comparison test (theorem 7). The limit of ratio of the general terms of these two series is

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{2n^3 - 4n^2 + 3n + 7}} : \frac{1}{n^{\frac{3}{2}}} \right) = \lim_{n\to\infty} \frac{n^{\frac{3}{2}}}{n^{\frac{3}{2}}\sqrt{2 - \frac{4}{n} + \frac{3}{n^2} + \frac{7}{n^3}}} = \lim_{n\to\infty} \frac{1}{\sqrt{2 - \frac{4}{n} + \frac{3}{n^2} + \frac{7}{n^3}}} = \frac{1}{\sqrt{2}}$$

The limit $k = 1/\sqrt{2}$ is positive finite number, and so the given series converges simultaneously with the convergent harmonic series.

Theorem 8 (D'Alembert[1] test). If for a positive term series (18) there exists the limit

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l, \qquad (24)$$

then the series converges for $l < 1$ and diverges for $l > 1$. In the case $l = 1$ one can't say anything about behavior of the series.

■1. Let at first

[1] D'Alembert, J. (1717 - 1783), a French mathematician and philosopher

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=l<1.$$

By the theory of limit for any $\varepsilon>0$ there exists a number N such that for any $n\geq N$ the next inequalities hold:

$$\left|\frac{u_{n+1}}{u_n}-l\right|<\varepsilon, -\varepsilon<\frac{u_{n+1}}{u_n}-l<\varepsilon, l-\varepsilon<\frac{u_{n+1}}{u_n}<l+\varepsilon, (l-\varepsilon)u_n<u_{n+1}<(l+\varepsilon)u_n.$$

Let ε is so small that $l+\varepsilon<1$. Taking successively $n=N, N+1, N+2,...$ in the inequality

$$u_{n+1}<(l+\varepsilon)u_n,$$

we'll obtain

$$u_{N+1}<(l+\varepsilon)u_N,$$

$$u_{N+2}<(l+\varepsilon)u_{N+1}<(l+\varepsilon)^2u_N,$$

$$u_{N+3}<(l+\varepsilon)u_{N+2}<(l+\varepsilon)^3u_N,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

We see that for any $n\geq N+1$ the terms of the given series are less than the corresponding terms of the converging geometric progression with the ratio $q=l+\varepsilon<1$. By the theorem 6 (case 1)) the given series is convergent one.

2. Let now

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=l>1.$$

In this case for sufficiently large n we'll have

$$\frac{u_{n+1}}{u_n}>1\Rightarrow u_n<u_{n+1}<u_{n+2}<u_{n+3}<...,$$

and the necessary condition for convergence of the series isn't fulfilled. The series diverges.■

Ex. 16. Investigate for convergence the series

$$\sum_{n=1}^{\infty}\frac{n^2}{2^n}.$$

Here

$$u_n = \frac{n^2}{2^n}, \quad u_{n+1} = \frac{(n+1)^2}{2^{n+1}},$$

and

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n\to\infty}\frac{2^n(n+1)^2}{2^{n+1}n^2} = \lim_{n\to\infty}\frac{2^n\left(n\left(1+\frac{1}{n}\right)\right)^2}{2^n\cdot 2n^2} = \lim_{n\to\infty}\frac{n^2\left(1+\frac{1}{n}\right)^2}{2n^2} =$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = \frac{1}{2} < 1.$$

By virtue of D'Alembert test the series converges.

Ex. 17. Solve the same problem for the series

$$\frac{1}{2} + \frac{4\cdot 7}{2\cdot 6} + \frac{4\cdot 7\cdot 10}{2\cdot 6\cdot 10} + \ldots\ .$$

The general term u_n and the next term u_{n+1} of the series are

$$u_n = \frac{4\cdot 7\cdot 10\cdot\ldots\cdot(4+3\cdot(n-1))}{2\cdot 6\cdot 10\cdot\ldots\cdot(2+4\cdot(n-1))} = \frac{4\cdot 7\cdot 10\cdot\ldots\cdot(3n+1)}{2\cdot 6\cdot 10\cdot\ldots\cdot(4n-2)},$$

$$u_{n+1} = \frac{4\cdot 7\cdot 10\cdot\ldots\cdot(3n+1)\cdot(3(n+1)+1)}{2\cdot 6\cdot 10\cdot\ldots\cdot(4n-2)\cdot(4(n+1)-2)} = \frac{4\cdot 7\cdot 10\cdot\ldots\cdot(3n+1)\cdot(3n+4)}{2\cdot 6\cdot 10\cdot\ldots\cdot(4n-2)\cdot(4n+2)}.$$

Hence

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\left(\frac{4\cdot 7\cdot 10\cdot\ldots\cdot(3n+1)\cdot(3n+4)}{2\cdot 6\cdot 10\cdot\ldots\cdot(4n-2)\cdot(4n+2)} : \frac{4\cdot 7\cdot 10\cdot\ldots\cdot(3n+1)}{2\cdot 6\cdot 10\cdot\ldots\cdot(4n-2)}\right) =$$

$$= \lim_{n\to\infty}\frac{4\cdot 7\cdot 10\cdot\ldots\cdot(3n+1)\cdot(3n+4)\cdot 2\cdot 6\cdot 10\cdot\ldots\cdot(4n-2)}{2\cdot 6\cdot 10\cdot\ldots\cdot(4n-2)\cdot(4n+2)\cdot 4\cdot 7\cdot 10\cdot\ldots\cdot(3n+1)} = \lim_{n\to\infty}\frac{3n+4}{4n+2} = \lim_{n\to\infty}\frac{3n}{4n} = \frac{3}{4} < 1.$$

The series is convergent by D'Alembert test.

Ex. 18. The same problem for the series

$$\sum_{n=1}^{\infty}\frac{(2n+1)!\sqrt[3]{3n-1}}{(3n)!}.$$

We have

$$u_n = \frac{(2n+1)!\sqrt[3]{3n-1}}{(3n)!},\ u_{n+1} = \frac{(2(n+1)+1)!\sqrt[3]{3(n+1)-1}}{(3(n+1))!} = \frac{(2n+3)!\sqrt[3]{3n+2}}{(3n+3)!};$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left(\frac{(2n+3)!\sqrt[3]{3n+2}}{(3n+3)!}:\frac{(2n+1)!\sqrt[3]{3n-1}}{(3n)!}\right)=\lim_{n\to\infty}\frac{(2n+3)!\,(3n)!\sqrt[3]{3n+2}}{(3n+3)!\,(2n+1)!\sqrt[3]{3n-1}}=$$

$$=\lim_{n\to\infty}\frac{(2n+1)!\,(2n+2)(2n+3)(3n)!}{(3n)!\,(3n+1)(3n+2)(3n+3)(2n+1)!}\cdot\lim_{n\to\infty}\sqrt[3]{\frac{3n+2}{3n-1}}=$$

$$=\lim_{n\to\infty}\frac{(2n+2)(2n+3)}{(3n+1)(3n+2)(3n+3)}\cdot\lim_{n\to\infty}\sqrt[3]{\frac{n\left(3+\frac{2}{n}\right)}{n\left(3-\frac{1}{n}\right)}}=$$

$$=\lim_{n\to\infty}\frac{n^2\left(2+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}{n^3\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)}\cdot\lim_{n\to\infty}\sqrt[3]{\frac{3+\frac{2}{n}}{3-\frac{1}{n}}}=$$

$$n\to\infty$$

$$\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}{\left(3+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{3}{n}\right)}\cdot 1=0\cdot\frac{4}{27}=0<1.$$

The series converges.

Ex. 19. D'Alembert test isn't applicable to the series of Ex. 11 that is to

$$\sum_{n=2}^{\infty}\frac{1}{n\ln^2 n},\quad \sum_{n=1}^{\infty}\frac{n}{n^2+1}.$$

■For the second series

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left(\frac{n+1}{(n+1)^2+1}:\frac{n}{n^2+1}\right)=\lim_{n\to\infty}\frac{(n+1)(n^2+1)}{n(n^2+2n+2)}=\lim_{n\to\infty}\frac{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n^2}\right)}{n^3\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}=1.$$

For the first series

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left(\frac{1}{(n+1)\ln^2(n+1)}:\frac{1}{n\ln^2 n}\right)=\lim_{n\to\infty}\frac{n\ln^2 n}{(n+1)\ln^2(n+1)}=$$

$$=\lim_{n\to\infty}\frac{n}{n+1}\cdot\lim_{n\to\infty}\left(\frac{\ln n}{\ln(n+1)}\right)^2=\lim_{n\to\infty}\frac{n}{n\left(1+\frac{1}{n}\right)}\cdot\lim_{n\to\infty}\left(\frac{\ln n}{\ln n\left(1+\frac{1}{n}\right)}\right)^2=$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \cdot \lim_{n\to\infty}\left(\frac{\ln n}{\ln n + \ln\left(1+\frac{1}{n}\right)}\right)^2 = \lim_{n\to\infty}\left(\frac{\ln n}{\ln n\left(1+\ln\left(1+\frac{1}{n}\right)\Big/\ln n\right)}\right)^2 =$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\ln\left(1+\frac{1}{n}\right)\Big/\ln n}\right)^2 = \left(\frac{1}{1+0}\right)^2 = 1.\blacksquare$$

Ex. 20. Prove that for any positive number a

$$\lim_{n\to\infty}\frac{a^n}{n!} = 0.$$

■Let's introduce the numerical series with the general term $a^n/n!$, namely

$$\sum_{n=1}^{\infty}\frac{a^n}{n!} = a + \frac{a^2}{2!} + \frac{a^3}{3!} + \ldots + \frac{a^n}{n!} + \ldots .$$

It converges on the base of D'Alembert test (verify!). Therefore by virtue of the necessary condition for convergence of a series the limit of its general term equals zero if $n \to \infty$.■

Theorem 9 (**Cauchy**[1] **radical** test)**.** If for a series (18) with positive terms there exists the limit

$$\lim_{n\to\infty}\sqrt[n]{u_n} = l, \tag{24}$$

then for $l<1$ the series converges and for $l>1$ diverges. The case $l=1$, similarly to D'Alembert test, is doubtful one.

Ex. 21. Prove convergence of the series

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2} .$$

The general term of the series

$$u_n = \left(1+\frac{1}{n}\right)^{-n^2},$$

[1] Cauchy, A.L. (1780 - 1859), an eminent French mathematician

and

$$\lim_{n\to\infty}\sqrt[n]{u_n}=\lim_{n\to\infty}\sqrt[n]{\left(1+\frac{1}{n}\right)^{-n^2}}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n}=e^{-1}=\frac{1}{e}<1.$$

By Cauchy radical test the series converges.

It is useful to observe that

$$\lim_{n\to\infty}\sqrt[n]{n}=1. \qquad (25)$$

■With the help of L'Hospital rule

$$\lim_{n\to\infty}\sqrt[n]{n}=\lim_{n\to\infty}n^{\frac{1}{n}}=\lim_{n\to\infty}e^{\frac{1}{n}\ln n}=e^{\lim\limits_{n\to\infty}\frac{\ln n}{n}}=\left|\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{(\ln x)'}{x'}=\lim_{x\to\infty}\frac{1}{x}=0\right|=e^0=1■$$

Prove yourselves that for any natural m

$$\lim_{n\to\infty}\sqrt[n]{n+m}=1. \qquad (26)$$

Ex. 22. The series

$$\sum_{n=1}^{\infty}\frac{2^n}{n}$$

diverges because of by (24) and (25)

$$\lim_{n\to\infty}\sqrt[n]{u_n}=\lim_{n\to\infty}\sqrt[n]{\frac{2^n}{n}}=\lim_{n\to\infty}\frac{2}{\sqrt[n]{n}}=2\cdot\frac{1}{\lim\limits_{n\to\infty}\sqrt[n]{n}}=2\cdot1=2>1.$$

Theorem 10 (Cauchy integral test)**.** If we substitute n by x in the general term u_n of the series (18) (with positive terms), we'll obtain a function $f(x)=u_x$. If this function is positive continuous non-increasing on the interval $[1,\infty)$, then the series (18) and the improper integral

$$\int_1^{\infty}f(x)dx \qquad (27)$$

both (simultaneously) converge or diverge.

■Let $k-1\le x\le k$; on the strength of non-increasing of the function $f(x)=u_x$ one gets successively

$$f(k) \le f(x) \le f(k-1), u_k \le f(x) \le u_{k-1}, \int_{k-1}^{k} u_k dx \le \int_{k-1}^{k} f(x)dx \le \int_{k-1}^{k} u_{k-1}dx,$$

$$u_k \le \int_{k-1}^{k} f(x)dx \le u_{k-1}. \qquad (28)$$

Putting successively $k = 2, 3, ..., n$ in (28) and adding termwise all the inequalities one obtains

$$u_2 + u_3 + ... + u_n \le \int_1^n f(x)dx \le u_1 + u_2 + ... + u_{n-1},$$

or

$$S_n - u_1 \le \int_1^n f(x)dx \le S_{n-1}. \qquad (29)$$

1. If the integral (27) converges, then

$$\int_1^\infty f(x)dx = \lim_{n\to\infty} \int_1^n f(x)dx.$$

By positivity [positiveness] of the function $f(x)$ the integral

$$\int_1^n f(x)dx$$

increases with *n*, and so by (29)

$$\int_1^n f(x)dx \le \int_1^\infty f(x)dx, S_n \le \int_1^n f(x)dx + u_1 \le \int_1^\infty f(x)dx + u_1.$$

Thus, the sequence of partial sums of the series is bounded above, and the series (18) convergers by virtue of the theorem 5.

2. If the series (18) converges, the inequality (29) permits to prove convergence of the integral (27).■

Ex. 23. Investigation of the harmonic series (8) on convergence.

In this case the general term of the series is

$$u_n = \frac{1}{n^p},$$

and corresponding function

$$f(x) = u_x = \frac{1}{x^p}.$$

It's known (see Lecture No. 23, Point 2, Ex. 2) that the improper integral

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{dx}{x^p}$$

converges for $p > 1$ and diverges for $p \le 1$. Therefore the harmonic series (8) converges for $p > 1$ and diverges for $p \le 1$.

Ex. 24. With the help of Cauchy integral test investigate for convergence the series

$$\text{a) } \sum_{n=2}^\infty \frac{1}{n \ln^2 n}; \qquad \text{b) } \sum_{n=1}^\infty \frac{n}{n^2+1}.$$

of Ex. 11.

a) For the first series

$$u_n = \frac{1}{n \ln^2 n},$$

and the corresponding function

$$f(x) = \frac{1}{x \ln^2 x}$$

is positive continuous non-increasing on the interval $[2, \infty)$. The improper integral

$$\int_2^\infty f(x)dx = \int_2^\infty \frac{dx}{x \ln^2 x} = \left| \begin{array}{l} \ln x = y. \\ \dfrac{dx}{x} = dy, \end{array} \begin{array}{c|c|c} x & 2 & \infty \\ \hline y & \ln 2 & \infty \end{array} \right| = \int_{\ln 2}^\infty \frac{dy}{y^2} = -\frac{1}{y}\bigg|_{\ln 2}^\infty = \frac{1}{\ln 2}$$

converges and therefore the series a) also converges.

b) The second series is divergent, because of

$$u_n = \frac{n}{n^2+1}, \quad f(x) = \frac{x}{x^2+1},$$

and the improper integral

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{xdx}{x^2+1} = \frac{1}{2}\int_1^\infty \frac{2xdx}{x^2+1} = \frac{1}{2}\ln\left(x^2+1\right)\bigg|_1^\infty = \infty$$

diverges.

