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CALCULATION OF DYNAMIC STRENGTH OF ELASTIC DEFORMED SYSTEMS

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Abstract

The paper considers the theory of calculating the dynamic strength of elastic rod systems with distributed and localized masses when the eigenfunctions of such systems are weighted orthogonal. The ways of defining weight functions and squared norm of eigenfunctions for transverse and longitudinal vibrations are suggested. Theoretical studies are illustrated with the example of double-step drill column descending.

Keywords: eigenfunctions; systems with step-variable section; weight functions, weighted orthogonal eigenfunctions; natural vibrations; unit function; eigenfunctions with weight, squared norm of eigenfunctions with weight

Deformed systems with distributed parameters are often used in the equipment of different branches of industry. These are rod structures with distributed masses which have loads from interaction with other objects as well as localized masses that determine inertial loads. The examples of such systems with distributed / localized masses are shafts of rolling mills, transport pipelines of ground equipment and deep-water mining complexes, airlifting and pumping systems, drilling rigs for oil and gas wells and of pit-shafts and special-purpose wells of big diameters, pipelines of drain and ventilating systems, pipelines of suction-tube dredge, and others.

Dynamic processes in such systems are described by differential equations in partial derivatives, their solutions being presented as eigenfunctions of respective boundary problems. The eigenfunctions are always orthogonal in the absence of localized masses and for the rods with uniform cross-sections, but they are weighted orthogonal for the systems with step-variable sections if localized masses are present. This fact essentially complicates the solution of such dynamic problems.

Transverse vibrations of homogeneous rods under different boundary conditions are specified in monograph [1]. Similar problems of natural vibrations are discussed in [2] for a double-step rod. Various problems related to dynamics of homogeneous rods with localized masses are described in [3].

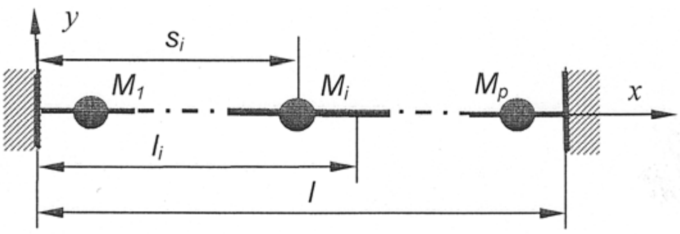

Figure 1. Calculation scheme of a rod system

Let us consider the general problem of transverse vibrations in the rod systems of step-variable rigidity with localized masses $M_i (i = 1,2,..., p)$ (Figure 1) [4].

Transverse vibrations of such system section are examined separately for each part and then conjugation conditions are used for the system parts.

$$\frac{\partial^2 y_i}{\partial t^2} + a_i^2 \frac{\partial^4 y_i}{\partial x^4} = 0, \quad i = 1, 2,..., p, \tag{1}$$

where $y_i(x,t)$ are transverse displacements of rod sections of the i^{th} part, $a_i^2 = \frac{E_i J_i}{m_i}$, $E_i J_i$, m_i are bending rigidity and linear mass of a part respectively.

First let us study natural vibrations of such a system ignoring the localized masses. Boundary conditions should be specified in order to solve equation (1)

$$L_1 y_1\Big|_{x=0} = 0;\ L_2 y_1\Big|_{x=0} = 0;\ L_3 y_p\Big|_{x=l} = 0;\ L_4 y_p\Big|_{x=l} = 0. \tag{2}$$

The type of liner differential operator $L_j (j = 1,...,4)$ under condition (2) corresponds to the following types of rod system ends fastening: anchorage, pinning and a loose end.

Moreover, it is necessary to specify the conjugation conditions when $x = l_i (i = 1, 2, 3..., p-1)$.

$$\begin{aligned} y_i(l_i,t) &= y_{i+1}(l_i,t); \\ y_i'(l_i,t) &= y_{i+1}'(l_i,t); \\ E_i J_i y_i''(l_i,t) &= E_{i+1} J_{i+1} y_{i+1}''(l_i,t); \\ E_i J_i y_i'''(l_i,t) &= E_{i+1} J_{i+1} y_{i+1}'''(l_i,t). \end{aligned} \tag{3}$$

Let us study the properties of the eigenfunctions required to solve natural and forced vibrations problem. The eigenfunctions of the boundary problem are specified as

$$X_n(x) = \sum_{i=1}^{p} (e(l_i - x) - e(l_{i-1} - x)) X_{n,i}(x), \tag{4}$$

where $e(x)$ is a unit function, $X_{n,i}(x)$ are the eigenfunctions of respective boundary problems (1).