Ex. 25. Apply Cauchy integral test to the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2+6n+18}.$$

Corresponding improper integral

$$\int_1^{\infty}\frac{dx}{x^2+6x+18}=\left|\begin{matrix}\frac{1}{2}\left(x^2+6x+18\right)'=t,\ x+3=t, \\ x=t-3,\ dx=dt,\ x^2+6x+18=t^2+9\end{matrix}\quad\begin{array}{c|c|c} x & 1 & \infty \\ \hline t & 4 & \infty\end{array}\right|=\int_4^{\infty}\frac{dt}{t^2+9}=$$

$$=\frac{1}{3}\arctan\frac{t}{3}\Big|_4^{\infty}=\frac{1}{3}\left(\lim_{t\to\infty}\arctan\frac{t}{3}-\arctan\frac{4}{3}\right)=\frac{1}{3}\left(\frac{\pi}{2}-\arctan\frac{4}{3}\right)<\infty$$

is convergent, and therefore the series converges.

POINT 3. NUMERICAL SERIES WITH ARBITRARY REAL TERMS. ABSOLUTE AND CONDITIONAL CONVERGENCE

In the Point 2 we have dealt with numerical series with positive terms [with positive term series]. Now we consider series whose terms are arbitrary real numbers both positive and negative, so-called real term series. By virtue of the theorem 2 we'll suppose that series in question contain infinitely many positive and negative terms. As the first example of such the series we'll consider that alternating.

Alternating series

Def. 7. Alternating series is called that of the next form:

$$\sum_{n=1}^{\infty}(-1)^{n-1}u_n=u_1-u_2+u_3-u_4+\ldots\ (\forall n:(u_n>0\,\text{or}\,u_n<0)), \qquad (30)$$

where all numbers u_n have the same sign. In other words an alternating series is a series with terms of alternating signs.

Theorem 11 (Leibniz[1] test). If in an alternating series (30)

[1] Leibniz, G. (1646 – 1717), the great German philosopher and mathematician

a) the necessary condition of convergence

$$\lim_{n\to\infty} u_n = 0 \qquad (31)$$

is fulfilled,

b) the terms don't increase in modulus,

then the series converges, and its sum S satisfies the inequality

$$|S| \le |u_1|. \qquad (32)$$

■Let for definiteness [to fix the idea] all numbers u_n are positive.

1. At first we consider partial sums of the series (30) with even number of terms. Let's represent the $2m$-th partial sum in two forms, namely

a) $S_{2m} = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ldots + (u_{2m-1} - u_{2m})$;

b) $S_{2m} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots - (u_{2m-2} - u_{2m-1}) - u_{2m}$.

If follows increasing of S_{2m} with m from the first representation and boundedness above of S_{2m} from the second. Therefore there exists the limit S of S_{2m} for $m \to \infty$,

$$\lim_{m\to\infty} S_{2m} = S.$$

2. To finish we must prove that the sequence of partial sums of the series (30) with odd number of terms converges to the same limit S. But by the condition (31)

$$\lim_{m\to\infty} S_{2m-1} = \lim_{m\to\infty}(S_{2m} + u_{2m}) = \lim_{m\to\infty} S_{2m} + \lim_{m\to\infty} u_{2m} = S + 0 = S.$$

We have proved that

$$\lim_{n\to\infty} S_n = S$$

for any n both even and odd, and so the series (30) converges. For the case of positiveness of all u_n we've obtained the inequality $S \le u_1$. In general we'll come to the inequality (32).■

Ex. 26. The alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n-1} \frac{1}{n} + \ldots$$

satisfies both conditions of Leibniz test:

$$\text{a) } 1 > \left|-\frac{1}{2}\right| > \frac{1}{3} > \left|-\frac{1}{4}\right| > \ldots, 1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \ldots \text{ ; b) } \lim_{n\to\infty}\frac{1}{n} = 0.$$

Therefore the series converges, and one has

$$|S| \le 1$$

for its sum *S*.

Ex. 27. Find approximate value of the sum *S* of the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n(2n+1)} = 1 - \frac{1}{4\cdot 3} + \frac{1}{4^2\cdot 5} - \frac{1}{4^3\cdot 7} + \frac{1}{4^4\cdot 9} - \ldots.$$

The series is alternating one, it satisfies the conditions of Leibniz test and therefore converges. By virtue of the formulas (12), (13) we have

$$S \approx S_n$$

with absolute error

$$\alpha = |R_n|.$$

1. Let $n = 3$. Then

$$R_3 = -\frac{1}{4^3\cdot 7} + \frac{1}{4^4\cdot 9} - \ldots$$

is the sum of the convergent alternating series, and by (32)

$$|R_3| \le \left|-\frac{1}{4^3\cdot 7}\right| = 0.002232142857\ldots < 0.002.$$

Further

$$S_3 = 1 - \frac{1}{4\cdot 3} + \frac{1}{4^2\cdot 5} \approx 1.000 - 0.083 + 0.013 = 0.930;$$

$$0.930 - 0.002 < S < 0.930 + 0.002, \quad 0.928 < S < 0.932,$$

and

$$S \approx 0.9,$$

where all digits are exact, or

$$S \approx 0.93$$

with the accuracy to 0.01.

2. Let now $n = 4$. In this case

$$R_4 = \frac{1}{4^4 \cdot 9} - \frac{1}{4^5 \cdot 11} + \ldots; |R_4| \le \frac{1}{4^4 \cdot 9} = 0.000434,$$

$$S_4 = 1 - \frac{1}{4 \cdot 3} + \frac{1}{4^2 \cdot 5} - \frac{1}{4^3 \cdot 7} =$$

$$= 1 - 0.083333 + 0.012500 - 0.002232 = 0.926935,$$

$$0.926935 - 0.000434 < S < 0.926935 + 0.000434,$$

$$0.926501 < S < 0.927369,$$

$S \approx 0.92$ with all exact digits or better

$S \approx 0.927$ with an accuracy to 0.001.

3. Let at last $n = 5$. By the same way we find

$$R_5 = -\frac{1}{4^5 \cdot 11} + \frac{1}{4^6 \cdot 13} - \ldots; |R_5| \le \frac{1}{4^5 \cdot 11} \approx 0.000089 < 0.0001;$$

$$S_5 = 1 - \frac{1}{4 \cdot 3} + \frac{1}{4^2 \cdot 5} - \frac{1}{4^3 \cdot 7} + \frac{1}{4^4 \cdot 9} \approx 1.0000 - 0.0833 + 0.0125 - 0.0022 +$$

$$+ 0.0004 \approx 0.9274,$$

$$0.9274 - 0.0001 < S < 0.9274 + 0.0001,$$

$$0.9273 < S < 0.9275,$$

$S \approx 0.927$, and all digits are exact.

Ex. 28. The alternating series

$$\sum_{n=2}^{\infty} \left(\frac{1}{\sqrt{n}-1} - \frac{1}{\sqrt{n}+1} \right) = \frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{3}+1} + \ldots + \frac{1}{\sqrt{n}-1} - \frac{1}{\sqrt{n}+1} + \ldots$$

is reduced to harmonic one with $p = 1$ and therefore diverges. Indeed,

$$\frac{1}{\sqrt{n}-1} - \frac{1}{\sqrt{n}+1} = \frac{(\sqrt{n}+1) - (\sqrt{n}-1)}{(\sqrt{n}-1)(\sqrt{n}+1)} = \frac{2}{n-1},$$

and

$$\sum_{n=2}^{\infty} \left(\frac{1}{\sqrt{n}-1} - \frac{1}{\sqrt{n}+1} \right) = 2\sum_{n=2}^{\infty} \frac{1}{n-1} = 2\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots \right).$$

For the given series the necessary condition (32) of convergence is fulfilled, but the second condition of Leibniz test isn't fulfilled. Indeed, for every n

$$\frac{1}{\sqrt{n}+1} < \frac{1}{\sqrt{n+1}-1} \text{ (verify!).}$$

Absolutely and conditionally convergent series

Let be given a numerical series with arbitrary real terms (it can be called as a real term series or a plus-and-minus series)

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots. \qquad (33)$$

We'll introduce the **series of moduli** of its terms, that is the series

$$\sum_{n=1}^{\infty} |u_n| = |u_1| + |u_2| + |u_3| + \ldots + |u_n| + \ldots \qquad (34)$$

Theorem 12. If the **series** (34) **of moduli** of terms of the series (33) converges, then the series (33) also converges.

■Let

$$\sigma_n = |u_1| + |u_2| + |u_3| + \ldots + |u_n|$$

is the *n*-th partial sum of the modulus series (34). By virtue of its convergence there is the limit

$$\sigma = \lim_{n\to\infty} \sigma_n,$$

and

$$\sigma_n \le \sigma$$

for any *n*. Let's write the *n*-th partial sum S_n of the series (33) in the next form:

$$S_n = S_n^+ - S_n^-$$

where S_n^+ is the sum of all positive terms of the series (33) in S_n and S_n^- is the sum of moduli of all negative terms. It is obviously that

$$S_n^- \le \sigma_n \le \sigma, S_n^+ \le \sigma_n \le \sigma \Rightarrow S_n^- \le \sigma, S_n^+ \le \sigma.$$

It means that the sums S_n^-, S_n^+ are bounded above by the number σ and so have the limits S^-, S^+ for $n \to \infty$

$$S^- = \lim_{n\to\infty} S_n^-,\ \ S^+ = \lim_{n\to\infty} S_n^+ .$$

Therefore there exists the limit S of the n-th partial sum S_n of the series (33),

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(S_n^+ - S_n^-\right) = \lim_{n\to\infty} S_n^+ - \lim_{n\to\infty} S_n^- = S^+ - S^- < \infty ,$$

that is the series converges.■

Def. 8. If the series of moduli (or the modulus series) of a real term series (33) converges, then this latter is called **absolutely convergent** (one says that it absolutely converges).

By virtue of this definition we can call the theorem 12 as that on absolute convergence of a plus-and-minus series.

Corollary. It follows from the proof of the theorem 12 that in absolutely convergent series the series of positive and negative terms are convergent (correspondingly to S^+ and $-S^-$).

Ex. 29. The series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots + \frac{(-1)^{n-1}}{n^2} + \ldots$$

absolutely converges because of the series of moduli of its terms

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} + \ldots$$

is the convergent harmonic series (8) with $p = 2 > 1$.

Def. 9. If a real term series (33) converges, but the series of moduli its terms diverges, then the series (33) is called **conditionally convergent** (one says that it conditionally converges).

Remark 2. On the base of the proof of the theorem 12 we can deduce that in a conditionally convergent series the series of its positive and negative terms are divergent.

Ex. 30. The series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + (-1)^{n-1}\frac{1}{n} + \ldots$$

(see Ex. 26) conditionally converges for its modulus series

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\left|(-1)^{n-1}\right|\frac{1}{n} = \sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots$$

(the harmonic series with $p=1$) diverges.

Tests on absolute convergence

To establish absolute convergence of a real term series (33) we can apply all sufficient tests of the Point 2.

We'll give some examples.

1. If there exists some **convergent** positive term series

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots, \quad \forall n : a_n > 0, \qquad (35)$$

such that

$$|u_n| \le a_n \qquad (36)$$

(at least for sufficiently large *n*), then the series (33) absolutely converges.

Remark 3. If for some **divergent** positive term series

$$b_1 + b_2 + b_3 + \ldots + b_n + \ldots, \quad \forall n : b_n > 0,$$

the inequality

$$|u_n| \ge b_n$$

holds (at least for sufficiently large *n*), then the series (33) can't absolutely converge. But it can converge conditionally.

Ex. 31. The series

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n^2} = \sin x + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \frac{\sin 4x}{4^2} + \ldots + \frac{\sin nx}{n^2} + \ldots$$

absolutely converges for any *x* because of the series of moduli of its terms

$$\sum_{n=1}^{\infty}\frac{|\sin nx|}{n^2} = |\sin x| + \frac{|\sin 2x|}{2^2} + \frac{|\sin 3x|}{3^2} + \frac{|\sin 4x|}{4^2} + \ldots + \frac{|\sin nx|}{n^2} + \ldots$$

converges for any *x* by the first comparison test:

$$\frac{|\sin nx|}{n^2} \le \frac{1}{n^2}$$

for each n, and the positive term series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}+\ldots$$

converges.

2. If for a series with **real** terms (33) there exists the limit

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=l\,, \tag{37}$$

then the series **absolutely** converges for $l<1$ and diverges for $l>1$.

■The limit (37) states D'Alembert sufficient test for convergence of the modulus series (34). In the case $l>1$ not only the series (34) diverges but also the series (33), for the necessary condition for convergence isn't fulfilled. See theorem 8.■

Ex. 32. Let be given a functional series

$$\sum_{n=1}^{\infty}\frac{1}{nx^n}$$

that is a series whose terms are functions. Investigate for which values of x it converges.

Def. 9. The set of all values of x for which a functional series converges is called the **domain of its convergence.**

By virtue of this definition we have to find the domain of convergence of the series of Ex. 32.

The general term of the series is a function of x, which we'll denote as $u_n(x)$,

$$u_n(x)=\frac{1}{nx^n}\,.$$

For fixed x D'Alembert test for convergence of the modulus series gives

$$\lim_{n\to\infty}\frac{|u_{n+1}(x)|}{|u_n(x)|}=\lim_{n\to\infty}\left(\frac{1}{(n+1)|x|^{n+1}}:\frac{1}{n|x|^n}\right)=\lim_{n\to\infty}\frac{n|x|^n}{(n+1)|x|^{n+1}}=\lim_{n\to\infty}\frac{n|x|^n}{n\left(1+\frac{1}{n}\right)|x|^n|x|}=$$

$$\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)|x|}=\frac{1}{|x|}\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}=\frac{1}{|x|}.$$

The series absolutely converges if the obtained limit is less than 1,

$$\frac{1}{|x|}<1 \Rightarrow |x|>1 \Rightarrow x \in (-\infty, -1) \cup (1, \infty);$$

it diverges if

$$\frac{1}{|x|}>1 \Rightarrow |x|<1 \Rightarrow x \in (-1, 1).$$

It's remaining to investigate behavior of the series at two points $x=1,\ x=-1$.

At the point $x=1$ the series becomes

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots$$

and diverges as the harmonic series with $p=1$. At the point $x=-1$ the series takes on the form

$$\sum_{n=1}^{\infty}\frac{1}{n(-1)^n}=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\ldots=-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\right)$$

and converges by Leibniz test (see Ex. 26).

Thus the given series converges for $x \in (-\infty, -1] \cup (1, \infty)$. In the other words its **domain of convergence** is the point set $(-\infty, -1] \cup (1, \infty)$ that is the union of two intervals $(-\infty, -1]$ and $(1, \infty)$.

Ex. 33. Prove yourselves that the domain of convergence of the functional series

$$\sum_{n=1}^{\infty}\frac{1}{(x+2)^n}$$

is the next point set $(-\infty, -3) \cup (-1, \infty)$.

Ex. 34. Find the domain of convergence of the series

$$\sum_{n=1}^{\infty}\frac{x^{n-1}}{n^2}=1+\frac{x}{2^2}+\frac{x^2}{3^2}+\frac{x^3}{4^2}+\ldots+\frac{x^{n-1}}{n^2}+\ldots.$$

The nth, $(n+1)$th terms of the series and their moduli are respectively

$$u_n(x)=\frac{x^{n-1}}{n^2},\ u_{n+1}(x)=\frac{x^n}{(n+1)^2},\ |u_n(x)|=\left|\frac{x^{n-1}}{n^2}\right|=\frac{|x|^{n-1}}{n^2},\ |u_{n+1}(x)|=\left|\frac{x^n}{(n+1)^2}\right|=\frac{|x|^n}{(n+1)^2}.$$

On the base of D'Alembert test for the modulus series (for fixed x)

$$\lim_{n\to\infty}\frac{|u_{n+1}(x)|}{|u_n(x)|}=\lim_{n\to\infty}\left(\frac{|x|^n}{(n+1)^2}:\frac{|x|^{n-1}}{n^2}\right)=\lim_{n\to\infty}\frac{|x|^n n^2}{|x|^{n-1}(n+1)^2}=\lim_{n\to\infty}\frac{|x|^{n-1}|x|n^2}{|x|^{n-1}n^2(1+1/n)^2}=$$

$$=|x|\lim_{n\to\infty}\frac{1}{(1+1/n)^2}=|x|\cdot 1=|x|.$$

The series absolutely converges if $|x|<1$, that is if $-1<x<1$, or $x\in(-1,1)$.