For each part of the system we use the well known formula [1]

$$\left(\omega_n^2 - \omega_m^2\right) \int_{l_{i-1}}^{l_i} X_{n,i} X_{m,i} dx = a_i^2 \left(X_{m,i} X_{n,i}''' - X_{n,i} X_{m,i}''' + X_{n,i}' X_{m,i}'' - X_{m,i}' X_{n,i}''\right)\Big|_{l_{i-1}}^{l_i}, \tag{5}$$

where ω_n is the natural frequency of vibration.

Integrals (5) are summed over the whole system. Due to boundary conditions (2) the extreme terms in this sum are equal to zero. Using conjugation condition (3) we obtain

$$\begin{aligned} \left(\omega_n^2 - \omega_m^2\right) \int_0^l X_n X_m dx = \sum_{i=1}^{p-1} E_i J_i \left(\frac{1}{m_i} - \frac{1}{m_{i+1}}\right) (X_{m,i}(l_i) X_{n,i}'''(l_i) - X_{n,i}(l_i) X_{m,i}'''(l_i) + \\ + X_{n,i}'(l_i) X_{m,i}''(l_i) - X_{m,i}'(l_i) X_{n,i}''(l_i)). \end{aligned} \tag{6}$$

It is evident from (6) that eigenfunctions orthogonality is possible only if linear masses are equal $m_i = m_{i+1}$; else they will be weighted orthogonal.

$$\rho_1(x) = \sum_{i=1}^{p} m_i (e(l_i - x) - e(l_{i-1} - x)). \tag{7}$$

Now let us study the influence of local masses on the orthogonality of the eigenfunctions for the given boundary problem. The presence of local mass M_i if $x = s_i$ leads to changing the last equation in (3)

$$E_i J_i \left(y'''(s_i + 0, t) - y'''(s_i - 0, t)\right) = M_i \ddot{y}(s_i, t). \tag{8}$$

Then condition (8) is presented as follows

$$X'''_{n,i}(s_i+0)-X'''_{n,i}(s_i-0)=-\frac{\omega_n^2 M_i}{E_i J_i}X_{n,i}(s_i)\cdot \qquad (9)$$

If integral (5) is presented as the sum of two integrals of intervals $[l_{i-1};s_i-0]$ and $[s_i+0;l_i]$, we obtain the following equation from formulas (5) and (9) (if $\omega_n \neq \omega_m$)

$$\int_{l_{i-1}}^{l_i} X_{n,i}X_{m,i}dx=-\frac{M_i a_i^2}{E_i J_i}X_{m,i}(s_i)X_{n,i}(s_i)\cdot \qquad (10)$$

Equation (10) signifies the weighted orthogonality of the eigenfunctions for the given case on the segment $[l_{i-1};l_i]$

$$\rho_{2,i}(x)=m_i+M_i\delta(x-s_i), \qquad (11)$$

where $\delta(x)$ - Dirac delta function.

The weighted orthogonality of the given boundary problem eigenfunctions is obtained by means of combining results (7) and (11)

$$\rho(x)=\sum_{i=1}^{p}(m_i+M_i\delta(x-s_i))(e(l_i-x)-e(l_{i-1}-x))\cdot \qquad (12)$$

Formula (12) for weight corresponds to the general theory of eigenfunctions [5].

The squared norm with weight is defined as

$$\int_0^l \rho(x)X_n^2(x)dx=\sum_{i=1}^{p}m_i\int_{l_{i-1}}^{l_i}X_{n,i}^2(x)dx+M_iX_{n,i}^2(s_i)\cdot \qquad (13)$$

The well known equations for calculating the squared norm of eigenfunctions [1] can be applied if we introduce wave numbers $k_{n,i}^4=\omega_n^2/a_i^2$ and pass to differentiation by $z=k_{n,i}x$. Then conjugation conditions (3) for the eigenfunctions assume the form

$$\begin{aligned} &X_{n,i}(l_i)=X_{n,i+1}(l_i);\\ &k_{n,i}X'_{n,i}(l_i)=k_{n,i+1}X'_{n,i+1}(l_i);\\ &k_{n,i}^2E_iJ_iX''_{n,i}(l_i)=k_{n,i+1}^2E_{i+1}J_{i+1}X''_{n,i+1}(l_i);\\ &k_{n,i}^3E_iJ_iX'''_{n,i}(l_i)=k_{n,i+1}^3E_{i+1}J_{i+1}X'''_{n,i+1}(l_i), \end{aligned} \qquad (14)$$

with variable *z* differentiation.

The case of the rod system ends anchorage is examined below as an example. Taking into account the boundary conditions and the expression for squared norm (14) we obtain

$$\begin{aligned} \int_0^l \rho(x)X_n^2(x)dx=&\sum_{i=1}^{p-1}\frac{3}{4}X_{n,i}(l_i)\left(\frac{m_i}{k_{n,i}}X'''_{n,i}(l_i)-\frac{m_{i+1}}{k_{n,i+1}}X'''_{n,i+1}(l_i)\right)-\frac{l_i}{2}(m_iX'_{n,i}(l_i)X'''_{n,i}(l_i)-\\ &-m_{i+1}X'_{n,i+1}(l_i)X'''_{n,i+1}(l_i))+\frac{l_i}{4}(m_iX_{n,i}^2(l_i)-m_{i+1}X_{n,i+1}^2(l_i))-\\ &-\frac{1}{4}\left(\frac{m_i}{k_{n,i}}X'_{n,i}(l_i)X''_{n,i}(l_i)-\frac{m_{i+1}}{k_{n,i+1}}X'_{n,i+1}(l_i)X''_{n,i+1}(l_i)\right)+\\ &+\frac{l_i}{4}\left(m_i\left(X''_{n,i}(l_i)\right)^2-m_{i+1}\left(X''_{n,i+1}(l_i)\right)^2\right)+\frac{lm_p}{4}\left(X''_{n,p}(l_i)\right)^2+\sum_{i=1}^{p}M_iX_{n,i}^2(s_i). \end{aligned} \qquad (15)$$

If we use conjugation conditions (14) and pass to x differentiation, formula (15) will become as follows

$$\int_0^l \rho(x)X_n^2(x)dx = \sum_{i=1}^{p-1}\frac{E_iJ_il_i}{4\omega_n^2}\left(1-\frac{E_iJ_i}{E_{i+1}J_{i+1}}\right)\left(\left(X''_{n,i}(l_i)\right)^2-2X'_{n,i}(l_i)X'''_{n,i}(l_i)\right)+ \tag{16}$$
$$+\frac{m_il_i}{4}X_{n,i}^2(l_i)\left(1-\frac{m_{i+1}}{m_i}\right)+\frac{E_pJ_pl}{4\omega_n^2}\left(X''_{n,p}(l)\right)^2+\sum_{i=1}^{p}M_iX_{n,i}^2(s_i).$$

The squared norm of the eigenfunctions is defined similarly for other types of system ends fixing. For example the penultimate term in (16) should be replaced by

$-\frac{E_pJ_pl}{2\omega_n^2}X'_{n,p}(l)X'''_{n,p}(l)$ in the case of pinning and by $\frac{1}{4}m_plX_{n,p}^2(l)$ for the loose end.

It is easy to obtain from (16) the well-known equation for the squared norm of the eigenfunctions with weight $\rho(x)=1$ [1] for the case of a homogenous rod.

In the same way it is possible to consider corresponding problems for longitudinal vibrations of a step-variable rod system with localized masses. In this case the equation of longitudinal vibrations of i^{th} part is

$$\frac{\partial^2 u_i}{\partial t^2}=a_i^2\frac{\partial^2 u_i}{\partial x^2},$$

where $u(x,t)$ are longitudinal displacements, $a_i^2=E_iF_i/m_i$, E_iF_i, m_i are longitudinal rigidity and linear mass of the system respectively.

The equation for the eigenfunctions is obtained from conjugation conditions if $x=l_i$

$$X_{n,i}(l_i)=X_{n,i+1}(l_i);$$
$$E_iF_iX'_{n,i}(l_i)=E_{i+1}F_{i+1}X'_{n,i+1}(l_i). \tag{17}$$

In this case eigenfunctions properties are the following

$$\left(\omega_n^2-\omega_m^2\right)\int_0^l X_nX_mdx=\sum_{i=1}^{p-1}E_iF_i\left(\frac{1}{m_i}-\frac{1}{m_{i+1}}\right)\left(X_{n,i}(l_i)X'_{m,i}(l_i)-X_{m,i}(l_i)X'_{n,i}(l_i)\right). \tag{18}$$

From equation (18) and conjugation conditions (17) it follows that the eigenfunctions will also be weighted orthogonal (12) if we take into account the presence of localized masses.