The series diverges if $|x|>1$, that is if $x<-1$ or $x>1$, or $x\in(-\infty,-1)\cup(1,\infty)$.

It's necessary to study the case $|x|=1$ or $x=\pm 1$.

For $x=-1$ we have the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots+\frac{(-1)^{n-1}}{n^2}+\ldots$$

which converges by Leibniz test.

For $x=1$ the corresponding series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}+\ldots$$

converges as harmonic one with $p=2>1$.

Answer: the domain of convergence of the series is the segment $[-1,1]$.

Ex. 35. The same problem for the functional series

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

Answer: $(-1,1]$.

Ex. 36. Find the domain of convergence of the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x-3)^{n-1}}{n^2\cdot 4^{n-1}}=1-\frac{x-3}{2^2\cdot 4}+\frac{(x-3)^2}{3^2\cdot 4^2}-\frac{(x-3)^3}{4^2\cdot 4^3}+\ldots+(-1)^{n-1}\frac{(x-3)^{n-1}}{n^2\cdot 4^{n-1}}+\ldots.$$

Here

$$u_n(x)=(-1)^{n-1}\frac{(x-3)^{n-1}}{n^2\cdot 4^{n-1}},\ |u_n(x)|=\left|(-1)^{n-1}\frac{(x-3)^{n-1}}{n^2\cdot 4^{n-1}}\right|=\frac{|x-3|^{n-1}}{n^2\cdot 4^{n-1}},\ |u_{n+1}(x)|=\frac{|x-3|^n}{(n+1)^2\cdot 4^n},$$

and for fixed x by D'Alembert test for the modulus series

$$\lim_{n\to\infty}\frac{|u_{n+1}(x)|}{|u_n(x)|}=\lim_{n\to\infty}\left(\frac{|x-3|^n}{(n+1)^2\cdot 4^n}:\frac{|x-3|^{n-1}}{n^2\cdot 4^{n-1}}\right)=\lim_{n\to\infty}\frac{|x-3|^n n^2\cdot 4^{n-1}}{|x-3|^{n-1}(n+1)^2\cdot 4^n}=$$

$$=\lim_{n\to\infty}\frac{|x-3|^{n-1}|x-3|n^2\cdot 4^{n-1}}{|x-3|^{n-1}(n+1)^2\cdot 4^{n-1}\cdot 4}=\frac{|x-3|}{4}\lim_{n\to\infty}\frac{n^2}{(n+1)^2}=\frac{|x-3|}{4}\cdot 1=\frac{|x-3|}{4}.$$

The given series absolutely converges if

$$\frac{|x-3|}{4}<1,\ |x-3|<4,\ -4<x-3<4,\ -1<x<7,\ x\in(-1,7)$$

and diverges if

$$\frac{|x-3|}{4}>1,\ |x-3|>4,\ \left[\begin{array}{l}x-3<-4,\\ x-3>4,\end{array}\right.\ \left[\begin{array}{l}x<-1,\\ x>7,\end{array}\right.\quad x\in(-\infty,-1)\cup(7,\infty).$$

Thus we know behavior of the series for all values of x but $x=-1,\ x=7$.

For $x=-1$ the series takes on the form

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-4)^{n-1}}{n^2\cdot 4^{n-1}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^{n-1}4^{n-1}}{n^2\cdot 4^{n-1}}=\sum_{n=1}^{\infty}\frac{(-1)^{2n-2}}{n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$$

and converges as harmonic series (8) with $p=2>1$.

For $x=7$ the series becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4^{n-1}}{n^2\cdot 4^{n-1}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$$

and absolutely converges (its convergence follows also from Leibniz test, see theorem 11).

Therefore the domain of convergence of our series is the segment $[-1,7]$. It has the length 8 and the center $x=x_0=3$ and can be written in the form $[3-4,3+4]$.

Some properties of real term series

Theorem 4 of the Point 1 states some "arithmetical" properties of series. They are similar to corresponding properties of finite sums. But series aren't finite sums, and there are some peculiarities in their properties. In particular it concerns associativity, commutativity and multiplication of series.

Theorem 13. One can parenthesize [put in parentheses] arbitrary groups of a convergent series. The sum of the series doesn't change.

But it isn't permissible in general to remove parentheses in a convergent series.

Ex. 37. The series $(1-1)+(1-1)+(1-1)+\ldots+(1-1)+\ldots$ converges to zero (why?). But removal of parentheses leads to the divergent series

$$1-1+1-1+\ldots+1-1+\ldots.$$

Absolutely convergent series possess the commutative property.

Theorem 14. One can interchange terms of absolute convergent series. Its sum doesn't change.

For conditionally convergent series such the property isn't valid. That states the next theorem.

Theorem 15 (Riemann[1]**).** Interchanging terms of conditionally convergent series we can obtain the convergent series with arbitrary sum and even the divergent series.

Validity of the theorems 14, 15 is based on the fact that in an absolutely convergent series the series of its positive and negative terms are convergent, but in a conditionally convergent one such the series simultaneously diverge.

Theorem 16. The product of two absolutely convergent to *S* and *T* series absolutely converges to the product *S*·*T*.

Let for example

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots = S, \quad \sum_{n=1}^{\infty} v_n = v_1 + v_2 + v_3 + \ldots + v_n + \ldots = T$$

be mentioned absolutely convergent series. Theorem 16 means that

$$\sum_{n=1}^{\infty} u_n \cdot \sum_{n=1}^{\infty} v_n = (u_1 + u_2 + u_3 + \ldots + u_n + \ldots)\cdot(v_1 + v_2 + v_3 + \ldots + v_n + \ldots) = S \cdot T .$$

By virtue of absolute convergence of the product of the series its terms can be arranged in different ways. We can in particular write

$$S \cdot T = (u_1 + u_2 + u_3 + \ldots + u_n + \ldots)\cdot(v_1 + v_2 + v_3 + \ldots + v_n + \ldots) =$$

[1] Riemann G.F.B. (1826 - 1866), an eminent German mathematician

$$= u_1v_1 + (u_1v_2 + u_2v_2 + u_2v_1) + (u_1v_3 + u_2v_3 + u_3v_3 + u_3v_2 + u_3v_1) + \ldots$$

(see the table 1) or better

$$S \cdot T = u_1v_1 + (u_1v_2 + u_2v_1) + (u_1v_3 + u_2v_2 + u_3v_1) + (u_1v_4 + u_2v_3 + u_3v_2 + u_4v_1) + \ldots \quad (38)$$

(see the table 2).

The development (38) retains correct if only one of series absolutely converges and the other simply converges.

Table 1

u_1v_1	u_2v_1	u_3v_1	u_4v_1	...
u_1v_2	u_2v_2	u_3v_2	u_4v_2	...
u_1v_3	u_2v_3	u_3v_3	u_4v_3	...
u_1v_4	u_2v_4	u_3v_4	u_4v_4	...
......	......	......	......	...

Table 2

u_1v_1	u_2v_1	u_3v_1	u_4v_1	…
u_1v_2	u_2v_2	u_3v_2	u_4v_2	…
u_1v_3	u_2v_3	u_3v_3	u_4v_3	…
u_1v_4	u_2v_4	u_3v_4	u_4v_4	…
………………………….....				

Ex. 38. Find the product of series of the Ex. 34, 35 which both absolutely converge on the interval $(-1, 1)$.

By the theorem 16 the product in question absolutely converges on the interval $(-1, 1)$. We'll find four first its terms arranging them correspondent with the formula (38) (see the table 2) that is in ascending powers of x (see the table 3),

$$\left(1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \frac{x^3}{4^2} + \ldots + \frac{x^{n-1}}{n^2} + \ldots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots\right) =$$

$$= x + \left(-\frac{1}{2} + \frac{1}{2^2}\right)x^2 + \left(\frac{1}{3} - \frac{1}{2^2}\cdot\frac{1}{2} + \frac{1}{3^2}\right)x^3 + \left(-\frac{1}{4} + \frac{1}{2^2}\cdot\frac{1}{3} - \frac{1}{3^2}\cdot\frac{1}{2} + \frac{1}{4^2}\right)x^4 + \ldots =$$

$$= x - \frac{1}{4}x^2 + \frac{23}{72}x^3 - \frac{23}{144}x^4 + \ldots.$$

Table 3

x	$x^2/2^2$	$x^3/3^2$	$x^4/4^2$	...
$-x^2/2$	$-x^3/(2^2\cdot 2)$	$-x^4/(3^2\cdot 2)$	$-x^5/(4^2\cdot 2)$	...
$x^3/3$	$x^4/(2^2\cdot 3)$	$x^5/(3^2\cdot 3)$	$x^6/(4^2\cdot 3)$	...
$-x^4/4$	$-x^5/(2^2\cdot 4)$	$-x^6/(3^2\cdot 4)$	$-x^7/(4^2\cdot 4)$	...
....	....	....	....	...

LECTURE NO. 29. POWER SERIES

POINT 1. POWER SERIES AND PROPERTIES OF ITS SUM

POINT 2. DEVELOPEMENT OF FUNCTIONS INTO POWER SERIES

POINT 3. SOME APPLICATIONS OF POWER SERIES

POINT 1. POWER SERIES AND PROPERTIES OF ITS SUM

We've dealt with functional series in Lecture No. 28 (see examples 31 - 35).

Def. 1. A power series is called a functional series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \qquad (1)$$

where real numbers

$$a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$$

are coefficients and

$$1 = x^0, x = x^1, x^2, x^3, \ldots, x^n, \ldots$$

power functions with integer non-negative indices [indexes].

Ex. 1. The series

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2} = 1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \frac{x^3}{4^2} + \ldots + \frac{x^{n-1}}{n^2} + \ldots,$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1} \frac{x^n}{n} + \ldots$$

which was considered in Ex. 34, 35 of the preceding Lecture are those power.

One often considers a power series of more general form, namely

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + a_3 (x - x_0)^3 + \ldots + a_n (x - x_0)^n + \ldots \qquad (2)$$

Ex. 2. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (x-3)^{n-1}}{n^2 \cdot 4^{n-1}} = 1 - \frac{x-3}{2^2 \cdot 4} + \frac{(x-3)^2}{3^2 \cdot 4^2} - \frac{(x-3)^3}{4^2 \cdot 4^3} + \ldots + (-1)^{n-1} \frac{(x-3)^{n-1}}{n^2 \cdot 4^{n-1}} + \ldots$$

of Ex. 36 of the same Lecture No. 28 is a power one of the type (2).

The series (1) is a particular case of the series (2) for $x_0 = 0$. On the other hand we can reduce the series (2) to the form (1) putting for example

$$x - x_0 = y$$

whence

$$\sum_{n=0}^{\infty} a_n y^n = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \ldots + a_n y^n + \ldots .$$

By this reason we can consider only the theory of the series (1).

Radius and interval of convergence of a power series

Def. 2. If a power series (1) converges at a point $x = r$, that is a numerical series

$$\sum_{n=0}^{\infty} a_n r^n = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \ldots + a_n r^n + \ldots$$

converges, then this point $x = r$ is called a **convergence point** (or a point of convergence) of the series.

Such the definition is valid for any functional series. It follows that the domain of convergence of a functional series is the set of all its convergence points.

Our óbject is the domain of convergence of a power series.

The power series (1) always converges at the point $x = 0$ because of it takes the form

$$\sum_{n=0}^{\infty} a_n 0^n = a_0 + a_1 0 + a_2 0^2 + a_3 0^3 + \ldots + a_n 0^n + \ldots = a_0 + 0 + 0 + 0 + \ldots = a_0$$

for $x = 0$.

a) There are power series which converge only at the point $x = 0$.

Ex. 3. The power series

$$\sum_{n=1}^{\infty} n!\, x^n = 0! + 1!\, x + 2!\, x^2 + 3!\, x^3 + \ldots = 1 + x + 2!\, x^2 + 3!\, x^3 + \ldots$$

has unique convergence point $x = 0$, for

$$\lim_{n\to\infty}\frac{|u_{n+1}(x)|}{|u_n(x)|}=\lim_{n\to\infty}\frac{|(n+1)!\,x^{n+1}|}{|n!\,x^n|}\lim_{n\to\infty}\frac{(n+1)!\,|x|^{n+1}}{n!\,|x|^n}=\lim_{n\to\infty}\frac{n!\,(n+1)|x|^n|x|}{n!\,|x|^n}=|x|\lim_{n\to\infty}(n+1),$$

and this limit is less than 1 (namely is equal to 0) only for $x=0$.

b) There are power series the domain of convergence of which is the set of all real numbers.

Ex. 4. The power series

$$\sum_{n=1}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}+\ldots$$

absolutely converges at any point *x*, because of for any fixed *x*

$$u_n(x)=\frac{x^n}{n!},\ |u_n(x)|=\left|\frac{x^n}{n!}\right|=\frac{|x|^n}{n!},\ |u_{n+1}(x)|=\frac{|x|^{n+1}}{(n+1)!},$$

and

$$\lim_{n\to\infty}\frac{|x|^n|x|n!}{|x|^n n!(n+1)}\underset{n\to\infty}{\frac{|u_{n+1}(x)|}{|u_n(x)|}}=\lim_{n\to\infty}\left(\frac{|x|^{n+1}}{(n+1)!}:\frac{|x|^n}{n!}\right)=\lim_{n\to\infty}\frac{|x|^{n+1}n!}{|x|^n(n+1)!}=|x|\lim_{n\to\infty}\frac{1}{(n+1)}=0<1.$$

c) There are power series whose domain of convergence is some part of the set of all reals.

For example the series of Ex. 1 converge on the intervals $[-1,1]$, $(-1,1]$ and diverge on the point sets $(-\infty,-1)\cup(1,\infty)$, $(-\infty,-1]\cup(1,\infty)$ respectively (see Ex. 34, 35 of the lecture No. 28).

Theorem 1 (Abel[1] theorem)**.** If a power series (1) converges at a point $x=x'$, then it absolutely converges on the interval $(-|x'|,|x'|)$. If it diverges at a point $x=x''$, then it diverges outside the interval $(-|x''|,|x''|)$.

■Let for example the series (1) converges at a point $x=x'$ that is the numerical series

$$\sum_{n=0}^{\infty}a_n(x')^n=a_0+a_1x'+a_2(x')^2+a_3(x')^3+\ldots+a_n(x')^n+\ldots$$

converges. By virtue of the necessary condition of convergence the general term of this series (namely $a_n(x')^n$) tends to zero with $n \to \infty$. Therefore for sufficiently large n (let for $n \geq N$ where N is some natural number) it is bounded: there is a number A such that

$$\left|a_n(x')^n\right| < A$$

for $n \geq N$.

Let now x is an arbitrary point of the interval $(-|x'|, |x'|)$, that is $|x| < |x'|$ and so

$$\frac{|x|}{|x'|} < 1.$$

Is this case for the general term $a_n x^n$ of the series (1) we have (if $n \geq N$)

$$\left|a_n x^n\right| = \left|a_n(x')^n \frac{x^n}{(x')^n}\right| = \left|a_n(x')^n\right|\left|\frac{x^n}{(x')^n}\right| = \left|a_n(x')^n\right|\left|\frac{x}{x'}\right|^n < A \cdot \left|\frac{x}{x'}\right|^n.$$

It means that for $n \geq N$ the moduli of the terms of the series (1) are less then the corresponding terms of the convergent geometrical progression with the ratio

$$q = \left|\frac{x}{x'}\right| < 1.$$

Therefore the series (1) absolutely converges on the interval $(-|x'|, |x'|)$.■

It can be deduced from Abel theorem that for a power series (1) of the case c) (when it has as convergence as divergence points) there is a positive number R (**convergence radius** or the radius of convergence) such that the series absolutely converges in the interval (**convergence interval** or the interval of convergence)

$$(-R, R)$$

and diverges outside the segment $[-R, R]$.