The squared norm of the eigenfunctions is calculated as

$$\int_0^l \rho(x)X_n^2(x)dx=\frac{1}{2}\sum_{i=1}^{p-1}\frac{E_iF_i}{\omega_n^2}X'_{n,i}(l_i)\left(l_iX'_{n,i}(l_i)\left(1-\frac{m_i}{m_{i+1}}\right)-X_{n,i}(l_i)\left(1-\frac{a_{i+1}}{a_i}\right)\right)+ \tag{19}$$
$$+m_il_iX_{n,i}^2(l_i)\left(1-\frac{m_{i+1}}{m_i}\right)+\frac{E_pF_pl}{2\omega_n^2}\left(X'_{n,p}(l)\right)^2+\sum_{i=1}^{p}M_iX_{n,i}^2(s_i).$$

Equation (19) corresponds to the case when rod system ends are fixed.

The penultimate member in (19) should be replaced by $\frac{l}{2}X_{n,p}^2(l)$ in the case of loose ends.

The mathematical model for torsional vibrations coincides completely with the model for longitudinal vibrations. So in this case equations (18) and (19) remain true if tension (compression) strength is substituted for corresponding torsional strength.

In order to study the natural vibrations of the considered systems (after the eigenfunctions had been defined) we need to use conjugation conditions (3) (for longitudinal and torsional vibrations – (17)) and boundary conditions. As a result we obtain a homogeneous system of linear algebraic equations with $4p$ $(2p)$ unknowns. By means of equating the determinant to zero we obtain an equation for defining natural vibration frequencies as it had been done in [2].

Forced vibration problems can be solved by means of Fourier method for eigenfunctions with weight (12), and the use of (16) and (19) simplifies this task.

Obtained results can be used for rough calculation of variable section rod systems dynamics; in this case the form of rod longitudinal section is approximated as a step figure. The dynamic calculation of an airplane wing can serve as an example.

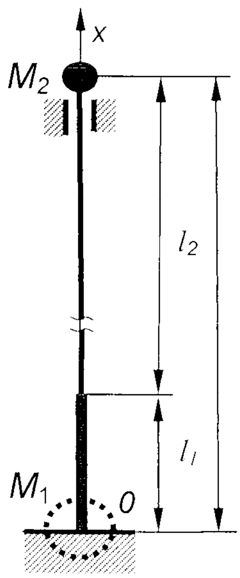

Figure 2. Calculation scheme of a drill column

To exemplify the theory let us dwell upon the problem of longitudinal impact in case of double-step drill column descending (fig. 2) [6].

The boundary (2) and initial conditions will be the following in this case

$$M_2\ddot{u}(l,t)+E_2F_2u'(l,t)=0; \quad u(0,t)=0. \tag{20}$$

$$u(x,0)=0; \quad \dot{u}(x,0)=-V_o. \tag{21}$$

Here M_2 is tackle system mass; v_o is descending velocity of the column in the coalface: $k_{ni}=\omega_n/a_i$.

The weight function and the eigenfunctions have the forms:

$$\rho(x)=\begin{cases} m_1, & 0\le x\le l_1, \\ m_2+M_2\delta(x-l), & l_1<x\le l; \end{cases}$$

$$X_{n,i}(x)=A_{n,i}\cos k_{n,i}x+B_{n,i}\sin k_{n,i}x, \quad i=1,2, \tag{22}$$

From the boundary and conjugation conditions (17) of two parts of the drill column it follows that

$$\begin{cases} A_{n,1}=0; \\ A_{n,1}\cos k_{n,1}l_1+B_{n,1}\sin k_{n,1}l_1=A_{n,2}\cos k_{n,2}l_1+B_{n,2}\sin k_{n,2}l_1; \\ E_1F_1k_{n,1}(-A_{n,1}\sin_{n,1}l_1+B_{n,1}\cos k_{n,1}l_1)=E_2F_2k_{n,2}(-A_{n,2}\sin_{n,2}l_1+B_{n,2}\cos k_{n,2}l_1); \\ -M_2\omega_n^2(A_{n,2}\cos k_{n,2}l+B_{n,2}\sin k_{n,2}l)+E_2F_2k_{n,2}(-A_{n,2}\sin_{n,2}l+B_{n,2}\cos k_{n,2}l)=0. \end{cases} \tag{23}$$