Behavior of the series (1) at the end points $\pm R$ of the convergence interval must be tested apárt (separately, singly, extra *лат.*).

The convergence radius and convergence interval of the series of Ex. 1 are respectively $R = 1, (-1, 1)$. The first series converges at the end points $x = \pm 1$ of the

[1] Abel, N.H. (1802 - 1829), the famous Norwegian mathematician

convergence interval and the second one converges at the right end point $x=1$ and diverges at the left end point $x=-1$.

A general power series (2) always converges at the point x_0. Its convergence interval (in the case c) of existing convergence and divergence points) is of the form

$$(x_0-R, x_0-R).$$

For example the convergence radius and convergence interval of the series in Ex. 2 are respectively $R=4, (x_0-R, x_0+R)=(3-4, 3+4)=(-1, 7)$. The series converges at both the end points $x=-1, x=7$ of the convergence interval (see Ex. 35 of preceding Lecture).

If a power series ((1) or (2)) converges at unique point ($x=1$ or $x=x_0$ respectively) we as usually say that $R=0$.

For the series of Ex. 3 we $R=0$.

In the case of convergence of the series at all points we say that $R=\infty$.

We have $R=\infty$ in Ex. 4.

We can seek a convergence radius *R* of a power series by the same way as in Ex. 34 - 36 that is with the help of D'Alembert test for the modulus series of the given series. In one particular case we can give a corresponding formula, namely

$$R=\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}, \qquad (3)$$

if the limit (3) exists (prove this fact yourselves!).

Ex. 5. For the first series of Ex. 1 we have

$$a_n=|a_n|=\frac{1}{n^2},\ a_{n+1}=|a_{n+1}|=\frac{1}{(n+1)^2},\ R=\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}=\lim_{n\to\infty}\left(\frac{1}{n^2}:\frac{1}{(n+1)^2}\right)=\lim_{n\to\infty}\frac{(n+1)^2}{n^2}=1$$

for the second series

$$a_n=(-1)^{n-1}\frac{1}{n},\ |a_n|=\frac{1}{n},\ |a_{n+1}|=\frac{1}{n+1},\ R=\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}=\lim_{n\to\infty}\left(\frac{1}{n}:\frac{1}{n+1}\right)=\lim_{n\to\infty}\frac{n+1}{n}=1.$$

Ex. 6. For a series of Ex. 2

$$a_n=\frac{(-1)^{n-1}}{n^2\cdot 4^{n-1}},\ a_{n+1}=\frac{(-1)^n}{(n+1)^2\cdot 4^n},\ R=\lim_{n\to\infty}\frac{|a_n|}{|a_{n+1}|}=\lim_{n\to\infty}\left(\frac{1}{n^2\cdot 4^{n-1}}:\frac{1}{(n+1)^2\cdot 4^n}\right)=$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \cdot 4^n}{n^2 \cdot 4^{n-1}} = \lim_{n\to\infty} \frac{(n+1)^2 \cdot 4^{n-1} \cdot 4}{n^2 \cdot 4^{n-1}} = 4 \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 4 \cdot 1 = 4 .$$

Ex. 7. The formula (3) isn't applicable for the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \ldots$$

(in this case $a_0 = 0, a_1 = 1, a_2 = 0, a_3 = -\frac{1}{3!}, a_4 = 0, a_5 = \frac{1}{5!}, \ldots$), and we apply D'Alembert test. Namely for any fixed x

$$u_n(x) = (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!},\ |u_n(x)| = \frac{|x|^{2n-1}}{(2n-1)!},\ |u_{n+1}(x)| = \frac{|x|^{2n+1}}{(2n+1)!},$$

$$\lim_{n\to\infty} \frac{|u_{n+1}(x)|}{|u_n(x)|} = \lim_{n\to\infty} \left(\frac{|x|^{2n+1}}{(2n+1)!} : \frac{|x|^{2n-1}}{(2n-1)!} \right) = \lim_{n\to\infty} \frac{(2n-1)!\,|x|^{2n+1}}{(2n+1)!\,|x|^{2n-1}} =$$

$$= \lim_{n\to\infty} \frac{(2n-1)!\,|x|^{2n-1} |x|^2}{(2n-1)!\,2n(2n+1)|x|^{2n-1}} = |x|^2 \lim_{n\to\infty} \frac{1}{2n(2n+1)} = 0 < 1 .$$

The series absolutely converges on the set of all reals, and we have $R = \infty$.

Ex. 8. Prove that for the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

the convergence radius equals $R = \infty$.

As conclusion we'll say that the domain of convergence of a power series can be: a) the unique point ($x = 0$ for the series (1) and $x = x_0$ for (2)); b) the set of all real numbers; c) some finite interval ($(-R, R)$ for (1) and $(x_0 - R, x_0 + R)$ for (2)) inclu-ding or excluding one or both of its end points.

Properties of the sum of a power series

Def. 3. Let a point set X is the domain of convergence of a power series. For any $x \in X$ we denote by $S(x)$ the sum of corresponding numerical series. The function $S(x)$ with the domain of definition $D(S) = X$ is called the **sum of a power series**.

Let for definiteness we consider a series (1), and so for any $x \in X$ we can write

$$S(x)=\sum_{n=0}^{\infty}a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots . \quad (4)$$

The sum of a power series possesses some important properties which we'll state without proving.

1. The sum $S(x)$ of a power series is continuous in a convergence interval.

Remark 1. If a power series converges at some end point of the convergence interval, then the sum $S(x)$ is continuous and at this end point.

Let the series (4) converges at the end point $x = R$. The remark means that

$$S(R)=\sum_{n=0}^{\infty}a_n R^n = a_0 + a_1 R + a_2 R^2 + a_3 R^3 + \ldots + a_n R^n + \ldots .$$

2. The sum $S(x)$ of a power series is integrable in a convergence interval and can be integrated by termwise integration of the series.

For the case of the series (1) integration over an interval $[0, x] \subset X$ gives

$$\int_0^x S(x)dx = \int_0^x \left(\sum_{n=0}^{\infty}a_n x^n\right)dx = a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \ldots + \frac{a_n}{n+1}x^{n+1} + \ldots . \quad (5)$$

The series (5) is a power one and has the same convergence radius *R* as the series (4).

This last fact can be easy proved in condition of existing of the limit (3). Indeed the convergence radius R' of the series (5) by the formula (3) equals

$$R' = \lim_{n\to\infty}\left(\left|\frac{a_{n-1}}{n}\right| : \left|\frac{a_n}{n+1}\right|\right) = \lim_{n\to\infty}\frac{|a_{n-1}|(n+1)}{|a_n| n} = \lim_{n\to\infty}\frac{|a_{n-1}|}{|a_n|}\cdot\lim_{n\to\infty}\frac{n+1}{n} = R\cdot 1 = R .$$

3. The sum $S(x)$ of a power series is differentialbe in a convergence interval and can be differentiated by termwise differentiation of the series without changing the convergence radius *R*.

For the series (4) the property means

$$S'(x)=\left(\sum_{n=0}^{\infty}a_n x^n\right)' = 1a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots + na_n x^{n-1} + \ldots . \quad (6.1)$$

The series (6) is a power one and has the same convergence radius *R* as the series (4). Its invàriabílity [inalterability] can be proved by the same way as in the property 2 (if the limit (3) exists). Do it yourselves.

3. Applying the property 3 infinitely many times we'll get

$$S''(x)=1\cdot 2a_2+2\cdot 3a_3x+3\cdot 4a_4x^2+4\cdot 5a_5x^3+\ldots+(n-1)na_nx^{n-2}+\ldots \qquad (6.2)$$

$$S'''(x)=1\cdot 2\cdot 3a_3+2\cdot 3\cdot 4a_4x+3\cdot 4\cdot 5a_5x^2+\ldots+(n-2)(n-1)na_nx^{n-3}+\ldots \qquad (6.3)$$

……………………………………………………………………………

$$S^{(n)}(x)=1\cdot 2\cdot 3\cdot\ldots\cdot na_n+2\cdot 3\cdot\ldots\cdot n(n+1)a_{n+1}+\ldots \qquad (6.n)$$

……………………………………………………………………………..

Putting $x=0$ in the formulas (4), (6.1), (6.2), (6.3), …, (6.*n*),… we'll find the coefficients $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$ of the series (4),

$$a_0=S(0)=\frac{1}{0!}S(0),\ a_1=S'(0)=\frac{1}{1!}S'(0),\ a_2=\frac{1}{2!}S''(0),\ a_3=\frac{1}{3!}S'''(0),\ldots,$$

$$a_n=\frac{1}{n!}S^{(n)}(0),\ldots.$$

Now the series (4) can be written as follows

$$S(x)=S(0)+S'(0)x+\frac{S''(0)}{2!}x^2+\frac{S'''(0)}{3!}x^3+\ldots+\frac{S^{(n)}(0)}{n!}x^n+\ldots, \qquad (7)$$

or briefly

$$S(x)=\sum_{n=0}^{\infty}\frac{S^{(n)}(0)}{n!}x^n$$

The series (7) is called Maclaurin[1] series for a function $S(x)$.

Analogously if a function $S(x)$ is a sum of a power series (2),

$$S(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\ldots+a_n(x-x_0)^n+\ldots, \qquad (8)$$

then

$$S(x)=S(x_0)+S'(x_0)(x-x_0)+\frac{S''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{S^{(n)}(x_0)}{n!}(x-x_0)^n+\ldots \qquad (9)$$

$$S(x)=\sum_{n=0}^{\infty}\frac{S^{(n)}(x_0)}{n!}(x-x_0)^n$$

The series (9) is called Taylor[1] series for a function $S(x)$.

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[1] Maclaurin, C. (1698 - 1746), a Scotch mathematician

Ex. 9. Find the sum of the series

$$f(x)=\frac{x^3}{3}+\frac{x^7}{7}+\frac{x^{11}}{11}+\ldots+\frac{x^{4n-1}}{4n-1}+\ldots.$$

If we termwise differentiate the series, we'll obtain the geometrical progression with the ratio $q=x^4$ and the convergence radius $R=1$. Indeed,

$$f'(x)=x^2+x^6+x^{10}+\ldots+x^{4n-2}+\ldots.$$

By the formula (7) of the sum of a geometrical progression

$$f'(x)=\frac{x^2}{1-x^4}.$$

After integration

$$f(x)=\int\frac{x^2dx}{1-x^4}+C=-\frac{1}{2}\int\left(\frac{1}{x^2-1}+\frac{1}{x^2+1}\right)dx+C=-\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\arctan x+C.$$

On the base of the obvious condition $f(0)=0$ we find $C=0$ and get the sought sum

$$f(x)=-\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\arctan x.$$

Ex. 10. By the same method seek the sum of the series

$$f(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots+(-1)^{n-1}\frac{x^{2n-1}}{2n-1}+\ldots.$$

Ex. 11. Find the sum of the series

$$f(x)=1-3x^2+5x^4-7x^6+\ldots+(-1)^n(2n+1)x^{2n}+\ldots=\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{2n}.$$

The problem is solved with the help of termwise integration. Namely

$$\int f(x)dx=x-x^3+x^5-x^7+\ldots+(-1)^n x^{2n+1}+\ldots=\sum_{n=0}^{\infty}(-1)^n x^{2n+1}=\frac{x}{1+x^2}+C,\ |x|<1.$$

After differentiation

$$f(x)=\left(\int f(x)dx\right)'=\left(\frac{x}{1+x^2}\right)'=\frac{1\cdot(1+x^2)-x\cdot 2x}{(1+x^2)^2}=\frac{1-x^2}{(1+x^2)^2}\text{ for }|x|<1.$$

Ex. 12. Seek the sum of the series

[1] Taylor, B. (1685 - 1731), an English mathematician

$$f(x)=1+2x+3x^2+4x^3+\ldots+nx^{n-1}+\ldots=\sum_{n=1}^{\infty}nx^{n-1}.$$

POINT 2. DEVELOPEMENT OF FUNCTIONS INTO POWER SERIES

Here we'll study developements of functions into Maclaurin power series.

Let be given an infinitely differentiable function $f(x)$. It can be assigned [it can be associated with] its Maclaurin series, namely

$$f(x)\sim f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n. \quad (10)$$

It is necessary to solve the next two problems: under which conditions the series (10) converges and has the sum $f(x)$. In other words we find conditions of developability of functions into Maclaurin series.

The necessary and sufficient condition of developability gives us Maclaurin formula for a function $f(x)$ (see Lecture No. 16, Point 4)

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+r_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k+r_n(x) \quad (11)$$

in which the polynomial coincides with the *n*-th partial sum of Maclaurin series (10) and $r_n(x)$ is the remainder [the remainder term, the residual member]. For example Lagrangian remainder form is

$$r_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1},\ c\in(0,x). \quad (12)$$

Comparison the formulas (10) and (11) obviously leads to the nest theorem.

Theorem 2. An infinitely differentiable function is developable into Maclaurin series (10) on some interval $(-a,a)$ if and only if for $n\to\infty$ the remainder of its Maclaurin formula (11) goes to zero on $(-a,a)$,

$$\lim_{n\to\infty}r_n(x)=0 \quad \text{for} \quad x\in(-a,a)^{1}. \quad (13)$$

[1] It's évident that the interval $(-a,a)$ must lie on the convergence interval of the series.

In the case when the condition (13) is fulfilled we can write the development of the function into Maclaurin series if we'll change the sign of correspondence ~ by the equality sign is (10), namely

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \quad (14)$$

Theorem 3. If all derivatives of an infinitely differentiable function are bounded by the same number on some interval $(-a, a)$, then the function is developable into Maclaurin series (10) on this interval.

■Let there are an interval $(-a, a)$ and a number C such that

$$\left|f^{(n)}(x)\right|\le C$$

for any n ($n=0,1,2,3,\ldots$) and arbitrary $x\in(-a, a)$ (that is $|x|<a$). With the help of Lagrangian remainder form (12) we have

$$|r_n(x)|=\left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right|=\frac{\left|f^{(n+1)}(c)\right|}{(n+1)!}|x|^{n+1}<\frac{C}{(n+1)!}|a|^{n+1}=C\cdot\frac{|a|^{n+1}}{(n+1)!}\to 0$$

by virtue of Ex. 20 of the Lecture 28. It's retains to apply the theorem 2.■

Ex. 13. The functions $\sin x, \cos x$ satisfy conditions of the theorem 3 on the set of all reals $(-\infty, \infty)$. The function e^x satisfies them on arbitrary finite interval $(-a, a)$ ($C=e^a$). Therefore we at once obtain their expandings into Maclaurin series from the results of the Lecture No. 16, Point 4, namely

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^n}{n!}+\ldots, \quad (15)$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\ldots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+\ldots, \quad (16)$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+(-1)^n\frac{x^{2n}}{(2n)!}+\ldots. \quad (17)$$

All the series converge on $(-\infty, \infty)$ (see Ex. 4, 7, 8), and so the formulas (15), (16), (17) are valid for any x.

Remark 2. We can reduce the series (17) by termwise differentiation of the series (16).

Ex. 14 (the **binomial series**). Let's expand into Maclaurin series the next function

$$f(x) = (1+x)^m, \quad m \in (-\infty, \infty), \tag{18}$$

where m is some real (no natural) number.