Simplifying the determinant of the homogeneous system (23) and equating it to zero, we obtain an equation for defining the eigenvalues and the natural frequencies of drill column vibration

$$\sin\lambda_n(\lambda_n\alpha\xi\cos\eta\lambda_n+\alpha^2\sin\eta\lambda_n)+\cos\lambda_n(\lambda_n\xi\sin\eta\lambda_n-\alpha\cos\eta\lambda_n)=0; \tag{24}$$

where $\lambda_n=\lambda_{n,1}=\dfrac{\lambda_{n,2}a_2l_1}{a_1l_2}$, $\eta=\dfrac{a_1l_2}{a_2l_1}$, $\alpha=\sqrt{\dfrac{E_2F_2m_2}{E_1F_1m_1}}$, $\xi=\dfrac{M_2}{m_1l_1}$.

A well-known particular case of a homogeneous column [7] follows from (24) if $\alpha=1$, $l_2=0,\ l_1=l$.

Longitudinal impact occurs in rotary drilling rigs or in spindle drills in case of block-and-tackle system slackening (it does not interact with the column) and it is possible to assume $\xi=0$ in (24), there fore

$$\alpha\sin\lambda_n\sin\eta\lambda_n-\cos\lambda_n\cos\eta\lambda_n=0 \tag{25}$$

For the particular case of a homogeneous column $(\alpha=1,\ \eta=0)$ it is possible to obtain well-known frequency equation [1].

The coefficients $B_{n,1}$, $A_{n,1}$, $B_{n,2}$ are defined from system (23). Taking into account the fact that the eigenfunctions are defined accurate to the constant, we assume $B_{n,1}=1$ and then the following equation can be defined from system (23)

$$\begin{cases} A_{n,2}\cos k_{n,2}l_1+B_{n,2}\sin k_{n,2}l_1=\sin k_{n,1}l_1; \\ -A_{n,2}\alpha\sin k_{n,2_{n,2}}l_1+B_{n,2}\alpha\cos k_{n,2}l_1=\cos k_{n,1}l_1. \end{cases} \tag{26}$$

The eigenfunctions are obtained from system (26) solution

$$X_n(x)=\begin{cases} \sin k_{n,1}x; & 0\le x\le l_1; \\ A_{n,2}\cos k_{n,2}x+B_{n,2}\sin k_{n,2}x, & l_1<x\le l, \end{cases}$$

where:

$$A_{n,2}=\sin\lambda_n\cos\frac{\lambda_n a_1}{a_2}-\frac{1}{\alpha}\sin\frac{\lambda_n a_1}{a_2}\cos\lambda_n;$$

$$B_{n,2}=\frac{1}{\alpha}\cos\frac{\lambda_n a_1}{a_2}\cos\lambda_n+\sin\lambda_n\sin\frac{\lambda_n a_1}{a_2},$$

and proper numbers λ_n are defined from the system of equations (25).

Taking into account the first initial condition (21) we present longitudinal displacements expanded into eigenfunctions

$$u(x,t)=\sum_{n=1}^{\infty}C_nX_n(x)\sin\omega_nt. \tag{27}$$

Then the second initial condition (21) should be satisfied. The following equation is defined by Fourier method with weight according to scheme

$$C_n\Delta_n^2\omega_n = -v_o\left(m_1\int_0^{l_1}\sin k_{n,1}x dx + m_2\int_{l_1}^{l}\left(A_{n,2}\cos k_{n,2}x + B_{n,2}\sin k_{n,2}x\right)dx\right), \qquad (28)$$

where

$$\Delta_n^2 = \frac{m_1 l_1}{2}\left(\cos^2\lambda_n\left(1-\frac{E_1F_1}{E_2F_2}\right)+\sin^2\lambda_n\left(1-\frac{m_2}{m_1}\right)\right)+\frac{m_2 l}{2}\left(A_{n,2}\cos\left(\frac{\lambda_n a_1 l}{a_2 l_1}\right)+B_{n,2}\sin\left(\frac{\lambda_n a_1 l}{a_2 l_1}\right)\right)^2$$

is the squared norm of the eigenfunctions calculated by formula (19).