At first we find the derivatives of the function (18),

$$f'(x) = m(1+x)^{m-1},\ f''(x) = m(m-1)(1+x)^{m-2},\ f'''(x) = m(m-1)(m-2)(1+x)^{m-3},\ldots,$$

$$f^{(n)}(x) = m(m-1)(m-2)\ldots(m-(n-1))(1+x)^{m-n},\ldots.$$

Then we find the values of the function and its derivatives at the point $x = 0$,

$$f(0) = 1,\ f'(0) = m,\ f''(0) = m(m-1),\ f'''(0) = m(m-1)(m-2),$$

$$f^{(n)}(0) = m(m-1)(m-2)\ldots(m-n+1),\ f^{(n+1)}(0) = m(m-1)(m-2)\ldots(m-n+1)(m-n),\ldots.$$

Now with the help of the formula (10) we obtain

$$(1+x)^m \sim$$

$$\sim 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots + \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}x^n + \ldots.$$

The convergence radius of this last series $R = 1$ because of by the formula (3)

$$R = \lim_{n\to\infty}\left(\frac{|m(m-1)(m-2)\ldots(m-n+1)|}{n!} : \frac{|m(m-1)(m-2)\ldots(m-n+1)(m-n)|}{(n+1)!}\right) =$$

$$= \lim_{n\to\infty}\frac{n+1}{|m-n|}\,\frac{|m(m-1)(m-2)\ldots(m-n+1)|(n+1)!}{|m(m-1)(m-2)\ldots(m-n+1)(m-n)|n!} = \lim_{n\to\infty}\frac{n+1}{|m-n|} =$$

$$\lim_{n\to\infty}\frac{n(1+1/n)}{n|m/n-1|} = \lim_{n\to\infty}\frac{1+1/n}{|m/n-1|} = \frac{1+0}{|0-1|} = 1.$$

It can be proved that on the convergence interval $(-1, 1)$ the conditions of the theorem 2 are fulfilled, and so we get the next development (so-called **binomial series**) on this interval

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots + \frac{m(m-1)(m-2)\ldots(m-n+1)}{n!}x^n + \ldots. \tag{19}$$

The binomial series (19) is the source of many other developments.

Ex. 15. Putting $m=-1$ in the binomial series (19) we'll have

$$(1+x)^{-1}=1+(-1)x+\frac{(-1)((-1)-1)}{2!}x^2+\frac{(-1)((-1)-1)((-1)-2)}{3!}x^3+\ldots=,$$

$$=1-x+\frac{1\cdot 2}{2!}x^2-\frac{1\cdot 2\cdot 3}{3!}x^3+\ldots=1-x+x^2-x^3+\ldots,$$

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots+(-1)^n x^n+\ldots=\sum_{n=0}^{\infty}(-1)^n x^n \quad (-1<x<1). \qquad (20)$$

Remark 3. The series (20) can be obtained at once as the sum of convergent geometrical progression (with the ratio $q=-x$ if $|q|=|-x|=|x|<1$). It diverges at the end points of the convergence interval $(-1,1)$.

Ex. 16. Integrating termwise the series (20) over the segment $[0,x]$, $|x|<1$, we obtain the expand of the natural logarithm. Indeed

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \quad (-1<x\le 1). \quad (21)$$

The series (21) converges at the end point $x=1$ (as alternating one by Leibniz test) and diverges at the point $x=-1$ (why?).

Ex. 17. Let's substitute x by x^2 in the series (20),

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\ldots+(-1)^n x^{2n}+\ldots=\sum_{n=0}^{\infty}(-1)^n x^{2n} \quad (x^2<1,-1<x<1).$$

After termwise integration of this last series we'll obtain the arctangent development with the same convergence interval $(-1,1)$, namely

$$\arctan x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots+(-1)^{n-1}\frac{x^{2n-1}}{2n-1}+\ldots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}. \qquad (22)$$

The convergence domain of this series is the segment $[-1,1]$ because of it converges at both the end points of the convergence interval (verify with the help of Leibniz test!).

Ex. 18. Let's take $m=-1/2$ in the binomial series (19),

$$\frac{1}{\sqrt{1+x}}=1-\frac{1}{2}x+\frac{(-1/2)((-1/2)-1)}{2!}x^2+\frac{(-1/2)((-1/2)-1)((-1/2)-2)}{3!}x^3+\ldots,$$

$$\frac{1}{\sqrt{1+x}}=1-\frac{1}{2}x+\frac{1\cdot 3}{2^2\cdot 2!}x^2-\frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^3+\frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!}x^4-\ldots, \qquad (23)$$

and then substitute x by $-x^2$ ($\left|-x^2\right|=\left|x^2\right|<1$, whence $|x|<1, -1<x<1$),

$$\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2}x^2+\frac{1\cdot 3}{2^2\cdot 2!}x^4+\frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^6+\frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!}x^8+\ldots.$$

Termwise integration gives the arcsine development with the same convergence interval $(-1,1)$

$$\arcsin x=x+\frac{1}{2\cdot 3}x^3+\frac{1\cdot 3}{2^2\cdot 5\cdot 2!}x^5+\frac{1\cdot 3\cdot 5}{2^3\cdot 7\cdot 3!}x^7+\frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 9\cdot 4!}x^9+\ldots. \qquad (24)$$

Remark 4. We often deal with developments of functions into Taylor series

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\ldots, \qquad (25)$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n .$$

Corresponding theory is similar to that stated above for Maclaurin series.

In Ex. 16 – 18 we've found developments of several functions knowing those of the others. Let's consider some additional examples.

Ex. 19. Expand into Maclaurin series the function

$$\ln\left(15+x^2\right).$$

Using the standard development (21) we do as follows

$$\ln\left(15+x^2\right)=\ln 15\left(1+\frac{x^2}{15}\right)=\ln 15+\ln\left(1+\frac{x^2}{15}\right)=$$

$$=\ln 15+\frac{x^2}{15}-\frac{x^4}{2\cdot 15^2}+\frac{x^6}{3\cdot 15^3}-\frac{x^8}{4\cdot 15^4}+\ldots+(-1)^n\frac{x^{2n}}{n\cdot 15^n}+\ldots.$$

The convergence interval $\left(-\sqrt{15},\sqrt{15}\right)$ of the series is determined by the inequalities

$$\left|\frac{x^2}{15}\right|=\frac{x^2}{15}<1,\ x^2<15,\ |x|<\sqrt{15},\ -\sqrt{15}<x<\sqrt{15}.$$

The domain of convergence is the segment $\left[-\sqrt{15},\sqrt{15}\right]$ (why?).

Ex. 20. Taking into account that

$$\ln\left(x^2-7x+12\right)=\ln\left((x-3)(x-4)\right)=\ln\left((3-x)(4-x)\right)=\ln(3-x)+\ln(4-x),$$

expand the function $\ln\left(x^2-7x+12\right)$ into Maclaurin series and find its domain of convergence.

Ex. 21. Expand into Maclaurin series the next function:

$$f(x)=\frac{\sin x-x\cos x}{x}.$$

We'll obtain the required development taking into account the standard expansions (16), (17) of the sine and cosine and absolute convergence of both series.

$$f(x)=\frac{\sin x-x\cos x}{x}=\frac{1}{x}\cdot\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots-x\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\right)\right)=$$

$$=\frac{1}{x}\cdot\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots-x+\frac{x^3}{2!}-\frac{x^5}{4!}+\frac{x^7}{6!}-\ldots\right)=$$

$$=\frac{1}{x}\cdot\left(\left(\frac{1}{2!}-\frac{1}{3!}\right)x^3-\left(\frac{1}{4!}-\frac{1}{5!}\right)x^5+\left(\frac{1}{6!}-\frac{1}{7!}\right)x^7-\left(\frac{1}{8!}-\frac{1}{9!}\right)x^9+\ldots\right)=$$

$$=\left(\frac{1}{2!}-\frac{1}{3!}\right)x^2-\left(\frac{1}{4!}-\frac{1}{5!}\right)x^4+\left(\frac{1}{6!}-\frac{1}{7!}\right)x^6-\left(\frac{1}{8!}-\frac{1}{9!}\right)x^8+\ldots.$$

The obtained expansion is valid for any $x\neq 0$.

POINT 3. SOME APPLICATIONS OF POWER SERIES

Power series can be successively applied to integration of differential equations, evaluating integrals which aren't expressible in terms of elementary functions and to approximate calculations.

1. Approximate integration of differential equations

a) Method of Taylor (Maclaurin) series

Let it's necessary to solve Cauchy problem

$$y'=f(x,y),\quad y(x_0)=y_0. \qquad (26)$$

We seek the solution of the problem in the form of Taylor series

$$y = y(x_0) + y'(x_0)(x - x_0) + \frac{1}{2!}y''(x_0)(x - x_0)^2 + \frac{1}{3!}y'''(x_0)(x - x_0)^3 + \ldots, \quad (27)$$

in which we have to determine the values of the unknown function and its derivatives at the point x_0. By the initial condition and the given equation we have

$$y(x_0) = y_0,\ y'(x_0) = f(x_0, y(x_0)) = f(x_0, y_0).$$

To find further values of the derivatives we successfully differentiate the given equation and equations obtained after differentiation,

$$y'' = F_1(x, y, y'),\ \text{where } F_1(x, y, y') = f'_x + f'_y \cdot y'$$

$$y''' = F_2(x, y, y', y''),\ y^{(4)} = F_3(x, y, y', y'', y'''),\ldots$$

and then put $x = x_0$ in all these equations,

$$y''(x_0) = F_1(x_0, y_0, y'(x_0)),\ y'''(x_0) = F_2(x_0, y_0, y'(x_0), y''(x_0)),$$
$$y^{(4)}(x_0) = F_3(x_0, y_0, y'(x_0), y''(x_0), y'''(x_0)),\ldots$$

By this way we can find arbitrary number of terms of Taylor series.

Ex. 22. Solve Cauchy problem

$$y' = y\sin x + 1,\ y\left(\frac{\pi}{2}\right) = 1.$$

Corresponding Taylor series is

$$y = y\left(\frac{\pi}{2}\right) + y'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + \frac{1}{2!}y''\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{3!}y'''\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right)^3 + \ldots .$$

Acting by the theory we'll obtain

$$y'\left(\frac{\pi}{2}\right) = y\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2} + 1 = 1\cdot 1 + 1 = 2,$$

$$y'' = y'\sin x + y\cos x,\ y''' = y''\sin x + 2y'\cos x - y\sin x,$$

$$y^{(4)} = y'''\sin x + 3y''\cos x - 3y'\sin x - y\cos x, \ldots .$$

$$y''\left(\frac{\pi}{2}\right) = y'\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2} + y\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2} = 2\cdot 1 + 1\cdot 0 = 2,$$

$$y'''\left(\frac{\pi}{2}\right) = y''\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2} + 2y'\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2} - y\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2} = 2\cdot 1 + 2\cdot 2\cdot 0 - 1\cdot 1 = 1,$$

$$y^{(4)}\left(\frac{\pi}{2}\right) = y'''\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2} + 3y''\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2} - 3y'\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2} - y\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2} = -5,$$

………………………………………………………………………………..

In this way we can write first five nonzero terms of the solution of the problem

$$y = 1 + 2\left(x - \frac{\pi}{2}\right) + \frac{1}{2!}\cdot 2\cdot\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{3!}\cdot 1\cdot\left(x - \frac{\pi}{2}\right)^3 + \frac{1}{4!}(-5)\left(x - \frac{\pi}{2}\right)^4 + \ldots,$$

$$y = 1 + 2\left(x - \frac{\pi}{2}\right) + \left(x - \frac{\pi}{2}\right)^2 + \frac{1}{6}\left(x - \frac{\pi}{2}\right)^3 - \frac{5}{24}\left(x - \frac{\pi}{2}\right)^4 + \ldots.$$

Ex. 23. The same problem for the second order differential equation

$$y'' = xy' - x^2 y + \sin x, \quad y(0) = 1,\ y'(0) = -1.$$

Here $x_0 = 0$, so we find the solution in the form of Maclaurin series

$$y = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + \frac{1}{4!}y^{(4)}(0)x^4 + \frac{1}{5!}y^{(5)}(0)x^5 + \ldots.$$

Initial conditions and the given equation give

$$y(0) = 1,\ y'(0) = -1,\ y''(0) = 0\cdot y'(0) - 0^2\cdot y(0) + \sin 0 = 0.$$

After differentiation of the given equation and its corollaries

$$y''' = y' + xy'' - 2xy - x^2 y' + \cos x,\ y^{(4)} = 2y'' + xy''' - 2y - 4xy' - x^2 y'' - \sin x,$$

$$y^{(5)} = 2y''' + xy^{(4)} - 6y' - 6xy'' - x^2 y''' - \cos x, \ldots$$

we get the values of the derivatives

$$y'''(0) = 0,\ y^{(4)}(0) = -2,\ y^{(5)}(0) = 5, \ldots$$

and first four nonzero terms of Maclaurin series for the unknown function

$$y = 1 - x - \frac{1}{12}x^4 + \frac{5}{120}x^5 + \ldots$$

b) Method of undetermined coefficients for linear equations

We'll illustrate this method with the help of two examples

Ex. 24. Solve the same Cauchy problem as in Ex. 23

$$y'' - xy' + x^2 y = \sin x, \quad y(0) = 1,\ y'(0) = -1.$$

We seek the solution of the problem in the form of a power series with undetermined coefficients $c_0, c_1, c_2, \ldots$.

$$\begin{array}{r|l} y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots & x^2 \\ y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots & -x \\ y'' = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \ldots & 1 \end{array}$$

The initial conditions give

$$y(0)=c_0=1,\ y'(0)=c_1=-1 \quad c_0=1,\ c_1=-1.$$

Then we substitute the series in the left side of the given equation and take the series (16) instead $\sin x$ in the right-hand member. We obtain the equality of two series

$$2c_2+(-c_1+6c_3)x+(c_0-2c_2+12c_4)x^2+(c_1-3c_3+20c_5)x^3+...=x-\frac{x^3}{3!}+...$$

Equating coefficients of the same power of x leads us to a system of equations in the coefficients $c_2, c_3, c_4, \ldots$.

$$\begin{array}{r|ccc c}
x^0 & 2c_2 & = & 0 & c_2=0, \\
x & -c_1+6c_3 & = & 1 & c_3=1/6(1+c_1)=0, \\
x^2 & c_0-2c_2+12c_4 & = & 0 & c_4=1/12(2c_2-c_0)=-1/12, \\
x^3 & c_1-3c_3+20c_5 & = & -1/6 & c_5=1/20(-1/6-c_1+3c_3)=5/120,
\end{array}$$

……………………………………………………………………..

First four nonzero terms of the series, which gives the solution of the problem,

$$y=1-x-\frac{1}{12}x^4+\frac{5}{120}x^5+...$$

coincide with those obtained in Ex. 23.