The coefficients in (27) are calculated from (28)

$$C_n = -\frac{V_o l_1^2}{\Delta_n^2 a_1\lambda_n^2}\left(m_1(1-\cos\lambda_n)+\frac{m_2 a_2}{a_1}\left(\begin{array}{l} A_{n,2}\left(\sin\left(\frac{\lambda_n a_1 l}{a_2 l}\right)-\sin\left(\frac{\lambda_n a_1}{a_2}\right)\right)- \\ -B_{n,2}\left(\cos\left(\frac{\lambda_n a_1 l}{a_2 l_1}\right)-\cos\left(\frac{\lambda_n a_1}{a_2}\right)\right)\end{array}\right)\right).$$

Then the stresses in the column are calculated from formula

$$\sigma(x,t) = \begin{cases}\sum_{n=1}^{\infty} E_1 C_n k_{n,1}\cos k_{n,1}x\sin\omega_n t, & 0\le x\le l_1; \\ \sum_{n=1}^{\infty} E_2 C_n k_{n,2}(-A_{n,2}\sin k_{n,2}x + B_{n,2}\cos k_{n,2}x)\sin\omega_n t, & l_1 < x \le l.\end{cases} \qquad (29)$$

We should take into account that calculation stresses consist of two components: stresses from motion velocity variation $\sigma^{(1)}$ - (29) and stresses from sudden application of column weight - $\sigma^{(2)}$. Maximum value of the second component of stresses does not exceed doubled value of static stresses [1] and their calculation is not much complicated.

The variation of maximum non-dimensional stresses $\bar{\sigma}_1 = \frac{\sigma_{1,\max} a_1}{E_1 v_o}$ (if $x = 0$) at drilling depth 200 m is presented on Fig. 3.

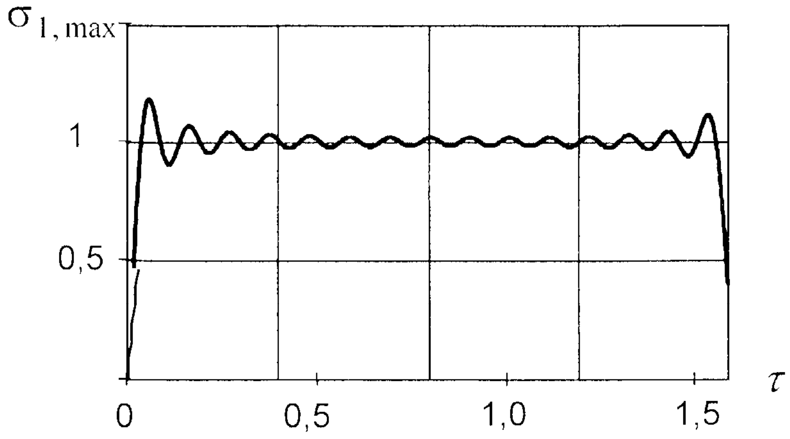

Figure 3. Variation of maximum nondimensional stresses $\bar{\sigma}_{\max}^{(1)}$ at drilling depth 200 m

These stresses appear at the joint of drill column and cutter. The parameters of the spindle drill column were used for their calculation.

It is evident from Fig. 3 that the behavior of column stresses corresponds to impact processes nature.

The variation of total maximum dynamic stresses $\sigma_{max} = \sigma_{1,max} + \sigma_{2,max}$ at the velocity of column descending $v_0 = 2\ m/s$ and drilling depth 200 m is presented in Fig. 4.

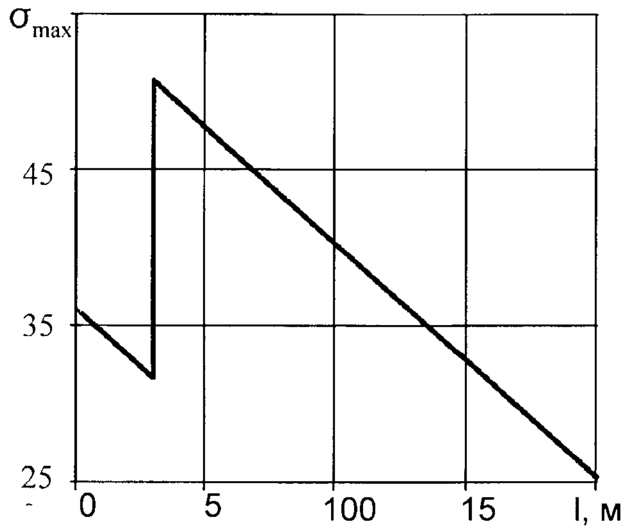

Figure 4. Distribution of the maximum stresses

So the use of the above dependences helps to investigate the stressed-deformed state of double-step drill columns under impact loads and to define permissible operation conditions for drilling rigs.

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