Ex. 25. Find the general solution of the differential equation

$$y''+k^2y=0.$$

Acting by analogy with Ex. 24 we at first obtain

$$\begin{array}{r|l}
y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6+c_7x^7+c_8x^8+... & k^2 \\
y'=c_1+2c_2x+3c_3x^2+4c_4x^3+5c_5x^4+6c_6x^5+7c_7x^6+8c_8x^7+9c_9x^8+... & 0 \\
y''=2!c_2+2\cdot3c_3x+3\cdot4c_4x^2+4\cdot5c_5x^3+6c_6x^5+6\cdot7c_7x^6+76c_6x^5+7c_7x^6+8c_8x^7+... & 1
\end{array}$$

$$(2!c_2+k^2c_0)+(2\cdot3c_3+k^2c_1)x+(3\cdot4c_4+k^2c_2)x^2+(4\cdot5c_5+k^2c_3)x^3+$$
$$+(5\cdot6c_6+k^2c_4)x^3+(6\cdot7c_7+k^2c_5)x^4+(7\cdot8c_8+k^2c_6)x^5+(8\cdot9c_9+k^2c_8)x^6+...=0.$$

Then we equate all the coefficients of the left series to zero

$$2!c_2+k^2c_0=0,\ 2\cdot3c_3+k^2c_1=0,\ 3\cdot4c_4+k^2c_2=0,\ 4\cdot5c_5+k^2c_3=0,\ 5\cdot6c_6+k^2c_4=0,$$

$$6\cdot7c_7+k^2c_5=0,\ 7\cdot8c_8+k^2c_6=0,\ 8\cdot9c_9+k^2c_8=0,...$$

and express $c_2, c_4, c_6, c_8,...$ through c_0, $c_3, c_5, c_7, c_9,...$ through c_1.

$$c_2 = -\frac{k^2}{2!}c_0,\ c_4 = \frac{k^4}{4!}c_0,\ c_6 = -\frac{k^6}{6!}c_0,\ c_8 = \frac{k^8}{8!}c_0, ...,$$

$$c_3 = -\frac{k^2}{3!}c_1,\ c_5 = \frac{k^4}{5!}a_1,\ c_7 = -\frac{k^6}{7!}c_1,\ c_9 = \frac{k^8}{9!}c_1, ...$$

There aren't initial conditions in our problem, therefore we must suppose the coefficients c_0, c_1 to be arbitrary numbers. After some clear steps with taking into account the series (16), (17) we'll get the final result in the finite form

$$y = c_0 + c_1x - \frac{k^2}{2!}c_0x^2 - \frac{k^2}{3!}c_1x^3 + \frac{k^4}{4!}c_0x^4 + \frac{k^4}{5!}c_1x^5 - \frac{k^6}{6!}c_0x^6 - \frac{k^6}{7!}c_1x^7 + \frac{k^8}{8!}c_0x^8 + ... =$$

$$= c_0\left(1 - \frac{k^2}{2!}x^2 + \frac{k^4}{4!}x^4 - \frac{k^6}{6!}x^6 + \frac{k^8}{8!}x^8 - ...\right) + c_1\left(x - \frac{k^2}{3!}x^3 + \frac{k^4}{5!}x^5 - \frac{k^6}{7!}x^7 + \frac{k^8}{9!}x^9 + ...\right) =$$

$$= c_0\left(1 - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} - \frac{(kx)^6}{6!} + \frac{(kx)^8}{8!} - ...\right) + \frac{c_1}{k}\left(kx - \frac{(kx)^3}{3!} + \frac{(kx)^5}{5!} - \frac{(kx)^7}{7!} + \frac{(kx)^9}{9!} + ...\right).$$

$$y = c_0 \cos kx + \frac{c_1}{k}\sin kx.$$

2. Calculation of integrals which can't be expressed in terms of elementary functions

Here we'll confine ourselves to two interesting examples.

Ex. 26. It's of very importance for the probability theory so-called Laplace function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}}dt. \qquad (28)$$

The primitive of the integrant doesn't express through elementary functions. Nevertheless we can represent Laplace function by series. Substituting x by $-t^2/2$ in the expansion (15) and termwise integrating the obtained series we'll get

$$e^{-\frac{t^2}{2}} = 1 - \frac{t^2}{2} + \frac{t^4}{2^2\cdot 2!} - \frac{t^6}{2^3\cdot 3!} + \frac{t^8}{2^4\cdot 4!} - \frac{t^{10}}{2^5\cdot 5!} + ... + (-1)^n\frac{t^{2n}}{2^n\cdot n!} + ... = \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{2^n\cdot n!},$$

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}}dt=\frac{1}{\sqrt{2\pi}}\left(x-\frac{x^3}{2\cdot 3\cdot 1!}+\frac{x^5}{2^2\cdot 5\cdot 2!}-\ldots+(-1)^n\frac{x^{2n+1}}{2^n\cdot(2n+1)\cdot n!}+\ldots\right).(29)$$

Ex. 27. With the help of the development (16) we find the series for so-called integral sine

$$Six=\int_0^x\frac{\sin x}{x}dx=\int_0^x\frac{1}{x}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+\ldots\right)dx=$$

$$=\int_0^x\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^5}{7!}+\ldots+(-1)^{n-1}\frac{x^{2n-2}}{(2n-1)!}+\ldots\right)dx,$$

$$Six=\int_0^x\frac{\sin x}{x}dx=x-\frac{x^3}{3\cdot 3!}+\frac{x^5}{5\cdot 5!}-\frac{x^7}{7\cdot 7!}+\ldots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)(2n-1)!}+\ldots. \quad (30)$$

3. Approximate calculations

Let's limit ourselves to some examples. See also Ex. 27 which in detail was studied in the Lecture No. 28.

Ex. 28. Find approximate value of $\sqrt{3}$.

We'll apply the binomial series (19) for $m=1/2$, that is (verify!)

$$\sqrt{1+x}=(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1}{2^2\cdot 2!}x^2+\frac{1\cdot 3}{2^3\cdot 3!}x^3-\frac{1\cdot 3\cdot 5}{2^4\cdot 4!}x^4+\ldots \quad (31)$$

If we represent the number $\sqrt{3}$ as follows (with 8 decimal places)

$$\sqrt{3}=\sqrt{1.73^2+0.0071}=\sqrt{1.73^2\left(1+\frac{0.0071}{2.9929}\right)}=1.73\sqrt{1+\frac{0.0071}{2.9929}}\approx 1.73\sqrt{1+0.00237228}$$

we can apply the series (31) for $x=0.00237228$. Namely

$$\sqrt{3}=$$

$$=1.73\left(1+\frac{0.00237228}{2}-\frac{0.00237228^2}{2^2\cdot 2!}+\frac{1\cdot 3\cdot 0.00237228^3}{2^3\cdot 3!}-\frac{1\cdot 3\cdot 5\cdot 0.00237228^4}{2^4\cdot 4!}+\ldots\right)=$$

$$=1.73000000+0.00205202-0.00000122+0.0000000014-\ldots\approx$$

$$\approx 1.73000000+0.00205202-0.00000122=1.73205080$$

with absolute error

$$\alpha=|0.0000000014-\ldots|\le 0.0000000014$$

which isn't greater in modulus then 0.0000000014. Therefore we can assert that

$$\sqrt{3} = 1.73205080.$$

with all exact digits.

Ex. 29. Calculate approximate value of π .

For this purpose we take into account that

$$\frac{\pi}{6} = \arctan\frac{1}{\sqrt{3}}$$

and make use of the formula (22) for $x = \frac{1}{\sqrt{3}}$ and (see Ex.…) $\sqrt{3} \approx 1.732051$.

$$\pi = 6\arctan\frac{1}{\sqrt{3}} = 6\left(\frac{1}{\sqrt{3}} - \frac{1}{3\cdot 3\sqrt{3}} + \frac{1}{5\cdot 3^2\sqrt{3}} - \frac{1}{7\cdot 3^3\sqrt{3}} + \frac{1}{9\cdot 3^4\sqrt{3}} - \frac{1}{11\cdot 3^5\sqrt{3}} + \ldots\right) =$$

$$= \frac{6}{\sqrt{3}}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{9\cdot 3^4} - \frac{1}{11\cdot 3^5} + \frac{1}{13\cdot 3^6} - \frac{1}{15\cdot 3^7} + \ldots\right) =$$

$$= 2\sqrt{3}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \frac{1}{9\cdot 3^4} - \frac{1}{11\cdot 3^5} + \frac{1}{13\cdot 3^6} - \frac{1}{15\cdot 3^7} + \frac{1}{17\cdot 3^8} - \frac{1}{19\cdot 3^9} + \ldots\right) =$$

$$= 3.464102 - 0.384900 + 0.076980 - 0.018329 + 0.004753 - 0.001296 + 0.000367 -$$

$$- 0.000104 + 0.000031 - 0.000010 + 0.000003 - \ldots \approx 3.141594 + 0.000003 - \ldots.$$

Thus,

$$\pi = 3.141594$$

with absolute error

$$\alpha = |0.000003 - \ldots| \le 0.000003.$$

Therefore,

$$3.141594 - 0.000003 < \pi < 3.141594 + 0.000003, 3.141591 < \pi < 3.141597,$$

$$\pi \approx 3.14159,$$

and all the digits are exact. More exact value of π is $\pi = 3.1415926$

Ex. 30. Find approximate value of $\cos 3°$.

Expressing the angle in radians and using the formula (17) we'll have

$$\cos 3° = \cos\frac{\pi}{60} = \cos\frac{3.1415926}{60} = \cos 0.0523599 =$$

$$=1.0000000-\frac{0.0523599^2}{2!}+\frac{0.0523599^4}{4!}-\frac{0.0523599^6}{6!}+...=$$

$$=1.0000000-0.0013708+0.0000003-...=0.9986292+0.0000003-...$$

Thus

$$\cos 3° = 0.9986292$$

with absolute error

$$\alpha = |0.0000003-...| \le 0.0000003.$$

Hence

$$0.9986292 - 0.0000003 \le \cos 3° \le 0.9986292 + 0.0000003,$$

$$0.9986289 \le \cos 3° \le 0.9986295,$$

and

$$\cos 3° \approx 0.998629$$

within to 0.000001.

Remark 5. In Ex. 28 - 30 and Ex. 27 of Lecture 28 we've dealt with alternating series when it's easy to estimate the error. In other cases Taylor formula can be more convenient. See Lecture No. 16.

LECTURE NO. 30. FOURIER SERIES

POINT 1. FOURIER SERIES BY ARBITRARY ORTHOGONAL FUNCTIONAL SYSTEM

POINT 2. FOURIER SERIES BY TRIGONOMETRICAL SYSTEM

POINT 1. FOURIER SERIES BY ARBITRARY ORTHOGONAL FUNCTIONAL SYSTEM

Def. 1. Two non-zero functions $f(x), g(x)$ are called orthogonal on some segment $[a, b]$ if

$$\int_a^b f(x)g(x)dx = 0. \qquad (1)$$

Def. 2. A system of non-zero functions is called orthogonal on a segment $[a, b]$ if its functions are pairwise orthogonal on $[a, b]$.

Let

$$\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_n(x), \ldots \qquad (2)$$

be a functional system which is orthogonal on a segment $[a, b]$, namely

$$\int_a^b \varphi_i(x)\varphi_j(x)dx = 0 \text{ for } i \neq j \text{ and } \int_a^b \varphi_i^2(x)dx \neq 0 \text{ for } j = i, \qquad (3)$$

and a function $f(x)$ is expanded into a series in this system, that is

$$f(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + c_3\varphi_3(x) + \ldots + c_n\varphi_n(x) + \ldots. \qquad (4)$$

It is necessary to determine the coefficients $c_1, c_2, c_3, \ldots, c_n, \ldots$ of this expansion.

To find c_n we multiply both sides of (4) by $\varphi_n(x)$ and termwise integrate the result over $[a, b]$ (if it's possible),

$$f(x)\varphi_n(x) = c_1\varphi_1(x)\varphi_n(x) + c_2\varphi_2(x)\varphi_n(x) + \ldots + c_n\varphi_n(x)\varphi_n(x) + \ldots$$

$$\int_a^b f(x)\varphi_n(x)dx = c_1\int_a^b \varphi_1(x)\varphi_n(x)dx + c_2\int_a^b \varphi_2(x)\varphi_n(x)dx + \ldots + c_n\int_a^b \varphi_n^2(x)dx + \ldots.$$

By virtue of orthogonality of the system (2) all integrals from the right equal zero but one, an so

$$\int_a^b f(x)\varphi_n(x)dx = c_n \int_a^b \varphi_n^2(x)dx,$$

$$c_n = \frac{1}{\int_a^b \varphi_n^2(x)dx} \int_a^b f(x)\varphi_n(x)dx. \qquad (5)$$

The series (4) with coefficients (5) is called Fourier series for a function $f(x)$ in the orthogonal functional system (3). The coefficients (5) are called Fourier coefficients.

We can write the series (4) and coefficients (5) for highly wide class of functions and orthogonal functional systems but we can't in general assert the validity of the equality (5).

By this reason one as usually writes

$$f(x) \sim c_1\varphi_1(x) + c_2\varphi_2(x) + c_3\varphi_3(x) + \ldots + c_n\varphi_n(x) + \ldots \qquad (6)$$

and says that the series (6) corresponds to the function $f(x)$.

The problem is to find conditions to substitute the correspondence (6) by the exact equality (4). Such the problem is solved for trigonometric functional system which will be studied in the next point.

POINT 2. FOURIER SERIES BY TRIGONOMETRICAL SYSTEM

Let be given the next trigonometric system of functions

$$\frac{1}{2}, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, \cos\frac{2\pi x}{l}, \sin\frac{2\pi x}{l}, \ldots, \cos\frac{n\pi x}{l}, \sin\frac{n\pi x}{l}, \ldots. \qquad (7)$$

All the functions of this system are periodic with a common period $2l$.

■If for example we take $x + 2l$ instead x, we'll obtain

$$\cos\frac{n\pi(x+2l)}{l} = \cos\frac{n\pi x + 2n\pi l}{l} = \cos\left(\frac{n\pi x}{l} + 2n\pi\right) = \cos\frac{n\pi x}{l}. \ ■$$

Theorem 1. Trigonometric functional system is orthogonal one on the segment $[-l, l]$ (and on arbitrary segment of the length $2l$).

■It's enough to prove the theorem for the segment $[-l, l]$. It must be proved that

$$\int_{-l}^{l}\frac{1}{2}\cos\frac{n\pi x}{l}dx=\frac{1}{2}\int_{-l}^{l}\cos\frac{n\pi x}{l}dx=0,\ \int_{-l}^{l}\frac{1}{2}\sin\frac{n\pi x}{l}dx=\frac{1}{2}\int_{-l}^{l}\sin\frac{n\pi x}{l}dx=0 \text{ for any } n; \quad (8)$$

$$\int_{-l}^{l}\cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}dx=0,\ \int_{-l}^{l}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx=0 \text{ for any different } m \text{ and } n; \quad (9)$$

$$\int_{-l}^{l}\cos\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx=0 \text{ for any } m \text{ and } n, \quad (10)$$

and in addition

$$\int_{-l}^{l}\left(\frac{1}{2}\right)^2dx=\frac{l}{2},\ \int_{-l}^{l}\left(\cos\frac{n\pi x}{l}\right)^2dx=l,\ \int_{-l}^{l}\left(\sin\frac{n\pi x}{l}\right)^2dx=l. \quad (11)$$

But

$$\int_{-l}^{l}\cos\frac{n\pi x}{l}dx=\frac{l}{n\pi}\sin\frac{n\pi x}{l}\Big|_{-l}^{l}=\frac{l}{n\pi}(\sin n\pi-\sin(-n\pi))=0,$$

$$\int_{-l}^{l}\sin\frac{n\pi x}{l}dx=-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\Big|_{-l}^{l}=\frac{l}{n\pi}(\cos n\pi-\cos(-n\pi))=\frac{l}{n\pi}(\cos n\pi-\cos n\pi)=0,$$

from which one gets (8) and (after transformation the products of trigonometric functions into algebraic sums) (9) - (10). The first of the formulas (11) is evident, to get the other it's well to apply power reduction formulas. For example

$$\int_{-l}^{l}\left(\cos\frac{n\pi x}{l}\right)^2dx=\int_{-l}^{l}\frac{1+\cos\frac{2n\pi x}{l}}{2}dx=\frac{1}{2}\int_{-l}^{l}\left(1+\cos\frac{2n\pi x}{l}\right)dx=\frac{1}{2}\left(\int_{-l}^{l}dx+\int_{-l}^{l}\cos\frac{2n\pi x}{l}dx\right)=$$

$$=\frac{1}{2}\left(x\Big|_{-l}^{l}+\frac{l}{2n\pi}\sin\frac{2n\pi x}{l}\Big|_{-l}^{l}\right)=\frac{1}{2}(2l+0)=l.■$$

Let's now establish a correspondence between an arbitrary function $f(x)$ and a series in the trigonometric system (7), namely

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right) \quad (12)$$

where on the base of the formulas (5)

$$a_0=\frac{1}{\int_{-l}^{l}\left(\frac{1}{2}\right)^2}\int_{-l}^{l}f(x)\frac{1}{2}dx=\frac{1}{l}\int_{-l}^{l}f(x)dx,$$

$$a_n = \frac{1}{\int\limits_{-l}^{l}\left(\cos\frac{n\pi x}{l}\right)^2}\int\limits_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \frac{1}{l}\int\limits_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

and similarly

$$b_n = \frac{1}{\int\limits_{-l}^{l}\left(\sin\frac{n\pi x}{l}\right)^2}\int\limits_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{l}\int\limits_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx.$$

Finally Fourier coefficients of the trigonometric series (12) are given by the next formulas:

$$a_0 = \frac{1}{l}\int\limits_{-l}^{l} f(x)dx,\ a_n = \frac{1}{l}\int\limits_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx,\ b_n = \frac{1}{l}\int\limits_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx. \qquad (13)$$

Remark 1. The sum of the series (12) is a periodic function with the period *2l* (in other words it is a *2l*-periodic function). Therefore a function $f(x)$ which is developed in such the series on the set of all reals must be *2l*-periodic.

Def. 3. A function $f(x)$ is called **piecewise monotone** on a segment $[a, b]$ if the segment can be divided into finite number of parts (subintervals) such that the function is monotone on every of these parts.

Theorem 2 (Dirichlet[1] expansibility theorem)**.** If a *2l*-periodic function $f(x)$ is bounded and piecewise monotone on the segment $[-l, l]$, then its Fourier series (12), (13) converges at each point *x*. The sum of the series equals the function

$$S(x) = f(x) \qquad (14)$$

at every its continuity point *x*. If x_0 is a discontinuity point of the function, then the sum of Fourier series at x_0 equals the half sum of the left and right limits of the function at this point

$$S(x_0) = \frac{f(x_0 - 0) + f(x_0 + 0)}{2} \qquad (15)$$

where $f(x_0 - 0) = \lim\limits_{x \to x_0 - 0} f(x), \quad f(x_0 + 0) = \lim\limits_{x \to x_0 + 0} f(x).$

[1] Dirichlet, Peter Gustav Lejeune (1805 - 1859), a German mathematician

As resume we can write

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)=\begin{cases} f(x) & \text{in a continuity point } x, \\ \dfrac{f(x-0)+f(x+0)}{2} & \text{in a discontinuity point } x \end{cases} \qquad (16)$$

(if the coefficients of the series are determined by the formula (13)).

Remark 2. It can be proved that for even or odd functions the formula (13) for Fourier coefficients take on some other form. Namely for an even function

$$a_0=\frac{2}{l}\int_0^l f(x)dx,\ a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx,\ b_n=0 \qquad (17)$$

and for an odd function

$$a_0=0,\ a_n=0,\ b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx\,. \qquad (18)$$

Ex. 1. A function is given by the formula

$$f(x)=x^2$$

on the segment $[-5,5]$. Develop it into Fourier series.

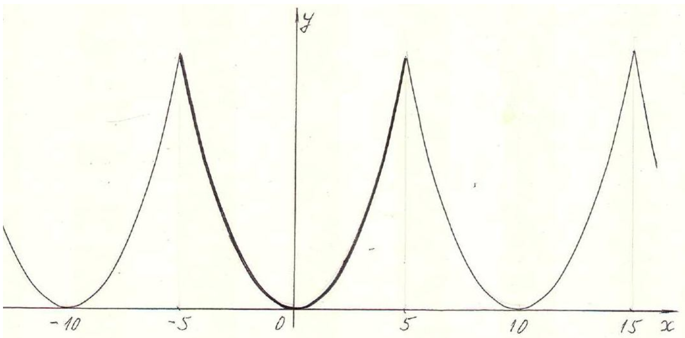

Fig. 1

Let's consider the 10-periodic function $f^*(x)$ which is determined by the given formula on the segment $[-5,5]$ (see fig. 1)[1]. It is associated with its Fourier series (12), (13) (for the case $2l=10$ that is $l=5$), namely

$$f^*(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{5}+b_n\sin\frac{n\pi x}{5}\right).$$

The function $f^*(x)$ is even one, and so we take Fourier coefficients in the form (17),

$$a_0=\frac{2}{5}\int_0^5 f(x)dx=\frac{2}{5}\int_0^5 x^2dx=\frac{2}{5}\cdot\frac{x^3}{3}\Bigg|_0^5=\frac{50}{3},\quad b_n=0,$$

[1] Such the function $f^*(x)$ is called a periodic continuation [periodic extension, periodic prolongation] of the given function $f(x)$ from the segment $[-5,5]$ onto the whole set of reals.

$$a_n = \frac{2}{5}\int_0^5 f(x)\cos\frac{n\pi x}{5}dx = \frac{2}{5}\int_0^5 x^2\cos\frac{n\pi x}{5}dx = \left| \begin{matrix} u = x^2 & dv = \cos\frac{n\pi x}{5}dx \\ du = 2xdx & v = \frac{5}{n\pi}\sin\frac{n\pi x}{5} \end{matrix} \right| =$$

$$= \frac{2}{5}\left(\frac{5}{n\pi}\left(x^2\sin\frac{n\pi x}{5} \right)\Bigg|_0^5 - \int_0^5 2x\cdot\frac{5}{n\pi}\sin\frac{n\pi x}{5}dx \right) = -\frac{4}{n\pi}\int_0^5 x\sin\frac{n\pi x}{5}dx =$$

$$= \left| \begin{matrix} u = x & dv = \sin\frac{n\pi x}{5}dx \\ du = dx & v = -\frac{5}{n\pi}\cos\frac{n\pi x}{5} \end{matrix} \right| = -\frac{4}{n\pi}\left(-\frac{5}{n\pi}\left(x\cos\frac{n\pi x}{5} \right)\Bigg|_0^5 + \frac{5}{n\pi}\int_0^5 \cos\frac{n\pi x}{5}dx \right) =$$

$$= \frac{20}{n^2\pi^2}\left(5\cos n\pi - \int_0^5 \cos\frac{n\pi x}{5}dx \right) = \frac{20}{n^2\pi^2}\left(5\cdot(-1)^n - \frac{5}{n\pi}\sin\frac{n\pi x}{5}\Bigg|_0^5 \right) = \frac{(-1)^n\cdot 100}{n^2\pi^2}.$$

The function $f^*(x)$ satisfies the conditions of Dirichlet expansibility theorem: it's bounded and piecewise monotone on the segment $[-5, 5]$ ($0 \le x^2 \le 25$, x^2 decreases on $[-5, 0]$ and increases on $[0, 5]$). In addition it's continuous on the set of all reals and so its Fourier series converges to it at any points. In particular it converges to the function $f(x) = x^2$ on the segment $[-5, 5]$. Therefore

$$\forall x \in [-5, 5] \quad x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{5} = \frac{25}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n\cdot 100}{n^2\pi^2}\cos\frac{n\pi x}{5}.$$

Ex. 2. A function is given by the formula

$$f(x) = \frac{1}{3}x$$

on the interval $(-\pi, \pi)$. Decompose it into Fourier series.

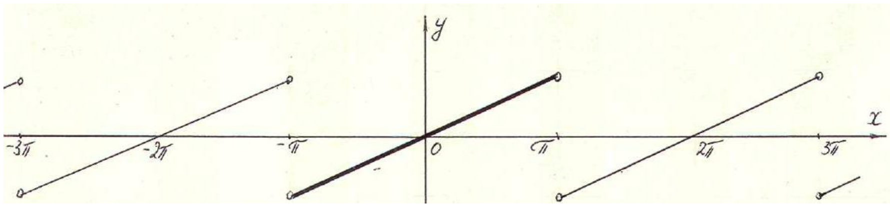

Fig. 2

Let's consider the 2π-periodic function $f^*(x)$ which is determined by the given formula on the interval $(-\pi, \pi)$ (see fig. 2)[1]. It is assigned its Fourier series, namely

[1] A periodic continuation of the function $f(x) = 1/3\,x$ from the interval $(-\pi, \pi)$ onto the set of all reals.

(for the case $2l = 2\pi$ that is $l = \pi$)

$$f^*(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

The function $f^*(x)$ is odd one, and so we take Fourier coefficients in the form (18),

$$a_0 = 0, a_n = 0,$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nxdx = \frac{2}{\pi}\cdot\frac{1}{3}\int_0^{\pi} x\sin nxdx = \begin{vmatrix} u = x & dv = \sin nxdx \\ du = dx & v = -\frac{1}{n}\cos nx \end{vmatrix} =$$

$$= \frac{2}{3\pi}\left(-\frac{1}{n}(x\cos nx)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nxdx \right) = \frac{2}{3\pi n}\left(-\pi\cos n\pi + \frac{1}{n^2}\sin nx\Big|_0^{\pi} \right) = -\frac{2\cos n\pi}{3n}.$$

The function $f^*(x)$ satisfies all the conditions of Dirichlet theorem (it is bounded by the numbers $-\pi/3, -\pi/3$ and increases on the interval $(-\pi,\ \pi)$) and is continuous on the set of all real numbers excepting the points $x = 2\pi k, k \in Z$. So its Fourier series converges to $f^*(x)$ at any point $x \neq 2\pi k, k \in Z$. In particular it converges to the function $f(x) = 1/3\, x$ on the interval $(-\pi, \pi)$, that is

$$\forall x \in (-\pi, \pi) \quad \frac{1}{3}x = -\frac{2}{3}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}\sin nx = -\frac{2}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx = \frac{2}{3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx.$$

The sum of Fourier series at the points $\pm\pi$ equals 0. For the point $x = \pi$ we can reasen as follows:

$$S(\pi) = \frac{1}{2}\left(f^*(\pi - 0) + f^*(\pi + 0)\right) = \frac{1}{2}\left(f(\pi - 0) + f(-\pi + 0)\right) = \frac{1}{2}\left(\frac{1}{3}\pi + \frac{1}{3}(-\pi)\right) = 0;$$

by analogous way we get

$$S(-\pi) = \frac{1}{2}\left(f^*(-\pi - 0) + f^*(-\pi + 0)\right) = \frac{1}{2}\left(f(\pi - 0) + f(-\pi + 0)\right) = \frac{1}{2}\left(\frac{1}{3}\pi + \frac{1}{3}(-\pi)\right) = 0$$

Ex. 3. Let be given a function

$$f(x) = \begin{cases} 1 & \text{on} \quad (-\pi, 0), \\ \frac{3}{\pi}x - 2 & \text{on} \quad [0, \pi]. \end{cases}$$

Expend it into Fourier series.

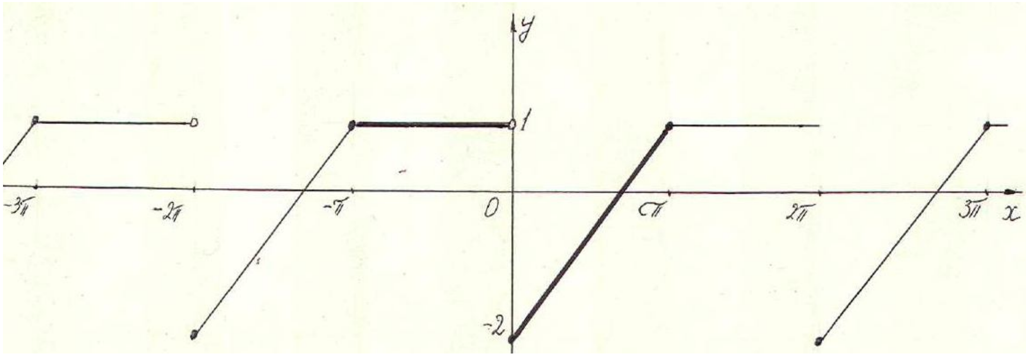

Fig. 3

Let's consider the 2π-periodic function $f^*(x)$ which is determined by the given formula on the interval $(-\pi,\ \pi]$ (fig. 3). It is assigned [it is associated with] its Fourier series (12), (13) (for the case $2l = 2\pi$ that is $l = \pi$). Namely

$$f^*(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx\,.$$

The function $f(x)$ is given by different formulas in two different intervals $(-\pi, 0)$, $[0, \pi]$, and therefore we take the integrals over $(-\pi,\ \pi]$ as the sums of integrals over $(-\pi, 0)$ and $[0, \pi]$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right) = \frac{1}{\pi}\left(\int_{-\pi}^{0} 1\cdot dx + \int_{0}^{\pi}\left(\frac{3}{\pi}x - 2\right)dx\right) =$$

$$= \frac{1}{\pi}\left(x\Big|_{-\pi}^{0} + \left(\frac{3}{\pi}\cdot\frac{x^2}{2} - 2x\right)\Bigg|_{0}^{\pi}\right) = \frac{1}{\pi}\left(\pi + \frac{3}{\pi}\cdot\frac{\pi^2}{2} - 2\pi\right) = \frac{1}{2};$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\left(\int_{-\pi}^{0} 1\cdot\cos nxdx + \int_{0}^{\pi}\left(\frac{3}{\pi}x - 2\right)\cos nxdx\right) =$$

$$= \begin{vmatrix} u = \dfrac{3}{\pi}x - 2 & dv = \cos nxdx \\ du = \dfrac{3}{\pi}dx & v = \dfrac{1}{n}\sin nx \end{vmatrix} = \frac{1}{\pi}\left(\frac{1}{n}\sin nx\Big|_{-\pi}^{0} + \frac{1}{n}\left(\left(\frac{3}{\pi}x - 2\right)\sin nx\right)\Bigg|_{0}^{\pi} -\right.$$

$$-\frac{3}{\pi n}\int_0^{\pi}\sin nxdx\Bigg)=\frac{3}{\pi^2 n}\int_0^{\pi}\sin nxdx=\frac{3}{\pi^2 n^2}\cos nx\Bigg|_0^{\pi}=\frac{3}{\pi^2 n^2}(\cos n\pi-1)=\frac{3}{\pi^2 n^2}\left((-1)^n-1\right);$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\left(\int_{-\pi}^{0}1\cdot\sin nxdx+\int_0^{\pi}\left(\frac{3}{\pi}x-2\right)\sin nxdx\right)=$$

$$=\begin{vmatrix}u=\frac{3}{\pi}x-2 & dv=\sin nxdx\\ du=\frac{3}{\pi}dx & v=-\frac{1}{n}\cos nx\end{vmatrix}=\frac{1}{\pi}\left(-\frac{1}{n}\cos nx\Bigg|_{-\pi}^{0}-\frac{1}{n}\left(\left(\frac{3}{\pi}x-2\right)\cos nx\right)\Bigg|_0^{\pi}+\right.$$

$$\left.+\frac{3}{\pi n}\int_0^{\pi}\cos nxdx\right)=\frac{1}{\pi}\left(-\frac{1}{n}(1-\cos(-n\pi))-\frac{1}{n}(\cos n\pi+2)+\frac{3}{\pi n^2}\sin nx\Bigg|_0^{\pi}\right)$$

$$\frac{1}{\pi n}(-1+\cos n\pi-\cos n\pi-2)=-\frac{3}{\pi n}.$$

The function $f^*(x)$ satisfies the conditions of Dirichlet theorem (it's bounded by the numbers -2 and 1, is constant on the interval $[-\pi,0]$ and increases on the interval $[-\pi,\pi]$). In addition it's continuous on the set of all real numbers excepting the points $x=2\pi k,\ k\in Z$. Its Fourier series converges to $f^*(x)$ at any point $x\neq 2\pi k,\ k\in Z$. In particular it converges to the given function $f(x)$ on the union of intervals $(-\pi,0)$ and $(0,\pi]$, that is

$$\forall x\in(-\pi,0)\cup(0,\pi]\quad f(x)=\frac{1}{4}+\sum_{n=1}^{\infty}\left(\frac{3}{\pi^2 n^2}\left((-1)^n-1\right)\cos nx-\frac{3}{\pi n}\sin nx\right).$$

The value of the sum of Fourier series at the discontinuity point x=0

$$S(0)=\frac{1}{2}(f(-0)+f(+0))=\frac{1}{2}(1+(-2))=-\frac{1}{2}\neq f(0).$$

It doesn't coincide with the value $f(0)=-2$ of the function $f(x)$ at this point.

SERIES: main terms (RUEtr)

1. Абсолютно сходящийся ряд	Абсолютно збіжний ряд	Ábsolutely convérgent séries	´æbsəlu:tlɪ, ´æbsəlju:tlɪ, kən´vɜ:ʤənt, ´sɪəri:z
2. Биномиальный ряд	Біномний ряд	Binómial séries	baɪ´nəʊmɪəl, ´sɪəri:z
3. Гармонический ряд	Гармонічний ряд	Harmónic séries	hɑ:´mɒnɪk, ´sɪəri:z
4. Геометрическая прогрессия	Геометрична прогресія	Geométric(al) progréssion	ʤɪə´metrɪk(l), prəʊ´greʃn [prə´gre ʃn]
5. Достаточное условие сходимости ряда	Достатня ознака збіжності ряду	Sufficient test [critérion, *pl* critéria] for/of convérgence of a séries	sə´fɪʃnt, test, kraɪ´tɪərɪən, kraɪ´tɪərɪə, kən´vɜ:ʤəns, ´sɪəri:z
6. Знакоположительный ряд, ряд с положительными членами	Знакододатний ряд, ряд з додатними членами	Séries with pósitive terms, pósitive term séries	´sɪəri:z, ´pɒzətɪv, tɜ:m
7. Знакочередующийся ряд	Знакопочережний [знакопереміжний, знакозмінний, альтернуючий] ряд	Álternating séries	´ɒltəneɪtɪŋ, ´sɪəri:z
8. Знаменатель геометрической прогрессии	Знаменник геометричної прогресії	Rátio [common rátio, quótient] of a geométric(al) progréssion	´reɪʃɪəʊ, ´kɒmən, ´kwəʊʃnt, ʤɪə´metrɪk(l), prəʊ´greʃn
9. Интегральный признак сходимости	Інтегральна ознака збіжності	Íntegral test [critérion, *pl* critéria] for convérgence	´ɪntɪgrəl, test, kraɪ´tɪərɪən, kraɪ´tɪərɪə, kən´vɜ:ʤəns
10.Интервал сходимости степенного ряда	Інтервал збіжності степеневого ряду	Ínterval of convérgence of a pówer séries	´ɪntəvl, kən´vɜ:ʤəns, ´paʊə, ´sɪəri:z
11.Исследовать ряд на (абсолютную, условную) сходимость	Дослідити ряд на (абсолютну, умовну) збіжність	Test [invéstigate, exámine] a séries for (ábsolute, condítional) convérgence	test, ɪn´vestɪgeɪt, ɪg¸zæmɪn, ´æbsəlu:t, kən´dɪʃənl, kən´vɜ:ʤəns

12.Маклорена ряд	Маклорена ряд	Maclaurin(´s) séries	´sɪəri:z
13.Необходимое условие сходимости ряда	Необхідна ознака збіжності ряду	Nécessary test [critérion *pl* critéria] for/of convérgence of a séries	´nesəsrɪ, ´nesəsərɪ, test, kraɪ´tɪərɪən,kraɪ´tɪərɪə, kən´vɜ:ʤəns,´sɪəri:z
14.Область сходимости	Область збіжності	Domáin of convérgence	də´meɪn,dəʊ´meɪn, kən´vɜ:ʤəns
15.Общий член ряда	Загальний член ряду	Géneral term of a séries	´ʤenrəl, tɜ:m, sɪəri:z
16.Остаток ряда (после *n*-го члена)	Залишок ряду (після *n*-го члена)	Remáinder [*n*-th remáinder] of a séries (áfter the *n*-th term)	rɪ´meɪndə, ´sɪəri:z, ɑ:ftə, tɜ:m
17.Признак Даламбера	Ознака Даламбера	D´Alembert´s test [critérion]	test, kraɪ´tɪərɪən
18.Признак Лейбница	Ознака Лейбніца	Leibniz´ test [critérion]	test, kraɪ´tɪərɪən
19.Признак сравнения (для рядов с положительными членами)	Ознака порівняння (для рядів з додатними членами)	Compárison test [critérion, *pl* critéria] (for séries with pósitive terms)	kəm´pærɪsn,test, kraɪ´tɪərɪən, kraɪ´tɪərɪə, ´sɪəri:z, ´pɒzətɪv, tɜ:m
20.Радиус сходимости степенного ряда	Радіус збіжності степеневого ряду	Rádius of convérgence of a pówer séries	´reɪdɪəs, kən´vɜ:ʤəns, ´paʊə, ´sɪəri:z
21.Разложение функции в ряд	Розвинення функції в ряд	Expánsion/devélopment of a fúnction in/into a séries	ɪk´spænʃn, dɪ´veləpmənt, ´fʌŋkʃn, ´sɪəri:z,
22.Разложить функцию в ряд	Розвинути функцію в ряд	Expánd/devélop a fúnction in/into a séries	ɪk´spænd, dɪ´veləp, ´fʌŋkʃn,´paʊə, ´sɪəri:z
23.Раскладываться в ряд	Розвиватися [розкладатися] в ряд	Be expándable/devélopable in/into a séries	ɪk´spændəbl, dɪ´veləpəbl,´sɪəri:z
24.Расходиться (о ряде)	Розбігатися (про ряд)	Divérge (about a séries)	daɪ´vɜ:ʤ, ´sɪəri:z
25.Расходящийся ряд	Розбіжний ряд	Divérgent séries	daɪ´vɜ:ʤənt,´sɪəri:z
26.Ряд абсолютных величин чле-	Ряд абсолютних величин членів	Séries of móduli of terms of a séries	´sɪəri:z, ´mɒdjʊlaɪ, tɜ:m,´ɑ:bɪtrərɪ,´rɪ-

но ряда с произвольными [вещественными] членами	ряду з довільними [дійсними] членами	with árbitrary [réal] terms [of a plus-and-mínus séries]	əl, plʌs ænd (ənd, ən, n) ´maɪnəs
27.Степенной ряд	Степеневий ряд	Pówer séries	´paʊə, ´sɪəri:z
28.Сумма ряда	Сума ряду	Sum of a séries	sʌm, ´sɪəri:z
29.Сходимость ряда	Збіжність ряду	Convérgence of a séries	kən´vɜ:ʤəns,´sɪəri:z
30.Сходиться (абсолютно, условно) (о ряде)	Збігатися (абсолютно, умовно) (про ряд)	Convérge (ábslutely, condítionly) (about a séries)	kən´vɜ:ʤ, ´æbsəlu:tlɪ, ´æbsəlju:tlɪ, kən´dɪʃənlɪ, ´sɪəri:z
31.Сходящийся ряд	Збіжний ряд	Convérgent séries	kən´vɜ:ʤənt,´sɪəri:z
32.Тейлора ряд	Тейлора ряд	Taylor(´s) séries	´sɪəri:z
33.Точка сходимости	Точка збіжності	Póint of convérgence	kən´vɜ:ʤəns, ´pɔɪnt
34.Условно сходящийся ряд	Умовно збіжний ряд	Condítionally convérgent séries	kən´dɪʃənlɪ, kən´vɜ:ʤənt, ´sɪəri:z
35.Функциональный ряд	Функціональний ряд	Fúnction(al) séries, séries of fúnctions	´fʌŋkʃn, ´fʌŋkʃənl, ´fʌŋkʃnəl, ´fʌŋkʃnl
36.Фурье ряд	Фур"є ряд	Fourier('s) séries	´sɪəri:z
37.Частичная сумма ряда (первая, вторая, третья, *n*-ая)	Часткова сума ряду (перша, друга, третя, *n*-на)	Pártial sum of a séries (first, second, third, *n*-th)	pɑ:ʃl, sʌm, ´sɪəri:z, fɜ:st,´sekənd, θɜ:d,
38.Числовой ряд	Числовий ряд	Nùmérical/número séries	njʊ´merɪkl, ´nʌmbə, ´sɪəri:z

SERIES: main terms (EtrRU)

1. Ábsolutely convérgent séries	´æbsəlu:tlɪ,´æbsəlju:tlɪ, kən´vɜ:-ʤənt, ´sɪəri:z	Абсолютно сходящийся ряд	Абсолютно збіжний ряд
2. Álternating séries	´ɒltəneɪtɪŋ,´sɪəri:z	Знакочередующийся ряд	Знакопочережний [знакопереміжний, знакозмінний, альтернуючий] ряд
3. Be expándable/ devélopable in/into a séries	ɪk´spændəbl, dɪ-´veləpəbl,´sɪəri:z	Раскладываться в ряд	Розвиватися [розкладатися] в ряд
4. Binómial séries	baɪ´nəʊmɪəl, ´sɪəri:z	Биномиальный ряд	Біномний ряд
5. Compárison test [critérion, *pl* critéria] (for séries with pósitive terms)	kəm´pærɪsn, test, kraɪ´tɪərɪən, kraɪ-´tɪərɪə, ´sɪəri:z, ´pɒzətɪv, tɜ:m	Признак сравнения (для рядов с положительными членами)	Ознака порівняння (для рядів з додатними членами)
6. Condítionally convérgent séries	kən´dɪʃənlɪ, kən-´vɜ:ʤənt, ´sɪəri:z	Условно сходящийся ряд	Умовно збіжний ряд
7. Convérge (ábslutely, condítionly) (about a séries)	kən´vɜ:ʤ, ´æbsəlu:tlɪ, ´æbsəlju:tlɪ, kən´dɪʃənlɪ, ´sɪəri:z	Сходиться (абсолютно, условно) (о ряде)	Збігатися (абсолютно, умовно) (про ряд)
8. Convérgence of a séries	kən´vɜ:ʤəns,´sɪəri:z	Сходимость ряда	Збіжність ряду
9. Convérgent séries	kən´vɜ:ʤənt,´sɪəri:z	Сходящийся ряд	Збіжний ряд
10. D´Alembert´s test [critérion]	test, kraɪ´tɪərɪən	Признак Даламбера	Ознака Даламбера
11. Divérge (about a séries)	daɪ´vɜ:ʤ, ´sɪəri:z	Расходиться (о ряде)	Розбігатися (про ряд)
12. Divérgent séries	daɪ´vɜ:ʤənt	Расходящийся ряд	Розбіжний ряд
13. Domáin of convérgence	də´meɪn,dəʊ´meɪn kən´vɜ:ʤəns	Область сходимости	Область збіжності
14. Expánd [devélop] a fúnction in/ into a séries	ɪk´spænd, dɪ´veləp, ´fʌŋkʃn,´paʊə, ´sɪəri:z	Разложить функцию в ряд	Розвинути функцію в ряд
15. Expánsion/devé-	ɪk´spænʃn, dɪ´ve-	Разложение функ-	Розвинення функ-

lopment of a fúnction in/into a séries	ləpmənt, ´fʌŋkʃn, ´sɪəri:z,	ции в ряд	ції в ряд
16.Fourier('s) séries	´sɪəri:z	Фурье ряд	Фур"є ряд
17.Fúnction(al) séries, séries of fúnctions	´fʌŋkʃn, ´fʌŋkʃənl, ´fʌŋkʃnəl, ´fʌŋkʃnl	Функциональный ряд	Функціональний ряд
18.Géneral term of a séries	´ʤenrəl, tɜ:m, ´sɪəri:z	Общий член ряда	Загальний член ряду
19.Geométric(al) progréssion	ʤɪə´metrɪk(l), prəʊ´greʃn [prə´greʃn]	Геометрическая прогрессия	Геометрична проґресія
20.Harmónic séries	hɑ:´mɒnɪk,´sɪəri:z	Гармонический ряд	Гармонічний ряд
21.Íntegral test [critérion, *pl* critéria] for convérgence	´ɪntɪgrəl, test, kraɪ´tɪərɪən, kraɪ´tɪərɪə, kən´vɜ:ʤəns	Интегральный признак сходимости	Інтеґральна ознака збіжності
22.Ínterval of convérgence of a pówer séries	´ɪntəvl, kən´vɜ:ʤəns, ´paʊə, ´sɪəri:z	Интервал сходимости степенного ряда	Інтервал збіжності степеневого ряду
23.Leibniz´ test [cri-térion]	test, kraɪ´tɪərɪən	Признак Лейбница	Ознака Лейбніца
24.Maclaurin(´s) séries	´sɪəri:z	Маклорена ряд	Маклорена ряд
25.Nécessary test [critérion *pl* critéria] for/of convérgence of a séries	´nesəsrɪ, ´nesəsərɪ, test, kraɪ´tɪərɪən, kraɪ´tɪərɪə, kən´vɜ:ʤəns, ´sɪəri:z	Необходимое условие сходимости ряда	Необхідна ознака збіжності ряду
26.Nùmérical/númber séries	njʊ´merɪkl, ´nʌmbə, ´sɪəri:z	Числовой ряд	Числовий ряд
27.Pártial sum of a séries (first, second, third, *n*-th)	fɜ:st, ´sekənd, θɜ:d, pɑ: ʃl, sʌm, ´sɪəri:z	Частичная сумма ряда (первая, вторая, третья, *n*-ая)	Часткова сума ряду (перша, друга, третя, *n*-на)
28.Póint of convérgence	kən´vɜ:ʤəns, ´pɔɪnt	Точка сходимости	Точка збіжності
29.Pówer séries	´paʊə, ´sɪəri:z	Степенной ряд	Степеневий ряд
30.Rádius of convérgence of a pówer séries	´reɪdɪəs, kən´vɜ:ʤəns, ´paʊə, ´sɪə-	Радиус сходимости степенного ряда	Радіус збіжності степеневого ряду

	ri:z		
31.Rátio [common) rátio, quótient] of a geométric(al) progréssion	´kɒmən, ´reɪʃɪəʊ ´kwəʊʃnt, ʤɪə-´metrɪk(l), prəʊ-´greʃn	Знаменатель геометрической прогрессии	Знаменник геометричної проґресії
32.Remáinder [*n*-th remáinder] of a séries (áfter the *n*-th term)	rɪ´meɪndə,´sɪəri:z, ɑ:ftə, tɜ:m	Остаток ряда (после *n*-го члена)	Залишок ряду (після *n*-го члена)
33.Séries of móduli of terms of a séries with árbitrary [réal] terms [of a plus-and-mínus séries]	´sɪəri:z, ´mɒdjʊlaɪ, ´negətɪv, tɜ:m, ´ɑ:bɪtrərɪ,´rɪəl, plʌs ænd (ənd, ən, n) ´maɪnəs	Ряд абсолютных величин членов ряда с произвольными [вещественными] членами	Ряд абсолютних величин членів ряду з довільними [дійсними] членами
34.Séries with pósitive terms, pósitive term séries	´sɪəri:z, ´pɒzətɪv, tɜ:m	Знакоположительный ряд, ряд с положительными членами	Знакододатний ряд, ряд з додатними членами
35.Suffícient test [critérion, *pl* critéria] for/of convérgence of a séries	sə´fɪʃnt, test, kraɪ-´tɪərɪən, kraɪ´tɪə-rɪə, kən´vɜ:ʤəns, ´sɪəri:z	Достаточное условие сходимости ряда	Достатня ознака збіжності ряду
36.Sum of a séries	sʌm, ´sɪəri:z	Сумма ряда	Сума ряду
37.Taylor(´s) séries	´sɪəri:z	Тейлора ряд	Тейлора ряд
38.Test [invéstigate, exámine] a séries for (ábsolute, condítional) convérgence	test, ɪn´vestɪgeɪt, ɪg¸zæmɪn,´sɪəri:z, ´æbsəlu:t, kən´dɪ-ʃənl, kən´vɜ:ʤəns	Исследовать ряд на (абсолютную, условную) сходимость	Дослідити ряд на (абсолютну, умовну) збіжність

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Series (Ряди).

Методичний посібник по вивченню розділу курсу вищої математики для студентів ДонНТУ (англійською мовою)

УКЛАДАЧ: Косолапов Юрій Федорович, кандидат фізико-математичних наук, професор

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83000, м. Донецьк, вул. Артема, 58, ДонНТУ